EVALUATING STRATEGIC DECISION-MAKING WITH ITERATIVE VOTING

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Submitted in Partial Fulfillment of the Requirements
for the Degree of

DOCTOR OF PHILOSOPHY

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[August 2024]

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ACKNOWLEDGEMENTS

First and foremost, I would like to thank my esteemed advisor, Lirong Xia. Professor Xia has been exceptionally supportive and confident of me, and he has fostered a positive environment for me to grow as a researcher. He provided countless opportunities for me to form research collaborations, network at conferences and workshops, TA in his courses, mentor an undergraduate student, and present my own research. Professor Xia strategically nudged me in the right direction more than a handful of times. Moreover, Professor Xia taught me how to critically analyze social choice theory through a computational lens. I look forward to carrying forward his academic legacy.

I am incredibly grateful for and thankful to my family: Shelly Loomus, Lon Zaback, Alex Kavner, Kyle Zaback, Rachel Sellars, Jerry Loomus, the late Betsy Loomus, and Charlotte Zaback. There is as much personal growth as professional development through completing a Ph.D. I would not have been able to achieve this without their endearing encouragement and support. The skills of leadership, drive, professionalism, perseverance, communication, writing, artistry, humbleness, conscientiousness, and kindness that my family taught me and exemplified have proven invaluable to my success.

I am thankful to my many committee members, co-authors, and colleagues for fruitful discussions and lessons about working through research problems from different perspectives. Elliot Anshelevich and Malik Magdon-Ismail taught me much through their courses on approximation algorithms and computational finance. Working with Francesca Rossi was especially fun on our convergence of multi-issue iterative voting project and during my IBM internship. It has been a pleasure collaborating with Qishen Han, Inwon Kang, Samarth Khanna, Ao Liu, Reshef Meir, Farhad Mohsin, Sikai Ruan, Sujoy Sikdar, Rohit Vaish, and Tomasz Wąs during the last four years. Hadi Hosseini, especially, has been an exemplary mentor and expanded my horizons as a researcher.

I am grateful to my undergraduate mentors and those who provided me with professional experiences, setting me up for success at RPI. This includes Grant Schoenebeck, Brian Lin, Patrick Grim, and Michael Tiller. My interest in studying computational social choice originated from a lecture in evolutionary game theory by Carl Simon and the late Charles Doering. I was especially inspired by my professors Elizabeth Bruch, Abigail Jacobs, Mark Newman, and Scott Page from the University of Michigan's Center for the Study of Complex

Systems.

I would be amiss if I didn't recognize the substantial role my communities and friends have played in inspiring and supporting me. This includes the RPI's computer science department with guidance from Tracy Hoffman and Shannon Carrothers, the computer science and RPI graduate councils, RPI Hillel, and the RPI ballroom team and it's amazing instructor Jennifer Volpe. To the Lindy Hopcats, I thank Mona Xue and Genny Kunkel. I am also grateful to Swing Albany, Hudson Valley Swing Dance, and Swing Ann Arbor. To my many friends, including Clare Arrington, Mackensie Freeman, Elizabeth Ho, Jaren Hubal, Deniz Koyuncu, Beth Legator, Anne Lin, Andrew Marogi, Alexia Nelson, James Oswald, Michele Ripka, Brandon Rozeck, Max Rotenberg, and Avery Schonberger. And finally, I am thankful for friends who inspired me to pursue a Ph.D. while I was an undergraduate, including Kevin Lieberman, Elena Luchina, Cecil McGhee, and Ethan Sprague.

ABSTRACT

Social choice theory is prolific with paradoxes and impossibilities that prevent cogent justifications for decisions made by groups of people. For example, Gibbard and Satterthwaite proved that no reasonable voting procedure exists that is non-dictatorial and immune to agents misrepresenting their preferences. Significant work has sought to overcome this impossibility by either restricting the domain of agents' preferences or dissuading this form of strategic behavior through computational hardness. The recent approach of iterative voting (IV), rather, aims to characterize the complex interactions ensuing from agents reporting their preferences strategically. In particular, agents may update their votes, given information about other agents' reports, prior to finalizing the group decision. Prior work has documented properties about IV equilibrium and conditions for convergence according to various social choice rules, information agents have access to, and agents' behavioral schemes. Still, only preliminary work has studied the effect IV has on social welfare of equilibrium outcomes relative to the truthful vote.

This thesis advances our understanding of strategic behavior in social choice, via IV, on two fronts. First, we study the effect iterative plurality has on the social welfare of the chosen outcome with respect to the worst-case preference profile and as agents have arbitrary rank-based utility. To overcome a poor worst-case result, we study expected performance when agents' preferences are independent and identically distributed according to the impartial culture. Our finding surprises us in that IV helps agents choose higher quality alternatives on average, regardless of the order of their strategic manipulations. We go on to characterize certain classes of preference distributions for which IV improves or degrades social welfare, thus helping to explain why prior experiments attained varying results. Second, we generalize iterative plurality to multiple issues while agents have uncertainty about each alternative's score. In this setting, we identify sufficient conditions for convergence, including $\mathcal{O}$-legal preferences and a novel model about what information agents have access to. Our study through both fronts characterizes agents' behavior given the opportunity to deliberate their votes. Our results provide insight for mechanism designers choosing whether or not to encourage such deliberation, and call for further study of non-incentive compatible mechanisms.

CHAPTER 1
INTRODUCTION

1.1 Background

What is the proper process for a group of people to make a decision out of several possibilities? In other words, what cogent and rational basis is there for societies to choose one alternative over another (Sen, 1999)? This may be a group of friends deciding which restaurant to enjoy for lunch, a company outlining its budget for the next fiscal year, or a nation determining its next president. If all n people (agents) have the same preferences $u \succ v$ over two alternatives u and v, then clearly the proper decision is to take the *unanimous* choice u over v. [1] However, what is proper if only $n-1$ agents prefer $u \succ v$ and one agent wants $v \succ u$? This may be the case, for example in The Hitchhiker's Guide to the Galaxy, where a community overwhelmingly wants to bulldoze someone's house to make space for a freeway (Adams, 1995). How do we gauge the extent to which, as Spock from Star Trek proclaims, the "needs of the many outweigh the needs of the few, or the one" (Meyer, N. (Director), 1982)?

Sen (1999) chalks this problem up to a lack of basis for comparing welfare across persons: "Every mind is inscrutable to every other mind and no common denominator of feelings is possible" (Robbins, 1938). This means that we cannot directly compare losses and gains in welfare between two parties because welfare is subjective and privately observed. In addition to monetary loss, the homeowner's costs for bulldozing their house includes sentimental value, opportunity cost for enjoying where they lived, and potential psychological harm. Making these comparisons requires additional assumptions about a common social welfare metric.

Portions of this chapter have previously appeared as:

Kavner, J., & Xia, L. (2021). Strategic behavior is bliss: iterative voting improves social welfare. In *Advances in neural information processing systems* (Vol. 34, pp. 19021–19032). Curran Associates, Inc.

Kavner, J., Meir, R., Rossi, F., & Xia, L. (2023, August). Convergence of multi-issue iterative voting under uncertainty. In *Proceedings of the 32nd international joint conference on artificial intelligence* (pp. 2783-2791). ©2023 IJCAI.

Kavner, J., & Xia, L. (2024). *Average-case analysis of iterative voting.* arXiv. https://arxiv.org/abs/2402.08144.

[1] Of course, v could be chosen arbitrarily or randomly via a *sortition* method (Headlam, 1891). This method, randomly choosing an alternative amongst all possible, is susceptible to *cloning* by interest groups who stack the deck with favorable alternatives (Tideman, 1987). As many decision problems should consider peoples' opinions, we study decision procedures that depend on at least one agent's preferences.

One such common measure is Bentham (1789)'s utilitarianism, by which agents derive measurable utility out of each social outcome and the aggregate social welfare is the sum total of individual utilities. The utilitarian thus appraises all non-monetary facets of each alternative before choosing the alternative that maximizes the utilitarian sum. They would argue in favor of building the freeway if the homeowner's loss doesn't offset the gain others procure from it. While utilitarianism forms a basis for modern welfare economics (Kahneman, Wakker, & Sarin, 1997; Mas-Colell, Whinston, & Green, 1995), there is too much critique for it to be the end-all-be-all of proper group decision-making. For example, Rawls (1971) advocated that the *egalitarian rule*, which selects the alternative that maximizes the minimum utility offered to any individual, is more fair. Both the utilitarian and egalitarian decision rules follow by *distributive justice* arguments, as the rules choose alternatives based on the properties of the alternatives' respective utility distributions (Rawls, 1971). Still, these rules only work in limited circumstances for which individual utilities are measurable and can be compared across persons.

In lieu of these strict assumptions, Arrow (1951) advanced *social choice theory* to study the admissible properties of social choice functions using only agents' ordinal preferences, rather than their cardinal utilities. This frames proper decision-making in terms of the properties (i.e., *axioms*) of the choice rule $f : \mathcal{L}(\mathcal{A})^n \to \mathcal{L}(\mathcal{A})$ for $n \in \mathbb{N}$ agents, discrete sets of alternatives $\mathcal{A}$, and permutation group $\mathcal{L}$ for preferences. [2] For example, we may desire that there is no *dictator* in the decision-making process. This property entails that no agent j completely determines the social choice outcome: $u \succ_j v \implies u \succ_f v$, where $\succ_j$ denotes j's preferences and $\succ_f$ denotes the social choice ordering. Axiomatic social choice thus follows a *procedural justice* approach by emphasizing the fair process by which decisions are made (Rawls, 1971). Unfortunately, Arrow (1951) proved that any social choice rule f that satisfies *unrestricted domain*, *Pareto efficiency*, and *independent of irrelevant alternatives* among at least three alternatives must be a dictatorship. [3] Hence, there is no proper way for groups to make decisions as defined by these properties.

[2] Consider $\mathcal{A} = \{u, v, w\}$. Then $\mathcal{L}(\mathcal{A}) = \{(u \succ v \succ w), (u \succ w \succ v), (v \succ u \succ w), (v \succ w \succ u), (w \succ u \succ v), (w \succ v \succ u)\}$. Technical preliminaries about social choice theory are cataloged in Chapter 2.

[3] *Unrestricted domain* entails that all preferences for agents in $\mathcal{L}(\mathcal{A})$ are allowed. *Pareto efficiency* states that if every agent j prefers $u \succ_i v$, then so will the social choice f. *Independent of irrelevant alternatives* states that the relative ranking of u and v by f only depends on the relative rankings of u and v provided by the agents, rather than how agents rank some third alternative w (Brandt, Conitzer, Endriss, Lang, & Procaccia, 2016).

It turns out that social choice theory is prolific in axiomatic paradoxes. For instance, May (1952) proved that *majority voting* over two alternatives is the unique rule satisfying *anonymity*, *neutrality*, and *positive responsiveness*. [4] This holds for the *irresolute* version of majority vote which selects the alternative that at least half of the agents prefer and may yield ties. On the other hand, if we also impose *resoluteness* so that f yields a single outcome, then there is no social choice rule that satisfies anonymity and neutrality (Moulin, 1983). Moreover, majority rule fails to extend to decision problems with more than two alternatives, as demonstrated by *Condorcet's paradox* (Condorcet, 1785). Consider three agents deciding which single ice cream flavor to buy, for example, where the first agent prefers $\textit{chocolate} \succ_1 \textit{vanilla} \succ_1 \textit{strawberry}$, the second agent prefers $\textit{vanilla} \succ_2 \textit{strawberry} \succ_2 \textit{chocolate}$, and the third agent prefers $\textit{strawberry} \succ_3 \textit{chocolate} \succ_3 \textit{vanilla}$. Hence, each alternative in $\mathcal{A} = \{\textit{chocolate}, \textit{vanilla}, \textit{strawberry}\}$ is preferred to the next in $\mathcal{A}$ by a majority of the agents. This demonstrates the majority cycle $\textit{chocolate} \succ_f \textit{vanilla} \succ_f \textit{strawberry} \succ_f \textit{chocolate}$ by the pair-wise majority rule f even though agents' preferences are transitive.

These paradoxes imply that implementing a social choice rule will necessarily involve trading-off desirable criteria. It is therefore up to the group to determine which axioms they value and would like to satisfy, and hence which social choice rule to use, when aggregating their preferences. Sen (1999) discussed some of the implications of these trade-offs in welfare economics. Separately, Brandt et al. (2016) and Endriss (2017) summarize which axioms are satisfied for many common social choice rules. [5]

One important factor groups must contend with when choosing their decision rule is the fact that voting is a game. Arrow (1951) and subsequent research characterized properties of choice rules with respect to the preferences that agents report (i.e., their votes). Agents' votes do not need to be truthful. Rather, agents can report their preferences however they want based on how they believe other agents will vote. For example, voting insincerely is typically seen in resolute elections where supporters of minor political parties tend to split off their votes in favor of one of two leading parties. This phenomenon is known by political scientists as *Duverger's law* (Duverger, 1964; Riker, 1982). As a technical example, consider the agents in our ice cream example voting via the *plurality* rule subject to lexicographical

[4] A decision rule f is *anonymous* if it is invariant under permutations of the agents (Endriss, 2017). It is *neutral* if permuting alternatives in each agent's preference ranking leads to permutations in f accordingly. It satisfies *positive responsiveness* if increasing an alternative's support cannot make it worse-off in f.

[5] A useful chart comparing axioms satisfied by common social choice rules may be found at https://en.wikipedia.org/wiki/Comparison_of_electoral_systems.

tie-breaking. Each agent contributes one point to the alternative it reportedly favors most and *chocolate* wins as the tie-breaker. Notice that the second agent can switch their vote from truthful vote of *vanilla* to *strawberry* and thereby update the winner to *strawberry* in their favor. Hence, plurality is not *strategyproof*: some agent can gain by voting insincerely according to their preferences.

There are several reasons why groups might want to implement strategyproof voting rules, as discussed by Dowding and Hees (2008) and Conitzer and Walsh (2016). First, strategic voting replaces information about agents' truthful preferences with a potentially arbitrary substitute. This substitute may be unrepresentative about what agents actually desire and could be arbitrarily worse – for example, if all agents prefer $u \succ v$ yet all are voting $v \succ u$. Second, strategic voting could advantage some agents that have more computational power, better access to information, and manipulate their votes first, over other agents without these benefits. This reduces fairness by, in effect, eliminating *anonymity* of the decision-making procedure. Furthermore, the effort that agents spend discovering beneficial ways to strategically update their vote may be wasted and better spent on other productive tasks. For these reasons it would be ideal for social choice rules to incentivize agents to report their preferences truthfully. Unfortunately, the celebrated *Gibbard-Satterthwaite theorem* proves that any non-dictatorial social choice rule with at least three alternatives is susceptible to this type of strategic voting (Gibbard, 1973; Satterthwaite, 1975).

From the Gibbard-Satterthwaite theorem it would seem that groups cannot make decisions that are representative of all agents' truthful preferences using any reasonable social choice rule. Fortunately, a number of approaches have been proposed by social choice theorists and, recently, computer scientists, to overcome this limitation. Traditional approaches include relaxing certain assumptions of the theorem, for example by restricting the domain of agents' preferences, using randomized voting rules, and using irresolute voting rules that may not select a unique winner (Conitzer & Walsh, 2016). Bartholdi, Tovey, and Trick (1989), rather, proposed that high computational complexity may be a sufficient deterrent against agents seeking to strategically report their votes. That is, if finding a vote to improve the outcome above the agent's truthful vote is computationally hard, then agents won't bother. Human agents will find this task too tedious and computational agents, voting as a proxy on behalf of humans, won't spend the effort. Research into the computational complexity of voting manipulation was taken up early on by Bartholdi III and Orlin (1991), Conitzer and

Sandholm (2002), and Conitzer and Sandholm (2003); see discussions from Faliszewski and Procaccia (2010), Faliszewski, Hemaspaandra, and Hemaspaandra (2010), and Conitzer and Walsh (2016) about this topic.

Rather than deter strategic behavior, as in this prior work, Meir, Polukarov, Rosenschein, and Jennings (2010) sought to study the complex interactions induced by agents misreporting their preferences. The authors formulated *iterative voting (IV)* as a game where agents have the opportunity to change their votes prior to finalizing the collective decision. In their model, agents update their votes myopically given information about other agents' votes, which might come about via repeated opinion polls (Reijngoud & Endriss, 2012) or while voting through an online platform (Zou, Meir, & Parkes, 2015). However, agents are not assumed to be rational nor have knowledge about others' private preferences (Aumann, 1995). Meir et al. (2010)'s aim was to determine whether, how fast, and on what alternatives the agents will agree. They found that IV converges from any starting vote profile when agents update their votes one-at-a-time and to the direct best response of all other votes. Hence, strategic behavior can be accounted for in social choice by the equilibrium outcomes found via IV negotiation.

IV advances social choice theory in two respects. First, it naturally describes how agent strategic behavior unfolds over time and incorporates information about agents' truthful preferences as agents update their votes. Whereas social choice theory historically only considered the axiomatic properties of choice rules, IV suggests how agents might actually interact with the mechanism. This was measured empirically via human subject experiments by Zou et al. (2015), Tal, Meir, and Gal (2015), and Meir, Gal, and Tal (2020). Second, Meir et al. (2010) posed IV as a social choice rule in and of itself. Although strategic behavior is inevitable by the Gibbard-Satterthwaite theorem, groups have a choice between regular social choice rules and their iterative counterpart. They can design decision-making procedures as either single-shot or sequential, in which agents have opportunities to repeatedly update their votes given up-to-date vote information. For both of these aspects, understanding how iterated strategic behavior affects social choice outcomes may help groups determine which choice rule to implement. It may also provide some explanatory power for why real-world electoral outcomes occur the way they do.

Following Meir et al. (2010), a series of work has investigated IV convergence under various social choice rules and relaxed assumptions about agents' behavioral and information schemes (Endriss, Obraztsova, Polukarov, & Rosenschein, 2016; Koolyk, Strangway, Lev, & Rosenschein, 2017; Lev & Rosenschein, 2012; Meir, 2015; Meir, Lev, & Rosenschein, 2014; Obraztsova, Markakis, Polukarov, Rabinovich, & Jennings, 2015; Rabinovich, Obraztsova, Lev, Markakis, & Rosenschein, 2015; Reyhani & Wilson, 2012; Tsang & Larson, 2016). Still, the effect strategic behavior has on the quality of chosen outcomes, in terms of utilitarian social welfare, has been largely overlooked. A notable exception is Brânzei, Caragiannis, Morgenstern, and Procaccia (2013), who took the first step to study the quality of IV-induced outcomes. They sought to answer how bad the resulting outcome could be given that strategic voting is inevitable. Brânzei et al. (2013) therefore defined the *additive dynamic price of anarchy* (ADPOA) as the difference in social welfare between the truthful vote profile and the worst-case equilibrium that is reachable via IV. This notion is with respect to the worst-case preference profile and a given voting rule, and refines the well-known *price of anarchy* (Roughgarden & Tardos, 2002) for a dynamical setting with myopic agents. They found the performance is "very good" for the plurality voting rule (with an ADPoA of 1), "not bad" for veto (with a DPoA of $\Omega(m)$ with m alternatives, $m \geq 4$), and "very bad" for Borda (with a DPoA of $\Omega(n)$ with n agents). Nevertheless, it is unclear whether these observations holds for other notions of social welfare.

1.2 Thesis Contributions and Outline

We advance the study of IV in this thesis on two fronts: understanding the effect IV has on the social welfare of the chosen outcome (Chapters 3 and 5), and discovering sufficient conditions for IV convergence in a generalized multi-issue setting under uncertainty (Chapter 4). Our preliminaries precede three technical chapters. In Chapter 2, we define and characterize our models of social choice and iterative voting. As we prove Theorems 3.1, 3.2, and 5.1 for large populations of agents n, we provide definitions that characterize the asymptotic growth of real-valued functions, such as the canonical Big-$\mathcal{O}$ notation. These theorems also significantly apply Xia (2021a)'s research about the *smoothed likelihood of ties.* We therefore also include Xia's preliminaries and main results for completeness.

In Chapter 3, we begin our study of strategic behavior's effect on electoral outcome quality. Our results naturally extend those of Brânzei et al. (2013) by differentiating the rank-based utility vector $\vec{u}$ from the iterative positional scoring rule $f_{\vec{s}}$. Our first main result (Theorem 3.1) states that, unfortunately, for any fixed $m \geq 3$ and utility vector $\vec{u}$, the ADPoA of iterative plurality is $\Theta(n)$ for n agents. Therefore, the minimum loss result attained for iterative plurality by Brânzei et al. (2013) is not upheld if $\vec{u}$ differs from plurality utility. To overcome this negative worst-case result, we introduce the notion of *expected additive dynamic price of anarchy* (EADPoA), which presumes agents' truthful preferences to be generated from a probability distribution. Our second main result (Theorem 3.2) is positive and surprises us: for any fixed $m \geq 3$ and utility vector $\vec{u}$, the EADPoA is $-\Omega(1)$ when agents' preferences are i.i.d. uniformly at random, known as *Impartial Culture (IC)* distribution. In particular, our result suggests that *strategic behavior is bliss* because iterative voting helps agents choose an alternative with higher expected social welfare, regardless of the order of agents' strategic manipulations.

In Chapter 4, we next explore how convergence of iterative plurality over a single issue, as found by Meir et al. (2010), Meir et al. (2014), and Meir (2015), extends to multiple referenda as agents have limited access to information. We find that for binary issues, the existence of cycles hinges on the interdependence of issues in agents' preference rankings. Specifically, once an agent j takes an improvement step on an issue i, they only subsequently revert their vote if their preference for i changes. This occurs in the event that the set of possible winning alternatives, among other issues that affect j's preference for i, changes. Agents don't have this interdependence if their preferences are $\mathcal{O}$-legal – i.e., if preferences for each issue is independent of later issues in an order $\mathcal{O}$, conditioned on the outcomes of earlier issues in the order. Agent preferences over individual issues then change only finite times, so IV converges (Theorem 4.1).

We also find that as uncertainty increases over issues other than the one agents are changing, fewer preference rankings admit IV improvement steps, eliminating cycles (Theorem 4.2). This result assumes agents have *alternating uncertainty* – i.e., agents may gather more information about the issue they're changing their vote over, thus reducing their uncertainty about that issue, prior to making the change. Finally, convergence does not extend to multi-alternative issues since IV may cycle if agents only have partial order preference information (Corollary 4.1).

Theorem 3.2 takes the first step at understanding IV beyond the worst-case analysis and toward more realistic preference distributions. Still, IC has significant limitations and is widely understood to be implausible (Regenwetter, 2006; Tsetlin, Regenwetter, & Grofman, 2003; Van Deemen, 2014). In Chapter 5, we therefore take the next step in understanding IV's effect on welfare by extending the EADPoA to a wider class of agents' preference distributions. Theorem 5.1 demonstrates a threshold for which IV improves or degrades expected welfare over the truthful vote. We contribute several novel binomial and multinomial lemmas that may be useful for future study of IV and apply Xia (2021a)'s theorems to expectations of random functions, rather than the likelihood of events. Furthermore, we continue Chapter 3's representation of agents' preferences as a Bayesian network to gain further insight in behavioral social choice.

We conclude in Chapter 6 by summarizing our main results and suggesting a handful of avenues of future research. These avenues include both theoretical directions, such as studying the performance of IV beyond the distributions covered in Chapter 5, and empirical directions, such as evaluating the extent to which artificial intelligence-powered recommendations can assist groups to make higher quality decisions (Xia, 2017). We further contextualize our work within the perspective of smoothed analysis (Xia, 2021a).

1.2.1 Cognition and Ethics in Social Choice

Since our work follows from social choice theory (Arrow, 1951; Brandt et al., 2016), it is worth noting a couple of assumptions often made implicitly by theorists. First, social choice theory is about aggregating preferences from a set of agents about a set of alternatives (i.e., the study of functions $f : \mathcal{L}(\mathcal{A})^n \to \mathcal{L}(\mathcal{A})$ for $n \in \mathbb{N}$ agents, discrete sets of alternatives $\mathcal{A}$, and permutation group $\mathcal{L}$ for preferences). It is a necessary precondition that agents know and can report their own preferences and, for this thesis, that preferences are strict, complete, and ordinal. In particular, our results depend on agents knowing their most-preferred (i.e., truthful) alternative and whether they prefer $u \succ v$ or $v \succ u$ for any two alternatives $u, v \in \mathcal{A}$. This may be difficult for real-world human agents, who have biases and are susceptible to *framing effects* that inform peoples' preferences (Plous, 1993; Tversky & Kahneman, 1981), and computational agents if there are continuous or an exponential number of alternatives. In this case, determining one's favored alternative could be an NP-hard problem in itself. How people actually form their preferences is a problem left for the cognitive scientists, anthropologists, and marketing firms (Wildavsky, 1987). Nevertheless, we assume that agents can answer questions about their preferences with ease. We further assume that preferences are fixed throughout the voting process. We do not model agents that work to convince others of their point of view or modify others' preferences.

Second, as previously discussed, group decision-making is trivial when agents have unanimous preferences. In all other cases, any decision will advantage certain agents over others. This begs the question: by what right does an individual have to impose their will on another? In the abstract we take this right for granted as a consequence of participating in a society with limited resources. Trade-offs must be made, although such trade-offs may be evaluated and optimized. In practice, one consequence of voting is the *tyranny of the majority*, by which a majority uses their power enabled by the voting mechanism to oppress minority groups. This problem was addressed by early democratic philosophers such as Rousseau (1762), Tocqueville and Reeve (1835), and Mill (1859), who wrote cognizantly about majorities abusing their power as a king, dictator, or tyrant might.

One way of limiting powers of the majority is to impose certain properties on the social choice function is itself, such as requiring a supermajority (i.e., some percentage greater than 50% of voters) to pass a law. Rousseau (1762, Book 4, Chapter 2) writes in favor of this practice: "The more serious and important the question that is being put to the vote, the nearer to unanimity the threshold should be set." Another approach is to separate powers endowed to governing bodies or to elevate certain rights of individuals as *inalienable*. These rights cannot be nullified by vote of a legislative body; hence, certain alternatives cannot be voted upon.

Both of these design choices concern the practical implementation of alternatives that agents in our social choice model vote for. Our focus in this thesis is on the mathematical function of aggregating agents' preferences in order to make a decision. Therefore aspects about the alternatives and agents that could affect the ethics of social choice, beyond their mathematical abstraction, is beyond the scope of this work. For example, it would be just to assume that those agents taking part in a decision are exactly the relevant stake-holders. Such facets should be taken into consideration when implementing any particular voting procedure.

1.3 Related Work

1.3.1 Iterative Voting

The present study of iterative voting (IV) was initiated by Meir et al. (2010) who identified that iterative plurality converges when agents apply best response updates in sequence. However, guaranteeing convergence appeared quite sensitive to its assumptions, as the authors found several counter-examples when allowing agents to manipulate simultaneously, using better- instead of best-replies, or weighing agents' votes unequally. This inspired a line of research on sufficient conditions for convergence. For example, Lev and Rosenschein (2012) and Reyhani and Wilson (2012) simultaneously found that iterative veto converges while no other positional scoring rule does. Gourves, Lesca, and Wilczynski (2016) and Koolyk et al. (2017) demonstrated similar negative results for other common voting rules, such as Maximin, Copeland, Bucklin, and STV. In lieu of these negative results, Grandi, Loreggia, Rossi, Venable, and Walsh (2013), Obraztsova et al. (2015), and Rabinovich et al. (2015) proved IV's convergence upon imposing stricter assumptions on agent behavior, such as truth-bias (Dutta & Sen, 2012; Obraztsova, Markakis, & Thompson, 2013; Thompson, Lev, Leyton-Brown, & Rosenschein, 2013) and voting with abstentions (Börgers, 2004; Desmedt & Elkind, 2010; Elkind, Markakis, Obraztsova, & Skowron, 2015).

Reijngoud and Endriss (2012) and Endriss et al. (2016) took a different approach by relaxing assumptions about what information agents have access to. Instead of being certain about alternatives scores, agents only have access to noisy or incomplete *poll information* that might arise from imprecise opinion polls or latency in an online voting system, such as Doodle (Zou et al., 2015). Agents then make *local dominance improvement (LDI)* steps that may myopically improve the outcome but cannot degrade the outcome, given their information (Conitzer, Walsh, & Xia, 2011). Endriss and colleagues provided some IV convergence results according to different voting rules and information functions. Meir et al. (2014) and Meir (2015) extended LDI dynamics to characterize iterative plurality, finding broad conditions for convergence to be guaranteed. The latter work studied a nonatomic model variation where agents have negligible impact on the outcome but multiple agents update their votes simultaneously, greatly simplifying the dynamics. We further extend LDI dynamics to multi-issue IV in Chapter 4 and discuss a nonatomic variant of our model in Section 4.5. Relatedly, Sina, Hazon, Hassidim, and Kraus (2015) and Tsang and Larson (2016) studied IV with agents embedded in social networks and determining their improvement steps only given their neighbors' votes. The authors demonstrate how network structure affects the quality of strategic outcomes.

While most IV research focuses on convergence and equilibrium properties, Brânzei et al. (2013) quantified the worst-case quality of IV via a refinement of the well-known *price of anarchy (PoA)* notion for several positional scoring rules (Koutsoupias & Papadimitriou, 2009). Separately, Elkind et al. (2015) studied the PoA of (non-IV) plurality with agents that are truth-biased or may abstain from voting. This theoretical research is an important first step in understanding the effect of strategic behavior on social choice welfare. Still, it needs refinement, as several synthetic and human subjects experiments have proved inconclusive about IV's effects on welfare (Bowman, Hodge, & Yu, 2014; Grandi, Lang, Ozkes, & Airiau, 2022; Grandi et al., 2013; Koolyk et al., 2017; Meir et al., 2020; Reijngoud & Endriss, 2012; Tal et al., 2015; Thompson et al., 2013; Tsang & Larson, 2016). Moreover, Brânzei et al. (2013)'s results notably depend on the score vector being the same as agents' rank-based utility vector. We relax this assumption and introduce the average-case analysis of IV in Chapter 3. In Chapter 5, we further distinguish certain classes of distributions of agent preferences that improve or degrade average welfare.

1.3.2 Sequential and Game-Theoretic Voting

Sequential voting incorporates mechanisms by which agents don't submit their votes all at the same time. Agents make multiple decisions about providing their preference information to the voting mechanism, in sequence, and may reason strategically about future world states at each decision point. This contrasts the myopic behavior assumed in the IV literature. For example, consider *multi-issue voting* where agents must decide on a number of independent issues with several alternatives for each issue. This is an extensively studied problem in economics and computer science with applications in direct democracy referendums, group planning and committee elections (see e.g., Lang and Xia (2016) for a survey). While a standard voting procedure could be used to decide on the issues simultaneously, the number of distinct alternatives is exponential (e.g., there are 2^p alternatives for p binary issues) and eliciting agents' preferences may be an infeasible task. One simpler mechanism elicits agents' preferences sequentially according to an order $\mathcal{O} = \{o_1, \ldots, o_p\}$ such that outcomes of each issue o_i are revealed to agents prior to voting on the next issue o_{i+1} (Lacy &

Niou, 2000). Still, this mechanism is not without its own problems. For example, an agent's preference over the alternatives in issue o_i may be dependent on the outcome of a later issue in $\mathcal{O}$. This agent may have trouble reporting their preferences in sequential voting settings. In Chapter 4, we study IV in multi-issue contexts and compare it to this sequential mechanism. Our work follows research in multi-issue strategic behavior by Lang (2007), Lang and Xia (2009), Conitzer, Lang, and Xia (2009), and Xia, Conitzer, and Lang (2011).

Another form of sequential voting considers agents who appear one-at-a-time and submit their votes in the order of their appearance. For example, Desmedt and Elkind (2010) characterized properties of the subgame perfect Nash equilibrium when agents may abstain from voting. Xia and Conitzer (2010) independently demonstrated that the quality of the backward-induction winner may be highly unfavorable in the worst case. A third type of sequential voting was proposed by Airiau and Endriss (2009), who considered agents that repeatedly voted on whether to update a common world state or not. The authors analyzed what parameters of their model enable or accelerate convergence. These works were discussed further by Meir (2018).

Game-theoretic analyses of social choice when agents have imperfect information about alternative scores was popularized by Myerson and Weber (1993). In this imperfect information setting, Chopra, Pacuit, and Parikh (2004), Conitzer et al. (2011), Reijngoud and Endriss (2012), and Van Ditmarsch, Lang, and Saffidine (2013) studied the susceptibility of various voting rules to agents' strategically manipulating their votes. This line of research has further studied the computational complexity of agents identifying successful manipulations (Conitzer & Walsh, 2016). Grandi, Hughes, Rossi, and Slinko (2019) and Elkind, Grandi, Rossi, and Slinko (2020) studied the Nash equilibrium of Gibbard-Satterthwaite games in which agents may simultaneously manipulate their votes. Best-response dynamics, introduced to social choice as iterative voting by Meir et al. (2010), has been studied in the game theory literature as far back as Brown (1951)'s fictitious play procedure. Games for which sequences of best responses converge are known as *weakly acyclic* (Young, 1993). The *finite improvement property*, for which every improvement sequence converges, was first studied for potential games by Monderer and Shapley (1996a) and Monderer and Shapley (1996b). This property was further characterized by Fabrikant, Jaggard, and Schapira (2010), Andersson, Gurvich, and Hansen (2010), Kukushkin (2011), and Apt and Simon (2015). The relationship between IV, weakly acyclic games, and the finite improvement property was described by Meir (2016).

1.3.3 Smoothed Analysis

Spielman and Teng (2004) introduced *smoothed analysis* as a combination of worst- and average-case analyses to address the issue that average-case analysis distributions themselves may not be realistic. Their idea was to measure an algorithm's performance with respect to a worst-case instance subject to a random perturbation. Hence, even if an algorithm has exponential worst-case complexity, it may be unlikely to encounter such an instance in practice. This perspective has since been applied toward a large body of problems (see e.g., surveys by Spielman and Teng (2009) and Roughgarden (2021)). For example, Deng, Gao, and Zheng (2017), Gao and Zhang (2019), and Deng, Gao, and Zhang (2022) studied the smoothed performance of the random priority mechanism in matching problems. Extensions into social choice were independently proposed by Baumeister, Hogrebe, and Rothe (2020) and Xia (2020). The latter inspired a series of research extending prior results through this lens (e.g., Xia (2021b), Xia and Zheng (2021), Liu and Xia (2022), Xia and Zheng (2022), Xia (2023) and references within). Notably, Xia (2021a) studied the smoothed likelihood of ties in elections, which contributes meaningfully toward our primary results in Chapter 5. We describe in our conclusion, Chapter 6, how that chapter's contributions may be framed within this perspective.

CHAPTER 2
PRELIMINARIES

This chapter introduces preliminaries for the social welfare and convergence analysis of iterative voting that we study throughout this thesis. Section 2.1 introduces notation and common definitions for single-issue social choice. Section 2.2 extends these definitions to describe the iterative voting procedure, first introduced by Meir et al. (2010). Section 2.3 discusses the asymptotic analysis technique we use to describe our results. Finally, Section 2.4 describes the smoothed likelihood of ties framework, introduced by Xia (2021a), which we apply in Chapter 5. Preliminaries related to multi-issue social choice, which we study in Chapter 4, are reserved for Section 4.2.

2.1 Social Choice

Basic setting. For any integer $k \in \mathbb{N}$, let $[k] = \{1, \ldots, k\}$. We denote by $\mathcal{A} = [m]$ the set of *alternatives* and $n \in \mathbb{N}$ the number of agents. Each agent $j \leq n$ is endowed with a preference *ranking* $R_j \in \mathcal{L}(\mathcal{A})$, the set of strict linear orders over $\mathcal{A}$. A *preference profile* $P = (R_1, \ldots, R_n)$ is a collection of agents' preferences. For any pair of alternatives, $u, v \in \mathcal{A}$, we denote by $P[u \succ v]$ the number of agents that prefer u to v in P.

Plurality voting. A resolute *positional scoring rule* $f_{\vec{s}}$ maps vote profiles onto a unique outcome and is characterized by a scoring vector $\vec{s} = (s_1, \ldots, s_m) \in \mathbb{R}^m_{\geq 0}$ with $s_1 \geq s_2 \geq \ldots \geq s_m \geq 0$ and $s_1 > s_m$. For example, plurality uses $(1, 0, \ldots, 0)$, veto uses $(1, \ldots, 1, 0)$, and Borda uses $(m-1, m-2, \ldots, 0)$. Agents *vote* by reporting their (possibly non-truthful) preferences, each contributing s_i points to the alternative they reportedly rank i^{th}. The alternative with the most points wins subject to a tie-breaking rule.

Throughout this thesis we focus on the *plurality* voting rule f with lexicographical tie-breaking, unless specified otherwise. According to this rule, agents vote by reporting only a

Portions of this chapter have previously appeared as:

Kavner, J., & Xia, L. (2021). Strategic behavior is bliss: iterative voting improves social welfare. In *Advances in neural information processing systems* (Vol. 34, pp. 19021–19032). Curran Associates, Inc.

Kavner, J., Meir, R., Rossi, F., & Xia, L. (2023, August). Convergence of multi-issue iterative voting under uncertainty. In *Proceedings of the 32nd international joint conference on artificial intelligence* (pp. 2783-2791). ©2023 IJCAI.

Kavner, J., & Xia, L. (2024). *Average-case analysis of iterative voting.* arXiv. https://arxiv.org/abs/2402.08144.

single alternative $a_j \in \mathcal{A}$ into the *vote profile* $a = (a_1, \ldots, a_n)$. Plurality is then defined as $f(a) = \arg\max_{c \in \mathcal{A}} s_c(a)$ where $s_c(a) = |\{j \leq n : a_j = c\}|$. A vote $a_j^* = top(R_j)$ is *truthful* if it is agent j's most-favored alternative. We denote the truthful vote profile as $a^* = top(P)$.

Rank-based additive utility. We suppose agents have additive utilities characterized by a *rank-based utility vector* $\vec{u} = (u_1, \ldots, u_m) \in \mathbb{R}^m_{\geq 0}$ with $u_1 \geq \ldots \geq u_m$ and $u_1 > u_m$. For example, plurality welfare has $(1, 0, \ldots, 0)$ while Borda welfare has $(m-1, m-2, \ldots, 0)$. Each agent j gets $\vec{u}(R_j, c) = u_i$ utility for the alternative $c \in \mathcal{A}$ ranked i^{th} in R_j. The additive *social welfare* of c according to preference profile P is $\text{SW}_{\vec{u}}(P, c) = \sum_{j=1}^n \vec{u}(R_j, c)$.

2.2 Iterative Voting

We implement the iterative voting (IV) procedure introduced by Meir et al. (2010) for the plurality rule f. Given agents' truthful preferences P and an initial vote profile $a(0)$, we consider an iterative process of vote profiles $a(t) = (a_1(t), \ldots, a_n(t))$ that describe agents' reported votes over time $t \geq 0$. For each round t, a *scheduler* chooses an agent j to make a myopic *improvement step* over their prior vote $a_j(t)$ by applying a specified *response function* $g_j : \mathcal{A}^n \to \mathcal{A}$. A scheduler is simply a mapping from sequences of vote profiles to an agent with an improvement step in the latest vote profile (Apt & Simon, 2015). Each agent's response implicitly depends on their preferences and belief about the current vote profile, but they are not aware of others' private preferences. An improvement step must be selected if one exists, while other votes remains unchanged.

The literature on game dynamics considers different types of response functions, schedulers, initial profiles, and other assumptions (see e.g., Fudenberg and Levine (2009), Marden, Arslan, and Shamma (2007), Bowling (2005), and Young (1993)). This means that there are multiple levels in which a voting rule may guarantee convergence to an *equilibrium*, a vote profile where no improvement step exists (i.e., $g_j(a) = a_j, \ \forall j \leq n$) (Meir, Polukarov, Rosenschein, & Jennings, 2017). In this thesis, we study two response functions and, unless stated otherwise, begin from the truthful vote profile $a(0) = a^*$. Under *direct best response (BR)* dynamics, each agent j updates their vote to the unique alternative that (i) yields the most-preferred outcome under f with respect to their preferences R_j, and (ii) will become the winner as a result. Specifically, we denote the set of *potential winning* alternatives as those who could become a winner if their plurality score were to increment by one, including the current winner:

$$\mathrm{PW}(a) = \left\{ c \in \mathcal{A} \ : \begin{cases} s_c(a) = s_{f(a)}(a) - 1, & c \text{ is ordered before } f(a) \\ s_c(a) = s_{f(a)}(a), & c \text{ is ordered after } f(a) \end{cases} \right\} \cup \{f(a)\} \tag{2.1}$$

where the ordering is lexicographical for tie-breaking (Reyhani & Wilson, 2012). We call these alternatives *approximately-tied.* BR dynamics stipulate that agents change their vote to their favorite alternative in $\mathrm{PW}(a)$, given full information about each alternative c's score $s_c(a)$. Following BR dynamics from the truthful vote profile, Brânzei et al. (2013) proved that no agent ever changes their vote from the prior winner. Put together, this means $f(a(t)) \neq a_j(t)$ and $f(a(t+1)) = a_j(t+1)$.

Furthermore, Reyhani and Wilson (2012) proved that the potential winning set is monotonic in t: that is, $\forall t \geq 0$, $\mathrm{PW}(a(t+1)) \subseteq \mathrm{PW}(a(t))$. Therefore every BR sequence converges to an equilibrium in $\mathcal{O}(nm)$ rounds; specifically, this is a *Nash equilibrium* (NE). Let $\mathrm{EW}(a)$ denote the set of *equilibrium winning* alternatives corresponding to any NE reachable from a via some BR sequence:

$$\mathrm{EW}(a) = \{f(\tilde{a}) \ : \ \exists \text{ a BR sequence from } a \text{ leading to the NE profile } \tilde{a}\}. \tag{2.2}$$

This definition entails that $\mathrm{EW}(a(t)) \subseteq \mathrm{PW}(a(0))$. As a result, iterative plurality voting acts like a sequential tie-breaking mechanism whose outcome follows from the scheduler.

In Chapter 4, we also study *local dominance improvements (LDI)* for agents with uncertainty over alternatives' scores. We defer preliminaries related to this response function to that chapter. The following example demonstrates this section's concepts.

Example 2.1. *Let there be $n = 9$ agents, $m = 3$ alternatives, and consider the preference profile P defined such that:*

- $R_1 = R_2 = R_3 = (1 \succ 3 \succ 2)$;
- $R_4 = R_5 = (2 \succ 3 \succ 1)$;
- $R_6 = (2 \succ 1 \succ 3)$;
- $R_7 = R_8 = R_9 = (3 \succ 2 \succ 1)$.

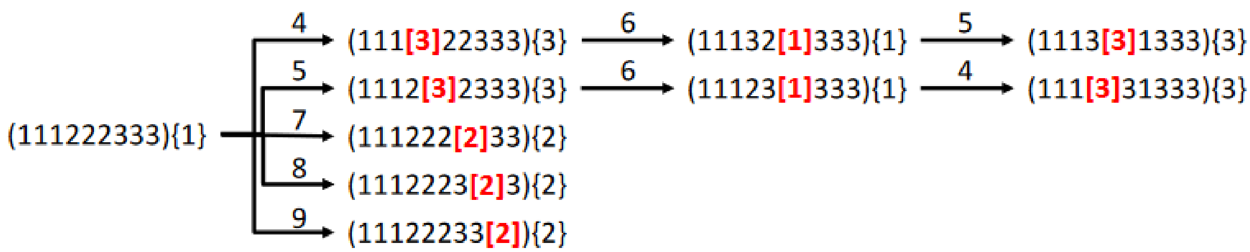

Figure 2.1: Five BR sequences in Example 2.1. The tuples denote agents' votes; the winner appears in curly brackets; arrows indicate which agent makes each BR step; the updated report is emphasized.

The truthful vote is $a^* = top(P) = (1,1,1,2,2,2,3,3,3)$. *We observe from alternatives' scores* $(s_1(a^*), s_2(a^*), s_3(a^*)) = (3,3,3)$ *that* $f(a^*) = 1$, *due to lexicographical tie-breaking, and* $PW(a^*) = \{1,2,3\}$, *representing a three-way tie. Figure 2.1 describes the five BR sequences from* a^*, *where we denote improvement steps by* $(a(t))\{f(a(t))\} \xrightarrow{j} (a(t+1))\{f(a(t+1))\}$. *We therefore conclude* $EW(a^*) = \{2,3\}$.

Finally, consider the utility vector $\vec{u} = (u_1, u_2, u_3)$. *The social welfare for each alternative is*

$$\begin{pmatrix} SW_{\vec{u}}(P,1) \\ SW_{\vec{u}}(P,2) \\ SW_{\vec{u}}(P,3) \end{pmatrix} = \begin{pmatrix} 3u_1 + 1u_2 + 5u_3 \\ 3u_1 + 3u_2 + 3u_3 \\ 3u_1 + 5u_2 + 1u_3 \end{pmatrix}. \tag{2.3}$$

2.3 Asymptotic Analysis

In this thesis we explore the long-term behavior of sequences in the limit of the number of agents $n \in \mathbb{N}$. We would like to be able to quantify how quickly sequences converge to certain values or diverge to $\pm\infty$, or if sequences are bounded, so that we may compare them. For example, the sequence $(\log n)_{n\in\mathbb{N}}$ diverges slower than $(n^2)_{n\in\mathbb{N}}$, which diverges slower than $(e^n)_{n\in\mathbb{N}}$. The nomenclature of Big-$\mathcal{O}$ notation enables us to make these comparisons.

Definition 2.1. *Let* f *and* g *be real-valued functions. We say that* $f(n) = \mathcal{O}(g(n))$ *if* $\exists N, C > 0$ *such that* $\forall n > N$, $0 \le f(n) \le Cg(n)$.

For example, $f(n) = n^2 + 2n = \mathcal{O}(n^2)$ since $f(n) \le 2n^2$, $\forall n > 2$. One useful application of big-$\mathcal{O}$ notation is to describe MacLaurin series. For example, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Hence,

$$e^{-\frac{1}{n}} = 1 - \frac{1}{n} + \mathcal{O}\left(\frac{1}{n^2}\right). \tag{2.4}$$

Big-$\mathcal{O}$ is often used to compare the asymptotic runtime of algorithms. In our case, we use it to describe the asymptotic economic efficiency of the iterative voting procedure. Hence, f may be non-positive. We use the following notation to describe combined positive and negative bounds on f.

Definition 2.2. *Let f and g be real-valued functions. We say that $f(n) = \pm\mathcal{O}(g(n))$ if $\exists N, C > 0$ such that $\forall n > N$, $|f(n)| \leq C|g(n)|$.*

Equivalently, we have that $|f(n)| = \mathcal{O}(g(n))$. For example, $f(n) = n \cdot cos(n) = \pm\mathcal{O}(n)$ since $-n \leq f(n) \leq n$, $\forall n > 0$. The next two definitions describe asymptotic lower-and tight-bounds on functions.

Definition 2.3. *Let f and g be real-valued functions. We say that $f(n) = \Omega(g(n))$ if $\exists N, C > 0$ such that $\forall n > N$, $f(n) \geq Cg(n) \geq 0$.*

For example, $f(n) = n^2 + 2n = \Omega(n)$ since $f(n) \geq 2n$, $\forall n > 0$. Notice also that saying $f(n) = -\Omega(g(n))$ is equivalent to $-f(n) = \Omega(g(n))$.

Definition 2.4. *Let f and g be real-valued functions. We say that $f(n) = \Theta(g(n))$ if $f(n) = \mathcal{O}(g(n))$ and $f(n) = \Omega(g(n))$.*

Notice that Big-$\mathcal{O}$ and Big-Ω notation do not describe smallest-upper-bounds or largest-lower-bounds like *supremum* and *infimum*. Hence, we have that $f(n) = n^2 + 2n = \Theta(n^2)$ since $f = \mathcal{O}(n^2)$ and $f = \Omega(n^2)$. Furthermore, we write $\exp(-\Theta(n))$ for $e^{f(n)}$, where $f(n) = -\Theta(n)$. We write $f(n) \sim g(n)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$.

2.3.1 Little-o Notation

Little-o notation compares the asymptotic rate of functions such that one pales in comparison to another.

Definition 2.5. *Let f and g be real-valued functions. We say that $f(n) = o(g(n))$ if $\forall \epsilon > 0$, $\exists N > 0$ such that $\forall n > N$, $|f(n)| \leq \epsilon|g(n)|$. When $g(n)$ does not vanish, we may write $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.*

For example, $f(n) = \sqrt{np(1-p)}$ for $p \in (0, 1)$ is $o(n)$ since $\lim_{n\to\infty} \frac{\sqrt{np(1-p)}}{n} = 0$.

2.3.2 Asymptotic Multiplication

Let $f_1(n) = \mathcal{O}(g_1(n))$, $f_2(n) = \mathcal{O}(g_2(n))$, $f_3(n) = \Theta(g_3(n))$ and $f_4(n) = \Theta(g_4(n))$. Then by these definitions we have

- $f_1(n) \cdot f_2(n) = \mathcal{O}(g_1(n) \cdot g_2(n))$,
- $f_3(n) \cdot f_4(n) = \Theta(g_3(n) \cdot g_4(n))$,
- $f_1(n) \cdot f_3(n) = \mathcal{O}(g_1(n) \cdot g_3(n))$.

To be more precise, we give the examples of $f_1(n) = n^2 + 2n$, $f_2(n) = \log(n)$, and $f_3(n) = \mathcal{O}(1)$. It is clear that $f_1(n) \cdot f_2(n) = \Theta(n^2 \log(n))$. We can say that $f_1(n) \cdot f_3(n) = \mathcal{O}(n^2)$ but not that it is $\Theta(n^2)$. This is because we do not have enough information about the lower-bound $\Omega(f_3(n))$. It holds that

$$f_1(n) \cdot f_3(n) = \begin{cases} \Theta(n^2), & f_3(n) = \Theta(1) \\ \Theta(n), & f_3(n) = \Theta\left(\frac{1}{n}\right) \\ \Theta(1), & f_3(n) = \Theta\left(\frac{1}{n^2}\right). \end{cases} \tag{2.5}$$

2.4 Smoothed Likelihood of Ties

A tied election is a characterization on the histogram of a preference profile satisfying certain criterion. With positional scoring rules $f_{\vec{s}}$, for instance, a W*-way tie* (i.e., a k*-way tie* between the alternatives $W \subseteq \mathcal{A}$, $|W| = k$) is the event that these alternatives have the same score and that this score is strictly greater than those of other alternatives. This may be characterized as a system of linear constraints on the multiplicity of rankings in P, as described by Xia (2021a) as follows.

Definition 2.6 (Score difference vector). *For any scoring vector $\vec{s}$ and pair $u, v \in \mathcal{A}$, let $Score^{\vec{s}}_{u,v}$ denote the $m!$-dimensional vector indexed by rankings in $\mathcal{L}(\mathcal{A})$ such that $\forall R \in \mathcal{L}(\mathcal{A})$, the R-component of $Score^{\vec{s}}_{u,v}$ is $s_{R[u]} - s_{R[v]}$, where $R[c]$ is the index of c in R.*

Let $Hist(P) = (x_R : R \in \mathcal{L}(\mathcal{A}))$ denote the vector of $m!$ variables, each of which represents the multiplicity of a linear order in a profile P. Therefore, $\mathrm{Score}^{\vec{s}}_{u,v} \cdot Hist(P)$ represents the score difference between u and v in P. For any $W \subseteq \mathcal{A}$, we define the polyhedron $\mathcal{H}^{\vec{s},W}$ as follows.

Definition 2.7. *Let $\mathbf{E}^{\vec{s},W}$ denote the matrix whose row vectors are $\{Score^{\vec{s}}_{u,v} : u \in W, v \in W, u \neq v\}$. Let $\mathbf{S}^{\vec{s},W}$ denote the matrix whose row vectors are $\{Score^{\vec{s}}_{u,v} : u \notin W, v \in W\}$. Let $\mathbf{A}^{\vec{s},W} = \begin{bmatrix} \mathbf{E}^{\vec{s},W} \\ \mathbf{S}^{\vec{s},T} \end{bmatrix}$, $\vec{b} = \begin{bmatrix} \vec{0} \\ -\vec{1} \end{bmatrix}$, and let $\mathcal{H}^{\vec{s},W} = \{\vec{x} \in \mathbb{R}^{m!} : \mathbf{A}^{\vec{s},W}\vec{x} \leq \vec{b}\}$ denote the corresponding polyhedron.*

It follows that the alternatives W are tied in $f_{\vec{s}}(P)$ (notwithstanding any tie-breaking) if and only if $Hist(P) \in \mathcal{H}^{\vec{s},W}$. The following example characterizes a plurality tie between alternatives 1 and 2 with this polyhedral representation. We denote the plurality score vector by $\vec{s}_{plu} = (1, 0, \ldots, 0)$.

Example 2.2 (Polyhedral representation of a $\{1,2\}$-way plurality tie)**.** *Let $m = 3$ and consider the vote profile $a^* = top(P)$ for preference profile P. Then a W-way tied plurality election of a^*, for $W = \{1,2\}$, occurs if and only if $Hist(P)$ is in a polyhedron $\mathcal{H}^{\vec{s}_{plu},W}$ represented by the following inequalities:*

$$x_{123} + x_{132} - x_{213} - x_{231} \leq 0; \tag{2.6}$$

$$-x_{123} - x_{132} + x_{213} + x_{231} \leq 0; \tag{2.7}$$

$$-x_{123} - x_{132} + x_{312} + x_{321} \leq -1; \tag{2.8}$$

$$-x_{213} - x_{231} + x_{312} + x_{321} \leq -1. \tag{2.9}$$

The variables are $\vec{x} = (x_{123}, x_{132}, x_{213}, x_{231}, x_{312}, x_{321})$ where x_{xyz} corresponds to the number of rankings in P with ranking $(x \succ y \succ z)$. The first two inequalities state that alternatives 1 and 2 have the same plurality score, while the last two inequality states that alternative 3 has a strictly smaller plurality score than alternatives 1 and 2. This suggests that $\mathcal{H}^{\vec{s}_{plu},W} = \{\vec{x} \in \mathbb{R}^6 : \mathbf{A}^{\vec{s}_{plu},W}\vec{x} \leq \vec{b}\}$ where

$$\mathbf{A}^{\vec{s}_{plu},W} = \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \end{bmatrix}. \tag{2.10}$$

Following this example, for the plurality score vector $\vec{s}_{plu}$, general $m \geq 3$, and $W \subseteq \mathcal{A}$, the polyhedron $\mathcal{H}^{\vec{s}_{plu},W}$ is represented by the following inequalities:

$$\forall \{i_1, i_2\} \subseteq W \text{ s.t. } i_1 \neq i_2,\ \sum\nolimits_{R:top(R)=i_1} x_R - \sum\nolimits_{R:top(R)=i_2} x_R \leq 0; \tag{2.11}$$

$$\forall i_1 \in W, i_2 \in \mathcal{A}\backslash W,\ \sum\nolimits_{R:top(R)=i_2} x_R - \sum\nolimits_{R:top(R)=i_1} x_R \leq -1. \tag{2.12}$$

These inequalities cover the case of $\text{PW}(a^*) = W$ such that all alternatives in W have the same score. In fact, there are $|W|$ possible cases depending on which alternative $f(a^*) \in \text{PW}(a^*)$ is the winner. The other cases may be characterized by modifying $\vec{b}$ accordingly. For example, consider $\text{PW}(a^*) = W$ with $W = \{1, 2, 3\}$ such that $s_1(a^*) + 1 = s_2(a^*) = s_3(a^*)$. Then $\mathcal{H}^{\vec{s}_{plu},W}$ would be represented by the inequalities:

$$\sum\nolimits_{R:top(R)=1} x_R - \sum\nolimits_{R:top(R)=2} x_R \leq -1; \tag{2.13}$$

$$\sum\nolimits_{R:top(R)=2} x_R - \sum\nolimits_{R:top(R)=1} x_R \leq 1; \tag{2.14}$$

$$\sum\nolimits_{R:top(R)=2} x_R - \sum\nolimits_{R:top(R)=3} x_R \leq 0; \tag{2.15}$$

$$\sum\nolimits_{R:top(R)=3} x_R - \sum\nolimits_{R:top(R)=2} x_R \leq 0; \tag{2.16}$$

$$\forall i \in [4, m],\ \sum\nolimits_{R:top(R)=i} x_R - \sum\nolimits_{R:top(R)=2} x_R \leq -1. \tag{2.17}$$

This polyhedral representation of agents' preferences is due to the smoothed analysis work of Xia (2020) and Xia (2021a). The latter work studied how likely large elections are tied according to several voting rules when agents' preferences are independently distributed: $P \sim \vec{\pi}$ where $\forall j \leq n$, it holds that $R_j \sim \vec{\pi}(j) \in \Delta(\mathcal{L}(\mathcal{A}))$, the probability simplex over all rankings $\mathcal{L}(\mathcal{A})$. This problem has been studied extensively in the public choice literature (see e.g., Beck (1975), Gillett (1977), Margolis (1977), Gillett (1980), Chamberlain and Rothschild (1981), and Marchant (2001)). Xia (2021a) solved this problem beyond the prior work by recognizing that the histogram of a randomly generated preference profile is a *Poisson multivariate variable (PMV)*. A tied election of the alternatives W, then, is that PMV occurring within the polyhedron $\mathcal{H}^{\vec{s},W}$. To determine the likelihood of this event, Xia (2021a) defined the *PMV-in-polyhedron problem* as $\Pr_{P\sim\vec{\pi}}(\mathit{Hist}(P) \in \mathcal{H})$ for any polyhedron $\mathcal{H}$, taken in supremum or infimum over distributions $\vec{\pi} \in \Pi^n$, and proved a dichotomy theorem for conditions on this likelihood. The following definitions are used to formally describe his main result.

Definition 2.8 (Poisson multivariate variables (PMVs)). *Given $\mu, n \in \mathbb{N}$ and distribution $\vec{\pi}$ over $[\mu]$, let $\vec{X}_{\vec{\pi}}$ denote the (n,μ)-PMV that corresponds to $\vec{\pi}$. That is, let $Y_1, \ldots, Y_n$ denote n identical random variables over $[\mu]$ such that for any $j \leq n$, Y_j is distributed as $\vec{\pi}(j)$. For any $i \in [\mu]$, the i-th component of $\vec{X}_{\vec{\pi}}$ is the number of Y_j's that take value i.*

Given $\mu, L, n \in \mathbb{N}$, an $L \times \mu$ matrix $\mathbf{A}$, and an L-dimensional vector $\vec{b}$, we define $\mathcal{H}, \mathcal{H}_{\leq 0}, \mathcal{H}_n$ and $\mathcal{H}_n^{\mathbb{Z}}$ as follows:

$$\mathcal{H} = \left\{\vec{x} \in \mathbb{R}^{\mu} \ : \ \mathbf{A}\vec{x} \leq \vec{b}\right\}; \tag{2.18}$$

$$\mathcal{H}_{\leq 0} = \left\{\vec{x} \in \mathbb{R}^{\mu} \ : \ \mathbf{A}\vec{x} \leq \vec{0}\right\}; \tag{2.19}$$

$$\mathcal{H}_n = \left\{\vec{x} \in \mathcal{H} \cap \mathbb{R}^{\mu}_{\geq 0} \ : \ \vec{x} \cdot \vec{1} = n\right\}; \tag{2.20}$$

$$\mathcal{H}_n^{\mathbb{Z}} = \mathcal{H}_n \cap \mathbb{Z}^{\mu}_{\geq 0}. \tag{2.21}$$

That is, $\mathcal{H}$ is the polyhedron represented by $\mathbf{A}$ and $\vec{b}$; $\mathcal{H}_{\leq 0}$ is the *characteristic cone* of $\mathcal{H}$, $\mathcal{H}_n$ consists of non-negative vectors in $\mathcal{H}$ whose L_1 norm is n, and $\mathcal{H}_n^{\mathbb{Z}}$ consists of non-negative integer vectors in $\mathcal{H}_n$. By definition, $\mathcal{H}_n^{\mathbb{Z}} \subseteq \mathcal{H}_n \subseteq \mathcal{H}$. Let $\dim(\mathcal{H}_{\leq 0})$ denote the dimension of $\mathcal{H}_{\leq 0}$, i.e., the dimension of the minimal linear subspace of $\mathbb{R}^{\mu}$ that contains $\mathcal{H}_{\leq 0}$. For a set Π of distributions over $[\mu]$, $CH(\Pi)$ denotes the convex hull of Π. Π is called *strictly positive* (by $\epsilon > 0$) if $\forall \vec{\pi} \in \Pi, \forall j \in [\mu], \vec{\pi}(j) > \epsilon$.

Theorem 2.3 (Xia (2021a), Theorem 1). *Given any $\mu \in \mathbb{N}$, any closed and strictly positive Π over $[\mu]$, and any polyhedron $\mathcal{H}$ characterized by a matrix $\mathbf{A}$, for any $n \in \mathbb{N}$,*

$$\sup_{\vec{\pi} \in \Pi^n} \Pr\left(\vec{X}_{\vec{\pi}} \in \mathcal{H}\right) = \begin{cases} 0, & \text{if } \mathcal{H}_n^{\mathbb{Z}} = \emptyset \\ \exp(-\Theta(n)), & \text{if } \mathcal{H}_n^{\mathbb{Z}} \neq \emptyset \text{ and } \mathcal{H}_{\leq 0} \cap CH(\Pi) = \emptyset \\ \Theta\left(n^{\frac{dim(\mathcal{H}_{\leq 0}) - \mu}{2}}\right), & \text{otw. (i.e., } \mathcal{H}_n^{\mathbb{Z}} \neq \emptyset \text{ and } \mathcal{H}_{\leq 0} \cap CH(\Pi) \neq \emptyset); \end{cases}$$

$$\inf_{\vec{\pi}\in\Pi^n} \Pr\left(\vec{X}_{\vec{\pi}} \in \mathcal{H}\right) = \begin{cases} 0, & \textit{if } \mathcal{H}_n^{\mathbb{Z}} = \emptyset \\ \exp(-\Theta(n)), & \textit{if } \mathcal{H}_n^{\mathbb{Z}} \neq \emptyset \textit{ and } CH(\Pi) \nsubseteq \mathcal{H}_{\leq 0} \\ \Theta\left(n^{\frac{dim(\mathcal{H}_{\leq 0})-\mu}{2}}\right), & \textit{otw. (i.e., } \mathcal{H}_n^{\mathbb{Z}} \neq \emptyset \textit{ and } CH(\Pi) \subseteq \mathcal{H}_{\leq 0}). \end{cases}$$

Xia (2021a) used this theorem to depict the likelihood of k-way ties according to several voting rules. In particular, the likelihood of k-way plurality ties with i.i.d. preferences corresponds to $\Pr_{P\sim\pi^n}(Hist(P) \in \mathcal{H}^k)$, where $\pi^n = (\pi, \pi, \ldots, \pi)$ and $\mathcal{H}^k = \bigcup_{W\subseteq 2^{\mathcal{A}}:|W|=k} \mathcal{H}^{\vec{s}_{plu},W}$. In this case, $\Pi = \{\pi\}$ consists of a single distribution $\pi \in \Delta(\mathcal{L}(\mathcal{A}))$ and the two probabilities of Theorem 2.3 coincide. The following corollary holds for either $\mathcal{H}^k$ or $\mathcal{H}^{\vec{s}_{plu},W}$ that corresponds to any case of $\text{PW}(a^*) = W$ with $|W| = k$.

Corollary 2.1 (Xia (2021a), Corollary 1)**.** *Fix $m \geq 3$ and let $n \in \mathbb{N}$ agents' preferences be i.i.d. according to IC. Then the likelihood of a k-way plurality tied election is $\Theta\left(n^{-\frac{k-1}{2}}\right)$.*

The probability of a 2- or 3-way tie with respect to IC is therefore $\Theta\left(\frac{1}{\sqrt{n}}\right)$ or $\Theta\left(\frac{1}{n}\right)$, respectively.

CHAPTER 3
STRATEGIC BEHAVIOR IS BLISS: ITERATIVE VOTING IMPROVES SOCIAL WELFARE

3.1 Introduction

Voting is one of the most popular methods for a group of agents to make a collective decision based on their preferences. Whether a decision is for a high-stakes presidential election or a routine luncheon, agents submit their preferences and a voting rule is applied to select a winning alternative.

One critical flaw of voting is its susceptibility to strategic manipulation. That is, agents may have an incentive to misreport their preferences (i.e. votes) to obtain a more favorable outcome. Unfortunately, manipulation is inevitable under any non-dictatorial single-round voting systems when there are three or more alternatives, as recognized by the celebrated Gibbard-Satterthwaite theorem (Gibbard, 1973; Satterthwaite, 1975). Consequently, decades of research sought to deter manipulation, especially by high computational barriers (Bartholdi et al., 1989; Faliszewski et al., 2010; Faliszewski & Procaccia, 2010); see Conitzer and Walsh (2016) for a recent survey of the field.

While there is a large body of literature on manipulation of single-round voting systems, sequential and iterative voting procedures are less understood. Indeed, these procedures occur in a variety of applications, such as Doodle or presidential election polls, where people finalize their votes after previewing others' responses (Desmedt & Elkind, 2010; Meir et al., 2010; Reijngoud & Endriss, 2012; Xia & Conitzer, 2010; Zou et al., 2015). Our key question is:

What is the effect of strategic behavior in sequential and iterative voting?

A series of work initiated by Meir et al. (2010) characterizes the dynamics and equilibria of *iterative voting*, where agents sequentially and myopically improve their reported preferences based on other agents' reports (Brânzei et al., 2013; Endriss et al., 2016; Grandi et al., 2013; Koolyk et al., 2017; Lev & Rosenschein, 2012; Meir, 2016; Meir et al., 2014;

Portions of this chapter have previously appeared as: Kavner, J., & Xia, L. (2021). Strategic behavior is bliss: iterative voting improves social welfare. In *Advances in neural information processing systems* (Vol. 34, pp. 19021–19032). Curran Associates, Inc.

Obraztsova et al., 2015, 2013; Rabinovich et al., 2015; Reyhani & Wilson, 2012; Tsang & Larson, 2016). While the convergence of iterative voting has been investigated for many commonly studied voting rules, the effect of strategic behavior, in terms of aggregate social welfare, remains largely unclear.

A notable exception is Brânzei et al. (2013)'s work that introduced and characterized the *additive dynamic price of anarchy* (ADPoA) of iterative voting with respect to the plurality, veto, and Borda social choice functions. The (additive) DPoA measures the social welfare (difference) ratio between the truthful winner and an iterative policy's equilibrium winners when an adversary minimizes aggregate social welfare by controlling both the order in which agents make their strategic manipulations and agents' truthful preferences altogether. In particular, Brânzei et al. (2013) proved that under iterative plurality, the number of agents whose top preference is an equilibrium winner is at most one less than that of the truthful plurality winner. Therefore, strategic behavior does not have a significant negative impact on the social welfare measured by the sum plurality score of the winner. Nevertheless, it is unclear whether this observation holds for other notions of social welfare.

3.1.1 Our Contributions

We address the key question discussed above in the iterative voting framework, first proposed by Meir et al. (2010), by characterizing Brânzei et al. (2013)'s ADPoA under plurality dynamics and *rank-based utility functions* that differ from the iteration method. Given $m \geq 3$ alternatives, a ranked-based utility function is characterized by a utility vector $\vec{u}$ such that each agent receives u_i utility if their i^{th} ranked alternative wins, although this alternative may differ for each agent. We study iterative plurality due to its simplicity and popularity in practice. Moreover, our results absolve the need for the mechanism's center to know $\vec{u}$ exactly, thus conserving agents' privacy. Still, we assume this is constant for all agents.

Our first main result (Theorem 3.1) states that, unfortunately, for any fixed $m \geq 3$ and utility vector $\vec{u}$, the ADPoA is $\Theta(n)$ for n agents. Therefore, the positive result achieved by Brânzei et al. (2013) is not upheld if $\vec{u}$ differs from plurality utility under the iterative plurality mechanism.

To overcome this negative worst-case result, we introduce the notion of *expected additive dynamic price of anarchy* (EADPoA), which presumes agents' truthful preferences to be generated from a probability distribution. Our second main result (Theorem 3.2) is positive and surprises us: for any fixed $m \geq 3$ and utility vector $\vec{u}$, the EADPoA is $-\Omega(1)$ when agents' preferences are i.i.d. uniformly at random, known as *Impartial Culture (IC)* distribution. In particular, our result suggests that *strategic behavior is bliss* because iterative voting helps agents choose an alternative with higher expected social welfare, regardless of the order of agents' strategic manipulations.

Techniques. We compute the EADPoA by partitioning the (randomly generated) profiles according to their *potential winners* – the alternatives that can be made to win by incrementing their plurality scores by at most one. Conditioned on profiles with two potential winners, we show that iterative plurality returns the alternative that beats the other in a head-to-head competition (Lemma 3.1). This type of "self selection" improves the expected social welfare over truthful plurality winner by an additive $\Omega(1)$ (Lemma 3.2). When there are three or more potential winners, we further show that the expected welfare loss is $o(1)$ (Lemmas 3.3 and 3.4). Since the likelihood of k-way ties is small (in fact, $\Theta\left(n^{-\frac{k-1}{2}}\right)$ by Corollary 2.1), the overall social welfare is improved in expectation. We provide an experimental justification of this main result in Section 3.4.

3.2 Additive Dynamic PoA Under General Utility Vectors

How bad are equilibrium outcomes, given that strategic manipulations inevitably occur by the Gibbard-Satterthwaite theorem (Gibbard, 1973; Satterthwaite, 1975)? Brânzei et al. (2013) sought to answer this question by defining the *additive dynamic price of anarchy (ADPoA)* as the *adversarial loss* – the difference in welfare between the truthful winner $f(a^*)$ and its worst-case equilibrium winner in $\mathrm{EW}(a^*)$ – according to the worst-case preference profile P. To motivate this concept, consider users of a website that can regularly log in and update their preferences for an election. Then the ADPoA bounds the welfare loss if a virtual assistant can recommend when users should make their changes.

Brânzei et al. (2013) originally chacterized the ADPoA for a given positional scoring rule $f_{\vec{s}}$ and an additive social welfare function respecting $\vec{u} = \vec{s}$. In this case, the ADPoA of plurality was found to be 1, while the (multiplicative) DPoA of veto is $\Omega(m)$ and Borda is $\Omega(n)$ for fixed $m \geq 4$. Although these results answer the authors' question and appear optimistic for plurality, they suggest more about the iteration mechanism than agents' collective welfare. For example, an ADPoA for plurality of 1 means that for any truthful profile, the difference in initial plurality scores of any equilibrium winner is at most one less that of the truthful winner. However, when we relax the utility vector $\vec{u}$ to differ from $\vec{s}$, we find in Theorem 3.1 that the ADPoA is quite poor at $\Theta(n)$. First, we recall Brânzei et al. (2013)'s definition of ADPoA using our notation and explicitly define the *adversarial loss* D^+ for a particular truthful vote profile a^*, before proceeding to our first main result.

Definition 3.1 (**Additive Dynamic Price of Anarchy (Brânzei et al., 2013)**). *Given positional scoring rule $f_{\vec{s}}$, utility vector $\vec{u}$, n agents, and preference profile P, the* adversarial loss *starting from the truthful vote profile a^* is*

$$D^+_{f_{\vec{s}},\vec{u}}(P) = SW_{\vec{u}}(P, f_{\vec{s}}(a^*)) - \min_{c \in EW(a^*)} SW_{\vec{u}}(P, c). \tag{3.1}$$

The additive dynamic price of anarchy (ADPoA) *is*

$$ADPoA(f_{\vec{s}}, \vec{u}) = \max_{P \in \mathcal{L}(\mathcal{A})^n} D^+_{f_{\vec{s}},\vec{u}}(P). \tag{3.2}$$

In what follows we may drop parameters and scripts from these definitions for ease of notation when the context is clear. [6] For example, we saw in Example 2.1 in Chapter 2 that $f(a^*) = 1$ and $EW(a^*) = \{2, 3\}$. Then

$$\begin{aligned}
\mathrm{D}^+(P) &= \max\{\ SW_{\vec{u}}(P,1) - SW_{\vec{u}}(P,2),\ SW_{\vec{u}}(P,1) - SW_{\vec{u}}(P,3)\ \} \\
&= \max\{\ (3u_1 + 1u_2 + 5u_3) - (3u_1 + 3u_2 + 3u_3)\,, \\
&\qquad\qquad (3u_1 + 1u_2 + 5u_3) - (3u_1 + 5u_2 + 1u_3)\ \} \qquad (3.3) \\
&= -2(u_2 - u_3) \\
&\leq 0.
\end{aligned}$$

Therefore the social welfare of both equilibrium winners is at least that of the truthful winner. In Theorem 3.2 below we'll see this conclusion hold in expectation. For the worst case preferences P, on the other hand, the following theorem proves this is not the case. Rather, the worst-case equilibrium winner of P has a social welfare linearly worse than the truthful winner.

[6] The superscript '+' denotes an additive measure instead of a multiplicative one, as in the classical definition of PoA.

Theorem 3.1. *Fix $m \geq 3$, the plurality rule f, and utility vector $\vec{u} = (u_1, \ldots, u_m)$. Then $ADPoA(f, \vec{u}) = \Theta(n)$.*

Proof. The ADPoA is trivially upper bounded by the maximum social welfare attainable by any truthful profile P. For example, let P be defined with $R_j = (1, 2, \ldots, m)\ \forall j \leq n$ and $a^* = top(P)$. Then $\forall \tilde{P} \in \mathcal{L}(\mathcal{A})^n$ with $\tilde{a}^* = top(\tilde{P})$,

$$\begin{aligned} \mathrm{D}^+(\tilde{a}^*) &= \mathrm{SW}_{\vec{u}}(\tilde{P}, f(\tilde{a}^*)) - \min\nolimits_{c \in \mathrm{EW}(\tilde{a}^*)} \mathrm{SW}_{\vec{u}}(\tilde{P}, c) \\ &\leq \mathrm{SW}_{\vec{u}}(\tilde{P}, f(\tilde{a}^*)) \\ &\leq \mathrm{SW}_{\vec{u}}(P, f(a^*)) \\ &= nu_1. \end{aligned}$$

To lower bound ADPoA, we will construct a profile P with a two-way tie between alternatives $1, 2 \in \mathcal{A}$ such that $\mathrm{D}^+(P) = (u_2 - u_m)\left(\frac{n}{m} - 2\right)$. This implies the desired lower bound of

$$\begin{aligned} \mathrm{ADPoA} &= \max\nolimits_{\tilde{P} \in \mathcal{L}(\mathcal{A})^n} \mathrm{D}^+(\tilde{P}) \\ &\geq \mathrm{D}^+(P) \\ &= (u_2 - u_m)\left(\frac{n}{m} - 2\right). \end{aligned}$$

Fix $m \geq 3$ and let $n > 2m$. Let $k = \arg\min_{\tilde{k} \in [2, m-1]}(u_{\tilde{k}} - u_{\tilde{k}+1})$ the position in $\vec{u}$ with the minimal difference in adjacent coordinates. We construct P such that there are:

- α agents that prefer 1 first and 2 last;
- α agents that prefer 2 first and 1 second;
- $(\frac{\beta}{2} - 1)$ agents that prefer 1 second and 2 last;
- $(\frac{\beta}{2} + 1)$ agents that prefer 2 in their ranking's k-th position and 1 in their ranking's $(k+1)$-th position.

where we define $\alpha = \frac{1}{m}(n + m - 2)$ and $\beta = (\alpha - 1)(m - 2)$. It is easy to see that $n = 2\alpha + \beta$. We can see here that $s_1(a^*) = s_2(a^*) = \alpha$, and $\forall c > 2$, $s_c(a^*) = \alpha - 1$, thus guaranteeing the two-way tie. Therefore $f(a^*) = 1$ and $P[2 \succ 1] = \alpha + \frac{\beta}{2} + 1 > \alpha + \frac{\beta}{2} - 1 = P[1 \succ 2]$. This implies $\mathrm{EW}(a^*) = \{2\}$ by the following lemma.

Lemma 3.1. *Let $m \geq 3$ and $u, v \in \mathcal{A}$ such that u is ordered before v in tie-breaking. Suppose $PW(a^*) = \{u, v\}$ for some preference profile P. Then $EW(a^*) = \{u\}$ if $P[u \succ v] \geq P[u \succ v]$; otherwise $EW(a^*) = \{v\}$.*

Proof. Without loss of generality, let $u = 1$ and $v = 2$. There are two cases of $\mathrm{PW}(a^*) = \{1, 2\}$: either alternatives 1 and 2 are exactly tied with maximal score, or they are almost tied for maximum score while alternative 2 has one more point that 1. First, consider the case where 1 and 2 are tied with $s_1(a^*) = s_2(a^*)$. Let

- $\mathrm{Id}^{(1)}(a; P) = \{j \leq n : a_j \notin \{1, 2\},\ \text{and } 1 \succ_j 2\}$
- $\mathrm{Id}^{(2)}(a; P) = \{j \leq n : a_j \notin \{1, 2\},\ \text{and } 2 \succ_j 1\}$

denote the indices of agents who don't rank 1 or 2 highest but prefer $1 \succ 2$ or $2 \succ 1$ respectively. Iterative voting proceeds by *third-party* agents, those in $\mathrm{Id}^{(1)}(a(t)) \cup \mathrm{Id}^{(2)}(a(t))$ that are not voting for the tied alternatives 1 and 2 in round t, alternately changing their votes to whichever of the two they favor. Since $f(a(0)) = 1$, an agent from $\mathrm{Id}^{(2)}(a(0))$ will first change their vote to 2, thus changing the outcome to $f(a(1)) = 2$. This enables an agent from $\mathrm{Id}^{(1)}(a(1))$ to change their vote to 1 and revert the outcome back to $f(a(2)) = 1$. This process continues until round t when either $\mathrm{Id}^{(1)}(a(t))$ or $\mathrm{Id}^{(2)}(a(t))$ are emptied of indices. If $|\mathrm{Id}^{(1)}(a^*)| \geq |\mathrm{Id}^{(2)}(a^*)|$, the last BR step will make 1 the unique equilibrium winner, whereas if $|\mathrm{Id}^{(1)}(a^*)| < |\mathrm{Id}^{(2)}(a^*)|$, the last BR step will make 2 the unique equilibrium winner.

Inverse reasoning holds if 1 and 2 differ by one initial plurality score such that $s_1(a^*) + 1 = s_2(a^*)$, implying $f(a^*) = 2$. In this case, the last BR step will make 1 the unique equilibrium winner only if $|\mathrm{Id}^{(1)}(a^*)| > |\mathrm{Id}^{(2)}(a^*)|$, since the plurality score of 1 is initially disadvantaged by 1. Otherwise, the unique equilibrium winner will be 2. We therefore conclude that if $P[1 \succ 2] \geq P[2 \succ 1]$ across all n agents, then $\mathrm{EW}(a^*) = \{1\}$; otherwise $\mathrm{EW}(a^*) = \{2\}$. $\square$

As a result of Lemma 3.1,

$$\begin{aligned}
\mathrm{D}^{+}(P) &= \mathrm{SW}_{\vec{u}}(P,1) - \mathrm{SW}_{\vec{u}}(P,2) \\
&= \alpha(u_2 - u_m) + \left(\frac{\beta}{2} - 1\right)(u_2 - u_m) - \left(\frac{\beta}{2} + 1\right)(u_k - u_{k+1}) \\
&\geq (u_2 - u_m)(\alpha - 2) \\
&= (u_2 - u_m)\left(\frac{n-m-2}{m}\right) \\
&\geq (u_2 - u_m)\left(\frac{n}{m} - 2\right)
\end{aligned}$$

where the first inequality holds because $(u_k - u_{k+1}) \leq (u_2 - u_m)$. □

3.3 Expected Additive Dynamic PoA

In this section we extend Brânzei et al. (2013)'s ADPoA notion to account for the average-case adversarial loss for a positional scoring rule f, rather than only the studying worst-case. This *expected additive dynamic price of anarchy* (EADPoA) bounds the adversarial loss of strategic manipulation according to more typical distributions of agents' rankings. Here we distribute agents' preference rankings i.i.d. uniformly over $\mathcal{L}(\mathcal{A})$, known as the *Impartial Culture* (IC) distribution.

Definition 3.2 (Expected Additive Dynamic PoA). *Given a positional scoring rule $f_{\vec{s}}$, a utility vector $\vec{u}$, n agents, and a distribution $\vec{\pi}$ over $\mathcal{L}(\mathcal{A})^n$ for agents' preferences, the* expected additive dynamic price of anarchy (EADPoA) *is*

$$EADPoA(f_{\vec{s}}, \vec{u}, \vec{\pi}) = \mathbb{E}_{P\sim\vec{\pi}}\left[D^{+}_{f_{\vec{s}},\vec{u}}(P)\right]. \tag{3.4}$$

Like before, we may drop parameters and scripts when the context is clear. We similarly fix a rank-based utility vector $\vec{u}$ that may differ from the scoring rule $\vec{s}$, but we will not presume in the following theorem that this is known by the iterative plurality mechanism.

Theorem 3.2. *Fix $m \geq 3$ and utility vector $\vec{u} = (u_1, \ldots, u_m)$. For any $n \in \mathbb{N}$ we have*

$$EADPoA(f, \vec{u}, IC) = -\Omega(1).$$

Proof. The key proof technique is to partition $\mathcal{L}(\mathcal{A})^n$ according to each profile's potential winner set. More precisely, for every $W \subseteq \mathcal{A}$ with $W \neq \emptyset$, we define:

$$\overline{\mathrm{PoA}}(W) = \Pr(\mathrm{PW}(a^*) = W) \times \mathbb{E}[\mathrm{D}^+(P) \mid \mathrm{PW}(a^*) = W].$$

By the law of total expectation, then,

$$\mathrm{EADPoA} = \mathbb{E}[\mathrm{D}^+(P)] = \sum\nolimits_{\alpha=1}^{m} \sum\nolimits_{W \subseteq \mathcal{A}:|W|=\alpha} \overline{\mathrm{PoA}}(W) \tag{3.5}$$

where α denotes the number of potential winners in a^*. It is straightforward to see that when $\alpha = 1$, any profile P with $|\mathrm{PW}(a^*)| = 1$ is already a NE, which implies $\mathrm{D}^+(P) = 0$. We demonstrate in Lemma 3.2, below, that for $\forall W \subseteq \mathcal{A}$ with $|W| = 2$ we have $\overline{\mathrm{PoA}}(W) = -\Omega(1)$. We will then demonstrate that $\overline{\mathrm{PoA}}(W) = o(1)$ holds $\forall W \subseteq \mathcal{A}$ with $|W| = 3$ (Lemma 3.3) and $|W| \geq 4$ (if $m \geq 4$; Lemma 3.4). Recalling that m is fixed, the total number of subsets of $\mathcal{A}$ is viewed as a constant. Finally, these results combine to conclude

$$\mathrm{EADPoA} = \underbrace{0}_{\alpha=1} - \underbrace{\Omega(1)}_{\alpha=2} + \underbrace{o(1)}_{\alpha\geq 3} = -\Omega(1).$$

□

Intuitively, profiles with two tied alternatives drive the EADPoA negative because of the self-selecting property of Lemma 3.1. For example, consider a truthful P with $\mathrm{PW}(a^*) = \{1, 2\}$ and $f(a^*) = 1$. When more agents prefer the non-winning alternative 2, in this setting, iterative plurality makes this correction by changing the winner to 2 and increases agents' social welfare on average. When more agents prefer the truthful winner 1, rather, iterative plurality doesn't change this outcome and the adversarial loss remains zero. Without a sufficient counter-balance to the former two-tied ($\alpha = 2$) case by any of the three-or-more-tied cases ($\alpha \geq 3$), the adversarial loss overall remains negative in expectation. The remainder of this section proves the theorem's three lemmas in sequence.

Lemma 3.2 (Two-alternative tied case). *Given $m \geq 3$ and a utility vector $\vec{u}$, for any $W \subseteq \mathcal{A}$ with $|W| = 2$ and any $n \in \mathbb{N}$, we have $\overline{PoA}(W) = -\Omega(1)$.*

Proof. Without loss of generality let $W = \{1, 2\}$ and suppose $u_2 > u_m$, since the case where $u_2 = u_m$ is covered by Brânzei et al. (2013). There are two possible cases of $\mathrm{PW}(a^*) = \{1, 2\}$:

$\mathcal{E}_1 = \mathbb{1}\{s_1(a^*) = s_2(a^*)\}$, where 1 is the truthful winner, and $\mathcal{E}_2 = \mathbb{1}\{s_1(a^*) + 1 = s_2(a^*)\}$, where 2 is the truthful winner. This suggests the following partition:

$$\overline{\text{PoA}}(W) = \Pr(\mathcal{E}_1) \times \mathbb{E}[\text{D}^+(P) \mid \mathcal{E}_1] + \Pr(\mathcal{E}_2) \times \mathbb{E}[\text{D}^+(P) \mid \mathcal{E}_2].$$

We'll focus on the former summand where 1 and 2 are tied and prove that $\Pr(\mathcal{E}_1) \times \mathbb{E}[\text{D}^+(P) \mid \mathcal{E}_1] = -\Omega(1)$. The latter summand can be proved similarly.

We believe this proof is challenging due to the dependence in agents' rankings once we condition on profiles that satisfy two-way ties (i.e. $\mathcal{E}_1$). As a result, standard approximation techniques that assume independence, such as the Berry-Esseen inequality, no longer apply and may also be too coarse to support our claim. Instead, we will use a Bayesian network to further condition agents' rankings based on two properties: the top ranked-alternative and which of the two tied alternatives the agents prefer. Once we guarantee agents' rankings' conditional independence, we can identify the expected utility they gain for each alternative and then compute $\mathbb{E}[\text{D}^+(P) \mid \mathcal{E}_1]$ efficiently.

At a high level, there are two conditions for a profile P to satisfy $\mathcal{E}_1$ and have non-zero adversarial loss. First, the profile must indeed be a two-way tie. This is represented in Step 1 below by identifying each agent j's top-ranked alternative $t_j \in \mathcal{A}$ and conditioning $\text{D}^+(P)$ on a specific vector of top-ranked alternatives $\vec{t} \in \mathcal{T}_2 \subseteq \mathcal{A}^n$, a set corresponding to all profiles satisfying $\mathcal{E}_1$. Second, by Lemma 3.1, the profile should satisfy $P[2 \succ 1] > P[1 \succ 2]$. This is represented in Step 1 by identifying an indicator $z_j \in \{1, 2\}$ to suggest whether $1 \succ_j 2$ or $2 \succ_j 1$ respectively. We further condition $\text{D}^+(P)$ on a specific vector $\vec{z} \in \mathcal{Z}_{\vec{t},k}$, a set corresponding to all profiles in $\mathcal{E}_1$ with $k = P[2 \succ 1] > P[1 \succ 2] = n - k$. Once we condition $\text{D}^+(P)$ to satisfy these two conditions, we identify the expected difference in welfare between the alternatives $\mathbb{E}_{t_j,z_j}$ for each agent j conditioned on t_j, z_j in Step 2, which follows from the impartial culture assumption. Finally, we compute $\text{D}^+(P)$ by summing over all profiles satisfying the above two conditions and solve in Step 3, making use of Stirling's approximation.

More precisely, for any $j \leq n$, we represent agent j's ranking distribution (i.i.d. uniform over $\mathcal{L}(\mathcal{A})$) by a Bayesian network of three random variables: T_j represents the top-ranked alternative, Z_j represents whether $1 \succ_j 2$ or $2 \succ_j 1$, conditioned on T_j, and Q_j represents the linear order conditioned on T_j and Z_j. Formally, we have the following definition.

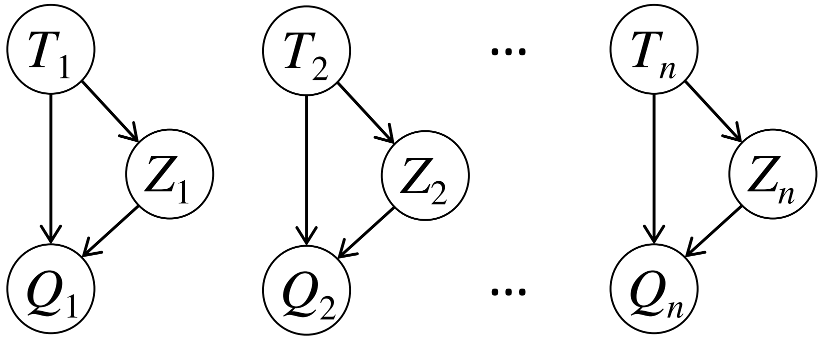

Figure 3.1: Bayesian network representation of P as $\vec{T}$, $\vec{Z}$, and $\vec{Q}$.

Definition 3.3. *For any $j \le n$, we define a Bayesian network with three random variables $T_j \in \mathcal{A}$, $Z_j \in \{1, 2\}$, and $Q_j \in \mathcal{L}(\mathcal{A})$, where T_j has no parent, T_j is the parent of Z_j, and T_j and Z_j are Q_j's parents (see Figure 3.1). Let $\vec{T} = (T_1, , \dots, T_n)$, $\vec{Z} = (Z_1, , \dots, Z_n)$, and $\vec{Q} = (Q_1, , \dots, Q_n)$. The (conditional) distributions are:*

- *T_j follows a uniform distribution over $\mathcal{A}$;*
- $\Pr(Z_j = 1 \mid T_j = t) = \begin{cases} 1, & t = 1 \\ 0, & t = 2 \\ 0.5, & t \in [3, m]; \end{cases}$
- *Q_j follows the uniform distribution over linear orders whose top alternative is T_j and $1 \succ_j 2$ if $Z_j = 1$, or $2 \succ_j 1$ if $Z_j = 2$.*

It is not hard to verify that (unconditional) Q_j follows the uniform distribution over $\mathcal{L}(\mathcal{A})$, which implies that $\vec{Q}$ follows the same distribution as P, namely IC. Notice that if alternative 1 or 2 is ranked at the top, then Z_j is deterministic and equals to T_j. Furthermore, if $T_j \in \{1, 2\}$, then Q_j follows the uniform distribution over $(m-1)!$ linear orders; otherwise Q_j follows the uniform distribution over $(m-1)!/2$ linear orders.

Example 3.1. *Let $m = 4$ and $W = \{1, 2\}$. For every $j \le n$, T_j is the uniform distribution over $\{1, 2, 3, 4\}$. We have that $\Pr(Z_j = 1 \mid T_j = 1) = \Pr(Z_j = 2 \mid T_j = 2) = 1$ and $\Pr(Z_j = 1 \mid T_j = 3) = \Pr(Z_j = 1 \mid T_j = 4) = 0.5$. Given $T_j = Z_j = 1$, Q_j is the uniform* *distribution over*

$$\{[1 \succ 2 \succ 3 \succ 4], [1 \succ 2 \succ 4 \succ 3], [1 \succ 3 \succ 2 \succ 4]$$
$$[1 \succ 3 \succ 4 \succ 2], [1 \succ 4 \succ 2 \succ 3], [1 \succ 4 \succ 3 \succ 2]\}.$$

Given $T_j = 4$ *and* $Z_j = 2$, Q_j *is the uniform distribution over*

$$\{[4 \succ 2 \succ 1 \succ 3], [4 \succ 2 \succ 3 \succ 1], [4 \succ 3 \succ 2 \succ 1]\}.$$

Step 1: Identify profiles that satisfy $\mathcal{E}_1$. Let $\mathcal{T}_2 \subseteq [m]^n$ denote the set of vectors $\vec{t} = (t_1, \ldots, t_n)$ such that alternatives 1 and 2 have the maximum plurality score:

$$\mathcal{T}_2 = \left\{\vec{t} \in [m]^n : \forall 3 \leq i \leq m, |\{j : t_j = 1\}| = |\{j : t_j = 2\}| > |\{j : t_j = i\}|\right\}.$$

$\mathcal{E}_1$ holds for $\vec{Q}$ if and only if $\vec{T}$ takes a value in $\mathcal{T}_2$, implying that

$$\Pr(\mathcal{E}_1) \times \mathbb{E}[\mathrm{D}^+(P) \mid \mathcal{E}_1] = \sum_{\vec{t} \in \mathcal{T}_2} \Pr\left(\vec{T} = \vec{t}\right) \times \mathbb{E}_{\vec{Q}}[\mathrm{D}^+(\vec{Q}) \mid \vec{T} = \vec{t}]. \tag{3.6}$$

Conditioned on agents' top-ranked alternatives being $\vec{t} \in \mathcal{T}_2$, we have by Lemma 3.1 that $\mathrm{D}^+(\vec{Q})$ is non-zero if and only if $\vec{Q}[2 \succ 1] > \vec{Q}[1 \succ 2]$ – thus $\mathrm{EW}(top(\vec{Q})) = \{2\}$ is unique. For any $\vec{t} \in \mathcal{T}_2$, let

- $\mathrm{Id}_1(\vec{t}) \subseteq [n]$ denote the indices j such that $t_j = 1$;
- $\mathrm{Id}_2(\vec{t}) \subseteq [n]$ denote the indices j such that $t_j = 2$;
- $\mathrm{Id}_3(\vec{t}) \subseteq [n]$ denote the indices j such that $t_j \notin \{1, 2\}$ – we call these *third-party* agents.

$\mathcal{E}_1$ implies $|\mathrm{Id}_1(\vec{t})| = |\mathrm{Id}_2(\vec{t})|$, so in order to uphold $\vec{Q}[2 \succ 1] > \vec{Q}[1 \succ 2]$ there must be more third-party agents that prefer $(2 \succ 1)$ than those that prefer $(1 \succ 2)$. Specifically, for every $\lceil \frac{|\mathrm{Id}_3(\vec{t})|+1}{2} \rceil \leq k \leq |\mathrm{Id}_3(\vec{t})|$, we define $\mathcal{Z}_{\vec{t},k} \subseteq \{1, 2\}^n$ as the vectors $\vec{z}$ where the number of 2's among indices in $\mathrm{Id}_3(\vec{t})$ is exactly k:

$$\mathcal{Z}_{\vec{t},k} = \{\vec{z} \in \{1, 2\}^n : \forall j \in \mathrm{Id}_1(\vec{t}) \cup \mathrm{Id}_2(\vec{t}), z_j = t_j, \text{ and } |\{j \in \mathrm{Id}_3(\vec{t}) : z_j = 2\}| = k\}.$$

Example 3.2. *Suppose* $m = 4$*,* $n = 9$*, and* $\vec{t} = (1,1,2,2,3,2,4,1,3)$*. Then,* $Id_1(\vec{t}) = \{1,2,8\}$*,* $Id_2(\vec{t}) = \{3,4,6\}$*,* $Id_3(\vec{t}) = \{5,7,9\}$*. Moreover, for* $k = 2$*, we have*

$$\mathcal{Z}_{\vec{t},2} = \left\{ \begin{array}{c} (1,1,2,2,1,2,2,1,2) \\ (1,1,2,2,2,2,1,1,2) \\ (1,1,2,2,2,2,2,1,1) \end{array} \right\}$$

where exactly two reports from agents 5*,* 7*, or* 9 *are* 2*'s:* $|\{z_j = 2 : j \in \{5,7,9\}\}| = 2$.

Continuing Equation (3.6), we have

$$\begin{aligned}
&\Pr(\mathcal{E}_1) \times \mathbb{E}[\mathrm{D}^+(P) \mid \mathcal{E}_1] \\
&= \sum_{\vec{t} \in \mathcal{T}_2} \sum_{k=\lceil \frac{|\mathrm{Id}_3(\vec{t})|+1}{2} \rceil}^{|\mathrm{Id}_3(\vec{t})|} \sum_{\vec{z} \in \mathcal{Z}_{\vec{t},k}} \Pr(\vec{T} = \vec{t}, \vec{Z} = \vec{z}) \times \mathbb{E}_{\vec{Q}}[\mathrm{D}^+(\vec{Q}) \mid \vec{T} = \vec{t}, \vec{Z} = \vec{z}] \\
&= \sum_{\vec{t} \in \mathcal{T}_2} \sum_{k=\lceil \frac{|\mathrm{Id}_3(\vec{t})|+1}{2} \rceil}^{|\mathrm{Id}_3(\vec{t})|} \sum_{\vec{z} \in \mathcal{Z}_{\vec{t},k}} \Pr(\vec{T} = \vec{t}, \vec{Z} = \vec{z}) \sum_{j=1}^{n} \mathbb{E}_{\vec{Q}_j}[\vec{u}(Q_j, 1) - \vec{u}(Q_j, 2) \mid \vec{T} = \vec{t}, \vec{Z} = \vec{z}] \\
&= \sum_{\vec{t} \in \mathcal{T}_2} \sum_{k=\lceil \frac{|\mathrm{Id}_3(\vec{t})|+1}{2} \rceil}^{|\mathrm{Id}_3(\vec{t})|} \sum_{\vec{z} \in \mathcal{Z}_{\vec{t},k}} \Pr(\vec{T} = \vec{t}, \vec{Z} = \vec{z}) \sum_{j=1}^{n} E_{t_j, z_j} \qquad (3.7)
\end{aligned}$$

where

$$E_{t_j,z_j} = \mathbb{E}_{\vec{Q}_j}[\vec{u}(Q_j, 1) - \vec{u}(Q_j, 2) \mid T_j = t_j, Z_j = z_j].$$

The last equation holds because of the Bayesian network structure: for any $j \le n$, given T_j and Z_j, Q_j is independent of other Q's.

Step 2: Compute expected welfare difference per agent. Notice that E_{t_j,z_j} only depends on the values of t_j, z_j but not j:

- If $t_j = z_j = 1$, then $E_{t_j,z_j} = u_1 - \frac{u_2 + \ldots + u_m}{m-1}$, the expected utility of alternative 2.
- If $t_j = z_j = 2$, then E_{t_j,z_j} is the expected utility of alternative 1, which is $\frac{u_2 + \ldots + u_m}{m-1}$, minus u_1. Notice that $E_{2,2} + E_{1,1} = 0$.
- If $t_j \notin \{1,2\}$ and $z_j = 1$, then $\eta = E_{t_j,1}$ is the expected utility difference of alternatives 1 minus 2, conditioned on third-party agents and $1 \succ 2$. Note that $\eta > 0$ since $u_2 > u_m$.

- If $t_j \notin \{1,2\}$ and $z_j = 2$, then $E_{t_j,2}$ is the expected utility difference of alternative 1 minus 2, conditioned on third-party agents and $2 \succ 1$. It follows that $E_{t_j,2} = -\eta$.

As a result, Equation (3.7) becomes

$$\sum_{\vec{t}\in\mathcal{T}_2} \sum_{k=\lceil\frac{|\mathrm{Id}_3(\vec{t})|+1}{2}\rceil}^{|\mathrm{Id}_3(\vec{t})|} \sum_{\vec{z}\in\mathcal{Z}_{\vec{t},k}} \Pr(\vec{T}=\vec{t},\vec{Z}=\vec{z})(|\mathrm{Id}_3(\vec{t})|-2k)\eta \tag{3.8}$$

where we've inserted

$$\sum_{j=1}^{n} E_{t_j,z_j} = |\mathrm{Id}_1(\vec{t})|E_{1,1} + |\mathrm{Id}_2(\vec{t})|E_{2,2} - k\eta + (|\mathrm{Id}_3(\vec{t})|-k)\eta.$$

Step 3: Simplify and solve. Note that $|\mathrm{Id}_3(\vec{T})|$ is equivalent to the sum of n i.i.d. binary random variables, each of which is 1 with probability $\frac{m-2}{m} \geq \frac{1}{3}$. By Hoeffding's inequality, with exponentially small probability we have $|\mathrm{Id}_3(\vec{T})| < \frac{1}{6}n$. We may therefore focus on the $\mathrm{Id}_3(\vec{T}) \geq \frac{1}{6}n$ case of Equation (3.8), which, by denoting $\beta = |\mathrm{Id}_3(\vec{t})|$ for ease of notation, becomes

$$\begin{aligned}
&\leq \exp^{-\Theta(n)} + \sum_{\vec{t}\in\mathcal{T}_2:\beta\geq\frac{1}{6}n} \sum_{k=\lceil\frac{\beta+1}{2}\rceil}^{\beta} \sum_{\vec{z}\in\mathcal{Z}_{\vec{t},k}} \Pr(\vec{T}=\vec{t},\vec{Z}=\vec{z})(\beta-2k)\eta \\
&= \exp^{-\Theta(n)} + \sum_{\vec{t}\in\mathcal{T}_2:\beta\geq\frac{1}{6}n} \sum_{k=\lceil\frac{\beta+1}{2}\rceil}^{\beta} (\beta-2k)\eta \sum_{\vec{z}\in\mathcal{Z}_{\vec{t},k}} \Pr(\vec{Z}=\vec{z} \mid \vec{T}=\vec{t})\Pr(\vec{T}=\vec{t}) \\
&= \exp^{-\Theta(n)} + \sum_{\vec{t}\in\mathcal{T}_2:\beta\geq\frac{1}{6}n} \sum_{k=\lceil\frac{\beta+1}{2}\rceil}^{\beta} (\beta-2k)\eta \left(\frac{1}{2}\right)^{\beta} \binom{\beta}{k} \Pr(\vec{T}=\vec{t}) && (3.9) \\
&= \exp^{-\Theta(n)} + \sum_{\vec{t}\in\mathcal{T}_2:\beta\geq\frac{1}{6}n} \left(\frac{1}{2}\right)^{\beta} \eta \Pr(\vec{T}=\vec{t}) \sum_{k=\lceil\frac{\beta+1}{2}\rceil}^{\beta} \binom{\beta}{k}(\beta-2k) \\
&= \exp^{-\Theta(n)} - \eta \sum_{\vec{t}\in\mathcal{T}_2:\beta\geq\frac{1}{6}n} \left(\frac{1}{2}\right)^{\beta} \left(\left\lceil\frac{\beta+1}{2}\right\rceil\right) \binom{\beta}{\lceil\frac{\beta+1}{2}\rceil} \Pr(\vec{T}=\vec{t}) && (3.10)
\end{aligned}$$

where Equation (3.9) follows from $\Pr(Z_j = 1 \mid T_j \notin \{1,2\}) = 0.5$ and Equation (3.10) follows from the following claim, plugging in $n \leftarrow \beta$ and $p \leftarrow \lceil\frac{\beta+1}{2}\rceil$.

Claim 3.1. *For any* $n \in \mathbb{N}$ *and any* $p \in [0, n]$*, we have*

$$\sum_{k=p}^{n} \binom{n}{k}(n-2k) = -p\binom{n}{p}.$$

Proof. We begin by considering a negated form of the objective:

$$\begin{aligned}
\sum_{k=p}^{n} \binom{n}{k}(n-2k) &= \sum_{k=0}^{n} \binom{n}{k}(n-2k) - \sum_{k=0}^{p-1} \binom{n}{k}(n-2k) \\
&= n2^n - 2(n2^{n-1}) - \sum_{k=0}^{p-1} \binom{n}{k}(n-2k) \\
&= -\sum_{k=0}^{p-1} \binom{n}{k}(n-2k).
\end{aligned}$$

The proof continues by induction. We want to show that for all $p \in [n]$,

$$\sum_{k=0}^{p-1} \binom{n}{k}(n-2k) = p\binom{n}{p}. \tag{3.11}$$

Base step. Substituting $p = 1$ into Equation (3.11) yields

$$\binom{n}{0}(n-0) = n = 1\binom{n}{1}.$$

<u>**Inductive step.**</u> Suppose Equation (3.11) holds for all $p \in [n']$ for some $n' < n$. We want to show this holds for $p+1$. We have

$$\begin{aligned}
\sum_{k=0}^{p} \binom{n}{k}(n-2k) &= \sum_{k=0}^{p-1} \binom{n}{k}(n-2k) + \binom{n}{p}(n-2p) \\
&= p\binom{n}{p} + \binom{n}{p}(n-2p) \\
&= (n-p)\binom{n}{p} \\
&= \frac{n!(n-p)}{p!(n-p)!} \\
&= \frac{n!(p+1)}{(p+1)!(n-p-1)!} \\
&= (p+1)\binom{n}{p+1}
\end{aligned}$$

as desired, where we've used the induction hypothesis to get the second equality. □

We next apply Stirling's approximation to simplify Equation (3.10). Recall that Stirling's approximation is that $\forall n \in \mathbb{N}$, $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. It is easy to show then that $\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{n\pi}}$. Since $\beta = \Theta(n)$, we get that $\left(\lceil\frac{\beta+1}{2}\rceil\right)\binom{\beta}{\lceil\frac{\beta+1}{2}\rceil} = \Theta(\sqrt{n}\ 2^n)$. Hence, Equation (3.10) becomes

$$\begin{aligned}
&\exp^{-\Theta(n)} - \eta \sum_{\vec{t} \in \mathcal{T}_2 : \beta \geq \frac{1}{6}n} \Theta(\sqrt{n}) \Pr(\vec{T} = \vec{t}) \\
&= \exp^{-\Theta(n)} - \Theta(\sqrt{n}) \Pr\left(\vec{T} \in \mathcal{T}_2,\ \mathrm{Id}_3(\vec{T}) \geq \frac{1}{6}n\right) \\
&= \exp^{-\Theta(n)} - \Theta(\sqrt{n}) \left(\Pr(\vec{T} \in \mathcal{T}_2) - \Pr\left(\vec{T} \in \mathcal{T}_2,\ \mathrm{Id}_3(\vec{T}) < \frac{1}{6}n\right)\right) \\
&\leq \exp^{-\Theta(n)} - \Theta(\sqrt{n}) \left(\Pr(\vec{T} \in \mathcal{T}_2) - \Pr\left(\mathrm{Id}_3(\vec{T}) < \frac{1}{6}n\right)\right) \\
&\leq \exp^{-\Theta(n)} - \Theta(\sqrt{n}) \left(\Theta\left(\frac{1}{\sqrt{n}}\right) - e^{-\Theta(n)}\right) \\
&= -\Omega(1)
\end{aligned}$$

where $\Pr(\vec{T} \in \mathcal{T}_2)$ is equivalent to the probability of a two-way tie under plurality w.r.t. IC, which is known to be $\Theta\left(\frac{1}{\sqrt{n}}\right)$ by Corollary 2.1. This proves Lemma 3.2. □

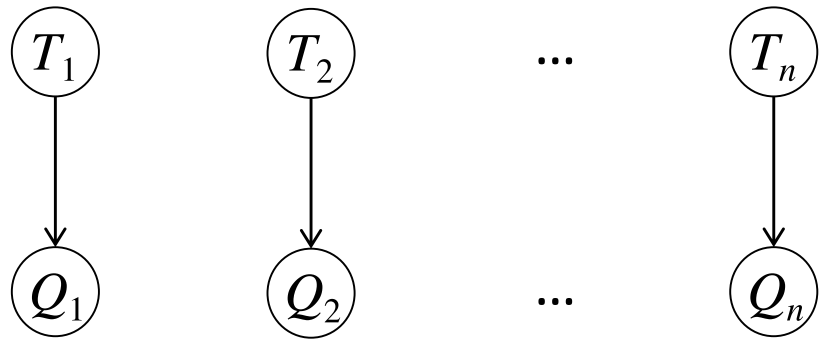

Figure 3.2: Bayesian network representation of P as $\vec{T}$ and $\vec{Q}$.

Lemma 3.3 **(Three-alternative tied case).** *Given $m \geq 3$ and utility vector $\vec{u}$, for any $W \subseteq \mathcal{A}$ with $|W| = 3$ and any $n \in \mathbb{N}$, we have $\overline{PoA}(W) = o(1)$.*

Proof. The proof uses a similar yet simpler technique than that of Lemma 3.2. Without loss of generality, suppose $W = \{1, 2, 3\}$ and consider the case where the plurality scores for 1, 2, and 3 are equal, denoted $\mathcal{E}$. The proofs for cases with alternatives 2 or 3 being truthful winners are similar. We first prove that conditioned on the vector $\vec{t}$ of all agents' top preferences that satisfy $\mathcal{E}$, the maximum score difference between any pair of alternatives in $\{1, 2, 3\}$ is $o(n)$ with high probability that is close to 1. Secondly, it is clear that $\Pr(\text{PW}(a^*) = W) = \Theta\left(\frac{1}{n}\right)$ by Corollary 2.1. Put together, this yields

$$\overline{\text{PoA}}(W) = \Pr(\text{PW}(a^*) = W) \times \mathbb{E}[\text{D}^+(P) \mid \text{PW}(a^*) = W] = \Theta\left(\frac{1}{n}\right) o(n) = o(1).$$

More precisely, for every $j \leq n$, we represent agent j's ranking distribution (i.i.d. uniform over $\mathcal{L}(\mathcal{A})$) by a Bayesian network of two random variables: T_j represents agent j's top-ranked alternative, and Q_j represents j's ranking conditioned on T_j. Formally, we have the following definition.

Definition 3.4. *For any $j \leq n$, we define a Bayesian network with two random variables $T_j \in \mathcal{A}$ and $Q_j \in \mathcal{L}(\mathcal{A})$, where T_j has no parent and is the parent of Q_j (see Figure 3.2). Let $\vec{T} = (T_1, , \ldots, T_n)$ and $\vec{Q} = (Q_1, , \ldots, Q_n)$. The (conditional) distributions are:*

- *T_j follows a uniform distribution over $\mathcal{A}$;*
- *Q_j follows the uniform distribution over linear orders whose top alternative is T_j.*

It is not hard to verify that (unconditional) Q_j follows the uniform distribution over $\mathcal{L}(\mathcal{A})$. Therefore, $\vec{Q}$ follows the same distribution as P, which is IC.

Example 3.3. *Let $m = 4$ and $W = \{1, 2, 3\}$. For every $j \leq n$, T_j is the uniform distribution over $[4]$. Given $T_j = 1$, Q_j is the uniform distribution over*

$$\begin{aligned}&\{[1 \succ 2 \succ 3 \succ 4], [1 \succ 2 \succ 4 \succ 3], [1 \succ 3 \succ 2 \succ 4],\\&\ [1 \succ 3 \succ 4 \succ 2], [1 \succ 4 \succ 2 \succ 3], [1 \succ 4 \succ 3 \succ 2]\}.\end{aligned}$$

Given $T_j = 4$, Q_j is the uniform distribution over

$$\begin{aligned}&\{[4 \succ 1 \succ 2 \succ 3], [4 \succ 1 \succ 3 \succ 2], [4 \succ 3 \succ 1 \succ 2],\\&\ [4 \succ 2 \succ 1 \succ 3], [4 \succ 2 \succ 3 \succ 1], [4 \succ 3 \succ 2 \succ 1]\}.\end{aligned}$$

Step 1: Identify $\mathcal{E}$. Let $\mathcal{T}_3 \subseteq [m]^n$ denote the set of vectors $\vec{t} = (t_1, \ldots, t_n)$ such that alternatives 1, 2, and 3 have the maximum plurality score. Formally,

$$\mathcal{T}_3 = \left\{\vec{t} \in [m]^n : \forall 4 \leq i \leq m, |\{j : t_j = 1\}| = |\{j : t_j = 2\}| = |\{j : t_j = 3\}| > |\{j : t_j = i\}|\right\}.$$

$\mathcal{E}$ holds for $\vec{Q}$ if and only if $\vec{T}$ takes a value in $\mathcal{T}_3$, implying the following equality.

$$\begin{aligned}&\overline{\mathrm{PoA}}(\{1, 2, 3\}) && (3.12)\\&= \Pr\left(\mathrm{PW}(top(\vec{Q})) = \{1, 2, 3\}\right) \times \mathbb{E}[\mathrm{D}^+(\vec{Q}) \mid \mathrm{PW}(top(\vec{Q})) = \{1, 2, 3\}]\\&= \sum_{\vec{t} \in \mathcal{T}_3} \Pr(\vec{T} = \vec{t}) \times \mathbb{E}[\mathrm{D}^+(\vec{Q}) \mid \vec{T} = \vec{t}] && (3.13)\end{aligned}$$

Step 2: Upper-bound the conditional adversarial loss. We next employ the law of total expectation on Equation (3.13) by further conditioning on $\mathbb{1}\{\mathrm{D}^+(\vec{Q}) > n^{0.6}\}$. This event represents whether the adversarial loss scales positively and at least sub-linearly in n. We will show this holds with high probability and establish the following conditional expectation to be $o(n)$, term-by-term:

$$\begin{aligned}
&\mathbb{E}[\mathrm{D}^+(\vec{Q}) \mid \vec{T} = \vec{t}] \\
&= \mathbb{E}[\mathrm{D}^+(\vec{Q}) \mid \vec{T} = \vec{t}, \mathrm{D}^+(\vec{Q}) > n^{0.6}] \times \Pr(\mathrm{D}^+(\vec{Q}) > n^{0.6} \mid \vec{T} = \vec{t}) \\
&+ \mathbb{E}[\mathrm{D}^+(\vec{Q}) \mid \vec{T} = \vec{t}, \mathrm{D}^+(\vec{Q}) \leq n^{0.6}] \times \Pr(\mathrm{D}^+(\vec{Q}) \leq n^{0.6} \mid \vec{T} = \vec{t}).
\end{aligned}$$

First, trivially, we note that

$$\mathbb{E}[\mathrm{D}^+(\vec{Q}) \mid \vec{T} = \vec{t}, \mathrm{D}^+(\vec{Q}) \leq n^{0.6}] \leq n^{0.6}. \tag{3.14}$$

Second, for any $t \in [m]$ and $i_1, i_2 \in \{1, 2, 3\}$ with $i_1 \neq i_2$, we denote by $D^t_{i_1,i_2}$ the random variable representing the utility difference between alternatives i_1 and i_2 in Q_j, conditioned on $T_j = t$:

$$D^t_{i_1,i_2} = \vec{u}(Q_j, i_1) - \vec{u}(Q_j, i_2)$$

For any $\vec{t} \in [m]^n$ and $j \leq n$, $D^{t_j}_{i_1,i_2} \in [u_m - u_1, u_1 - u_m]$, which implies $\mathrm{D}^+(\vec{Q}) \leq (u_1 - u_m)n$, and henceforth

$$\mathbb{E}[\mathrm{D}^+(\vec{Q}) \mid \vec{T} = \vec{t}, \mathrm{D}^+(\vec{Q}) > n^{0.6}] \leq (u_1 - u_m)n. \tag{3.15}$$

Third, we observe that:

- $\mathbb{E}[D^{t_j}_{i_1,i_2}] > 0$ if $t_j = i_1$;
- $\mathbb{E}[D^{t_j}_{i_1,i_2}] = -\mathbb{E}[D^{i_1}_{i_1,i_2}] < 0$ if $t_j = i_2$;
- $\mathbb{E}[D^{t_j}_{i_1,i_2}] = 0$ otherwise.

Let $D^{\vec{t}}_{i_1,i_2} = \sum_{j=1}^{n} D^{t_j}_{i_1,i_2}$. It follows that for any $\vec{t} \in \mathcal{T}_3$ we have $\mathbb{E}[D^{\vec{t}}_{i_1,i_2}] = 0$, since $\mathcal{E}$ implies $|\{j : t_j = i_1\}| = |\{j : t_j = i_2\}|$. Recalling that $D^{t_j}_{i_1,i_2}$ is bounded, it follows from Hoeffding's inequality that

$$\Pr(|D^{\vec{t}}_{i_1,i_2}| > n^{0.6}) = \exp(-\Theta(n^{0.2})).$$

Recall that the equilibrium winner must be among the initial potential winners of any truthful profile (Reyhani & Wilson, 2012). Therefore, for any $\vec{t} \in \mathcal{T}_3$, following the law of total probability, we have

$$\Pr\left(\mathrm{D}^+(\vec{Q}) > n^{0.6} \mid \vec{T} = \vec{t}\right) \leq 6 \exp(-\Theta(n^{0.2})). \tag{3.16}$$

Combining Equations (3.14), (3.15), and (3.16) with Equation (3.13) yields our claim:

$$\begin{aligned}
&\overline{\mathrm{PoA}}(\{1,2,3\}) \\
&= \sum_{\vec{t}\in\mathcal{T}_3} \Pr(\vec{T}=\vec{t}) \times \mathbb{E}[\mathrm{D}^+(\vec{Q}) \mid \vec{T}=\vec{t}] \\
&\leq \sum_{\vec{t}\in\mathcal{T}_3} \Pr(\vec{T}=\vec{t}) \left[6n(u_1-u_m)\exp(-\Theta(n^{0.2})) + n^{0.6}(1-6\exp(-\Theta(n^{0.2})))\right] \\
&= \Pr(\vec{T}\in\mathcal{T}_3)o(n) \\
&= o(1)
\end{aligned}$$

where $\Pr(\vec{T}\in\mathcal{T}_3)$ is the probability of a three-way tie, known to be $\Theta\left(\frac{1}{n}\right)$ by Corollary 2.1. $\square$

Lemma 3.4 (Four-or-more alternative tied case). *Given $m \geq 4$ and a utility vector $\vec{u}$, for any $W \subseteq \mathcal{A}$ with $|W| \geq 4$ and any $n \in \mathbb{N}$, we have $\overline{PoA}(W) = o(1)$.*

Proof. The lemma follows after noticing that $\Pr(\mathrm{PW}(a^*) = W) = \mathcal{O}\left(\frac{1}{n^{1.5}}\right)$ by Corollary 2.1 and for any profile P, $\mathrm{D}^+(P) = \mathcal{O}(n)$. $\square$

3.4 Experiments

Figures 3.3 and 3.4 were generated by fixing $m = 4$ alternatives with Borda utility $\vec{u} = (3,2,1,0)$ and varying the number of agents. For each $n \in \{100, 200, \ldots, 1000\}$, we sampled 10 million profiles uniformly at random and determined, for each $P \sim IC$, its equilibrium winning set $\mathrm{EW}(a^*)$. We then computed each profile's adversarial loss $\mathrm{D}^+(P)$ and averaged their values across all profiles with the same n. Experiments were run on an Intel Core i7-7700 CPU running Windows with 16.0 GB of RAM.

Figure 3.3 demonstrates the sample average adversarial loss using these parameters. Figure 3.4 partitions the loss based on 2-, 3-, and 4-way ties. We note the average adversarial loss decreases as n increases and takes the trend of the 2-way tie loss. Since a significant proportion of profiles have no BR dynamics, the overall trend keeps close to zero. Therefore these results support our main theorem in this paper, that the welfare of the worst-case strategic equilibrium winner is greater than that of the truthful winner when agents' preferences are distributed according to IC.

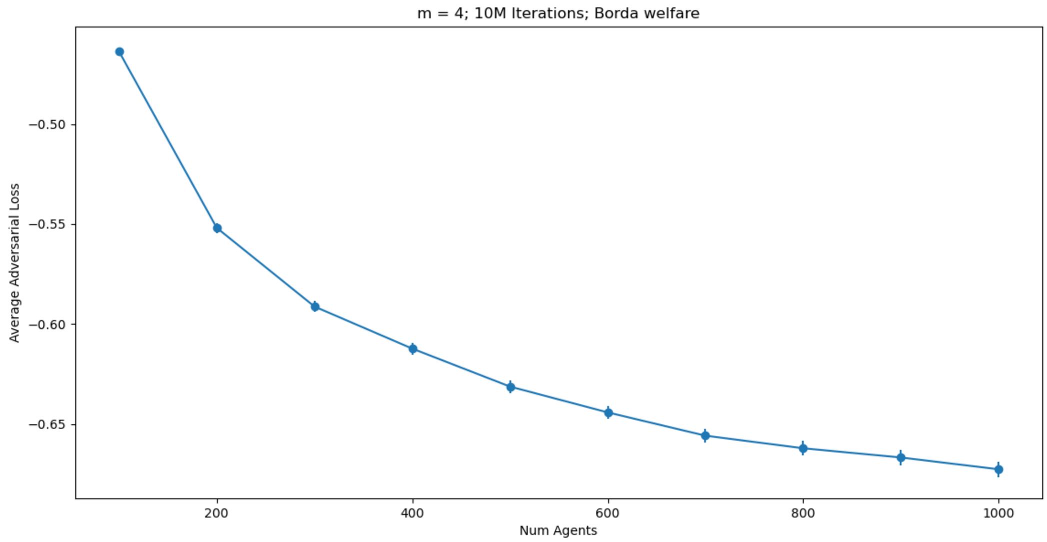

Figure 3.3: Average adversarial loss with $m = 4$, Borda $\vec{u}$, and 10M samples. Error bars represent 95% confidence intervals, too small to see.

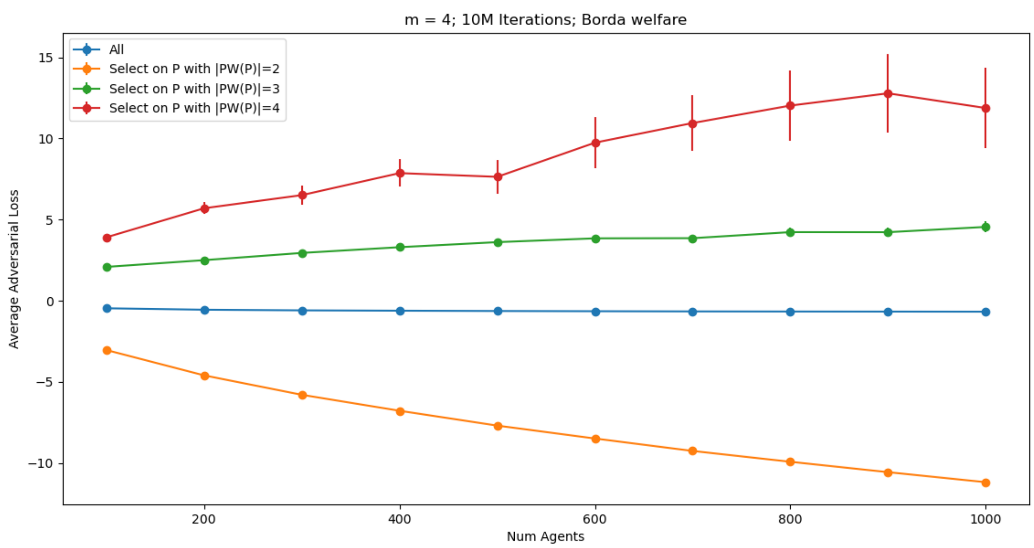

Figure 3.4: Average adversarial loss conditioned on 2-, 3-, and 4-way ties. Error bars represent 95% confidence intervals.

CHAPTER 4
CONVERGENCE OF MULTI-ISSUE ITERATIVE VOTING UNDER UNCERTAINTY

4.1 Introduction

Consider a wedding planner who is deciding a wedding's banquet and wants to accommodate the party invitees' preferences. There are three issues with two alternatives each: the main course (chicken or beef), the paired wine (red or white), and the cake flavor (chocolate or vanilla). How should the planner proceed? On the one hand, they could request each attendee's (agent's) full preference ranking over the 2^p alternatives, for p binary issues. However, aggregating these preferences is computationally prohibitive and eliciting them imposes a high cognitive cost for agents. On the other hand, the planner could solicit only agents' votes and decide each issue independently. Although simpler, this option admits *multiple election paradoxes* whereby agents can collectively select each of their least favored outcomes. For example, suppose three agents prefer $(1,1,0)$, $(1,0,1)$, and $(0,1,1)$ first, respectively on the issues, and all prefer $(1,1,1)$ last. Then the agents select $(1,1,1)$ by majority rule on each issue independently (Lacy & Niou, 2000). A third approach is to decide the issues in sequence and have agents vote for their preferred alternative amongst the *current* issue given the previously chosen outcomes. Still, the joint outcome may depend on the voting agenda and agents may be uneasy voting on the current issue if their preference depends on the outcomes of later issues (Conitzer et al., 2009).

In this chapter, we study *iterative voting* (IV) as a different yet natural method for deciding multiple issues (Meir et al., 2010). We elicit agents' most preferred alternatives and, given information about others' votes, allow agents to update their reports before finalizing the group decision. This approach combines the efficiency of simultaneous voting with the dynamics of sequential voting, thus incorporating information about agents' lower-ranked preferences without directly eliciting them. Like the former approach, agents only report their most preferred alternative. Like the latter approach, agents only update one issue at a time but are unrestricted in the order of improvements.

Portions of this chapter have previously appeared as: Kavner, J., Meir, R., Rossi, F., & Xia, L. (2023, August). Convergence of multi-issue iterative voting under uncertainty. In *Proceedings of the 32nd international joint conference on artificial intelligence* (pp. 2783-2791).

IV is an effective framework for its adaptability to various information and behavioral schemes. First, we consider agents with full information about the real vote profile, such as in online Doodle polls (Zou et al., 2015), who update their votes to the *best response* of all others. Second, we consider agents with access only to a noisy signal about the real vote profile, such as from imprecise opinion polls (Reijngoud & Endriss, 2012) or latency in a networked system if they can only periodically retrieve accurate vote counts. These agents update their votes to those that *locally dominate* their prior reports – votes that achieve weakly better outcomes for all possible vote profiles and strictly better outcomes for some possible vote profile (Meir et al., 2014). We ask two primary questions:

1. *Under what conditions does multi-issue IV converge?*

2. *How does introducing and increasing uncertainty affect the rate of convergence?*

Prior work in single-issue IV offers mixed answers, as iterative plurality and veto have strong convergence guarantees but many other rules do not (Meir et al., 2017). This leaves us with mixed hope in the multi-issue plurality case, and if so, that it can solve other problems like multiple election paradoxes. Furthermore, in contrast to prior work, uncertainty for multiple issues plays a double role. First, like the single-issue case, agents consider themselves as possibly pivotal on any issue that is sufficiently close to a tie. Second—and this part is new—agents may be uncertain whether changing their vote on an issue would improve or degrade the outcome, as this may depend on the outcomes of other uncertain issues.

4.1.1 Our Contribution

On the conceptual side, we introduce a novel model that synthesizes prior work in local dominance strategic behavior, iterative plurality voting, and simultaneous voting over multiple issues. This generalized model naturally captures both types of uncertainty discussed above.

On the technical side, we first show that IV with or without uncertainty may not converge. We then present two model refinements that prove sufficient to guarantee convergence for binary issues: restricting agent preferences to have $\mathcal{O}$-legal preferences and alternating uncertainty, in which agents are more certain about the current issue than others. The former converges because agents' preferences on issues are not interdependent; the latter because fewer preference rankings yield valid improvement steps. These convergence results do not extend to the multi-alternative issues setting, as IV may cycle if agents have partial order preference information. Our convergence results for binary issues also hold for a nonatomic variant of plurality IV in which agents are part of a large population and arbitrary subsets of agents may change their vote simultaneously, establishing more general convergence results. This is discussed in Section 4.5.

We conclude with empirical evidence corroborating our findings that introducing uncertainty eliminates almost all cycles in IV for multiple binary issues. Our experiments further suggest IV improves the quality of equilibrium vote profiles relative to their respective truthful profiles, thus reducing multiple election paradoxes. Increasing uncertainty yields faster convergence but degrades this welfare improvement.

4.2 Preliminaries

In this section, we extend the notation and concepts about social choice and iterative improvement dynamics from Chapter 2 to the multi-issue setting. Some concepts may be reiterated in this context for completeness.

4.2.1 Multi-issue Social Choice

Basic model. Let $\mathcal{P} = \{1, 2, \dots, p\}$ be the set of p issues over the joint domain $\mathcal{D} = \times_{i=1}^{p} D^i$, where D^i is the finite domain of *alternatives* for issue i. We call the issues *binary* if $D^i = \{0, 1\}$ for each $i \in \mathcal{P}$ or *multi-alternative* otherwise. Each of $n \in \mathbb{N}$ agents is endowed with a preference *ranking* $R_j \in \mathcal{L}(\mathcal{D})$, the set of strict linear orders over the $m = \prod_{i=1}^{p} |D^i|$ alternatives. We call the collection of agents' preferences $P = (R_1, \dots, R_n)$ a *preference profile* and each agent's most preferred alternative their *truthful* vote. A *vote profile* $a = (a_1, \dots, a_n) \in \mathcal{D}^n$ is a collection of *votes*, where $a_j \in (a_j^1, \dots, a_j^p) \in \mathcal{D}$ collects agent j's single-alternative vote per issue. A resolute *voting rule* $f : \mathcal{L}(\mathcal{D})^n \to \mathcal{D}$ maps vote profiles onto a unique outcome. We call $a \in \mathcal{D}$ and $a^i \in D^i$ for $i \in \mathcal{P}$ an *alternative* or *outcome* synonymously.

Simultaneous plurality voting. A local voting rule, applied to each issue independently, is *simultaneous* if issues' outcomes are revealed to agents at the same time. It is *sequential* according to the order $\mathcal{O} = \{o_1, \dots, o_p\}$ if outcomes of each issue o_i are revealed to agents prior to voting on the next issue o_{i+1} (Lacy & Niou, 2000). We focus on simultaneous plurality voting and adapt the framework of Xia et al. (2011).

The plurality rule $f^i(a)$ applied to vote profile a on issue i only depends on the total number of votes for each of its alternatives. We define the *score tuple* $s(a) = (s^i(a))_{i\in\mathcal{P}}$ as a collection of *score vectors* $s^i(a) = (s^i(c;a))_{c\in D_i}$, which compose the score of a alternative $c \in D_i$ as $s^i(c;a) = |\{j \leq n \ : \ a^i_j = c\}|$. We use the plurality rule $f(a) = (f^i(a))_{i\in\mathcal{P}} \in \mathcal{D}$, where $f^i(a) = \arg\max_{c\in D_i} s^i(c;a)$, breaking ties lexicographically on each issue.

Let a_{-j} denote the vote profile without agent j and $(a_{-j}, \hat{a}_j)$ the profile a by replacing j's vote with the prospective vote $\hat{a}_j$. Then s_{-j} and $s_{-j} + \hat{a}_j$ denote corresponding adjusted score tuples without j and upon replacing j's vote. We may interchange s, $s(a)$, and a for ease of notation.

Preferential dependence. Whenever there are $p \geq 2$ issues, agents have varying levels of expressiveness about their preference rankings. First, note that given any two alternatives $c, \tilde{c} \in \mathcal{D}$, agents can always answer whether they prefer c or $\tilde{c}$. Second, given two alternatives $c^i, \tilde{c}^i \in D^i$ and the outcomes of all other issues besides $i \in \mathcal{P}$, $\{f^p\}_{p\in\mathcal{P}\setminus\{i\}}$, agents can always answer whether they prefer c^i or $\tilde{c}^i$. However, agents may have problems reporting their preferences when solicited about specific issues if they are not given enough information about issues' outcomes. For instance, recall that for sequential voting, with respect to an order $\mathcal{O} = \{o_1, \ldots, o_p\}$ over the issues, agents are given the outcomes of prior issues $\{o_1, o_2, \ldots, o_{i-1}\}$ and must subsequently report their votes over the alternatives D^{o_i} about the issue o_i (Lang & Xia, 2009). Agents whose preferences about D^{o_i} depend on the outcome of an issue o_k later in the order, $k > i$, may not be able to precisely report their votes. Rather, only agents with $\mathcal{O}$-*legal* preferences can report their votes for sequential voting rules. We formalize the possibility of eliciting an agent's preference over alternatives about a single issue through the following definitions, due to Lang and Xia (2009).

Definition 4.1 (Conditional preferential independence)**.** *Let $\mathcal{Q} \subsetneq \mathcal{P}$ and $\{f^q\}_{q\in\mathcal{Q}}$ be outcomes of those issues. The issue $i \in \mathcal{P}\backslash\mathcal{Q}$ is conditionally independent of the issues $\mathcal{P}\backslash(\mathcal{Q} \cup \{i\})$ with respect to ranking R if and only if, given $\{f^q\}_{q\in\mathcal{Q}}$, the relative ordering of alternatives in D^i is constant in R for any combination of outcomes $\{f^p\}_{p\in\mathcal{P}\backslash(\mathcal{Q}\cup\{i\})}$.*

Definition 4.2 ($\mathcal{O}$-legal preferences)**.** *Given an order $\mathcal{O} = \{o_1, \ldots, o_p\}$ over the issues, the ranking R is called* $\mathcal{O}$-legal *if $\forall i \in \mathcal{P}$, o_i is preferentially conditionally independent of*

$o_{i+1}, \ldots, o_p$ *given* $o_1, \ldots, o_{i-1}$*. The preference profile* P *is* $\mathcal{O}$-legal *if every ranking is* $\mathcal{O}$*-legal for the same order* $\mathcal{O}$*;* R *is* separable *if it is* $\mathcal{O}$*-legal for* any *order* $\mathcal{O}$*.*

Separable rankings have the advantage that agents may express their preferences on individual issues and avoid multiple-election paradoxes, but it is a very demanding assumption (Hodge, 2002; Lacy & Niou, 2000; Xia et al., 2011). Relaxing rankings to be $\mathcal{O}$-legal maintains representation compactness without permitting arbitrary preferential dependencies.

Example 4.1. *Consider* $p = 2$ *binary issues and* $n = 3$ *agents with preference profile* $P = (R_1, R_2, R_3)$ *such that:*

- $R_1 : (1,0) \succ_1 (0,0) \succ_1 (0,1) \succ_1 (1,1)$*;*
- $R_2 : (1,1) \succ_2 (0,0) \succ_2 (0,1) \succ_2 (1,0)$*;*
- $R_3 : (0,0) \succ_3 (0,1) \succ_3 (1,0) \succ_3 (1,1)$*.*

The truthful vote profile $a = ((1,0), (1,1),\ (0,0))$ *consists of each agent's most preferred alternative. The score tuple is* $s(a) = \{(1,2),(2,1)\}$*, so the plurality outcome is* $f(a) = (1,0)$*.*

Notice that R_1 *is* $\mathcal{O}$*-legal for* $\mathcal{O} = \{2,1\}$*. That is, the agent always prefers* $0 \succ 1$ *on the second issue, yet their preference for the first issue depends on* f^2*. Next,* R_3 *is separable, as the agent prefers* $0 \succ 1$ *on each issue independent of the other issue's outcome. Third,* R_2 *is neither separable nor* $\mathcal{O}$*-legal for any* $\mathcal{O}$*.*

Finally, notice that agent 2 can improve the outcome for themselves by voting for $\hat{a}_2 = (0,1)$ *instead of* $a_2 = (1,1)$*. The adjusted score tuple is* $s_{-2} = \{(1,1),(2,0)\}$*, so* $s_{-2} + \hat{a}_2 = \{(2,1),(2,1)\}$ *and* $f(s_{-2} + \hat{a}_2) = (0,0) \succ_2 (1,0) = f(a)$*.*

4.2.2 Improvement Dynamics

We implement the iterative voting (IV) procedure introduced by Meir et al. (2010) for the plurality choice rule f and refined for uncertainty by Meir et al. (2014) and Meir (2015). Most definitions carry-over from the single-issue setting, introduced in Chapter 2, to the multi-issue setting of this chapter. However, there are notable differences in agents' improvement dynamics due to our relaxations of implementing IV over multiple issues and of what information agents have access to when performing their updates. In this chapter, we study two response functions: *best response (BR)* dynamics, where agents know the *real* score tuple $s(a)$, and *local dominance improvement (LDI)* dynamics, where agents have uncertainty over $s(a)$. For both dynamics, we restrict agents to only changing their vote on a single *current* issue $i \in \mathcal{P}$ per round, as determined by the scheduler. We do not necessarily assume IV begins from the truthful vote profile, unlike the other chapters. We therefore have the following form of convergence, as described by Kukushkin (2011), Monderer and Shapley (1996b), and Milchtaich (1996).

Definition 4.3. *An IV dynamic has the* restricted-finite improvement property *if every improvement sequence is finite from any initial vote profile for a given response function.*

Under BR dynamics, each agent j has full information about $s(a(t))$ and chooses the vote $\hat{a}_j$ that yields the best possible outcome $f(a(t+1))$ in the resulting vote profile with respect to their preferences R_j, subject to changing one issue $i \in \mathcal{P}$ at a time. This is formally put, as follows.

Definition 4.4 (Direct best response)**.** *Given the vote profile a, $g_j(a) = \hat{a}_j$ which yields agent j's most preferred outcome of the set $\{f(a_{-j}, \tilde{a}_j) : \tilde{a}^i_j \in D^i, \tilde{a}^k_j = a^k_j \ \forall k \neq i\}$ such that the update is direct (i.e., $\hat{a}^i_j = f^i(a^i_{-j}, \hat{a}^i_j)$). If there is no change in the outcome, then $g_j(a) = a_j$.*

LDI dynamics are based on the notions of *strict uncertainty* and *local dominance* (Conitzer et al., 2011; Reijngoud & Endriss, 2012). Let $S \subseteq \times_{i=1}^{p} \mathbb{N}^{|D^i|}$ be a set of score tuples that, informally, describes agent j's uncertainty about the real score tuple $s(a)$. An LDI step to a prospective vote $\hat{a}_j$ is one that is weakly better off than their original a_j for every $v \in S$ and strictly better off for some $v \in S$, as follows.

Definition 4.5. *The vote $\hat{a}_j$ S*-beats *a_j if there is at least one score tuple $v \in S$ such that $f(v + \hat{a}_j) \succ_j f(v + a_j)$. The vote $\hat{a}_j$ S*-dominates *a_j if both (I) $\hat{a}_j$ S-beats a_j; and (II) a_j does not S-beat $\hat{a}_j$.*

Definition 4.6 (Local dominance improvement)**.** *Given the vote profile a and agent j, let LD^i_j be the set of votes that S-dominate a_j, only differ from a_j on the i^{th} issue, and are not themselves S-dominated by any other vote differing from a_j only on the i^{th} issue. Then $g_j(a) = a_j$ if $LD^i_j = \emptyset$ and $\hat{a}_j \in LD^i_j$ otherwise.*

This definition distinguishes from (weak) LDI in Meir (2015) in that agents may change their votes consecutively but only on different issues. Note that S-dominance is transitive and antisymmetric, but not complete, so an agent j may not have an improvement step. To fully define the model, we need to specify S for every a. For example, if $S = \{s(a_{-j})\}$ and each j has no uncertainty about the real score tuple, then LDI coincides with BR and an equilibrium coincides with *Nash equilibrium*. Therefore, LDI broadens BR dynamics.

4.2.3 Distance-based Uncertainty

Agents in the single-issue model construed their uncertainty sets using *distance-based uncertainty*, in which all score vectors close enough to the current profile were believed possible (Meir, 2015; Meir et al., 2014). We adapt this to the multi-issue setting by assuming agents uphold alternative-wise distance-based uncertainty over score vectors for each issue independently.

For any issue $i \in \mathcal{P}$, let $\delta(s^i(a), \tilde{s}^i(a))$ be a distance measure for score vectors for any vote profile a. This measure is *alternative-wise* if it can be written as $\delta(s^i(a), \tilde{s}^i(a)) = \max_{c \in D^i} \hat{\delta}(s^i(c; a), \tilde{s}^i(c; a))$ for some monotone function $\hat{\delta}$. For example, the ℓ_∞ metric, where $\hat{\delta}(s, \tilde{s}) = |s - \tilde{s}|$, is alternative-wise.

Given the vote profile a and issue $i \in \mathcal{P}$, we model agent j's uncertainty about the adjusted score vector s^i_{-j} by the *uncertainty score set* $\tilde{S}^i_{-j}(s; r^i_j) = \left\{v^i : \delta(v^i, s^i_{-j}) \leq r^i_j\right\}$ with an *uncertainty parameter* r^i_j. That is, given other votes a^i_{-j}, agent j is not sure what the real score vector is within $\tilde{S}^i_{-j}(s; r^i_j)$. We define $\tilde{S}_{-j}(s, r_j) = \times_{i=1}^{p} \tilde{S}^i_{-j}(s; r^i_j)$ for $r_j = (r^i_j)_{i \in \mathcal{P}}$, and drop the parameters if the context is clear.

Example 4.2. *Consider $p = 2$ binary issues and $n = 13$ agents with the vote profile a defined such that:*

- *seven agents vote* $(0, 0)$*;*
- *three agents vote* $(1, 1)$*;*
- *two agents vote* $(1, 0)$*;*
- *the last agent, which we label j, votes $a_j = (0, 1)$.*

The score tuple is then $s(a) = \{(8, 5), (9, 4)\}$, so $f(a) = (0, 0)$.

Under BR dynamics, j has complete information about $s(a)$ and can compute $s_{-j}(a) = \{(7, 5), (9, 3)\}$. Clearly, no prospective vote $\hat{a}_j$ can change the outcome $f(a_{-j}, \hat{a}_j)$.

Under LDI dynamics, agent j has incomplete information about $s(a)$. Suppose that j uses the ℓ_∞ uncertainty metric with uncertainty parameters $(r_j^1, r_j^2) = (1,1)$. By the above definitions, the uncertainty score set for issue $i \in \{1,2\}$ is

$$\tilde{S}^i_{-j}(s; r^i_j) = \{v^i \ : \ |v^i - s^i_{-j}| \leq r^i_j\} = \{(6,7,8) \times (4,5,6)\} \times \{(8,9,10) \times (2,3,4)\} \tag{4.1}$$

which is a bandwidth of $r^i_j = 1$ around each real score s^i_{-j}. Finally, consider the prospective vote $\hat{a}_j = (1,1)$. Then

$$\tilde{S}_{-j} + \hat{a}_j = \{(6,7,8) \times (5,6,7)\} \times \{(8,9,10) \times (3,4,5)\} \tag{4.2}$$

so that $\{f(v + \hat{a}_j) \ : \ v \in \tilde{S}_{-j}\} = \{(0,0),(1,0)\}$.

4.3 Convergence Under Best Response Dynamics

Given the vote profile a, consider agent j changing their vote a_j on issue i to the prospective vote $\hat{a}_j$. Under BR dynamics, without uncertainty, j changes their vote only if they can feasibly improve the outcome $f(a)$ to one more favorable with respect to R_j. This happens under two conditions. First, j must be *pivotal* on the i^{th} issue, meaning that changing their vote will necessarily change the outcome. Second, j must be preferential to change i by voting for $\hat{a}^i_j$ over a^i_j given the outcomes of the other issues $\mathcal{P}\backslash\{i\}$. Agent j's preferences are always well-defined since they know every issue's real outcome. Thus BR dynamics behave similar to the single-issue setting, which we recall converges (Meir et al., 2010). However, in the multi-issue setting, agents' preferences on each issue may change as other issues' outcomes change. This entails the possibility of a cycle, as declared in the following proposition and proved with the subsequent example.

Proposition 4.1. *BR dynamics for multiple issues may not converge, even if issues are binary.*

Example 4.3. *Let there be $p = 2$ binary issues and $n = 3$ agents without uncertainty and the following preferences:*

- $R_1 : (0,1) \succ_1 (1,1) \succ_1 (1,0) \succ_1 (0,0)$;
- $R_2 : (0,0) \succ_2 (0,1) \succ_2 (1,1) \succ_2 (1,0)$;

Table 4.1: Agents' votes for $a(0)$ (truthful), $a(1)$, $a(2)$, and $a(3)$.

Agent j	$a_j(0)$	$a_j(1)$	$a_j(2)$	$a_j(3)$
1	$(0,1)$	$(1,1)$	$(1,1)$	$(0,1)$
2	$(0,0)$	$(0,0)$	$(0,1)$	$(0,1)$
3	$(1,0)$	$(1,0)$	$(1,0)$	$(1,0)$
$f(a)$	$(0,0)$	$(1,0)$	$(1,1)$	$(0,1)$

- $R_3 : (1,0) \succ_3 (1,1) \succ_3 (0,0) \succ_3 (0,1)$.

Table 4.1 demonstrates a cycle via BR dynamics from the truthful vote profile $a(0)$. The order of improvement steps is $j = (1,2,1,2)$. No other BR step is possible from any profile in the cycle, so no agent scheduler can lead to convergence.

4.4 Convergence Under Local Dominance Improvement Dynamics

LDI dynamics broadens best response since agents' uncertainty score sets contain the true score tuple, by definition, but it is initially unclear how uncertainty affects the possibility of cycles. Seemingly, greater uncertainty over an agent's current issue increases the possibility of having LDI steps over that issue, whereas greater uncertainty over other issues decreases this possibility. We demonstrate in Section 4.4.1 below that this relationship holds only for binary issues, but it does not eliminate the possibility of cycles, as declared in the following proposition and proved with Example 4.4.

Proposition 4.2. *LDI dynamics with multiple issues may not converge, even if agents have the same constant uncertainty parameters and issues are binary.*

Example 4.4. *Consider $p = 2$ binary issues and $n = 13$ agents who each use the ℓ_∞ uncertainty metric with common fixed uncertainty parameters $(r_j^1, r_j^2) = (1,2)\ \forall j \leq n$. Suppose that agents' preferences abide by the following four types:*

- *(Type 1) three agents have rankings $(0,1) \succ (1,1) \succ (1,0) \succ (0,0)$;*
- *(Type 2) five agents have rankings $(0,0) \succ (0,1) \succ (1,1) \succ (1,0)$;*
- *(Type 3) four agents have rankings $(1,0) \succ (1,1) \succ (0,0) \succ (0,1)$;*
- *(Type 4) one agent has ranking $(1,1) \succ (1,0) \succ (0,1) \succ (0,0)$.*

Table 4.2: Agents' votes for $a(0)$ (truthful), $a(3)$, $a(8)$, and $a(11)$.

Agent Type j	$a_j(0)$	$a_j(3)$	$a_j(8)$	$a_j(11)$
1	$(0,1)$	$(1,1)$	$(1,1)$	$(0,1)$
2	$(0,0)$	$(0,0)$	$(0,1)$	$(0,1)$
3	$(1,0)$	$(1,0)$	$(1,0)$	$(1,0)$
4	$(1,1)$	$(1,1)$	$(1,1)$	$(1,1)$

There is a cycle passing through the four vote profiles $a(0)$ (which is truthful), $a(3)$, $a(8)$, and $a(11)$ listed in Table 4.2, in which every agent of the same type has the same vote. There are four parts of the cycle between these profiles:

- *from $a(0) - a(3)$, all agents of Type 1 make LDI steps on the first issue $(0,1) \xrightarrow{1} (1,1)$;*
- *from $a(3) - a(8)$, all agents of Type 2 make LDI steps on the second issue $(0,0) \xrightarrow{2} (0,1)$;*
- *from $a(8) - a(11)$, all agents of Type 1 make LDI steps on the first issue $(1,1) \xrightarrow{1} (0,1)$;*
- *from $a(11) - a(16)$, all agents of Type 2 make LDI steps on the second issue $(0,1) \xrightarrow{2} (0,0)$, where $a(16) = a(0)$.*

Notice that no agent of Types 3 or 4 make LDI steps, as they vote truthfully and have separable preferences. We claim these are valid LDI steps as follows. For any vote profile in $\{a(0), a(1), a(2)\}$, let t be the number of Type 1 agents who have made LDI steps. Then

$$s(a(t)) = \{(8-t, 5+t), (9,4)\} \tag{4.3}$$

and for any agent j of Type 1 who has not made their first LDI step yet,

$$s_{-j} = \{(7-t, 5-t), (9,3)\}. \tag{4.4}$$

This entails

$$\tilde{S}^1_{-j} = \{(6-t, 7-t, 8-t) \times (4+t, 5+t, 6+t)\}; \tag{4.5}$$

$$\tilde{S}^2_{-j} = \{(7,8,9,10,11) \times (2,3,4,5,6)\}. \tag{4.6}$$

Define $\hat{a}_j = (1,1)$. Then $\forall t \in \{0,1,2\}$,

$$\tilde{s} = \{(6,6), (9,4)\} \in \tilde{S}_{-j}(a(t); r_j) \tag{4.7}$$

so that

$$f(\tilde{s} + \hat{a}_j) = (1, 0) \succ_j (0, 0) = f(\tilde{s} + a_j). \tag{4.8}$$

This entails that $\hat{a}_j$ $\tilde{S}_{-j}$-beats a_j. It is easy to see that a_j does not $\tilde{S}_{-j}$-beat $\hat{a}$ and that $\hat{a}$ is not $\tilde{S}_{-j}$-dominated since issues are binary. Thus $LD^1_j = \{\hat{a}_j\}$.

The other LDI steps follow similar reasoning, yielding the cycle presented in the table. It can be verified that this represents all possible LDI sequences from the truthful vote profile.

This finding contrasts convergence guaranteed in the single-issue setting with uncertainty (Meir, 2015). After explaining the effect of uncertainty on LDI steps, we conclude the section with two model refinements that prove sufficient to guarantee convergence for binary issues: $\mathcal{O}$-legal preferences and a form of dynamic uncertainty.

4.4.1 Effect of Uncertainty on LDI Steps

Given the vote profile a among binary issues, consider agent j changing their vote a_j on issue i to the prospective vote $\hat{a}_j$. Under LDI dynamics, j changes their vote only if two conditions hold, similar to BR dynamics: if (I) they believe they may be pivotal on issue i and (II) they can improve the outcome with respect to R_j. Notice that if the agent is pivotal on the binary issue i with respect to an uncertainty parameter r^i_j, it is pivotal with respect to all larger parameters $\tilde{r}_j \ : \ \tilde{r}^i_j > r^i_j$ over i. Furthermore, recall that j's preference over alternatives of issue i may depend on the outcomes of other issues, which j may be uncertain about. It stands to reason that the more uncertainty j has over other issues, the less clarity the agent has over their own preference for issue i's alternatives.

We realize the following monotonic relationships between the magnitude of agents' uncertainty parameters and whether they have an LDI step over an issue: increasing uncertainty on issue i (i) may only add LDI steps over issue i, but (ii) may only eliminate LDI steps over each other issue. This is stated technically in the following proposition. First, we define three uncertainty parameters α_j, r_j, and β_j such that:

- r_j and α_j only differ on issue $k \neq i$ such that $r^k_j < \alpha^k_j$;
- r_j and β_j only differ on issue i such that $r^i_j < \beta^i_j$.

For each $t_j \in \{\alpha_j, r_j, \beta_j\}$, let $LD^i_j(t_j)$ denote agent j's possible LDI steps as in Definition 4.6 with respect to the uncertainty parameter t_j.

Proposition 4.3. *Given binary issues, consider agent j changing their vote on issue i in vote profile a with one of three uncertainty parameters as defined above: α_j, r_j, or β_j. Then $LD_j^i(\alpha_j) \subseteq LD_j^i(r_j) \subseteq LD_j^i(\beta_j)$.*

The relationship $LD_j^i(r_j) \subseteq LD_j^i(\beta_j)$ characterizes point (i) above, in that increasing uncertainty over issue i only adds LDI steps on issue i. Meanwhile, the relationship $LD_j^i(\alpha_j) \subseteq LD_j^i(r_j)$ characterizes point (ii) above, in that increasing uncertainty over issue k only removes LDI steps over issue i. The proposition is proved in two parts by demonstrating that if a vote $\hat{a}_j$ $\tilde{S}_{-j}(a;\alpha_j)$-dominates a_j, then it must hold that $\hat{a}_j$ $\tilde{S}_{-j}(a;r_j)$-dominates a_j; likewise, this implies that $\hat{a}_j$ $\tilde{S}_{-j}(a;\beta_j)$-dominates a_j. Each of these relationships arise as a result of $\tilde{S}_{-j}(a;r_j) \subseteq \tilde{S}_{-j}(a;\alpha_j)$ and $\tilde{S}_{-j}(a;r_j) \subseteq \tilde{S}_{-j}(a;\beta_j)$, which result from how we defined α_j, r_j, and β_j. This is sufficient to prove since issues are binary. The full proof is as follows.

Proof. We prove the theorem by proving a weaker claim that holds for the more general multi-alternative issues setting. Specifically, for each $t_j \in \{\alpha_j, r_j, \beta_j\}$, let $D_j^i(t_j)$ denote the set of votes that S-dominate a_j and only differ on the i^{th} issue, where $S = \tilde{S}_{-j}(a;t_j)$. We show that $D_j^i(\alpha_j) \subseteq D_j^i(r_j) \subseteq D_j^i(\beta_j)$. The theorem follows because $LD_j^i(t_j) = D_j^i(t_j)$ for binary issues.

<u>**Step 1:** $D_j^i(\alpha_j) \subseteq D_j^i(r_j)$**.**</u> Without loss of generality let $D_j^i(r_j) \neq \emptyset$. Suppose $\hat{a}_j \in D_j^i(\alpha_j)$ so that $\hat{a}_j$ $\tilde{S}_{-j}(a;\alpha_j)$-dominates a_j. We prove that $\hat{a}_j$ $\tilde{S}_{-j}(a;r_j)$-dominates a_j by the 2-part definition of S-dominate. By definition, we have that:

1. $\exists \tilde{s} \in \tilde{S}_{-j}(a;\alpha_j)$ such that $f(\tilde{s} + \hat{a}_j) \succ_j f(\tilde{s} + a_j)$, and
2. $\nexists \tilde{s}' \in \tilde{S}_{-j}(a;\alpha_j)$ such that $f(\tilde{s}' + \hat{a}_j) \prec_j f(\tilde{s}' + a_j)$.

First, by construction of the uncertainty sets,

$$\tilde{S}^k_{-j}(a;r_j) \subseteq \tilde{S}^k_{-j}(a;\alpha_j) \text{ and } \tilde{S}^h_{-j}(a;r_j) = \tilde{S}^h_{-j}(a;\alpha_j)$$

for all $h \neq k$; therefore $\tilde{S}_{-j}(a;r_j) \subseteq \tilde{S}_{-j}(a;\alpha_j)$. It follows that

$$\nexists \tilde{s}' \in \tilde{S}_{-j}(a;r_j) \text{ such that } f(\tilde{s}' + \hat{a}_j) \prec_j f(\tilde{s}' + a_j)$$

by point (2) above. Hence, a_j does not $\tilde{S}_{-j}(a;r_j)$-beat $\hat{a}_j$.

Second, define a score tuple $\tilde{v}$ such that $\tilde{v}^h = \tilde{s}^h$ for each $h \neq k$ and $\tilde{v}^k \in \tilde{S}^k(a;r_j)$ arbitrarily. It is the case that $\tilde{v} \in \tilde{S}(a;r_j)$ since $\tilde{S}^h_{-j}(a;r_j) = \tilde{S}^h_{-j}(a;\alpha_j)$ for all $h \neq k$. Since

$$f^i(\tilde{s}+\hat{a}_j) \neq f^i(\tilde{s}+a_j) \text{ and } f^h(\tilde{s}+\hat{a}_j) = f^h(\tilde{s}+a_j)$$

for all $h \neq i$, we have $f(\tilde{v}+\hat{a}_j) \neq f(\tilde{v}+a_j)$. It follows from the first above argument that $f(\tilde{v}+\hat{a}_j) \succ_j f(\tilde{v}+a_j)$.

Therefore $\hat{a}_j$ $\tilde{S}_{-j}(a;r_j)$-beats a_j and $\hat{a}_j \in D^i_j(r_j)$.

Step 2: $D^i_j(r_j) \subseteq D^i_j(\beta_j)$. Without loss of generality let $D^i_j(\beta_j) \neq \emptyset$. Suppose $\hat{a}_j \in D^i_j(r_j)$, so that $\hat{a}_j$ $\tilde{S}_{-j}(a;r_j)$-dominates a_j. We prove that $\hat{a}_j$ $\tilde{S}_{-j}(a;\beta_j)$-dominates a_j by the 2-part definition of S-dominate. By definition, we have that:

1. $\exists \tilde{s} \in \tilde{S}_{-j}(a;r_j)$ such that $f(\tilde{s}+\hat{a}_j) \succ_j f(\tilde{s}+a_j)$, and
2. $\nexists \tilde{s}' \in \tilde{S}_{-j}(a;r_j)$ such that $f(\tilde{s}'+\hat{a}_j) \prec_j f(\tilde{s}'+a_j)$.

By construction of the uncertainty sets,

$$\tilde{S}^i_{-j}(a;r_j) \subseteq \tilde{S}^i_{-j}(a;\beta_j) \text{ and } \tilde{S}^h_{-j}(a;r_j) = \tilde{S}^h_{-j}(a;\beta_j)$$

for all $h \neq i$; therefore $\tilde{S}_{-j}(a;r_j) \subseteq \tilde{S}_{-j}(a;\beta_j)$. It immediately follows that $\tilde{s} \in \tilde{S}_{-j}(a;\beta_j)$, so $\hat{a}_j$ $\tilde{S}_{-j}(a;\beta_j)$-beats a_j.

Furthermore, $\nexists \tilde{s}' \in \tilde{S}_{-j}(a;\beta_j)$ such that $f(\tilde{s}'+\hat{a}_j) \prec_j f(\tilde{s}'+a_j)$ by point (2) above. Hence, a_j does not $\tilde{S}_{-j}(a;\beta_j)$-beat $\hat{a}_j$.

Therefore $\hat{a}_j \in D^i_j(\beta_j)$, concluding our proof. □

We find that this relationship between agents' uncertainty parameters and their LDI steps is not monotonic in the generalized case of multi-alternative issues, as Example 4.5 demonstrates that different sets of prospective votes LD^i_j may not be comparable for an agent j with different uncertainty parameters even from the same vote profile a.

Example 4.5. *The purpose of this example is to demonstrate that, when issues have multiple alternatives each, there may not be a monotonic relationship between an agent's possible LDI steps and the magnitude of their uncertainty parameters. Specifically, consider $p = 2$ issues* *with four alternatives each, labeled* $\{A^i, B^i, C^i, D^i\}$ *for* $i \leq 2$, *and* $n = 21$ *agents. Let agent* j *have* ℓ_∞ *uncertainty metric and a* $\mathcal{O} = (2,1)$*-legal preference ranking* R_j *that satisfies:*

- *If* $f^2 = A^2$, *then over issue 1:* $D^1 \succ_j A^1 \succ_j B^1 \succ_j C^1$;
- *If* $f^2 = B^2$, *then over issue 1:* $B^1 \succ_j A^1 \succ_j D^1 \succ_j C^1$;
- *If* $f^2 = C^2$, *then over issue 1:* $A^1 \succ_j B^1 \succ_j D^1 \succ_j C^1$.

The remaining definition of R_j *does not matter to illustrate this example. Suppose* $a_j = (D^1, C^2)$ *in vote profile* a *with score tuple* $s(a) = \{(6,6,4,5),(5,7,9,0)\}$.

Part 1: increasing r_j^1.

Step 1. *If* $r_j = (0,0)$ *then it is easy to see that agent* j *has no BR steps: the unique winner is* $f(a) = (A^1, C^2)$ *and, although* j *can make* B^1 *win, they prefer* $A^1 \succ B^1$ *when* $f^2(a) = C^2$. *Hence* $LD_j^1 = \emptyset$.

Step 2. *If* $r_j = (1,0)$ *then*

$$\tilde{S}_{-j}^1 = \{(5,6,7) \times (5,6,7) \times (3,4,5) \times (3,4,5)\} \tag{4.9}$$

First, when

$$\tilde{s} = \{(5,5,5,5),(5,7,9,0)\} \in \tilde{S}_{-j} \tag{4.10}$$

then j *prefers to vote for* $\hat{a}_j = (A^1, C^2)$ *over* a_j *Hence* $\hat{a}_j$ $\tilde{S}_{-j}$*-beats* a_j.

Second, it is easy to see that there is no $\tilde{s}' \in \tilde{S}_{-j}$ *for which it is preferable for* j *to vote for either* a_j *or* $\hat{a}'_j = (B^1, C^2)$ *rather than* $\hat{a}_j$. *This follows since if* $f^1(\tilde{s}' + a_j) = B^1$ *then* $f^1(\tilde{s}' + \hat{a}_j) \in \{A^1, B^1\}$ *and if* $f^1(\tilde{s}' + a_j) = A^1$ *then* $f^1(\tilde{s}' + \hat{a}_j) = A^1$. *The same holds for* $\hat{a}'_j$. *Thus* j *cannot achieve better by voting for* a_j *or* $\hat{a}'_j$ *than* $\hat{a}_j$ *for any* $\tilde{s}' \in \tilde{S}_{-j}$. *Hence,* $\hat{a}_j$ *is not* $\tilde{S}_{-j}$*-beaten, so* $LD_j^1 = \{(A^1, C^2)\}$.

Step 3. *Now consider* $r_j = (2,0)$. *Then*

$$\tilde{S}_{-j}^1 = \{(4,5,6,7,8) \times (4,5,6,7,8) \times (2,3,4,5,6) \times (2,3,4,5,6)\} \tag{4.11}$$

Since

$$\tilde{s} = \{(5,5,5,5),(5,7,9,0)\} \in \tilde{S}_{-j} \tag{4.12}$$

then $\hat{a}_j$ *still* $\tilde{S}_{-j}$*-beats* a_j*. However, as*

$$\tilde{s}'' = \{(4,4,6,6),(5,7,9,1)\} \in \tilde{S}_{-j} \tag{4.13}$$

we have

$$f^1(\tilde{s}'' + a_j) = (D^1, C^2) \succ_j (C^1, C^2) = f^1(\tilde{s}'' + \hat{a}_j) \tag{4.14}$$

so a_j $\tilde{S}_{-j}$*-beats* $\hat{a}_j$*. Hence,* $LD_j^1 = \emptyset$.

Part 2: increasing r_j^2.

Step 1. *Recall that when* $r_j = (1,0)$*, we found that* $LD_j^1 = \{(A^1, C^2)\}$.

Step 2. *If* $r_j = (1,1)$*, then*

$$\tilde{S}^1_{-j} = \{(5,6,7) \times (5,6,7) \times (3,4,5) \times (3,4,5)\}; \tag{4.15}$$

$$\tilde{S}^2_{-j} = \{(4,5,6) \times (6,7,8) \times (7,8,9) \times (0,1)\}. \tag{4.16}$$

For

$$\tilde{s} = \{(5,5,5,5),(4,7,8,3)\} \in \tilde{S}_{-j} \tag{4.17}$$

then j *prefers to vote for* $\hat{a}_j$ *rather than* $\hat{a}'_j = (B^1, C^2)$*, whereas for*

$$\tilde{s}' = \{(5,5,5,5),(4,8,7,3)\} \in \tilde{S}_{-j} \tag{4.18}$$

it is preferable for j *to vote for* $\hat{a}'_j$ *rather than* $\hat{a}_j$*. Then both* $\hat{a}_j$ *and* $\hat{a}'_j$ $\tilde{S}_{-j}$*-dominate* a_j *but neither are* $\tilde{S}_{-j}$*-dominated. Therefore* $LD_j^1 = \{(A^1, C^2), (B^1, C^2)\}$.

Step 3. *If* $r_j = (1,2)$*, then*

$$\tilde{S}^2_{-j} = \{(3,4,5,6,7) \times (5,6,7,8,9) \times (6,7,8,9,10) \times (0,1,2)\} \tag{4.19}$$

For

$$\tilde{s}'' = \{(5,5,5,5),(7,5,6,0)\} \in \tilde{S}_{-j} \tag{4.20}$$

it is preferable for j to vote for a_j rather than $\hat{a}_j$ or $\hat{a}'_j$, so neither $\hat{a}_j$ nor $\hat{a}'_j$ $\tilde{S}_{-j}$-dominate a_j. Thus $LD^1_j = \emptyset$.

4.4.2 Strategic Responses and $\mathcal{O}$-legal Preferences

We are motivated by observing in Examples 4.3 and 4.4 that cycles appear due to agents' interdependent preferences among the issues. Specifically, in Table 4.1, a cycle is formed as agents 1 and 2 switch their preferences among alternatives for one issue when the other issue changes outcomes, and this holds for opposite issues. It therefore stands to reason that eliminating interdependent preferences by fixing agents with a $\mathcal{O}$-legal preference profile would guarantee convergence.

We prove that this is the case in Theorem 4.1. To state this result technically, we first introduce a characterization about agents' strategic responses, extending a lemma from Meir (2015) to the multi-issue setting.

Definition 4.7. *Agent j believes a alternative c on issue i is a* possible winner *if there is some score vector where c wins:*

$$W^i_j(a) = \{c \in D^i \ : \ \exists v \in \tilde{S}_{-j}(a; r_j) \ s.t. \ f^i(v + a_j) = c\}. \tag{4.21}$$

In contrast, j calls c a potential winner *if there is some score vector in which they can vote to make c win:*

$$\begin{aligned} H^i_j(a) = \{c \in D^i \ : \exists v \in \tilde{S}_{-j}(a; r_j) \ and \ \hat{a}_j \\ s.t. \ \hat{a}^i_j = c, \hat{a}^k_j = a^k_j \ \forall k \neq i \ and \ f^i(v + \hat{a}_j) = c\}. \end{aligned} \tag{4.22}$$

The set of real potential winners is denoted:

$$H^i_0(a) = \{c \in D^i \ : \ f^i(s_{-j} + \hat{a}_j) = c \ where \ \hat{a}^i_j = c, \hat{a}^k_j = a^k_j \ \forall k \neq i\}. \tag{4.23}$$

By this definition, $W^i_j(a) \subseteq H^i_j(a)$. [7] Denote by $\mathcal{W}^{-i}(a; r_j) = \times_{k \in \mathcal{P} \setminus \{i\}} W^k_j(a)$ the

[7] Without uncertainty, $H^i_j(a)$ (or $H^i_j(a) \cup \{a^i_j\}$ if adding a vote to a^i_j makes it win) is also known as the *chasing set* (excluding $f(a)$) (Rabinovich et al., 2015) or *potential winner set* (including $f(a)$, in Chapter 3)

set of possible winning alternatives on all issues besides i, from agent j's perspective with uncertainty parameter r_j.

Lemma 4.1. *Consider an LDI step $a_j \xrightarrow{j} \hat{a}_j$ over issue i from vote profile a by agent j with uncertainty parameter r_j. Then either:*

(1) $a_j^i \notin H_j^i(a)$, or

(2) for every combination of possible winners in $\mathcal{W}^{-i}(a; r_j)$, either

(2a) $a_j^i \prec_j b$ for all $b \in H_j^i(a)$, or

(2b) $r_j^i = 0$, $\{a_j^i, \hat{a}_j^i\} \subseteq H_0^i(a)$ and $\hat{a}_j^i \succ_j a_j^i$.

Proof. The proof directly follows that of Lemma 3 in Meir (2015). Suppose that $a_j^i, b \in H_j^i(a)$ and $a_j^i \succ_j b$ whenever some combination of possible winners $(c^1, c^2, \ldots, c^{i-1}, c^{i+1}, \ldots, c^p) \in \mathcal{W}^{-i}(a; r_j)$ wins (i.e., (1) and (2a) are violated). Assume first that $\hat{a}_j^i \notin H_0^i(a)$. By Lemma 2 of Meir (2015), $\exists \tilde{s} \in \tilde{S}_{-j}(a; r_j)$ such that:

- a_j^i, b have maximal score (possibly with other alternatives), strictly above $\hat{a}_j^i$, in $\tilde{s}^i$;
- for each $k \neq i$, $\tilde{s}^k$ is such that $f^k(\tilde{s} + a_j) = c^k$ wins.

W.l.o.g. assume b is prior to a_j^i in tie-breaking (otherwise adjust $\tilde{s}$ so that $f^i(\tilde{s}) = b$). Thus $f^i(\tilde{s} + a_j) = a_j^i$ while $f^i(\tilde{s} + \hat{a}_j) = b$. Since $a_j^i \succ_j b$ given $\tilde{s}$, this implies a_j $\tilde{S}_{-j}(a; r_j)$-beats $\hat{a}_j$.

The remaining case is where $\hat{a}_j^i \in H_0^i(a)$ and $a_j^i \succ_j \hat{a}_j^i$ whenever some

$$(c^1, c^2, \ldots, c^{i-1}, c^{i+1}, \ldots, c^p) \in \mathcal{W}^{-i}(a; r_j)$$

wins. Then in $\tilde{s}$ where $a_j^i, \hat{a}_j^i$ are tied and $(c^k)_{k \in \mathcal{P} \setminus \{i\}}$ wins, it is better for j to vote for a_i.

In both cases we get $\hat{a}_j \notin LD_j^i$, which is a contradiction. □

Theorem 4.1. *LDI dynamics converge over binary issues when all agents have $\mathcal{O}$-legal preferences for the common order $\mathcal{O}$.*

on issue i. $H_j^i(a)$ coincides with Meir et al. (2014) and Meir (2015)'s definition of possible winner "$W_j(s)$."

Proof. Fix an initial vote profile $a(0)$. Suppose for contradiction that there is a cycle among the vote profiles $\mathcal{C} = \{a(t_1), \ldots, a(t_T)\}$, where $a(t_T + 1) = a(t_1)$ and $a(t_1)$ is reachable from $a(0)$ via LDI dynamics. Let i be the highest order issue in $\mathcal{O}$ for which any agent changes their vote in $\mathcal{C}$.

Let $t^* \in [t_1, t_T)$ be the first round that some agent j takes an LDI step on issue i, where $a_j \xrightarrow{j} \hat{a}_j$ from vote profile $a(t^*)$; let $t^{**} \in (t^*, t_T]$ be the last round that j switches their vote on i back to a^i_j. It must be the case that $a^i_j \in H^i_j(a(t^*))$, since issues are binary and otherwise, $|H^i_j(a(t^*))| = 1$ and j would not have an improvement step. Hence by Lemma 4.1, $\hat{a}^i_j \succ_j a^i_j$ for every combination of possible winners in $\mathcal{W}^{-i}(a(t^*); r_j)$. Likewise, on round t^{**}, $a^i_j \succ_j \hat{a}^i_j$ for every combination of possible winners in $\mathcal{W}^{-i}(a(t^{**}); r_j)$. Thus for some issue k and outcomes $x, y \in \{0, 1\}$, $x \neq y$, we have $W^k_j(a(t^*)) = \{x\}$ and $W^k_j(a(t^{**})) = \{y\}$.

Since j has $\mathcal{O}$-legal preferences, k must be prior to issue i in the order $\mathcal{O}$. However, no agent changed their vote on issue k between rounds t^* and t^{**} so it must be that $x \in W^k_j(a(t^{**}))$, even if j's uncertainty parameters changed. This forms a contradiction, so no such cycle can exist. □

The intuition behind Theorem 4.1 is that as an LDI sequence develops, there is some "foremost" issue i in which no LDI step takes place on any issue prior to i in the order $\mathcal{O}$. Agents' relative preferences for the alternatives in i are fixed because their preferences are $\mathcal{O}$-legal: score vectors for issues prior to i in $\mathcal{O}$ do not change, while scores of issues afterward do not affect agents' preferences for i. Hence, agents' improvement steps over the issue i converge, whereas any cycle must have a sub-sequence of vote profile whose votes for issue i cycles.

Note that $\mathcal{O}$-legality is not necessary for convergence, as BR dynamics induced from the truthful vote profile in Example 4.1 converge. Although $\mathcal{O}$-legality is a strict assumption, loosening this even slightly may lead to cycles. Example 4.3 demonstrates a cycle in which each agent has an $\mathcal{O}$-legal ranking but orders differ between agents.

Separately, the theorem describes that LDI steps over the issue i eventually terminate, thus enabling each subsequent issue in $\mathcal{O}$ to converge. This seems to suggest that IV under $\mathcal{O}$-legal preferences is the same as *truthful sequential voting*, where agents vote for their preferred alternative on each issue o_i given the known previous outcomes of $\{o_1, \ldots, o_{i-1}\}$ (Lang & Xia, 2009). Although the procedures' outcomes could be the same, there are two notable differences. First, the initial vote profile could have an issue whose outcome differs from the truthful sequential outcome and no agent has an improvement step on that issue. Second, depending on the scheduler, agents may not have further improvement steps over an issue intermediately before IV reaches the same outcome as in truthful sequential voting.

This convergence result does not extend to the multi-alternative case, as declared in the following proposition and proved with the subsequent example.

Proposition 4.4. *LDI dynamics may not converge for multiple issues, even if agents have the same constant uncertainty parameters and $\mathcal{O}$-legal preferences for the common order $\mathcal{O}$.*

Example 4.6. *Consider $p = 2$ issues and $n = 15$ agents who each use the ℓ_∞ uncertainty metric with common fixed uncertainty parameters $(r_j^1, r_j^2) = (2, 1)$ $\forall j \leq n$. Label the alternatives $\{0, 1\}$ and $\{a, b, c, d\}$ respectively. Agent j has preferences:*

- *if $f^1 = 0$ then $b \succ_j c \succ_j a \succ_j d$ on the second issue;*
- *otherwise $c \succ_j b \succ_j a \succ_j d$.*

Agent k always prefers $a \succ_k d \succ_k b \succ_k c$ on the second issue. These preferences are $\mathcal{O}$-legal for $\mathcal{O} = \{1, 2\}$.

Define $a(0)$ so $s(a(0)) = \{(7, 8), (3, 5, 5, 2)\}$ and $a_j(0) = a_k(0) = (0, a)$. There are four LDI steps involved in this cycle:

1. $(0, a) \xrightarrow{j} (0, d)$;
2. $(0, a) \xrightarrow{k} (0, d)$;
3. $(0, d) \xrightarrow{j} (0, a)$;
4. $(0, d) \xrightarrow{k} (0, a)$.

We will prove that these are valid LDI steps in turn, demonstrating that (i) the new vote S-beats the old vote, (ii) the old vote does not S-beat the new vote, and (iii) the new vote is not S-dominated. For any alternative $e \in \{a, b, c, d\}$, denote the vote switching the second alternative to e by $\hat{e} = (0, e)$. Recall that ties are broken in lexicographical order.

Step 1: $s^2(a(0)) = (3,5,5,2)$**.** *Since* $r_j^2 = 1$ *and* $a_j^2 = a$*, we have* $H_j^2(a(0)) = \{a,b,c\}$*.*

(i) Let $\tilde{s} = \{(6,8),(3,4,4,2)\} \in \tilde{S}_{-j}(a(0);r_j)$*. Then* $f(\tilde{s}+\hat{d}) = (1,b) \succ_j (1,a) = f(\tilde{s}+\hat{a})$*, so* $\hat{d}$ $\tilde{S}_{-j}$*-beats* $\hat{a}$*.*

(ii) For any $\tilde{s} \in \tilde{S}_{-j}(a(0);r_j)$*, we have that if* $f^2(\tilde{s}+\hat{d}) = a$ *then* $f^2(\tilde{s}+\hat{a}) = a$*; otherwise, if* $f^2(\tilde{s}+\hat{a}) \in \{b,c\}$ *then* $f^2(\tilde{s}+\hat{d}) = f^2(\tilde{s}+\hat{a})$*. Hence, it is never preferable for* j *to vote for* $\hat{a}$ *than* $\hat{d}$*.*

(iii) Neither $\hat{b}$ *nor* $\hat{c}$ $\tilde{S}_{-j}$*-dominate* $\hat{d}$ *since for* $\tilde{s}' = \{(6,8),(2,4,5,2)\}$ *and* $\tilde{s}'' = \{(7,7),(2,4,4,2)\}$ *it is preferable for* j *to vote for* $\hat{d}$ *than* $\hat{b}$ *and* $\hat{c}$*, respectively.*

Step 2: $s^2(a(1)) = (2,5,5,3)$**.** *Since* $r_k^2 = 1$ *and* $a_j^2 = a$*, we have* $H_k^2(a(1)) = \{b,c,d\}$*.*

(i) Let $\tilde{s} = \{(6,8),(2,4,4,4)\} \in \tilde{S}_{-k}(a(1);r_k)$*. Then* $f(\tilde{s}+\hat{d}) = (1,d) \succ_k (1,b) = f(\tilde{s}+\hat{a})$*, so* $\hat{d}$ $\tilde{S}_{-k}$*-beats* $\hat{a}$*.*

(ii) For any $\tilde{s} \in \tilde{S}_{-k}(a(1);r_k)$*, we have that if* $f^2(\tilde{s}+\hat{d}) = d$ *then* $f^2(\tilde{s}+\hat{a}) = b$*; otherwise* $f^2(\tilde{s}+\hat{d}) = f^2(\tilde{s}+\hat{a})$*. Hence, it is never preferable for* k *to vote for* $\hat{a}$ *than* $\hat{d}$*.*

(iii) Neither $\hat{b}$ *nor* $\hat{c}$ $\tilde{S}_{-k}$*-dominate* $\hat{d}$ *since for* $\tilde{s} = \{(6,8),(2,4,4,4)\}$ *(the same as in (i)), it is preferable for* j *to vote for* $\hat{d}$ *than either* $\hat{b}$ *or* $\hat{c}$*.*

Step 3: $s^2(a(2)) = (1,5,5,4)$**.** *Since* $r_j^2 = 1$ *and* $a_j^2 = d$*, we have* $H_j^2(a(2)) = \{b,c,d\}$*.*

(i) Let $\tilde{s} = \{(6,8),(1,4,4,4)\} \in \tilde{S}_{-j}(a(2);r_j)$*. Then* $f(\tilde{s}+\hat{a}) = (1,b) \succ_j (1,d) = f(\tilde{s}+\hat{d})$*, so* $\hat{a}$ $\tilde{S}_{-j}$*-beats* $\hat{d}$*.*

(ii) For any $\tilde{s} \in \tilde{S}_{-j}(a(2);r_j)$*, we have that if* $f^2(\tilde{s}+\hat{a}) = d$ *then* $f^2(\tilde{s}+\hat{d}) = d$*; otherwise, if* $f^2(\tilde{s}+\hat{d}) \in \{b,c\}$ *then* $f^2(\tilde{s}+\hat{a}) = f^2(\tilde{s}+\hat{d})$*. Hence, it is never preferable for* j *to vote for* $\hat{d}$ *than* $\hat{a}$*.*

(iii) Neither $\hat{b}$ *nor* $\hat{c}$ $\tilde{S}_{-j}$*-dominate* $\hat{a}$ *since for* $\tilde{s}' = \{(6,8),(2,4,5,3)\}$ *and* $\tilde{s}'' = \{(7,7),(2,4,4,3)\}$ *it is preferable for* j *to vote for* $\hat{a}$ *than* $\hat{b}$ *and* $\hat{c}$*, respectively.*

Step 4: $s^2(a(3)) = (2,5,5,3)$**.** *Since* $r_k^2 = 1$ *and* $a_j^2 = d$*, we have* $H_k^2(a(3)) = \{a,b,c\}$*.*

(i) Let $\tilde{s} = \{(6,8),(3,4,4,2)\} \in \tilde{S}_{-k}(a(3);r_k)$*. Then* $f(\tilde{s}+\hat{a}) = (1,a) \succ_k (1,b) = f(\tilde{s}+\hat{d})$*, so* $\hat{a}$ $\tilde{S}_{-k}$*-beats* $\hat{d}$*.*

(ii) For any $\tilde{s} \in \tilde{S}_{-k}(a(3);r_k)$*, we have that if* $f^2(\tilde{s}+\hat{a}) = a$ *then* $f^2(\tilde{s}+\hat{d}) = b$*; otherwise* $f^2(\tilde{s}+\hat{a}) = f^2(\tilde{s}+\hat{d})$*. Hence, it is never preferable for* k *to vote for* $\hat{d}$ *than* $\hat{a}$*.*

(iii) Neither $\hat{b}$ *nor* $\hat{c}$ $\tilde{S}_{-k}$*-dominate* $\hat{a}$ *since for* $\tilde{s} = \{(6,8),(3,4,4,2)\}$ *(the same as in (i)), it is preferable for* j *to vote for* $\hat{a}$ *than either* $\hat{b}$ *or* $\hat{c}$*.*

Note that $H_j^2(a(0)) = \{a, b, c\}$ and $H_j^2(a(2)) = \{b, c, d\}$. In contrast to the single-issue setting (see Lemma 4 of Meir (2015)), agent j takes LDI steps to alternatives not in the potential winning set. This results from j's uncertainty over whether b or c is most-preferred, even as both are preferable to a and d. Hence, we get the following corollary:

Corollary 4.1. *LDI dynamics may not converge for plurality over a single issue for agents with partial order preferences.*

4.4.3 Alternating Uncertainty

In Proposition 4.3 we found that for binary issues, agents may have fewer LDI steps over an issue i if that issue has less uncertainty and other issues have more. This suggests that LDI steps occur from a relative lack of information about the current issue's score vector than for other issues. If agents can gather more information about the current issue before changing their vote, thereby decreasing its uncertainty relative to other issues, then they may not have an LDI step.

We therefore consider a specific form of dynamics over agents' uncertainty parameters where agents can gather this information and consider themselves pivotal only with respect to the lowered uncertainty. Agents are assumed to subsequently forget this relative information since it may be outdated by the time they change their vote again. We show in the following theorem that this eliminates cycles.

Definition 4.8. *(Alternating Uncertainty.) Fix two parameters r_j^c, r_j^o for each agent j such that $r_j^c < r_j^o$. Define each agent j's uncertainty parameters such that whenever they are scheduled to change their vote on issue i, j's uncertainty for i is r_j^c and for each other issue $k \neq i$ the uncertainty is r_j^o.*

Theorem 4.2. *Given binary issues, LDI dynamics converges for agents with alternating uncertainty.*

Proof. Fix an initial vote profile $a(0)$ and uncertainty parameters r_j^c, r_j^o for each agent $j \leq n$. Suppose for contradiction that there is a cycle among the vote profiles $\mathcal{C} = \{a(t_1), \ldots, a(t_T)\}$, where $a(t_T + 1) = a(t_1)$ and $a(t_1)$ is reachable from $a(0)$ via LDI dynamics. Without loss of generality, suppose all issues and agents are involved in the cycle.

Consider the agent j with the largest $r_j^o = \max_{u \leq n} r_u^o$. Let $t^* \in [t_1, t_T)$ be the first round that j takes an LDI step on issue i, where $a_j \xrightarrow{j} \hat{a}_j$ from vote profile $a(t^*)$; let $t^{**} \in (t^*, t_T]$

be the last round that j switches their vote on i back to a_j^i. It must be the case that $a_j^i \in H_j^i(a(t^*))$, since issues are binary and otherwise, $|H_j^i(a(t^*))| = 1$ and j would not have an improvement step. Hence by Lemma 4.1, $\hat{a}_j^i \succ_j a_j^i$ for every combination of possible winners in $\mathcal{W}^{-i}(a(t^*); r_j)$. Likewise, on round t^{**}, $a_j^i \succ_j \hat{a}_j^i$ for every combination of possible winners in $\mathcal{W}^{-i}(a(t^{**}); r_j)$. Thus for some issue k and outcomes $x, y \in \{0, 1\}$, $x \neq y$, we have $W_j^k(a(t^*)) = \{x\}$ and $W_j^k(a(t^{**})) = \{y\}$.

Let $t' \in (t^*, t^{**})$ be the first round since t^* that some agent h changes their vote on issue k. Then $H_h^k(a(t')) = \{0, 1\}$. Since $W_j^k(a(t')) = W_j^k(a(t^*)) = \{x\} \subsetneq \{0, 1\}$ and distance functions are alternative-wise, $r_h^c \geq r_j^o$. This entails $r_h^o > r_j^o$ by definition of alternating uncertainty, which contradicts the assertion that j is the agent u with the largest r_u^o. □

This convergence result does not extend to the multi-alternative case, as Example 4.6 also covers this setting.

4.5 Nonatomic Model

Each of the results so far in this chapter were presented for binary issues and atomic agents, where each agent contributes one unit of influence to the population of n votes. We find that these results also generalize to a nonatomic variant of IV, in which agents are part of a very large population and have negligible influence over the outcome. Our model extends Meir (2015)'s iterative plurality voting for nonatomic agents to the multi-issue setting. Like this setting, our convergence results permit arbitrary subsets of agents to change their vote simultaneously. This differs from the finite case, which may not converge if multiple agents change their votes simultaneously (Meir et al., 2010).

In this section, we provide the necessary definitions for the nonatomic model, using our existing notation wherever possible. There are two major differences: (i) with the identify of an agent, and (ii) with the uncertainty score set. First, rather than treating each agent $j \leq n$ individually, there are groups of identical agents with ϵ mass for some small amount ϵ. "An agent of type j" then refers to a representative agent in this group. Second, as each agent has negligible influence, type j agents consider any score tuple in $\tilde{S}(a; r_j)$ possible; all agents agree of the same type agree what real score tuples are possible. Lemma 4.1, Proposition 4.3, and Theorems 4.1 and 4.2 are each upheld after applying this model redefinition. Similar non-convergence results also apply for the nonatomic variant of IV.

Basic notation. We do not have a finite set of agents. Rather, a *preference profile* $Q \in \Delta(\mathcal{L}(D))$ is a distribution over rankings that specifies the fraction of agents $Q(R)$ for each $R \in \mathcal{L}(D)$. We only consider moves by subsets of agents since each agent has negligible influence. To avoid infinite improvement sequence paths of sequentially smaller subsets of agents, we assume there is a minimum resolution ϵ, such that sets of agents of the same type with mass ϵ always move together (although in an uncoordinated manner; see Appendix IX of Meir (2015)). We denote the collection of these $1/\epsilon$ sets by J. Since all agents in set $j \in J$ are indistinguishable, we refer to "agent j" as an arbitrary agent in the set j. Then $R_j \in \mathcal{L}(D)$ is the preference, r_j is the uncertainty parameter, and a_j is the vote of an arbitrary agent in the set of vote profile a.

Winner determination. For any issue $i \in \mathcal{P}$, we define the *score vector* induced by the vote profile a as $s^i(a) = (s^i(c;a))_{c \in D^i}$, such that $s^i(c;a) = |\{f : a^i_j = c\}|\epsilon \in [0,1]$. Winner determination is exactly as in the atomic model with lexicographical tie-breaking.

Uncertainty and local dominance. We assume agents utilize the same alternative-wise distance uncertainty as the atomic model. Like the atomic model, each agent of type $j \in J$ selected by the scheduler to change their vote has uncertainty parameters $r_j = (r^i_j)_{i \in \mathcal{P}}$. Unlike the atomic model, agents have negligible influence in the score vector. Hence, they take their uncertainty score sets with respect to the real score tuple: $\tilde{S}^i(a; r^i_j) = \{v^i : \delta(v^i, s^i(a)) \leq r^i_j\}$. As before, $\tilde{S}(a; r_j) = \times_{i \in \mathcal{P}} \tilde{S}^i(a; r^i_j)$.

Given $v \in \tilde{S}(a; r_j)$, we define $f(v + a_j)$ to be the outcome an agent of type j expects by voting for a_j according to score vector s. That is, for each issue i, the extra vote a^i_j decides the winner if several alternatives are tied with maximal score in s, overriding the default tie-breaker (see Appendix X of Meir (2015)).

The definition of a local dominance improvement (LDI) step for a nonatomic agent of type $j \in J$ from vote a_j to $\hat{a}_j$ on issue i then is the same as in the atomic model of Definitions 4.5 and 4.6, applying this redefinition and using $\tilde{S}(a; r_j)$ in lieu of $\tilde{S}_{-j}(a; r_j)$. The response function g_j is also the same, except that its domain (all possible profiles) is now $\mathcal{D}^{|J|}$ rather than $\mathcal{D}^n$. The definition of equilibrium does not change.

4.6 Experiments

Our computational experiments investigate the effects of uncertainty and numbers of binary issues and agents on LDI dynamics. Specifically, we ask how often truthful vote profiles are themselves in equilibrium, how often LDI dynamics do not converge, and the path length to equilibrium given that LDI dynamics do converge. Our inquiry focuses on whether cycles are commonplace in practice even though convergence is not guaranteed.

We answer these questions for a broad cross-section of inputs, with $n \in \{7, 11, 15, 19\}$ agents, $p \in \{2, 3, 4, 5\}$ binary issues, and $r \in \{0, 1, 2, 3\}$ uncertainty that is constant for all agents, issues, and rounds. We generate $10,000$ preference profiles for each combination by sampling agents' preferences uniformly and independently at random. We simulate LDI dynamics from the truthful vote profile using a scheduler that selects profiles uniformly at random from the set of valid LDI steps among all agents and issues. If there are no such steps, we say the sequence has converged. Otherwise, we take $50,000$ rounds as a sufficiently large stopping condition to declare the sequence has cycled.

Our results are presented in Figures 4.1 – 4.3 with respect to n. As uncertainty is introduced and r increases, given $p = 5$, the availability of LDI steps diminishes significantly from the initial vote profile (Figure 4.1) and throughout the dynamics to eliminate (almost) all cycles and shorten the path length to convergence (Figure 4.3). Figure 4.2 presents the number of initial vote profiles whose LDI sequence cycles for $r = 0$, given that they are not themselves in equilibrium; only five of the sampled $r \geq 1$ profiles' sequences cycle. Therefore, cycles with uncertainty are the exception rather than the norm.

These findings corroborate our theoretical analysis. As uncertainty increases, more issues are perceived by agents to have more than one possible winner. Since issues are interdependent for many preference rankings, fewer agents have LDI steps. On the other hand, as n increases, more agents have rankings without these interdependencies, thus increasing the availability of LDI steps.

As an additional inquiry, we studied how IV affects the quality of outcomes by comparing the social welfare of equilibrium to truthful vote profiles, measured by the percent change in Borda welfare. Recall that the *Borda utility* of outcome a for ranking R is 2^p minus the index of a's position in R; the *Borda welfare* is the sum of utilities across agents. We find in Figure 4.4 that IV improves average welfare, but at a rate decreasing in r. This finding

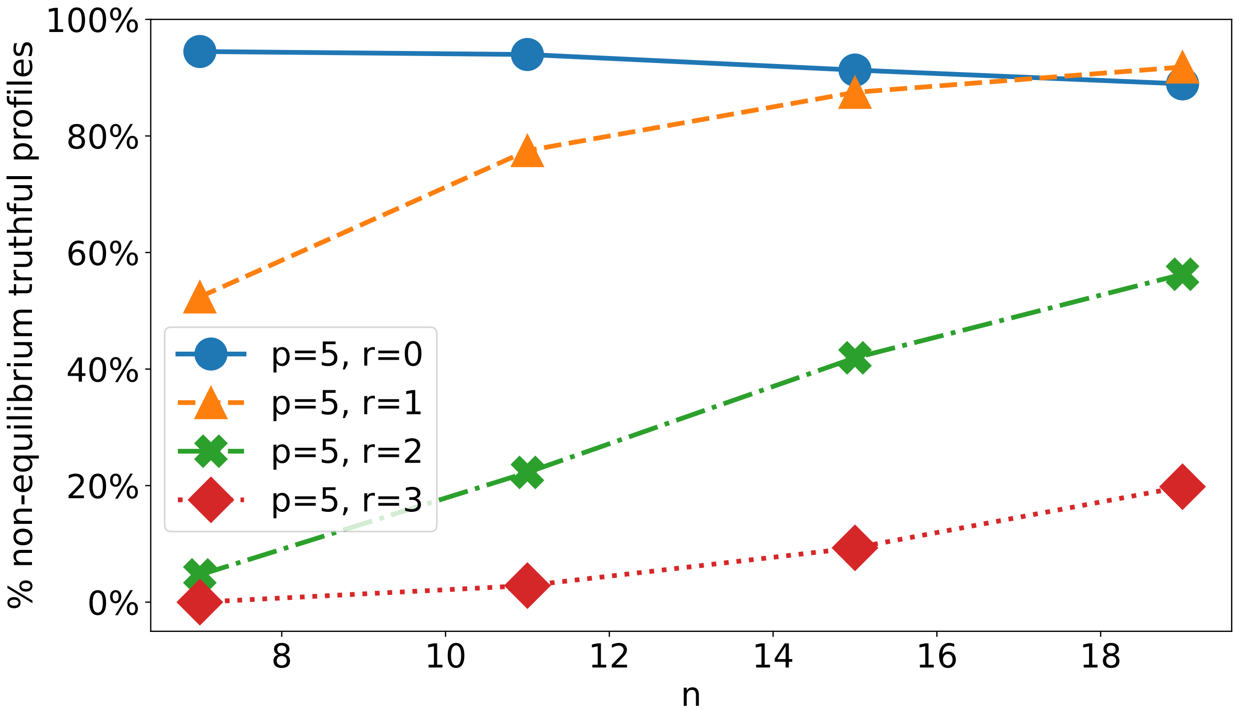

Figure 4.1: Percentage of truthful vote profiles not in equilibrium as n increases.

agrees with experiments by Bowman et al. (2014) and Grandi et al. (2022), suggesting that IV may reduce multiple-election paradoxes by helping agents choose better outcomes. However, further work will be needed to generalize this conclusion, as it contrasts experiments of single-issue IV by Meir et al. (2020) and Koolyk et al. (2017).

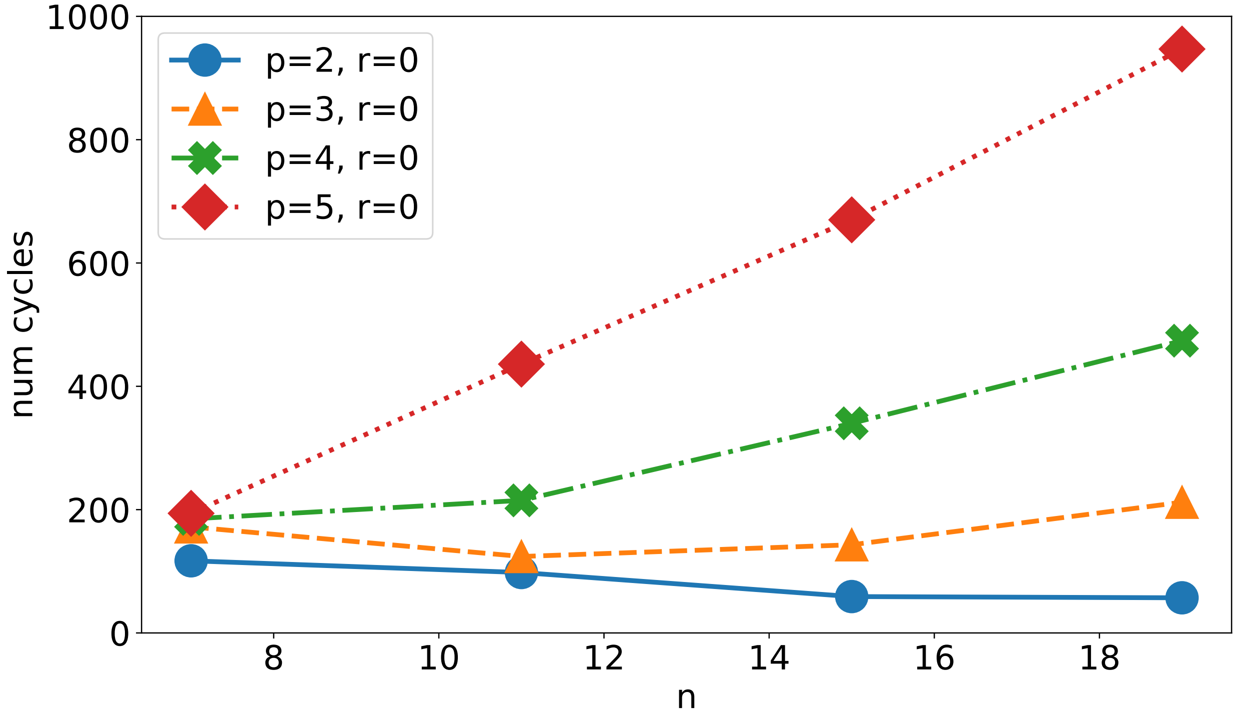

Figure 4.2: Number of truthful vote profiles whose LDI sequences cycle as n increases.

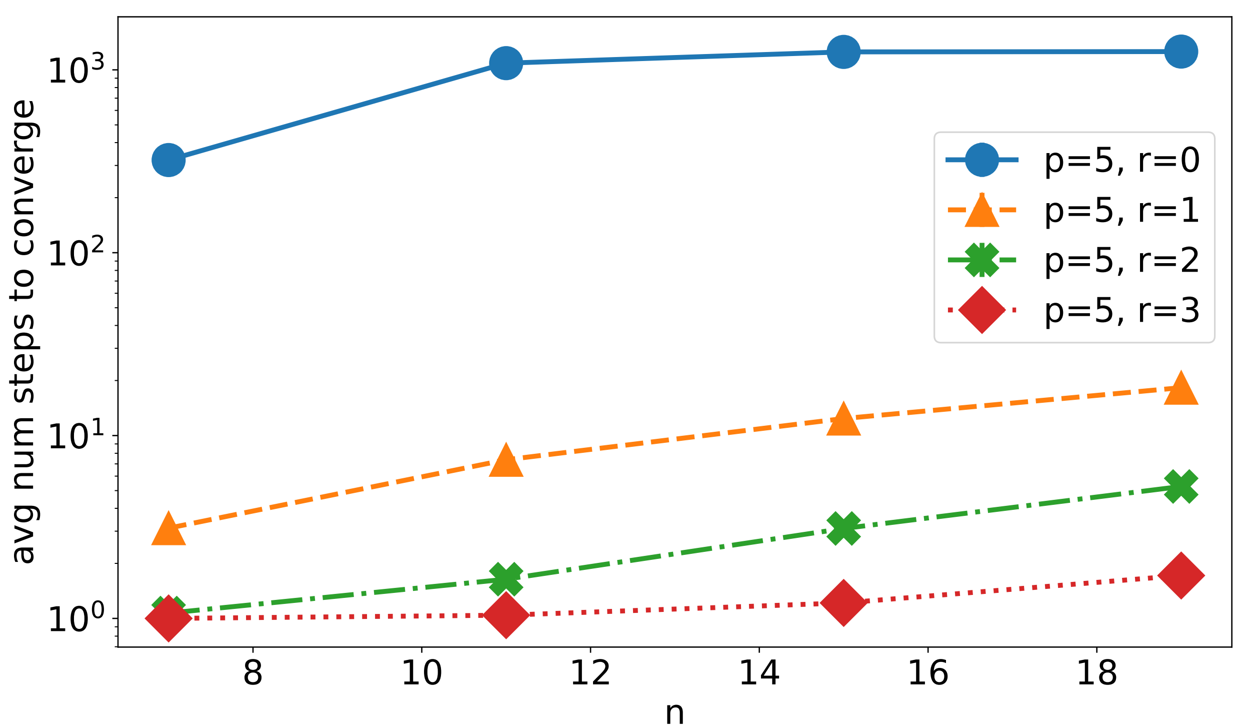

Figure 4.3: Average number steps for LDI sequences to converge as n increases; log scale; 95% CI (too small to show).

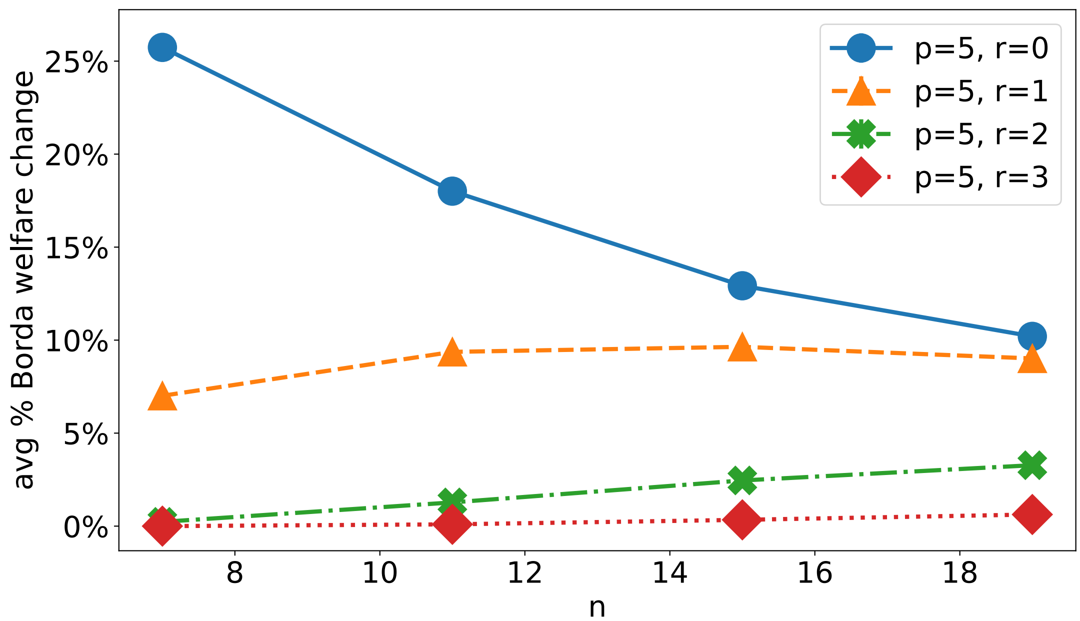

Figure 4.4: Average percent change in Borda welfare as n increases; 95% CI (too small to show).

CHAPTER 5
AVERAGE-CASE ANALYSIS OF ITERATIVE VOTING

5.1 Introduction

It is well-known in social choice theory that people may misreport their preferences to improve group decisions in their favor. Consider, for example, Alice, Bob, and Charlie deciding on which ice cream flavor to order for a party, and Charlie prefers strawberry to chocolate to vanilla. Given that Alice wants chocolate and Bob wants vanilla, Charlie would be better off voting for chocolate than truthfully (i.e., strawberry), by which vanilla may win as the tie-breaker. This form of strategic behavior is prolific in political science in narrowing the number of political parties (see e.g., Duverger's law (Riker, 1982)). Still, it is unclear what effect strategic behavior has on the social welfare of chosen outcomes.

Iterative voting (IV) is one model which naturally describes agents' strategic behavior – in misreporting their truthful preferences – over time. After agents reveal their preferences initially, they have the opportunity to repeatedly update their votes given information about other agents' votes, before the final decision is reached. Meir et al. (2010) first proposed iterative plurality voting and identified many sufficient conditions for IV to converge. This was followed up by a series of work examining various social choice rules, information and behavioral assumptions, and settings to determine when, to what outcomes, and how fast IV converges (see e.g., surveys by Meir (2017) and Meir (2018)).

While significant research has focused on the convergence and equilibrium properties of IV, only a few papers have analyzed its economic performance. Simulations and lab experiments by Reijngoud and Endriss (2012), Grandi et al. (2013), Bowman et al. (2014), and Grandi et al. (2022) found that IV improves outcome quality, while Koolyk et al. (2017) and Meir et al. (2020) found IV degrades quality. Given that strategic behavior is bound to occur, by the Gibbard-Satterthwaite theorem (Gibbard, 1973; Satterthwaite, 1975), Brânzei et al. (2013) investigated how bad the resulting outcome could be. Brânzei et al. (2013) defined the *additive dynamic price of anarchy* (ADPOA) as the difference in social welfare between the truthful vote profile and the worst-case equilibrium that is reachable via IV. This notion is with respect to the worst-case preference profile, any scheduler of agents'

Portions of this chapter have previously appeared as: Kavner, J., & Xia, L. (2024). *Average-case analysis of iterative voting.* arXiv. https://arxiv.org/abs/2402.08144.

improvement steps, and a given voting rule, and it refines the well-known *price of anarchy* (Roughgarden & Tardos, 2002) for a dynamical setting with myopic agents. They found the performance is "very good" for the plurality votng rule (with an ADPoA of 1), "not bad" for veto (with a DPoA of $\Omega(m)$ with m alternatives, $m \geq 4$), and "very bad" for Borda (with a DPoA of $\Omega(n)$ with n agents).

Notably, Brânzei et al. (2013)'s theorems assumed that the positional scoring voting rule had the same scoring vector as agents' additive utilities. In Chapter 3, we relaxed this assumption to arbitrary utility vectors with respect to iterative plurality. We found the additive DPoA worsened to $\Theta(n)$ in the worst-case. While this result bounds the theoretical consequences of IV, it provides little insight into how IV may perform realistically. Upon realizing this poor result, we took a first step in testing IV's practicality by exploring its average-case performance. By assuming that agents' preferences are distributed identically and uniformly at random, known as the *impartial culture* (IC) distribution, we found the expected additive DPoA to be $-\Omega(1)$. This suggests that IV actually *improves* social welfare over the truthful vote profile on average.

Average-case analysis is traditionally employed in computer science as a way around the intractability of NP-hard problems. This analysis is motivated by the possibility that worst-case results only occur infrequently in practice (Bogdanov & Trevisan, 2006). As seen with IV, average-case analysis hopes to provide a less pessimistic measure of an algorithm's performance. Still, the distribution used in the analysis may itself be unrealistic (Spielman & Teng, 2009). Indeed, IC used in Chapter 3 is widely understood to be implausible (Regenwetter, 2006; Tsetlin et al., 2003; Van Deemen, 2014), yet useful perhaps as a benchmark against other analytical results in social choice. This presents an opportunity to advance our understanding of iterative plurality voting beyond IC.

5.1.1 Our Contribution

We address the limitations of IC by analyzing the average-case performance of IV with respect to a larger class of input preferences. Our primary result is a characterization of certain classes of independent and identically distributed preferences for which IV improve or degrades social welfare. We describe to what extent welfare changes as the number of agents n increases, specified informally as follows.

Theorem 5.1 (Expected dynamic price of anarchy, informally put)**.** *There are classes of* *i.i.d. preferences such that, for even numbers of agents $n \in \mathbb{N}$, the expected performance of iterative plurality is* $\Theta(1)$, $-\Theta(1)$, *or* $\pm\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$*; for odd n, the performance is* $-\Theta(1)$.

Negative-valued performance according to our metric entails that equilibrium outcomes, resulting from IV, have *higher* welfare than their corresponding truthful winners. The differences in agents' preference distributions may therefore explain some of the variability of IV economic performance across experiments.

Our techniques begin similar to Theorem 3.2 in that we partition the expected performance of IV based on which set of alternatives are tied when agents vote truthfully, and we study each case separately. We further make use of a Bayesian network representation of agents' preferences in order to effectively group preference profiles by their economic performance and their likelihood of occurrence. Although our approach from Chapter 3 is applicable to our study, there are several places in their original proof which break down. Namely, their method only entails a bound on IV's expected performance of $\pm\mathcal{O}(\sqrt{n})$, which is insufficiently refined. Moreover, their is additional bias in the welfare each agent contributes to IV's expected performance, due to our assumptions about agents' preferences, that was not present in our study of IC. These factors make our analysis significantly more complicated and require a collection of novel binomial and multinomial lemmas to solve (see Sections 5.3.1 – 5.3.4).

Our analysis makes significant use of the *PMV-in-Polyhedron* theorem from (Xia, 2021a) to characterize the asymptotic likelihood of tied elections. Specifically, we capture the likelihood that the histogram of a preference profile, which is a Poisson multivariate variable, fits into a polyhedron that specify a tied election (Corollary 5.1 below) and with additional constraints (Lemma 5.6 below). Xia (2021a)'s techniques are not directly applicable in our setting because they characterize the likelihood of events occurring, whereas we study the expected performance of a protocol. Rather, we devise novel applications of their theorems in this chapter.

5.1.2 Preliminaries

The performance of IV is commonly measured by a worst-case comparison in social welfare between the truthful vote profile and the equilibrium that are reachable via the dynamics. This captures the impact that IV has against the outcome that would take place without agents' strategic manipulation of their votes. Moreover, it does not assume that the order agents make their improvement steps is controlled; the measure is over the worst-case scheduler. In the following definitions, we consider this performance measure according to the worst-case preference profile P as well as the average-case analysis when P is sampled from some distribution $\vec{\pi}$. Specifically, we recall those definitions from Chapters 2 and 3 that are pertinent to our present analysis.

We denote the set of *potential winning* alternatives as those who could become a winner if their plurality score $s_c(a)$ were to increment by one, including the current winner:

$$\mathrm{PW}(a) = \left\{ c \in \mathcal{A} \ : \begin{cases} s_c(a) = s_{f(a)}(a) - 1, & c \text{ is ordered before } f(a) \\ s_c(a) = s_{f(a)}(a), & c \text{ is ordered after } f(a) \end{cases} \right\} \cup \{f(a)\} \tag{5.1}$$

where the ordering is lexicographical for tie-breaking. We call these alternatives *approximately-tied*. We denote the set of *equilibrium winning* alternatives as those corresponding to any NE reachable from a via some BR sequence:

$$\mathrm{EW}(a) = \{f(\tilde{a}) : \exists \text{ a BR sequence from } a \text{ leading to the NE profile } \tilde{a}\}. \tag{5.2}$$

For a given positional scoring rule $f_{\vec{s}}$, a utility vector $\vec{u}$, n agents, preference profile P, and distribution of preferences $\vec{\pi}$, the *adversarial loss* starting from the truthful vote profile $a^* = top(P)$ is

$$\mathrm{D}^+_{f_{\vec{s}},\vec{u}}(P) = \mathrm{SW}_{\vec{u}}(P, f_{\vec{s}}(a^*)) - \min\nolimits_{c \in \mathrm{EW}(a^*)} \mathrm{SW}_{\vec{u}}(P, c). \tag{5.3}$$

The *additive dynamic price of anarchy (ADPoA)* is

$$\mathrm{ADPoA}(f_{\vec{s}}, \vec{u}) = \max\nolimits_{P \in \mathcal{L}(\mathcal{A})^n} \mathrm{D}^+_{f_{\vec{s}},\vec{u}}(P). \tag{5.4}$$

The *expected additive dynamic price of anarchy (EADPoA)* is

$$\mathrm{EADPoA}(f_{\vec{s}}, \vec{u}, \vec{\pi}) = \mathbb{E}_{P \sim \vec{\pi}} \left[\mathrm{D}^+_{f_{\vec{s}},\vec{u}}(P) \right]. \tag{5.5}$$

Throughout this chapter, we assume that there are $m = 3$ alternatives and that $f_{\vec{s}} = f$ is the plurality rule unless stated otherwise. We may drop parameters and scripts from these definitions for ease of notation when the context is clear.

5.2 Characterization of Average-Case Iterative Voting

Our main result extends the EADPoA beyond Chapter 3's study of IC toward a wider class of single-agent preference distributions. With IC, each agent has an equal probability of voting for each alternative, truthfully in a^*, and equal likelihood of preferring $u \succ v$ or $u \succ v$ for any two alternatives $u, v \in \mathcal{A}$. It was realized in Theorem 3.2 that these two concepts led IC to be concentrated around profiles that yielded a negative adversarial loss $\mathrm{D}^+(P)$, leading to an $EADPoA = -\Omega(1)$ conclusion. In contrast, our distribution π is characterized by the following assumption.

Assumption 5.1. *Consider the single-agent preference distribution $\pi = (\pi_1, \ldots, \pi_6)$ corresponding to the rankings:*

$$\begin{aligned} R_1 &= (1 \succ 2 \succ 3); & R_4 &= (3 \succ 1 \succ 2); \\ R_2 &= (2 \succ 3 \succ 1); & R_5 &= (1 \succ 3 \succ 2); \\ R_3 &= (3 \succ 2 \succ 1); & R_6 &= (2 \succ 1 \succ 3). \end{aligned} \tag{5.6}$$

We assume $\pi_1 = \pi_2 > 2\pi_3 = 2\pi_4 > 0$ and $\pi_5 = \pi_6 = 0$.

Like IC, π is designed to have equal probability for agents preferring alternatives 1 and 2 most and for preferring either $1 \succ 2$ or $2 \succ 1$. This maximizes the likelihood of a $\{1, 2\}$-tie and ensures that the likelihood of any other-way tie (i.e., $\mathrm{PW}(a^*) = W$ where $|W| \geq 2$ and $W \neq \{1, 2\}$) is exponentially small (see Corollary 5.1 below). With a $\{1, 2\}$-tie, IV will then be characterized by the *third-party* agents, those with rankings R_3 and R_4, alternatively switching their votes for alternatives 1 and 2 until convergence (recall Lemma 3.1).

The non-support of rankings R_5 and R_6 yields an important difference between π and IC that distinguishes our Theorem 5.1 result from Theorem 3.2. Notice in Assumption 5.1 that each agent with ranking R_1 adds $u_1 - u_2$ to $\mathrm{D}^+(P)$ while each agent with ranking R_2 subtracts $u_1 - u_3$ from $\mathrm{D}^+(P)$. Hence, we must keep track of how many agents have each of these rankings in our analysis (i.e., $\frac{n}{2} - q$ in Lemma 5.1). With IC (i.e., $\pi_1 = \ldots = \pi_6$), the average contribution that agents with ranking R_1 make to $\mathrm{D}^+(P)$ cancel out with those with ranking R_6; likewise, the contributions that agents with rankings R_2 and R_5 cancel out (recall Equation (3.8)). This distinction enables our different asymptotic values for different values of π_1 (see Lemma 5.4 below).

We present Theorem 5.1 next. Our main result is contributed by Lemmas 5.1 and 5.2 which characterize EADPoA given the two-way approximate-tie $\mathrm{PW}(a^*) = \{1, 2\}$ for even and odd numbers of agents n respectively. The primary techniques involve Xia (2021a)'s smoothed likelihood of ties (in Corollary 5.1 and Lemma 5.6), local central limit theorems (Petrov, 1975) (see Section 5.3.2), the Wallis product approximation for the central binomial coefficient (Galvin, 2018) (see Section 5.3.3), and a number of binomial theorems (in Lemma 5.3). Notice that by Assumption 5.1, we have $\pi_1 \in (\frac{1}{3}, \frac{1}{2})$ and $2(\pi_1 + \pi_3) = 1$. We denote agents' joint preference distribution by $P \sim \pi^n = (\pi, \pi, \ldots, \pi)$.

Theorem 5.1. *Suppose π follows Assumption 5.1. Given the plurality rule f and rank-based utility vector $\vec{u}$ with $u_2 > u_3$, $\exists N > 0$ such that $\forall n > N$ that are even,*

$$\mathit{EADPoA}(f, \vec{u}, \pi^n) = \begin{cases} \Theta(1), & 0.4 < \pi_1 < \frac{1}{2} \\ -\Theta(1), & \frac{1}{3} < \pi_1 < 0.4 \\ \pm\mathcal{O}\left(\frac{1}{\sqrt{n}}\right), & \pi_1 = 0.4 \end{cases}$$

while for odd n, $\mathit{EADPoA}(f, \vec{u}, \pi^n) = -\Theta(1)$. If $u_1 > u_2 = u_3$, then $\mathit{EADPoA}(f, \vec{u}, \pi^n) = \pm\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$; otherwise, if $u_1 = u_2 = u_3$, then $\mathit{EADPoA}(f, \vec{u}, \pi^n) = \pm\exp(-\Theta(n))$.

Proof. We prove the theorem by partitioning $\mathcal{L}(\mathcal{A})^n$ based on the possible potential winner sets and applying the law of total expectation to sum EADPoA across these partitions. Specifically, for every $W \subseteq \mathcal{A}$ we define

$$\overline{\mathrm{PoA}}(W) = \Pr_{P\sim\pi^n}(\mathrm{PW}(P) = W) \times \mathbb{E}_{P\sim\pi^n}[\mathrm{D}^+(P) \mid \mathrm{PW}(P) = W].$$

This entails

$$\mathrm{EADPoA} = \sum_{c\in\mathcal{A}} \overline{\mathrm{PoA}}(\{c\}) + \sum_{\substack{W\subseteq\mathcal{A}\setminus\{1,2\},\\ |W|\geq 2}} \overline{\mathrm{PoA}}(W) + \overline{\mathrm{PoA}}(\{1, 2\}). \tag{5.7}$$

First, it is clear that $\sum_{c\in\mathcal{A}} \overline{\mathrm{PoA}}(\{c\}) = 0$ since any profile P with $|\mathrm{PW}(P)| = 1$ is an equilibrium, so $\mathrm{D}^+(P) = 0$. Second, we get that

$$\begin{aligned}\sum_{\substack{W\subseteq\mathcal{A}\setminus\{1,2\},\\|W|\geq 2}} \left|\overline{\mathrm{PoA}}(W)\right| &= \sum_{\substack{W\subseteq\mathcal{A}\setminus\{1,2\},\\|W|\geq 2}} \Pr(\mathrm{PW}(a^*) = W)\cdot\left|\mathbb{E}[\mathrm{D}^+(P) \mid \mathrm{PW}(a^*) = W]\right| \\ &\leq \mathcal{O}(n) \sum_{\substack{W\subseteq\mathcal{A}\setminus\{1,2\},\\|W|\geq 2}} \Pr(\mathrm{PW}(a^*) = W) \qquad (5.8)\\ &= \exp(-\Theta(n)). \qquad (5.9)\end{aligned}$$

Equation (5.8) follows from $\max_P |\mathrm{D}^+(P)| = \mathcal{O}(n)$ since each agent contributes only a constant amount to $\mathrm{D}^+(P)$ (recall Theorem 3.1). Equation (5.9) follows from the following corollary and the fact that $|2^{\mathcal{A}}| = 2^m$ is constant for fixed m.

Corollary 5.1. *Fix $m \geq 3$ and distribution $\pi \in \Delta(\mathcal{L}(\mathcal{A}))$. Let $\lambda_i(\pi) = \sum_{j:top(R_j)=i} \pi_j$ be the likelihood of an agent truthfully voting for i, and $W^*(\pi) = \arg\max_{i\in[m]} \lambda_i(\pi)$ be a set. Then*

$$\Pr_{P\sim\pi^n}(PW(a^*) = W) = \begin{cases}\Theta\left(n^{-\frac{|W|-1}{2}}\right), & W \subseteq W^*(\pi)\\ \exp(-\Theta(n)), & W \not\subseteq W^*(\pi).\end{cases} \qquad (5.10)$$

Corollary 5.1 generalizes Corollary 2.1, the likelihood of k-way plurality ties, to distributions beyond IC. It follows directly from the proof of Theorem 3 of Xia (2021a), especially Claim 4(ii) in their appendix. For distributions π that abide by Assumption 5.1, we have $\lambda_1(\pi) = \lambda_2(\pi) > \lambda_3(\pi)$ which entails $W^* = \{1, 2\}$. It follows that $\Pr(\mathrm{PW}(a^*) = \{1, 2\}) = \Theta\left(\frac{1}{\sqrt{n}}\right)$ while $\Pr(\mathrm{PW}(a^*) = W) = \exp(-\Theta(n))$ for any other $W \subseteq \mathcal{A}\setminus\{1, 2\}, |W| \geq 2$.

Finally, Theorem 5.1 follows from the last term of Equation (5.7), entailed by Lemma 5.1 (n is even) and Lemma 5.2 (n is odd), below. The theorem's conditions on $\vec{u}$ are technical and represented in these lemmas' proofs (see the discussion following Equations (5.20) and (5.29)).

Lemma 5.1. *Given rank-based utility vector $\vec{u}$ with $u_2 > u_3$, $\exists N > 0$ such that $\forall n > N$ that* *are even,*

$$\overline{PoA}(\{1,2\}) = \begin{cases} \Theta(1), & 0.4 < \pi_1 < \frac{1}{2} \\ -\Theta(1), & \frac{1}{3} < \pi_1 < 0.4 \\ \pm\mathcal{O}\left(\frac{1}{\sqrt{n}}\right), & \pi_1 = 0.4. \end{cases}$$

Proof. For any preference profile $P \in \mathcal{L}(\mathcal{A})^n$ such that the potential winner set $\mathrm{PW}(P) = \{1, 2\}$, there are two possible cases: either alternative 1 or alternative 2 is the truthful winner. We denote these cases by $\mathcal{E}_1 = \mathbb{1}\{s_1(a^*) = s_2(a^*)\}$ and $\mathcal{E}_2 = \mathbb{1}\{s_1(a^*) + 1 = s_2(a^*)\}$ respectively. This suggests the following partition:

$$\begin{aligned} \overline{\mathrm{PoA}}(\{1,2\}) = &\Pr_{P\sim\pi^n}(\mathcal{E}_1) \times \mathbb{E}_{P\sim\pi^n}[\mathrm{D}^+(P) \mid \mathcal{E}_1] \\ &+ \Pr_{P\sim\pi^n}(\mathcal{E}_2) \times \mathbb{E}_{P\sim\pi^n}[\mathrm{D}^+(P) \mid \mathcal{E}_2]. \end{aligned} \tag{5.11}$$

Iterative plurality starting from a^* consists of agents changing their votes from alternatives that are not potentially-winning to alternatives that then become the winner (Brânzei et al., 2013). Hence, $\mathrm{EW}(a^*) \subsetneq \{1, 2\}$ is the unique alternative of the two with more agents preferring it (subject to lexicographical tie-breaking; recall Lemma 3.1). This means that the equilibrium winning alternative only differs from the truthful winner if either $\mathcal{E}_1 \wedge \{P[2 \succ 1] > P[1 \succ 2]\}$ or $\mathcal{E}_2 \wedge \{P[1 \succ 2] \geq P[2 \succ 1]\}$ holds. Notice that these conditions only depend on the histogram of agents' rankings and not on agent identities or the order of best response steps. Thus it is easy to identify when either condition occurs using this histogram. Still, given that one of these events occurs, $\mathrm{D}^+(P)$ also depends on the histogram of agents' rankings.

We prove Lemma 5.1 by grouping all vote profiles that satisfy the same pair of conditions and have the same adversarial loss, multiplying this loss with its likelihood of occurrence, and summing over all such groupings. Each group is defined by how many agent prefer each alternative most, captured by the set $\mathcal{T}^{1,q}$ (defined below), and how many agents prefer $1 \succ 2$ or $2 \succ 1$, captured by the set $\mathcal{Z}_{\vec{t},\beta}$ (defined below). To accomplish this, we utilize a Bayesian network to represent the agents' joint preference distribution and effectively condition the joint probability based of these groups.

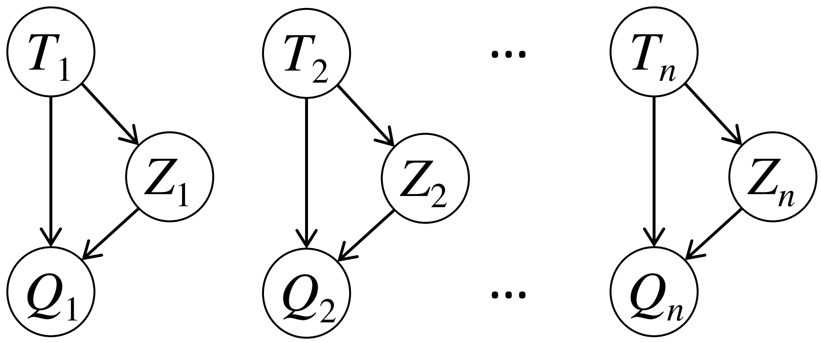

Figure 5.1: Bayesian network representation of P as $\vec{T}$, $\vec{Z}$, and $\vec{Q}$.

Step 1: Represent as Bayesian network. For each $j \leq n$, we represent agent j's ranking distribution by a Bayesian network of three random variables: T_j represents agent j's top-ranked alternative, Z_j represents whether $1 \succ_j 2$ or $2 \succ_j 1$, conditioned on T_j, and Q_j represents the linear order conditioned on T_j and Z_j.

Definition 5.1. *For any $j \leq n$, we define a Bayesian network with three random variables $T_j \in \{1, 2, 3\}$, $Z_j \in \{1, 2\}$, and $Q_j \in \{R_1, R_2, R_3, R_4\}$, where T_j has no parent, T_j is the parent of Z_j, and T_j and Z_j are Q_j's parents (see Figure 5.1). Let $\vec{T} = (T_1, \ldots, T_n)$, $\vec{Z} = (Z_1, \ldots, Z_n)$, and $\vec{Q} = (Q_1, \ldots, Q_n)$. The (conditional) distributions are:*

- *T_j is distributed with densities $\{\pi_1, \pi_2, \pi_3 + \pi_4\}$ over $\{1, 2, 3\}$;*
- $\Pr(Z_j = 1 \mid T_j = t) = \begin{cases} 1, & t = 1 \\ 0, & t = 2 \\ 0.5, & t \notin \{1, 2\}; \end{cases}$
- *Q_j is distributed uniformly over $\{R_1, R_2, R_3, R_4\}$ among those whose top alternative is T_j and $1 \succ_j 2$ if $Z_j = 1$, or $2 \succ_j 1$ if $Z_j = 2$.*

It is not hard to verify that (unconditional) Q_j follows the distribution π over $\{R_1, R_2, R_3, R_4\}$. Therefore $\vec{Q}$ follows the same distribution as P, which is π^n.

Step 2: Characterize the preference profiles. We first characterize the set of vote profiles such that alternative 1 is the truthful winner and alternative 2 is the unique equilibrium winner. This captures the event $\mathcal{E}_1 \wedge \{P[2 \succ 1] > P[1 \succ 2]\}$ discussed above, such that alternatives 1 and 2 are exactly tied and more agents prefer $2 \succ 1$ than otherwise. We define the set $\mathcal{T}^{1,q}$ below to describes how many agents vote for each alternative. Next, we define the set $\mathcal{Z}_{\vec{t},\beta}$ to detail how many agents prefer alternative $1 \succ 2$ or $2 \succ 1$, conditioned on $\vec{T}$. We use the Bayesian network framework from Step 1 and the law of total expectation to partition $\Pr(\mathcal{E}_1) \times \mathbb{E}[\mathrm{D}^+(P) \mid \mathcal{E}_1]$ further according to these events. Finally, we defer the other case, $\mathcal{E}_2 \wedge \{P[1 \succ 2] \geq P[2 \succ 1]\}$, such that alternative 2 is the truthful winner, alternative 1 is the equilibrium winner, and at least as many agents prefer $1 \succ 2$ than otherwise, to Step 4 below.

Our characterization for the former case consists of $q, \beta \in \mathbb{N}$ such that there are

- $\frac{n}{2} - q$ agents with rankings $R_1 = (1 \succ 2 \succ 3)$ and $R_2 = (2 \succ 3 \succ 1)$ each;
- β agents with ranking $R_3 = (3 \succ 2 \succ 1)$;
- $2q - \beta$ agents with ranking $R_4 = (3 \succ 1 \succ 2)$.

Formally, we define $\mathcal{T}^{1,q} \subseteq [3]^n$ as the set of vectors $\vec{t} = (t_1, \ldots, t_n)$ such that alternatives 1 and 2 have the maximal plurality score of $\frac{n}{2} - q$ and alternative 1 is the victor:

$$\mathcal{T}^{1,q} = \left\{\vec{t} \in [3]^n \ : |\{j : t_j = 1\}| = |\{j : t_j = 2\}| = \left(\frac{n}{2} - q\right) > |\{j : t_j = 3\}|\right\}.$$

The minimum of q is clearly 0, while its maximum is

$$q^* = \max\left\{q \in \mathbb{Z} : \left(\frac{n}{2} - q\right) > 2q\right\}$$

so that

$$q^* = \begin{cases} \left\lfloor \frac{n}{6} \right\rfloor - 1, & n \ mod\ 6 = 0 \\ \left\lfloor \frac{n}{6} \right\rfloor + 1, & n \ mod\ 6 = 2 \\ \left\lfloor \frac{n}{6} \right\rfloor + 3, & n \ mod\ 6 = 4. \end{cases}$$

It is easy to see that $\mathcal{E}_1$ holds for $\vec{Q}$ if and only if $\vec{T}$ takes a value in $\mathcal{T}^{1,q}$ for some $q \in [0, q^*]$. This implies the following equality:

$$\Pr(\mathcal{E}_1) \times \mathbb{E}[\mathrm{D}^+(P) \mid \mathcal{E}_1] = \sum_{q=0}^{q^*} \sum_{\vec{t} \in \mathcal{T}^{1,q}} \Pr(\vec{T} = \vec{t}) \times \mathbb{E}_{\vec{Q}}[\mathrm{D}^+(\vec{Q}) \mid \vec{T} = \vec{t}]. \tag{5.12}$$

Without loss of generality, we will assume for the duration of the proof that $q^* = \left\lfloor \frac{n}{6} \right\rfloor - 1$, taking the case that n is divisible by 6. It is easy to show that for a constant number of terms in Equation (5.12) such that $q = \Theta(n)$, the objective is exponentially small and hence does not affect the result of this lemma. The case where n is odd is handled by Lemma 5.2.

Given that $\vec{t} \in \mathcal{T}^{1,q}$, there are $2q$ agents with either ranking R_3 or R_4. We uphold the event $\mathcal{E}_1 \wedge \{P[2 \succ 1] > P[1 \succ 2]\}$ with respect to $\vec{Q}$ by having $\beta > q$ agents with ranking R_3 (recall that this event is necessary by Lemma 3.1). Specifically, for every $\beta \in [q+1, 2q]$, we define $\mathcal{Z}_{\vec{t},\beta} \subseteq \{1,2\}^n$ as the vectors $\vec{z}$ where the number of 2s among indices in $\{j \leq n : t_j \notin \{1,2\}\}$ is exactly β:

$$\mathcal{Z}_{\vec{t},\beta} = \{\vec{z} \in \{1,2\}^n \ : \ z_j = t_j \ \forall j \ : \ t_j \in \{1,2\} \text{ and } |\{j \ : \ t_j \notin \{1,2\}, z_j = 2\}| = \beta\}.$$

Continuing Equation (5.12) we get

$$\begin{aligned}
&\Pr(\mathcal{E}_1) \times \mathbb{E}[\mathrm{D}^+(P) \mid \mathcal{E}_1] \\
&= \sum_{q=0}^{\frac{n}{6}-1} \sum_{\vec{t} \in \mathcal{T}^{1,q}} \sum_{\beta=q+1}^{2q} \sum_{\vec{z} \in \mathcal{Z}_{\vec{t},\beta}} \Pr(\vec{T} = \vec{t}, \vec{Z} = \vec{z}) \mathbb{E}_{\vec{Q}}[\mathrm{D}^+(\vec{Q}) \mid \vec{T} = \vec{t}, \vec{Z} = \vec{z}] \\
&= \sum_{q=0}^{\frac{n}{6}-1} \sum_{\vec{t} \in \mathcal{T}^{1,q}} \sum_{\beta=q+1}^{2q} \sum_{\vec{z} \in \mathcal{Z}_{\vec{t},\beta}} \Pr(\vec{T} = \vec{t}, \vec{Z} = \vec{z}) \sum_{j=1}^{n} \mathbb{E}_{\vec{Q}_j}[\vec{u}(Q_j, 1) - \vec{u}(Q_j, 2) \mid \vec{T} = \vec{t}, \vec{Z} = \vec{z}] \\
&= \sum_{q=0}^{\frac{n}{6}-1} \sum_{\vec{t} \in \mathcal{T}^{1,q}} \sum_{\beta=q+1}^{2q} \sum_{\vec{z} \in \mathcal{Z}_{\vec{t},\beta}} \Pr(\vec{T} = \vec{t}, \vec{Z} = \vec{z}) \sum_{j=1}^{n} E^1_{t_j,z_j} \qquad (5.13)
\end{aligned}$$

where

$$E^1_{t_j,z_j} = \mathbb{E}_{Q_j}[\vec{u}(Q_j, 1) - \vec{u}(Q_j, 2) \mid T_j = t_j, Z_j = z_j].$$

This holds by the Bayesian network: for any $j \leq n$, given T_j and Z_j, Q_j is conditionally independent of other Q's.

Step 3: Substitute expected welfare per agent. Notice that $E^1_{t_j,z_j}$ only depends on the values of t_j and z_j, but not j. We interpret this summation as aggregating the amount of welfare each agent with given t_j and z_j values contributes to the adversarial loss $\mathrm{D}^+(\vec{Q})$. In particular, each agent j with $t_j = 1$ as their most-preferred alternative must have ranking $Q_j = R_1 = (1 \succ 2 \succ 3)$, by Definition 5.1. This entails that they contribute $\vec{u}(R_1, 1) - \vec{u}(R_1, 2) = u_1 - u_2$ welfare to $\mathrm{D}^+(\vec{Q})$; recall our use of rank-based utility with vector $\vec{u} =$

(u_1, u_2, u_3). The same argument holds to yield $-u_1 + u_3$ welfare for each agent with $t_j = 2$, corresponding with $R_2 = (2 \succ 3 \succ 1)$. Next, any agent with $t_j \notin \{1, 2\}$ entails that they have either ranking $R_3 = (3 \succ 2 \succ 1)$ or $R_4 = (3 \succ 1 \succ 2)$. If $z_j = 2$, the agent has ranking R_3 and contributes $-u_2 + u_3$ welfare to $\mathrm{D}^+(\vec{Q})$; otherwise, if $z_j = 1$, the agent has ranking R_4 and contributes $u_2 - u_3$ welfare. In sum, we have

$$E^1_{t_j,z_j} = \begin{cases} u_1 - u_2, & t_j = z_j = 1 \\ -u_1 + u_3, & t_j = z_j = 2 \\ -u_2 + u_3, & t_j \notin \{1,2\}, z_j = 2 \\ u_2 - u_3, & t_j \notin \{1,2\}, z_j = 1, \end{cases}$$

which yields

$$\begin{aligned} \sum_{j=1}^{n} E^1_{t_j,z_j} &= \begin{pmatrix} \frac{n}{2} - q \\ \frac{n}{2} - q \\ \beta \\ 2q - \beta \end{pmatrix} \cdot \begin{pmatrix} u_1 - u_2 \\ -u_1 + u_3 \\ -u_2 + u_3 \\ u_2 - u_3 \end{pmatrix} \\ &= \left(\frac{n}{2} - q\right)(-u_2 + u_3) + (2q - 2\beta)(u_2 - u_3) \\ &= (u_2 - u_3)\left(-\frac{n}{2} + 3q - 2\beta\right). \end{aligned}$$

The duration of this proof carries over $(u_2 - u_3)$ as a factor in front of Equation (5.13). Since we assumed $u_2 > u_3$, we will forego writing this factor for ease of notation. Equation (5.13) is therefore proportional to

$$\begin{aligned} &\sum_{q=0}^{\frac{n}{6}-1} \sum_{\vec{t} \in \mathcal{T}^{1,q}} \sum_{\beta=q+1}^{2q} \sum_{\vec{z} \in \mathcal{Z}_{\vec{t},\beta}} \Pr(\vec{T} = \vec{t}, \vec{Z} = \vec{z}) \left(-\frac{n}{2} + 3q - 2\beta\right) \\ &= \sum_{q=0}^{\frac{n}{6}-1} \sum_{\vec{t} \in \mathcal{T}^{1,q}} \Pr(\vec{T} = \vec{t}) \sum_{\beta=q+1}^{2q} \left(-\frac{n}{2} + 3q - 2\beta\right) \sum_{\vec{z} \in \mathcal{Z}_{\vec{t},\beta}} \Pr(\vec{Z} = \vec{z} \mid \vec{T} = \vec{t}) \\ &= \sum_{q=0}^{\frac{n}{6}-1} \sum_{\vec{t} \in \mathcal{T}^{1,q}} \Pr(\vec{T} = \vec{t}) \sum_{\beta=q+1}^{2q} \left(-\frac{n}{2} + 3q - 2\beta\right) \binom{2q}{\beta} \frac{1}{2^{2q}} \end{aligned} \tag{5.14}$$

where the last line follows from

$$\Pr(Z_j = 1 \mid T_j \notin \{1,2\}) = 0.5.$$

Case when $q = 0$. Notice that Equation (5.14) is zero when $q = 0$. This is because there are zero third-party agents, so there is no iterative plurality dynamics. By Lemma 3.1, the adversarial loss is zero.

Next, we apply the following binomial identities, which are proved in Section 5.3.4.

Lemma 5.3. *For $q \geq 1$, the following identities hold:*

1.
$$\sum_{\beta=q+1}^{2q} \binom{2q}{\beta} \frac{1}{2^{2q}} = \frac{1}{2} - \frac{1}{2^{2q}} \binom{2q-1}{q-1}$$

2.
$$\sum_{\beta=q+1}^{2q} \beta \binom{2q}{\beta} \frac{1}{2^{2q}} = \frac{q}{2}$$

By applying Equations 1 and 2 of Lemma 5.3, Equation (5.14) simplifies to

$$\begin{aligned} &\sum_{q=1}^{\frac{n}{6}-1} \sum_{\vec{t} \in \mathcal{T}^{1,q}} \Pr(\vec{T} = \vec{t}) \left(\left(-\frac{n}{2} + 3q \right) \left(\frac{1}{2} - \frac{1}{2^{2q}} \binom{2q-1}{q-1} \right) - 2 \left(\frac{q}{2} \right) \right) \\ &= \sum_{q=1}^{\frac{n}{6}-1} \Pr(\vec{T} \in \mathcal{T}^{1,q}) \left(\frac{1}{2} \left(-\frac{n}{2} + q \right) - \frac{\left(-\frac{n}{2} + 3q \right)}{2^{2q}} \binom{2q-1}{q-1} \right). \end{aligned} \tag{5.15}$$

Recall from Definition 5.1 that $\vec{T}$ is a multinomial distribution over $\{1,2,3\}$ with event probabilities $(\pi_1, \pi_2, \pi_3 + \pi_4)$. Furthermore, $\mathcal{T}^{1,q}$ is the event where there are $\frac{n}{2} - q$ of 1s and 2s each in $\vec{T}$, and $2q$ of 3s. Therefore

$$\begin{aligned} \Pr(\vec{T} \in \mathcal{T}^{1,q}) &= \binom{n}{\frac{n}{2}-q, \frac{n}{2}-q, 2q} \pi_1^{\frac{n}{2}-q} \pi_2^{\frac{n}{2}-q} (\pi_3 + \pi_4)^{2q} \\ &= \binom{n}{\frac{n}{2}-q, \frac{n}{2}-q, q, q} \pi_1^{n-2q} \pi_3^{2q} \cdot \frac{2^{2q}}{\binom{2q}{q}}. \end{aligned} \tag{5.16}$$

Hence, Equation (5.15) becomes

$$\sum_{q=1}^{\frac{n}{6}-1} \binom{n}{\frac{n}{2}-q, \frac{n}{2}-q, q, q} \pi_1^{n-2q} \pi_3^{2q} \left(\frac{\left(-\frac{n}{2}+q\right)}{2} \frac{2^{2q}}{\binom{2q}{q}} - \frac{\left(-\frac{n}{2}+3q\right)}{2} \right) \tag{5.17}$$

where we use the fact that $2\binom{2q-1}{q-1} = \binom{2q}{q}$.

<u>**Step 4: Repeat process for case $\mathcal{E}_2$.**</u> We next repeat the above process in Steps 2 and 3 for $\mathcal{E}_2 \wedge \{P[1 \succ 2] \geq P[2 \succ 1]\}$, such that alternative 2 is the truthful winner, alternative 1 is the equilibrium winner, and at least as many agents prefer $1 \succ 2$ than otherwise. In this case, we have

- $\frac{n}{2} - 1 - q$ agents with ranking $R_1 = (1 \succ 2 \succ 3)$;
- $\frac{n}{2} - q$ agents with ranking $R_2 = (2 \succ 3 \succ 1)$;
- β agents with ranking $R_3 = (3 \succ 2 \succ 1)$;
- $2q + 1 - \beta$ agents with ranking $R_4 = (3 \succ 1 \succ 2)$.

Define $\mathcal{T}^{2,q} \subseteq [3]^n$ as the set of vectors $\vec{t} = (t_1, \ldots, t_n)$ such that alternative 2 has the maximal plurality score of $\frac{n}{2} - q$ and alternative 1 has one fewer vote:

$$\mathcal{T}^{2,q} = \left\{ \vec{t} \in [3]^n \ : \ |\{j : t_j = 1\}| + 1 = |\{j : t_j = 2\}| = \left(\frac{n}{2} - q\right) > |\{j : t_j = 3\}| \right\}.$$

The number of third-party agents is $2q + 1$, so the maximum value of q is

$$q^* = \max \left\{ q \in \mathbb{Z} : \left(\frac{n}{2} - q\right) > 2q + 1 \right\}$$

so that

$$q^* = \begin{cases} \left\lfloor \frac{n}{6} \right\rfloor - 1, & n \ mod \ 6 = 0 \\ \left\lfloor \frac{n}{6} \right\rfloor, & n \ mod \ 6 \in \{2, 4\}. \end{cases}$$

It is easy to see that $\mathcal{E}_2$ holds for $\vec{Q}$ if and only if $\vec{T}$ takes a value in $\mathcal{T}^{2,q}$ for some $q \in [0, q^*]$. This implies the following equality:

$$\Pr(\mathcal{E}_2) \times \mathbb{E}[\mathrm{D}^+(P) \mid \mathcal{E}_2] = \sum_{q=0}^{q^*} \sum_{\vec{t} \in \mathcal{T}^{2,q}} \Pr(\vec{T} = \vec{t}) \times \mathbb{E}_{\vec{Q}}[\mathrm{D}^+(\vec{Q}) \mid \vec{T} = \vec{t}]. \tag{5.18}$$

Like in Step 2, we will assume $q^* = \left\lfloor \frac{n}{6} \right\rfloor - 1$ without loss of generality, taking the case that n is divisible by 6. Given that $\vec{t} \in \mathcal{T}^{2,q}$, there are $2q+1$ third-party agents with either ranking R_3 or R_4. We uphold the event $\mathcal{E}_2 \wedge \{P[1 \succ 2] \geq P[2 \succ 1]\}$ with respect to $\vec{Q}$ by having $\beta \in [0, q]$ agents with ranking R_3. Therefore, continuing Equation (5.18) we get

$$\begin{aligned}
&\Pr(\mathcal{E}_2) \times \mathbb{E}[\mathrm{D}^+(P) \mid \mathcal{E}_2] \\
&= \sum_{q=0}^{\frac{n}{6}-1} \sum_{\vec{t} \in \mathcal{T}^{2,q}} \sum_{\beta=0}^{q} \sum_{\vec{z} \in \mathcal{Z}_{\vec{t},\beta}} \Pr(\vec{T} = \vec{t}, \vec{Z} = \vec{z}) \mathbb{E}_{\vec{Q}}[\mathrm{D}^+(\vec{Q}) \mid \vec{T} = \vec{t}, \vec{Z} = \vec{z}] \\
&= \sum_{q=0}^{\frac{n}{6}-1} \sum_{\vec{t} \in \mathcal{T}^{2,q}} \sum_{\beta=0}^{q} \sum_{\vec{z} \in \mathcal{Z}_{\vec{t},\beta}} \Pr(\vec{T} = \vec{t}, \vec{Z} = \vec{z}) \sum_{j=1}^{n} \mathbb{E}_{\vec{Q}_j}[\vec{u}(Q_j, 2) - \vec{u}(Q_j, 1) \mid \vec{T} = \vec{t}, \vec{Z} = \vec{z}] \\
&= \sum_{q=0}^{\frac{n}{6}-1} \sum_{\vec{t} \in \mathcal{T}^{2,q}} \sum_{\beta=0}^{q} \sum_{\vec{z} \in \mathcal{Z}_{\vec{t},\beta}} \Pr(\vec{T} = \vec{t}, \vec{Z} = \vec{z}) \sum_{j=1}^{n} E^2_{t_j, z_j}
\end{aligned} \tag{5.19}$$

where

$$E^2_{t_j, z_j} = \mathbb{E}_{\vec{Q}_j}[\vec{u}(Q_j, 2) - \vec{u}(Q_j, 1) \mid T_j = t_j, Z_j = z_j]$$

which holds because of the Bayesian network structure. Notice that $E^2_{t_j, z_j}$ and only depends on the values of t_j and z_j, but not j. By Definition 5.1 and realizing that $E^1_{t_j, z_j} = -E^2_{t_j, z_j}$, we get

$$\begin{aligned}
\sum_{j=1}^{n} E^2_{t_j, z_j} &= -\begin{pmatrix} \frac{n}{2} - 1 - q \\ \frac{n}{2} - q \\ \beta \\ 2q + 1 - \beta \end{pmatrix} \cdot \begin{pmatrix} u_1 - u_2 \\ -u_1 + u_3 \\ -u_2 + u_3 \\ u_2 - u_3 \end{pmatrix} \\
&= -\left(\frac{n}{2} - q\right)(-u_2 + u_3) - (2q - 2\beta)(u_2 - u_3) - (-u_1 + 2u_2 - u_3) \\
&= -(u_2 - u_3)\left(-\frac{n}{2} + 3q - 2\beta\right) + (u_1 - 2u_2 + u_3).
\end{aligned} \tag{5.20}$$

First, notice that when $\vec{u}$ represents Borda welfare then $u_1 - 2u_2 + u_3 = 0$. Otherwise, for this constant term, we have that Equation (5.19) is proportional to the likelihood of two-way ties for plurality voting under i.i.d. preferences. By Corollary 5.1, this term is either $+\Theta\left(\frac{1}{\sqrt{n}}\right)$ or $-\Theta\left(\frac{1}{\sqrt{n}}\right)$. Second, like in Step 3 above, we will forego writing the factor $(u_2 - u_3)$ in front of Equation (5.19) for ease of notation. Equation (5.19) is therefore proportional to

$$\begin{aligned}
&-\sum_{q=0}^{\frac{n}{6}-1} \sum_{\vec{t}\in\mathcal{T}^{2,q}} \sum_{\beta=0}^{q} \sum_{\vec{z}\in\mathcal{Z}_{\vec{t},\beta}} \Pr(\vec{T}=\vec{t}, \vec{Z}=\vec{z}) \left(-\frac{n}{2} + 3q - 2\beta\right) \\
&= -\sum_{q=0}^{\frac{n}{6}-1} \sum_{\vec{t}\in\mathcal{T}^{2,q}} \Pr(\vec{T}=\vec{t}) \sum_{\beta=0}^{q} \left(-\frac{n}{2} + 3q - 2\beta\right) \sum_{\vec{z}\in\mathcal{Z}_{\vec{t},\beta}} \Pr(\vec{Z}=\vec{z} \mid \vec{T}=\vec{t}) \\
&= -\sum_{q=0}^{\frac{n}{6}-1} \sum_{\vec{t}\in\mathcal{T}^{2,q}} \Pr(\vec{T}=\vec{t}) \sum_{\beta=0}^{q} \left(-\frac{n}{2} + 3q - 2\beta\right) \binom{2q+1}{\beta} \frac{1}{2^{2q+1}}
\end{aligned} \tag{5.21}$$

where the last line follows from

$$\Pr(Z_j = 1 \mid T_j \notin \{1,2\}) = 0.5.$$

Case when $q = 0$. Unlike the prior case, now when $q = 0$ there is a single third-party agent with ranking R_4. This is illustrated in Equation (5.21), with $q = \beta = 0$, as

$$\begin{aligned}
&\sum_{\vec{t}\in\mathcal{T}^{2,0}} \Pr(\vec{T}=\vec{t}) \cdot \left(-\frac{n}{2}\right) \binom{1}{0} \frac{1}{2} \\
&= -\frac{n}{4} \Pr(\vec{T} \in \mathcal{T}^{2,0}) \\
&= -\frac{n}{4} \binom{n}{\frac{n}{2}-1, \frac{n}{2}, 1} \pi_1^{\frac{n}{2}-1} \pi_2^{\frac{n}{2}} (\pi_3 + \pi_4)^1 \\
&= -\frac{n\pi_3\pi_1^n}{2\pi_1} (n/2) \binom{n}{n/2} \\
&= -\mathcal{O}(n^{1.5}) \cdot (2\pi_1)^n \\
&= -\exp(-\Theta(n))
\end{aligned}$$

by Stirling's approximation.

Proposition 5.1 (Stirling's approximation)**.**

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

Hence,

$$\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{n\pi}}.$$

This proposition is discussed further in Section 5.3.3. Next, we make use of the following binomial identities, which are proved in Section 5.3.4.

Lemma 5.3. *For $q \geq 1$, the following identities hold:*

3.

$$\sum_{\beta=0}^{q} \binom{2q+1}{\beta} \frac{1}{2^{2q+1}} = \frac{1}{2}$$

4.

$$\sum_{\beta=0}^{q} \beta \binom{2q+1}{\beta} \frac{1}{2^{2q+1}} = \left(\frac{2q+1}{4}\right) - \frac{2q+1}{2^{2q+1}} \binom{2q-1}{q-1}.$$

By applying Equations 3 and 4 of Lemma 5.3, Equation (5.21) simplifies to

$$\begin{aligned}
&-\sum_{q=1}^{\frac{n}{6}-1} \sum_{\vec{t} \in \mathcal{T}^{2,q}} \Pr(\vec{T} = \vec{t}) \left(\left(-\frac{n}{2} + 3q\right)\left(\frac{1}{2}\right) - 2\left(\frac{2q+1}{4} - \frac{2q+1}{2^{2q+1}}\binom{2q-1}{q-1}\right)\right) \\
&= -\sum_{q=1}^{\frac{n}{6}-1} \left(\frac{1}{2}\left(-\frac{n}{2} + q - 1\right) + \frac{2q+1}{2^{2q}}\binom{2q-1}{q-1}\right) \Pr(\vec{T} \in \mathcal{T}^{2,q}). \qquad (5.22)
\end{aligned}$$

Recall that $\vec{T}$ is a multinomial distribution over $\{1,2,3\}$ with event probabilities $(\pi_1, \pi_2, \pi_3 + \pi_4)$. Furthermore, $\mathcal{T}^{2,q}$ is the event where there are $\frac{n}{2} - 1 - q$ of 1s in $\vec{T}$, $\frac{n}{2} - q$ of 2s, and $2q+1$ of 3s. Therefore

$$\begin{aligned}
\Pr(\vec{T} \in \mathcal{T}^{2,q}) &= \binom{n}{\frac{n}{2} - 1 - q, \frac{n}{2} - q, 2q+1} \pi_1^{\frac{n}{2}-1-q} \pi_2^{\frac{n}{2}-q} (\pi_3 + \pi_4)^{2q+1} \\
&= \binom{n}{\frac{n}{2} - q, \frac{n}{2} - q, q, q} \pi_1^{n-2q} \pi_3^{2q} \cdot \frac{\pi_3(\frac{n}{2} - q)}{\pi_1(2q+1)} \frac{2^{2q+1}}{\binom{2q}{q}}. \qquad (5.23)
\end{aligned}$$

Hence, Equation (5.22) becomes

$$-\sum_{q=1}^{\frac{n}{6}-1}\binom{n}{\frac{n}{2}-q,\frac{n}{2}-q,q,q}\pi_1^{n-2q}\pi_3^{2q}\left(\frac{\pi_3(-\frac{n}{2}+q-1)(\frac{n}{2}-q)}{\pi_1(2q+1)}\frac{2^{2q}}{\binom{2q}{q}}+\frac{\pi_3(\frac{n}{2}-q)}{\pi_1}\right) \tag{5.24}$$

where we again use the fact that $2\binom{2q-1}{q-1}=\binom{2q}{q}$.

<u>**Step 5: Put parts together.**</u> Recall that our original problem began as Equation (5.11) which we initially split into Equation (5.12) and (5.18). Through a sequence of steps we transformed these equations into Equations (5.17) and (5.24) and an additional $+$ or $-\Theta\left(\frac{1}{\sqrt{n}}\right)$ term (if $\vec{u}$ is not Borda). Recombining them yields

$$\begin{aligned}&\sum_{q=1}^{\frac{n}{6}-1}\binom{n}{\frac{n}{2}-q,\frac{n}{2}-q,q,q}\pi_1^{n-2q}\pi_3^{2q}\left(-\frac{1}{2}\left(-\frac{n}{2}+3q\right)+\frac{\pi_3}{\pi_1}\left(-\frac{n}{2}+q\right)\right)\\&+\sum_{q=1}^{\frac{n}{6}-1}\binom{n}{\frac{n}{2}-q,\frac{n}{2}-q,q,q}\pi_1^{n-2q}\pi_3^{2q}\frac{(-\frac{n}{2}+q)2^{2q}}{\binom{2q}{q}}\left(\frac{1}{2}+\frac{\pi_3(-\frac{n}{2}+q-1)}{\pi_1(2q+1)}\right).\end{aligned} \tag{5.25}$$

In Section 5.3.1 we introduce Lemma 5.4 to prove that the asymptotic rate of the first summation of Equation (5.25) is $\Theta(1)$ if $\pi_1<0.4$, $-\Theta(1)$ if $\pi_1>0.4$, and $\pm\mathcal{O}\left(\frac{1}{n}\right)$ otherwise (i.e., $\pi_1=0.4$). We further prove in Lemma 5.5, in that section, that the second summation of Equation (5.25) is $\pm\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$. This concludes the proof of Lemma 5.1. □

Lemma 5.2. *Given rank-based utility vector $\vec{u}$ with $u_2>u_3$, $\exists N>0$ such that $\forall n>N$ that are odd,*

$$\overline{PoA}(\{1,2\})=-\Theta(1).$$

Proof. This lemma's proof follows almost identically to that of Lemma 5.1, where n is even, except for how $\mathcal{T}^{1,q}$ and $\mathcal{T}^{2,q}$ are defined to account for n being odd. Our subsequent analysis yields a different conclusion than that lemma. We begin by defining two events as before, $\mathcal{E}_1=\mathbb{1}\{s_1(a^*)=s_2(a^*)\}$ and $\mathcal{E}_2=\mathbb{1}\{s_1(a^*)+1=s_2(a^*)\}$, and a similar partition of the objective by these cases:

$$\overline{\mathrm{PoA}}(\{1,2\}) = \Pr_{P\sim\pi^n}(\mathcal{E}_1) \times \mathbb{E}_{P\sim\pi^n}[\mathrm{D}^+(P) \mid \mathcal{E}_1] + \Pr_{P\sim\pi^n}(\mathcal{E}_2) \times \mathbb{E}_{P\sim\pi^n}[\mathrm{D}^+(P) \mid \mathcal{E}_2]. \tag{5.26}$$

We take care of the cases $\mathcal{E}_1 \wedge \{P[2 \succ 1] > P[1 \succ 2]\}$ and $\mathcal{E}_2 \wedge \{P[1 \succ 2] \geq P[2 \succ 1]\}$ separately, in Steps 1 and 2 below, and use Step 3 to tie these pieces back together.

Step 1: Case $\mathcal{E}_1$. In this case, we have

- $\frac{n-1}{2} - q$ agents with ranking $R_1 = (1 \succ 2 \succ 3)$;
- $\frac{n-1}{2} - q$ agents with ranking $R_2 = (2 \succ 3 \succ 1)$;
- β agents with ranking $R_3 = (3 \succ 2 \succ 1)$;
- $2q + 1 - \beta$ agents with ranking $R_4 = (3 \succ 1 \succ 2)$.

Define $\mathcal{T}^{1,q} \subseteq [3]^n$ as the set of vectors $\vec{t} = (t_1, \ldots, t_n)$ such that alternatives 1 and 2 have the maximal plurality score and alternative 1 is the victor:

$$\mathcal{T}^{1,q} = \left\{\vec{t} \in [3]^n \ : \ |\{j : t_j = 1\}| = |\{j : t_j = 2\}| = \left(\frac{n-1}{2} - q\right) > |\{j : t_j = 3\}|\right\}.$$

The number of third-party agents is $2q + 1$, so the maximum value of q is

$$q^* = \max\left\{q \in \mathbb{Z} : \left(\frac{n-1}{2} - q\right) > 2q + 1\right\}$$

so that

$$q^* = \begin{cases} \left\lfloor \frac{n}{6} \right\rfloor - 1, & n \textit{ mod } 6 \in \{1, 3\} \\ \left\lfloor \frac{n}{6} \right\rfloor, & n \textit{ mod } 6 = 5. \end{cases}$$

It is easy to see that $\mathcal{E}_1$ holds for $\vec{Q}$ if and only if $\vec{T}$ takes a value in $\mathcal{T}^{1,q}$ for some $q \in [0, q^*]$. This implies the following equality:

$$\Pr(\mathcal{E}_1) \times \mathbb{E}[\mathrm{D}^+(P) \mid \mathcal{E}_1] = \sum_{q=0}^{q^*} \sum_{\vec{t} \in \mathcal{T}^{1,q}} \Pr(\vec{T} = \vec{t}) \times \mathbb{E}_{\vec{Q}}[\mathrm{D}^+(\vec{Q}) \mid \vec{T} = \vec{t}]. \tag{5.27}$$

Without loss of generality, we will assume for the duration of the proof that $q^* = \left\lfloor \frac{n}{6} \right\rfloor - 1$. It is easy to show that for a constant number of terms in Equation (5.27) such that $q = \Theta(n)$, the objective is exponentially small and hence does not affect the result of this lemma.

Given that $\vec{t} \in \mathcal{T}^{1,q}$, there are $2q+1$ third-party agents with either ranking R_3 or R_4. We uphold the event $\mathcal{E}_1 \wedge \{P[2 \succ 1] > P[1 \succ 2]\}$ with respect to $\vec{Q}$ by having $\beta \in [q+1, 2q]$ agents with ranking R_3. Therefore, continuing Equation (5.27) we get

$$
\begin{aligned}
&\Pr(\mathcal{E}_1) \times \mathbb{E}[\mathrm{D}^+(P) \mid \mathcal{E}_1] \\
&= \sum_{q=0}^{\left\lfloor \frac{n}{6} \right\rfloor - 1} \sum_{\vec{t} \in \mathcal{T}^{1,q}} \sum_{\beta=q+1}^{2q+1} \sum_{\vec{z} \in \mathcal{Z}_{\vec{t},\beta}} \Pr(\vec{T} = \vec{t}, \vec{Z} = \vec{z}) \mathbb{E}_{\vec{Q}}[\mathrm{D}^+(\vec{Q}) \mid \vec{T} = \vec{t}, \vec{Z} = \vec{z}] \\
&= \sum_{q=0}^{\left\lfloor \frac{n}{6} \right\rfloor - 1} \sum_{\vec{t} \in \mathcal{T}^{1,q}} \sum_{\beta=q+1}^{2q+1} \sum_{\vec{z} \in \mathcal{Z}_{\vec{t},\beta}} \Pr(\vec{T} = \vec{t}, \vec{Z} = \vec{z}) \sum_{j=1}^{n} \mathbb{E}_{\vec{Q}_j}[\vec{u}(Q_j, 1) - \vec{u}(Q_j, 2) \mid \vec{T} = \vec{t}, \vec{Z} = \vec{z}] \\
&= \sum_{q=0}^{\left\lfloor \frac{n}{6} \right\rfloor - 1} \sum_{\vec{t} \in \mathcal{T}^{1,q}} \sum_{\beta=q+1}^{2q+1} \sum_{\vec{z} \in \mathcal{Z}_{\vec{t},\beta}} \Pr(\vec{T} = \vec{t}, \vec{Z} = \vec{z}) \sum_{j=1}^{n} E^1_{t_j, z_j} \qquad (5.28)
\end{aligned}
$$

where

$$E^1_{t_j, z_j} = \mathbb{E}_{Q_j}[\vec{u}(Q_j, 1) - \vec{u}(Q_j, 2) \mid T_j = t_j, Z_j = z_j].$$

This holds by the Bayesian network: for any $j \leq n$, given T_j and Z_j, Q_j is conditionally independent of other Q's. By Definition 5.1, we have

$$
\begin{aligned}
\sum_{j=1}^{n} E^1_{t_j, z_j} &= \begin{pmatrix} \frac{n-1}{2} - q \\ \frac{n-1}{2} - q \\ \beta \\ 2q + 1 - \beta \end{pmatrix} \cdot \begin{pmatrix} u_1 - u_2 \\ -u_1 + u_3 \\ -u_2 + u_3 \\ u_2 - u_3 \end{pmatrix} \\
&= \left(\frac{n-1}{2} - q \right) (-u_2 + u_3) + (2q - 2\beta)(u_2 - u_3) + (u_2 - u_3) \\
&= (u_2 - u_3) \left(-\frac{n-1}{2} + 3q - 2\beta \right) + (u_2 - u_3). \qquad (5.29)
\end{aligned}
$$

First, for this constant term, we have that Equation (5.28) is proportional to the likelihood of two-way ties for plurality voting under i.i.d. preferences. By Corollary 5.1, this term is $\Theta\left(\frac{1}{\sqrt{n}}\right)$. Second, the duration of this proof carries over $(u_2 - u_3)$ as a factor in front of

Equation (5.28). Since we assumed $u_2 > u_3$, we will forego writing this factor for ease of notation. Equation (5.28) is therefore proportional to

$$
\begin{aligned}
&\sum_{q=0}^{\lfloor \frac{n}{6} \rfloor -1} \sum_{\vec{t} \in \mathcal{T}^{1,q}} \sum_{\beta=q+1}^{2q+1} \sum_{\vec{z} \in \mathcal{Z}_{\vec{t},\beta}} \Pr(\vec{T} = \vec{t}, \vec{Z} = \vec{z}) \left(-\frac{n-1}{2} + 3q - 2\beta \right) \\
&= \sum_{q=0}^{\lfloor \frac{n}{6} \rfloor -1} \sum_{\vec{t} \in \mathcal{T}^{1,q}} \Pr(\vec{T} = \vec{t}) \sum_{\beta=q+1}^{2q+1} \left(-\frac{n-1}{2} + 3q - 2\beta \right) \sum_{\vec{z} \in \mathcal{Z}_{\vec{t},\beta}} \Pr(\vec{Z} = \vec{z} \mid \vec{T} = \vec{t}) \\
&= \sum_{q=0}^{\lfloor \frac{n}{6} \rfloor -1} \sum_{\vec{t} \in \mathcal{T}^{1,q}} \Pr(\vec{T} = \vec{t}) \sum_{\beta=q+1}^{2q+1} \left(-\frac{n-1}{2} + 3q - 2\beta \right) \binom{2q+1}{\beta} \frac{1}{2^{2q+1}} \qquad (5.30)
\end{aligned}
$$

where the last line follows from

$$\Pr(Z_j = 1 \mid T_j \notin \{1, 2\}) = 0.5.$$

<u>**Case when $q = 0$.**</u> The case for $q = 0$ is similar to the event $\mathcal{E}_2$ in Lemma 5.1. When $q = 0$ there is a single third-party agent with ranking R_4. This is illustrated in Equation (5.30), with $q = \beta = 0$, as

$$
\begin{aligned}
&\sum_{\vec{t} \in \mathcal{T}^{1,0}} \Pr(\vec{T} = \vec{t}) \cdot \left(-\frac{n-1}{2} \right) \binom{1}{0} \frac{1}{2} \\
&= -\frac{n-1}{4} \Pr(\vec{T} \in \mathcal{T}^{1,0}) \\
&= -\frac{n-1}{4} \binom{n}{\frac{n-1}{2}, \frac{n-1}{2}, 1} \pi_1^{\frac{n-1}{2}} \pi_2^{\frac{n-1}{2}} (\pi_3 + \pi_4)^1 \\
&= -\frac{(n-1)\pi_3 \pi_1^n}{2\pi_1} (n) \binom{n-1}{(n-1)/2} \\
&= -\mathcal{O}(n^{1.5}) \cdot (2\pi_1)^n \\
&= -\exp(-\Theta(n))
\end{aligned}
$$

by Stirling's approximation (Proposition 5.1).

Next, we apply the following binomial identities, which are proved in Section 5.3.4.

Lemma 5.3. *For $q \geq 1$, the following identities hold:*

5.

$$\sum_{\beta=q+1}^{2q+1} \binom{2q+1}{\beta} \frac{1}{2^{2q+1}} = \frac{1}{2}$$

6.

$$\sum_{\beta=q+1}^{2q+1} \beta \binom{2q+1}{\beta} \frac{1}{2^{2q+1}} = \left(\frac{2q+1}{4}\right) + \frac{2q+1}{2^{2q+1}} \binom{2q-1}{q-1}.$$

By applying Equations 5 and 6 of Lemma 5.3, Equation (5.30) simplifies to

$$\begin{aligned}
&\sum_{q=1}^{\lfloor \frac{n}{6} \rfloor -1} \left(\frac{-\frac{n-1}{2} + 3q}{2} - 2\left(\frac{2q+1}{4} + \frac{2q+1}{2^{2q+1}} \binom{2q-1}{q-1} \right) \right) \Pr(\vec{T} \in \mathcal{T}^{1,q}) \\
&= \sum_{q=1}^{\lfloor \frac{n}{6} \rfloor -1} \left(\frac{1}{2}\left(-\frac{n-1}{2} + q - 1 \right) - \frac{2q+1}{2^{2q}} \binom{2q-1}{q-1} \right) \Pr(\vec{T} \in \mathcal{T}^{1,q}). \qquad (5.31)
\end{aligned}$$

Recall that $\vec{T}$ is a multinomial distribution over [3] with event probabilities $(\pi_1, \pi_2, \pi_3 + \pi_4)$. Furthermore, $\mathcal{T}^{1,q}$ is the event where there are $\frac{n-1}{2} - q$ of $1s$ and $2s$ each in $\vec{T}$, and $2q+1$ of $3s$. Therefore

$$\begin{aligned}
\Pr(\vec{T} \in \mathcal{T}^{1,q}) &= \binom{n}{\frac{n-1}{2} - q, \frac{n-1}{2} - q, 2q+1} \pi_1^{\frac{n-1}{2}-q} \pi_2^{\frac{n-1}{2}-q} (\pi_3 + \pi_4)^{2q+1} \\
&= \binom{n-1}{\frac{n-1}{2} - q, \frac{n-1}{2} - q, q, q} \pi_1^{n-1-2q} \pi_3^{2q+1} \cdot \frac{n 2^{2q+1}}{(2q+1)\binom{2q}{q}}. \qquad (5.32)
\end{aligned}$$

Hence, Equation (5.31) becomes

$$n \sum_{q=1}^{\lfloor \frac{n}{6} \rfloor -1} \binom{n-1}{\frac{n-1}{2} - q, \frac{n-1}{2} - q, q, q} \pi_1^{n-1-2q} \pi_3^{2q+1} \left(\frac{\left(-\frac{n-1}{2} + q - 1\right)}{2q+1} \frac{2^{2q}}{\binom{2q}{q}} - 1 \right) \qquad (5.33)$$

where we use the fact that $2\binom{2q-1}{q-1} = \binom{2q}{q}$.

Step 2: Case $\mathcal{E}_2$. In this case, we have

- $\frac{n-1}{2} - q$ agents with ranking $R_1 = (1 \succ 2 \succ 3)$;
- $\frac{n+1}{2} - q$ agents with ranking $R_2 = (2 \succ 3 \succ 1)$;

- β agents with ranking $R_3 = (3 \succ 2 \succ 1)$;

- $2q - \beta$ agents with ranking $R_4 = (3 \succ 1 \succ 2)$.

Define $\mathcal{T}^{2,q} \subseteq [3]^n$ as the set of vectors $\vec{t} = (t_1, \ldots, t_n)$ such that alternative 2 has the maximal plurality score of $\frac{n}{2} - q$ and alternative 1 has one fewer vote:

$$\mathcal{T}^{2,q} = \left\{\vec{t} \in [3]^n \ : |\{j : t_j = 1\}| + 1 = |\{j : t_j = 2\}| = \left(\frac{n+1}{2} - q\right) > |\{j : t_j = 3\}|\right\}.$$

The number of third-party agents is $2q$, so the maximum value of q is

$$q^* = \max\left\{q \in \mathbb{Z} : \left(\frac{n+1}{2} - q\right) > 2q\right\}$$

so that $q^* = \left\lfloor \frac{n}{6} \right\rfloor$ for any $n \ mod \ 6 \in \{1, 3, 5\}$. It is easy to see that $\mathcal{E}_2$ holds for $\vec{Q}$ if and only if $\vec{T}$ takes a value in $\mathcal{T}^{2,q}$ for some $q \in [0, q^*]$. This implies the following equality:

$$\Pr(\mathcal{E}_2) \times \mathbb{E}[\mathrm{D}^+(P) \mid \mathcal{E}_2] = \sum_{q=0}^{q^*} \sum_{\vec{t} \in \mathcal{T}^{2,q}} \Pr(\vec{T} = \vec{t}) \times \mathbb{E}_{\vec{Q}}[\mathrm{D}^+(\vec{Q}) \mid \vec{T} = \vec{t}]. \tag{5.34}$$

To keep in line with the notation of the first case, in Step 1, and with Lemma 5.1, we will assume $q^* = \left\lfloor \frac{n}{6} \right\rfloor - 1$ without loss of generality. It is easy to show that the case of Equation (5.34) for $q = \left\lfloor \frac{n}{6} \right\rfloor$ is exponentially small.

Given that $\vec{t} \in \mathcal{T}^{2,q}$, there are $2q$ third-party agents with either ranking R_3 or R_4. We uphold the event $\mathcal{E}_2 \wedge \{P[1 \succ 2] \geq P[2 \succ 1]\}$ with respect to $\vec{Q}$ by having $\beta \in [0, q-1]$. Therefore, continuing Equation (5.34) we get

$$\begin{aligned}
&\Pr(\mathcal{E}_2) \times \mathbb{E}[\mathrm{D}^+(P) \mid \mathcal{E}_2] \\
&= \sum_{q=0}^{\left\lfloor \frac{n}{6} \right\rfloor - 1} \sum_{\vec{t} \in \mathcal{T}^{2,q}} \sum_{\beta=0}^{q-1} \sum_{\vec{z} \in \mathcal{Z}_{\vec{t},\beta}} \Pr(\vec{T} = \vec{t}, \vec{Z} = \vec{z}) \mathbb{E}_{\vec{Q}}[\mathrm{D}^+(\vec{Q}) \mid \vec{T} = \vec{t}, \vec{Z} = \vec{z}] \\
&= \sum_{q=0}^{\left\lfloor \frac{n}{6} \right\rfloor - 1} \sum_{\vec{t} \in \mathcal{T}^{2,q}} \sum_{\beta=0}^{q-1} \sum_{\vec{z} \in \mathcal{Z}_{\vec{t},\beta}} \Pr(\vec{T} = \vec{t}, \vec{Z} = \vec{z}) \sum_{j=1}^{n} \mathbb{E}_{\vec{Q}_j}[\vec{u}(Q_j, 2) - \vec{u}(Q_j, 1) \mid \vec{T} = \vec{t}, \vec{Z} = \vec{z}] \\
&= \sum_{q=0}^{\left\lfloor \frac{n}{6} \right\rfloor - 1} \sum_{\vec{t} \in \mathcal{T}^{2,q}} \sum_{\beta=0}^{q-1} \sum_{\vec{z} \in \mathcal{Z}_{\vec{t},\beta}} \Pr(\vec{T} = \vec{t}, \vec{Z} = \vec{z}) \sum_{j=1}^{n} E^2_{t_j, z_j}
\end{aligned} \tag{5.35}$$

where

$$E^2_{t_j,z_j} = \mathbb{E}_{\vec{Q}_j}[\vec{u}(Q_j,2) - \vec{u}(Q_j,1) \mid T_j = t_j, Z_j = z_j]$$

which holds because of the Bayesian network structure. Notice that $E^2_{t_j,z_j}$ and only depends on the values of t_j and z_j, but not j. By Definition 5.1 and realizing that $E^1_{t_j,z_j} = -E^2_{t_j,z_j}$, we get

$$\begin{aligned}
\sum_{j=1}^{n} E^2_{t_j,z_j} &= -\begin{pmatrix} \frac{n-1}{2} - q \\ \frac{n+1}{2} - q \\ \beta \\ 2q - \beta \end{pmatrix} \cdot \begin{pmatrix} u_1 - u_2 \\ -u_1 + u_3 \\ -u_2 + u_3 \\ u_2 - u_3 \end{pmatrix} \\
&= -\left(\frac{n+1}{2} - q\right)(-u_2 + u_3) - (2q - 2\beta)(u_2 - u_3) - (-u_1 + u_2) \\
&= -(u_2 - u_3)\left(-\frac{n+1}{2} + 3q - 2\beta\right) + (u_1 - u_2).
\end{aligned}$$

First, for this constant term, we have that Equation (5.35) is proportional to the likelihood of two-way ties for plurality voting under i.i.d. preferences. By Corollary 2.1 and Assumption 5.1, this term is $\Theta\left(\frac{1}{\sqrt{n}}\right)$. Second, like in Step 1 above, we will forego writing the factor $(u_2 - u_3)$ in front of Equation (5.35) for ease of notation. Equation (5.35) is therefore proportional to

$$\begin{aligned}
&- \sum_{q=0}^{\lfloor \frac{n}{6} \rfloor - 1} \sum_{\vec{t} \in \mathcal{T}^{2,q}} \sum_{\beta=0}^{q-1} \sum_{\vec{z} \in \mathcal{Z}_{\vec{t},\beta}} \Pr(\vec{T} = \vec{t}, \vec{Z} = \vec{z}) \left(-\frac{n+1}{2} + 3q - 2\beta\right) \\
&= - \sum_{q=0}^{\lfloor \frac{n}{6} \rfloor - 1} \sum_{\vec{t} \in \mathcal{T}^{2,q}} \Pr(\vec{T} = \vec{t}) \sum_{\beta=0}^{q-1} \left(-\frac{n+1}{2} + 3q - 2\beta\right) \sum_{\vec{z} \in \mathcal{Z}_{\vec{t},\beta}} \Pr(\vec{Z} = \vec{z} \mid \vec{T} = \vec{t}) \\
&= - \sum_{q=0}^{\lfloor \frac{n}{6} \rfloor - 1} \sum_{\vec{t} \in \mathcal{T}^{2,q}} \Pr(\vec{T} = \vec{t}) \sum_{\beta=0}^{q-1} \left(-\frac{n+1}{2} + 3q - 2\beta\right) \binom{2q}{\beta} \frac{1}{2^{2q}} \qquad (5.36)
\end{aligned}$$

since

$$\Pr(Z_j = 1 \mid T_j \notin \{1,2\}) = 0.5.$$

<u>**Case when $q = 0$.**</u> The case for $q = 0$ is similar to the event $\mathcal{E}_1$ in Lemma 5.1. Namely, Equation (5.36) is zero in this case because there are zero third-party agents, so there is no

iterative plurality dynamics. By Lemma 3.1, the adversarial loss is zero.

Next, we apply the following binomial identities, which are proved in Section 5.3.4.

Lemma 5.3. *For $q \geq 1$, the following identities hold:*

7.

$$\sum_{\beta=0}^{q-1} \binom{2q}{\beta} \frac{1}{2^{2q}} = \frac{1}{2} - \frac{1}{2^{2q+1}} \binom{2q}{q}$$

8.

$$\sum_{\beta=0}^{q-1} \beta \binom{2q}{\beta} \frac{1}{2^{2q}} = \frac{q}{2} - \frac{q}{2^{2q}} \binom{2q}{q}.$$

By applying Equations 7 and 8 of Lemma 5.3, Equation (5.36) simplifies to

$$\begin{aligned}
&- \sum_{q=1}^{\lfloor \frac{n}{6} \rfloor - 1} \left(\left(-\frac{n+1}{2} + 3q \right) \left(\frac{1}{2} - \frac{1}{2^{2q+1}} \binom{2q}{q} \right) - 2 \left(\frac{q}{2} - \frac{q}{2^{2q}} \binom{2q}{q} \right) \right) \Pr(\vec{T} \in \mathcal{T}^{2,q}) \\
&= - \sum_{q=1}^{\lfloor \frac{n}{6} \rfloor - 1} \left(\frac{1}{2} \left(-\frac{n+1}{2} + q \right) - \frac{\left(-\frac{n+1}{2} - q \right)}{2^{2q+1}} \binom{2q}{q} \right) \Pr(\vec{T} \in \mathcal{T}^{2,q}). \qquad (5.37)
\end{aligned}$$

Recall that $\vec{T}$ is a multinomial distribution over $[3]$ with event probabilities $(\pi_1, \pi_2, \pi_3 + \pi_4)$. Furthermore, $\mathcal{T}^{2,q}$ is the event where there are $\frac{n-1}{2} - q$ of 1s in $\vec{T}$, $\frac{n+1}{2} - q$ of 2s, and $2q$ of 3s. Therefore

$$\begin{aligned}
\Pr(\vec{T} \in \mathcal{T}^{2,q}) &= \binom{n}{\frac{n-1}{2} - 1 - q, \frac{n+1}{2} - q, 2q} \pi_1^{\frac{n-1}{2} - q} \pi_2^{\frac{n+1}{2} - q} (\pi_3 + \pi_4)^{2q} \\
&= \binom{n-1}{\frac{n-1}{2} - q, \frac{n-1}{2} - q, q, q} \pi_1^{n-2q} \pi_3^{2q} \frac{n 2^{2q}}{\left(\frac{n+1}{2} - q \right) \binom{2q}{q}}. \qquad (5.38)
\end{aligned}$$

Hence, Equation (5.37) becomes

$$\begin{aligned}
&- n \sum_{q=1}^{\lfloor \frac{n}{6} \rfloor - 1} \binom{n-1}{\frac{n-1}{2} - q, \frac{n-1}{2} - q, q, q} \pi_1^{n-2q} \pi_3^{2q} \left(\frac{\left(-\frac{n+1}{2} + q \right) 2^{2q}}{2 \left(\frac{n+1}{2} - q \right) \binom{2q}{q}} - \frac{\left(-\frac{n+1}{2} - q \right)}{2 \left(\frac{n+1}{2} - q \right)} \right) \\
&= -n \sum_{q=1}^{\lfloor \frac{n}{6} \rfloor - 1} \binom{n-1}{\frac{n-1}{2} - q, \frac{n-1}{2} - q, q, q} \pi_1^{n-2q} \pi_3^{2q} \left(-\frac{2^{2q-1}}{\binom{2q}{q}} + \frac{1}{2} + \frac{q}{\left(\frac{n+1}{2} - q \right)} \right) \qquad (5.39)
\end{aligned}$$

Step 3: Put parts together. Recall that our original problem began as Equation (5.26) which we initially split into Equations (5.27) and (5.34). Through a sequence of steps we transformed these equations into Equations (5.33) and (5.39). Recombining them yields

$$\begin{aligned}
&n \sum_{q=1}^{\lfloor \frac{n}{6} \rfloor -1} \binom{n-1}{\frac{n-1}{2}-q, \frac{n-1}{2}-q, q, q} \pi_1^{n-1-2q} \pi_3^{2q} \\
&\quad \times \left(\pi_3 \left(\frac{\left(-\frac{n-1}{2}+q-1\right)}{2q+1} \frac{2^{2q}}{\binom{2q}{q}} - 1 \right) - \pi_1 \left(-\frac{2^{2q-1}}{\binom{2q}{q}} + \frac{1}{2} + \frac{q}{\left(\frac{n+1}{2}-q\right)} \right) \right) \\
&= \pi_1 n \sum_{q=1}^{\lfloor \frac{n}{6} \rfloor -1} \binom{n-1}{\frac{n-1}{2}-q, \frac{n-1}{2}-q, q, q} \pi_1^{n-1-2q} \pi_3^{2q} \frac{2^{2q}}{\binom{2q}{q}} \left(\frac{1}{2} + \frac{\pi_3 \left(-\frac{n-1}{2}+q-1\right)}{\pi_1 (2q+1)} \right) \\
&\quad - n \sum_{q=1}^{\lfloor \frac{n}{6} \rfloor -1} \binom{n-1}{\frac{n-1}{2}-q, \frac{n-1}{2}-q, q, q} \pi_1^{n-1-2q} \pi_3^{2q} \left(\pi_3 + \frac{\pi_1}{2} + \frac{\pi_1 q}{\left(\frac{n-1}{2}-q\right)} \right)
\end{aligned} \tag{5.40}$$

In Section 5.3.1 we introduce Lemma 5.10 to prove that the first summation of Equation (5.40) is $\pm\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$. The second summation of Equation (5.40) is $-\Theta(1)$ by Lemma 5.6, which follows after realizing that $\left(\pi_3 + \frac{\pi_1}{2} + \frac{\pi_1 q}{\left(\frac{n-1}{2}-q\right)}\right) = \Theta(1)$ for each q in its domain. This concludes the proof of Lemma 5.2. $\square$

This concludes the proof of Theorem 5.1. $\square$

In Theorem 5.1, we found that the likelihood of a tie between alternatives $W \neq \{1, 2\}$ is exponentially small by Corollary 5.1. This corollary may also be applied to the EADPoA for distributions beyond Assumption 5.1. Namely, if π is concentrated around rankings with the same leading alternative, then the likelihood of any tie is exponentially small (Xia, 2021a); hence, so is the EADPoA. Let $m \geq 3$ and $\lambda_i(\pi) = \sum_{j:top(R_j)=i} \pi_j$ be the likelihood of an agent truthfully voting for i.

Proposition 5.2. *For any π where $\{\lambda_1(\pi), \ldots, \lambda_m(\pi)\}$ has a unique maximum,*

$$EADPoA(f, \vec{u}, \pi^n) = \pm \exp(-\Theta(n)). \tag{5.41}$$

This proposition holds by proving $|\overline{\text{PoA}}(W)| \leq \exp(-\Theta(n))$ for every $W \subseteq \mathcal{A}$, similar to the proof of Theorem 5.1.

5.3 Proof of Technical Lemmas

5.3.1 Multinomial Lemmas

Lemma 5.4.

$$\sum_{q=1}^{\frac{n}{6}-1}\binom{n}{\frac{n}{2}-q,\frac{n}{2}-q,q,q}\pi_1^{n-2q}\pi_3^{2q}\left(-\frac{1}{2}\left(-\frac{n}{2}+3q\right)+\frac{\pi_3}{\pi_1}\left(-\frac{n}{2}+q\right)\right)$$
$$=\begin{cases}\Theta\left(1\right), & \pi_1<0.4\\ -\Theta\left(1\right), & \pi_1>0.4\\ \pm\mathcal{O}\left(\frac{1}{n}\right), & \pi_1=0.4.\end{cases}$$

Proof. Notice that the objective takes the same form as the evaluation of a discrete expected value: it is a summation over the range of q of a multinomial likelihood, indexed by q, multiplied by the score function

$$f_n(q)=-\frac{1}{2}\left(-\frac{n}{2}+3q\right)+\frac{\pi_3}{\pi_1}\left(-\frac{n}{2}+q\right).$$

This function is bounded by $\pm\mathcal{O}(n)$ and is monotonic in q for fixed n. We prove the lemma by partitioning the summation into different ranges of q and handling each case differently. We see that for small enough q, $f_n(q)=-\Theta(n)$, so we can factor it out of the objective and evaluate the sum of probabilities. We make use Lemma 5.6, which applies a theorem from (Xia, 2021a), to evaluate this remaining sum as either $\mathcal{O}\left(\frac{1}{n}\right)$ or $\exp(-\Theta(n))$ depending on where we partition the objective. A similar case holds for large enough q, where $f_n(q)=\Theta(n)$.

Now, the place at which we partition the objective summation depends on π_1. When $\pi_1>0.4$ we ntoice that $f_n(q)>0 \iff q>\epsilon\pi_3 n$ for some $\epsilon>1$. Hence, we can partition the summation at the point $\frac{(\epsilon+1)}{2}\pi_3 n$ and employ Lemma 5.6 on these segments separately. A similar process holds for $\pi_1<0.4$ to partition the summation at $\epsilon\pi_3 n$ for $\epsilon<1$. However, a different technique must be used when $\pi_1=0.4$. We first recognize that $f_n(q)=\frac{5}{4}(\pi_3 n-q)$. We have $f_n(q)\in[-\mathcal{O}(n),\mathcal{O}(n)]$, so we cannot immediately factor it out of the objective summation. We treat this case separately as Lemma 5.7. The technical details are as follows.

Initial observations. Notice that

$$f_n(q) > 0 \iff q > \frac{n\left(\frac{1}{2} - \frac{\pi_3}{\pi_1}\right)}{2\left(\frac{3}{2} - \frac{\pi_3}{\pi_1}\right)}.$$

It is easy to verify that $\frac{n\left(\frac{1}{2} - \frac{\pi_3}{\pi_1}\right)}{2\left(\frac{3}{2} - \frac{\pi_3}{\pi_1}\right)} < \frac{n}{6}$ since $\pi_3 > 0$. Let $\epsilon = \frac{1}{2\pi_3}\left(\frac{1-6\pi_3}{3-10\pi_3}\right)$ such that

$$\epsilon \pi_3 n = \frac{n\left(\frac{1}{2} - \frac{\pi_3}{\pi_1}\right)}{2\left(\frac{3}{2} - \frac{\pi_3}{\pi_1}\right)}.$$

Recall that $\pi_3 \in (0, \frac{1}{6})$ and $\pi_1 = \frac{1}{2} - \pi_3$ by Assumption 5.1, so $\epsilon > 0$. Hence:

- $\epsilon > 1$ if and only if $\pi_3 < 0.1$,
- $\epsilon < 1$ if and only if $\pi_3 > 0.1$,
- $\epsilon = 1$ if and only if $\pi_3 = 0.1$.

This entails three cases to the lemma.

Case 1: ($\pi_3 < 0.1$). Suppose first that $\pi_3 < 0.1$ so that $\epsilon > 1$. Then $f_n(q) > 0 \iff q > \epsilon\pi_3 n$. We partition the summation

$$\begin{aligned}
&\sum_{q=1}^{\frac{n}{6}-1} \binom{n}{\frac{n}{2}-q, \frac{n}{2}-q, q, q} \pi_1^{n-2q}\pi_3^{2q} f_n(q) \\
&= -\Theta(n) \sum_{q=1}^{\left\lceil \frac{(\epsilon+1)}{2}\pi_3 n \right\rceil} \binom{n}{\frac{n}{2}-q, \frac{n}{2}-q, q, q} \pi_1^{n-2q}\pi_3^{2q} \\
&\quad - \Theta(n) \sum_{q=\left\lceil \frac{(\epsilon+1)}{2}\pi_3 n \right\rceil + 1}^{\lceil \epsilon\pi_3 n \rceil} \binom{n}{\frac{n}{2}-q, \frac{n}{2}-q, q, q} \pi_1^{n-2q}\pi_3^{2q} \\
&\quad + \Theta(n) \sum_{q=\lceil \epsilon\pi_3 n \rceil + 1}^{\frac{n}{6}-1} \binom{n}{\frac{n}{2}-q, \frac{n}{2}-q, q, q} \pi_1^{n-2q}\pi_3^{2q} \\
&= -\Theta(n)\Theta\left(\frac{1}{n}\right) - \exp(-\Theta(n)) + \exp(-\Theta(n)) \\
&= -\Theta(1)
\end{aligned}$$

by Lemma 5.6. This lemma is a direct application of Theorem 1 from Xia (2021a) and is proved in Section 5.3.4.

Lemma 5.6. *Fix $a, b \in (0, \frac{1}{6})$, $a < b$. Then*

$$\sum_{q=\lfloor an \rfloor}^{\frac{n}{6}-1} \binom{n}{\frac{n}{2}-q, \frac{n}{2}-q, q, q} \pi_1^{n-2q} \pi_3^{2q} = \begin{cases} \Theta\left(\frac{1}{n}\right), & \pi_3 \geq a \\ \exp(-\Theta(n)), & \textit{otherwise} \end{cases}$$

and

$$\sum_{q=1}^{\lceil bn \rceil} \binom{n}{\frac{n}{2}-q, \frac{n}{2}-q, q, q} \pi_1^{n-2q} \pi_3^{2q} = \begin{cases} \Theta\left(\frac{1}{n}\right), & \pi_3 \leq b \\ \exp(-\Theta(n)), & \textit{otherwise.} \end{cases}$$

Case 2: ($\pi_3 > 0.1$). Now suppose that $\pi_3 > 0.1$ so that $\epsilon < 1$. Then $f_n(q) < 0 \iff q < \epsilon \pi_3 n$. Like the first case, we get by applying Lemma 5.6 that

$$\begin{aligned}
&\sum_{q=1}^{\frac{n}{6}-1} \binom{n}{\frac{n}{2}-q, \frac{n}{2}-q, q, q} \pi_1^{n-2q} \pi_3^{2q} f_n(q) \\
&= -\Theta(n) \sum_{q=1}^{\lfloor \epsilon \pi_3 n \rfloor - 1} \binom{n}{\frac{n}{2}-q, \frac{n}{2}-q, q, q} \pi_1^{n-2q} \pi_3^{2q} \\
&\quad + \Theta(n) \sum_{q=\lfloor \epsilon \pi_3 n \rfloor}^{\left\lfloor \frac{(\epsilon+1)}{2} \pi_3 n \right\rfloor - 1} \binom{n}{\frac{n}{2}-q, \frac{n}{2}-q, q, q} \pi_1^{n-2q} \pi_3^{2q} \\
&\quad + \Theta(n) \sum_{q=\left\lfloor \frac{(\epsilon+1)}{2} \pi_3 n \right\rfloor}^{\frac{n}{6}-1} \binom{n}{\frac{n}{2}-q, \frac{n}{2}-q, q, q} \pi_1^{n-2q} \pi_3^{2q} \\
&= -\exp(-\Theta(n)) + \exp(-\Theta(n)) + \Theta(n)\Theta\left(\frac{1}{n}\right) \\
&= \Theta(1).
\end{aligned}$$

Case 3: ($\pi_3 = 0.1$). In the final case, in which $\pi_3 = 0.1$, we have

$$f_n(q) = \frac{1}{4}\left(\frac{n}{2} - 5q\right) = \frac{5}{4}\left(\pi_3 n - q\right).$$

Therefore the objective becomes

$$\begin{aligned}
&\frac{5}{4}\sum_{q=1}^{\frac{n}{6}-1}\binom{n}{\frac{n}{2}-q,\frac{n}{2}-q,q,q}\pi_1^{n-2q}\pi_3^{2q}(\pi_3 n-q)\\
&=-\frac{5\binom{n}{\frac{n}{2}}\sqrt{2n\pi_3\pi_1}}{2^{n+2}}\sum_{q=1}^{\frac{n}{6}-1}\left(\binom{\frac{n}{2}}{q}(2\pi_1)^{\frac{n}{2}-q}(2\pi_3)^q\right)^2\frac{(q-\pi_3 n)}{\sqrt{2n\pi_3\pi_1}} \qquad (5.42)
\end{aligned}$$

by Proposition 5.2, which is proved in Section 5.3.4.

Proposition 5.2. *Let* $q \in \left[1, \frac{n}{6}-1\right]$. *Then*

$$\binom{n}{\frac{n}{2}-q,\frac{n}{2}-q,q,q}\pi_1^{n-2q}\pi_3^{2q}=\frac{\binom{n}{\frac{n}{2}}}{2^n}\left(\binom{\frac{n}{2}}{q}(2\pi_1)^{\frac{n}{2}-q}(2\pi_3)^q\right)^2.$$

This proposition is useful for transforming the multinomial likelihood to a squared-binomial equivalence. The factor out-front is $\Theta\left(\frac{1}{\sqrt{n}}\right)$ by Stirling's approximation (see Proposition 5.1 in Section 5.3.3). Notice that the multinomial domain of q is $[0, \frac{n}{6}]$ while the binomial domain of q is $[0, \frac{n}{2}]$. We may therefore extend the range of Equation (5.42) by adding a summation that is exponentially small by Hoeffding's inequality (Proposition 5.3). That is, Equation (5.42) can be written as

$$\begin{aligned}
&\Theta(1)\sum_{q=\frac{n}{6}}^{\frac{n}{2}}\left(\binom{\frac{n}{2}}{q}(2\pi_1)^{\frac{n}{2}-q}(2\pi_3)^q\right)^2\frac{(q-\pi_3 n)}{\sqrt{2n\pi_3\pi_1}}\\
&\quad+\Theta(1)\left(\left(\binom{\frac{n}{2}}{0}(2\pi_1)^{\frac{n}{2}-0}(2\pi_3)^0\right)^2\frac{(0-\pi_3 n)}{\sqrt{2n\pi_3\pi_1}}\right)\\
&\quad-\Theta(1)\sum_{q=0}^{\frac{n}{2}}\left(\binom{\frac{n}{2}}{q}(2\pi_1)^{\frac{n}{2}-q}(2\pi_3)^q\right)^2\frac{(q-\pi_3 n)}{\sqrt{2n\pi_3\pi_1}}\\
&=\pm\exp(-\Theta(n))-\Theta(1)\sum_{q=0}^{\frac{n}{2}}\left(\binom{\frac{n}{2}}{q}(2\pi_1)^{\frac{n}{2}-q}(2\pi_3)^q\right)^2\frac{(q-\pi_3 n)}{\sqrt{2n\pi_3\pi_1}}. \qquad (5.43)
\end{aligned}$$

Proposition 5.3 (Hoeffding's Inequality)**.** *Let* $p \in (0,1)$ *and* $q = 1-p$*; let* $a, b \in \mathbb{R}$ *such that* $0 \le a < b \le 1$. *If* $p \notin [a,b]$ *then*

$$\sum_{k=\lfloor an\rfloor}^{\lceil bn\rceil}\left(\binom{n}{k}p^{n-k}q^k\right)^2=\exp(-\Theta(n)).$$

Our specific use of this inequality is proved in Section 5.3.4. Finally, Equation (5.43) leads to

$$\pm\exp(-\Theta(n)) \pm \Theta(1)\mathcal{O}\left(\frac{1}{n}\right) = \pm\mathcal{O}\left(\frac{1}{n}\right)$$

by Lemma 5.7, which is proved in Section 5.3.2. In that section, we discuss the necessary change of variables in order to apply the lemma. Simply put, we exchange $\frac{n}{2} \mapsto n$ and $2\pi_3 \mapsto p$.

Lemma 5.7. *Let $p \in (0, \frac{2}{3})$ and $S_n \sim Bin(n,p)$. Then*

$$\left|\sum_{k=0}^{n}\left(\frac{k-np}{\sqrt{np(1-p)}}\right)\Pr(S_n = k)^2\right| = \mathcal{O}\left(\frac{1}{n}\right).$$

This concludes the proof of Lemma 5.4. □

Lemma 5.5.

$$\left|\sum_{q=1}^{\frac{n}{6}-1}\binom{n}{\frac{n}{2}-q, \frac{n}{2}-q, q, q}\pi_1^{n-2q}\pi_3^{2q}\frac{(-\frac{n}{2}+q)2^{2q}}{\binom{2q}{q}}\left(\frac{1}{2}+\frac{\pi_3(-\frac{n}{2}+q-1)}{\pi_1(2q+1)}\right)\right| = \mathcal{O}\left(\frac{1}{\sqrt{n}}\right).$$

Proof. Throughout this proof we will use the fact that

$$\begin{aligned}
&\left(-\frac{n}{2}+q\right)\left(\frac{1}{2}+\frac{\pi_3(-\frac{n}{2}+q-1)}{\pi_1(2q+1)}\right)\\
&= \frac{1}{2\pi_1(2q+1)}\left(-\frac{n}{2}+q\right)(q-\pi_3 n+\pi_1-2\pi_3)\\
&= \frac{1}{2\pi_1(2q+1)}\left((q-\pi_3 n)^2+(q-\pi_3 n)(-\pi_1 n+\pi_1-2\pi_3)+(-\pi_1 n)(\pi_1-2\pi_3)\right) \qquad (5.44)
\end{aligned}$$

where $\pi_1 - 2\pi_3 \in (0, \frac{1}{2})$ by Assumption 5.1. Let

$$f_n(q) = \left(-\frac{n}{2}+q\right)(q-\pi_3 n+\pi_1-2\pi_3)$$

so that, in the objective,

$$\frac{(-\frac{n}{2}+q)2^{2q}}{\binom{2q}{q}}\left(\frac{1}{2}+\frac{\pi_3(-\frac{n}{2}+q-1)}{\pi_1(2q+1)}\right)=\frac{f_n(q)}{2\pi_1}\frac{2^{2q}}{(2q+1)\binom{2q}{q}}.$$

It is clear that $f_n(q)$ is $+\Theta(n^2)$ or $-\mathcal{O}(n^2)$ when $q-\pi_3 n$ is $-\Theta(n)$ or $+\Theta(n)$, respectively. Therefore we may expect to be able to factor out $f_n(q)$ from the objective when q is either small- or large-enough. However, it is initially unclear how to treat $f_n(q)$ when q is near the "center" of the summation since $f_n(q)$ switches signs. Recall from Proposition 5.2 that the likelihood term in the objective, $\binom{n}{\frac{n}{2}-q,\frac{n}{2}-q,q,q}\pi_1^{n-2q}\pi_3^{2q}$, is equivalent to $\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\times$ the squared-binomial probability centered at $\pi_3 n$. The "center" of the summation, therefore, is when q is $o(n)$ away from $\pi_3 n$.

To handle these separate cases, we partition the objective summation region $[1,\frac{n}{6})$ into three regions defined below:

1. a "small-q" region, A_n, covering $[1,\frac{\pi_3 n}{2})$,

2. a "large-q" region, B_n, covering (the negation of) $[\frac{n}{6},\frac{n}{2}]$,

3. the remainder, C_n, covering $[\frac{\pi_3 n}{2},\frac{n}{2}]$.

Both A_n and B_n are $\Theta(n)$ in size but do not cover the center $\pi_3 n$. Therefore, by Hoeffding's inequality (Proposition 5.3) and Lemma 5.6, these terms are exponentially small. Analyzing C_n requires a different technique using the definition of $f_n(q)$. From Equation (5.44) above, we see that there are three terms with varying powers of $(q-\pi_3 n)^k$, $k\in\{2,1,0\}$. These three terms split up C_n into three further sub-equations, D_n, E_n, and F_n, respectively. We make use of Lemmas 5.8 and 5.9 below to analyze two of these equations, while the third is analyzed using properties of the additional term $\frac{2^{2q}}{(2q+1)\binom{2q}{q}}\times\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$ from Proposition 5.2, and Lemma 5.6. The technical details are as follows.

We start off by splitting up the objective into three parts:

$$A_n+B_n+C_n \tag{5.45}$$

where we define

$$A_n = \sum_{q=1}^{\lfloor \frac{\pi_3 n}{2} \rfloor - 1} \binom{n}{\frac{n}{2}-q, \frac{n}{2}-q, q, q} \pi_1^{n-2q} \pi_3^{2q} \frac{f_n(q)}{2\pi_1} \frac{2^{2q}}{(2q+1)\binom{2q}{q}},$$

$$B_n = -\frac{\binom{n}{\frac{n}{2}}}{2^n} \sum_{q=\frac{n}{6}}^{\frac{n}{2}} \left(\binom{\frac{n}{2}}{q} (2\pi_1)^{\frac{n}{2}-q} (2\pi_3)^q \right)^2 \frac{f_n(q)}{2\pi_1} \frac{2^{2q}}{(2q+1)\binom{2q}{q}},$$

$$C_n = \frac{\binom{n}{\frac{n}{2}}}{2^n} \sum_{q=\lfloor \frac{\pi_3 n}{2} \rfloor}^{\frac{n}{2}} \left(\binom{\frac{n}{2}}{q} (2\pi_1)^{\frac{n}{2}-q} (2\pi_3)^q \right)^2 \frac{f_n(q)}{2\pi_1} \frac{2^{2q}}{(2q+1)\binom{2q}{q}},$$

making use of Proposition 5.2, while adding and subtracting the same terms over the regions $q \in [\frac{n}{6}, \frac{n}{2}]$ to make B_n and C_n.

Consider the first summation of Equation (5.45). Notice that $f_n(q) = \Theta(n^2)$ along the domain $q \in [1, \lfloor \frac{\pi_3 n}{2} \rfloor - 1]$, that $\frac{2^{2q}}{\binom{2q}{q}(2q+1)} \in (0, 1)$, $\forall q > 0$, and that $\frac{2^{2q}}{\binom{2q}{q}(2q+1)}$ is decreasing in q by Stirling's approximation (Proposition 5.1). Therefore

$$\begin{aligned}
A_n &= \sum_{q=1}^{\lfloor \frac{\pi_3 n}{2} \rfloor - 1} \binom{n}{\frac{n}{2}-q, \frac{n}{2}-q, q, q} \pi_1^{n-2q} \pi_3^{2q} \frac{f_n(q)}{2\pi_1} \frac{2^{2q}}{(2q+1)\binom{2q}{q}} \\
&\leq \frac{1}{2\pi_1} \sum_{q=1}^{\lfloor \frac{\pi_3 n}{2} \rfloor - 1} \binom{n}{\frac{n}{2}-q, \frac{n}{2}-q, q, q} \pi_1^{n-2q} \pi_3^{2q} f_n(q) \\
&= \Theta(n^2) \sum_{q=1}^{\lfloor \frac{\pi_3 n}{2} \rfloor - 1} \binom{n}{\frac{n}{2}-q, \frac{n}{2}-q, q, q} \pi_1^{n-2q} \pi_3^{2q} \\
&= \exp(-\Theta(n))
\end{aligned}$$

by Lemma 5.6; we evaluated $q=1$ for $\frac{2^{2q}}{\binom{2q}{q}(2q+1)}$. Likewise

$$\begin{aligned}
A_n &= \sum_{q=1}^{\lfloor\frac{\pi_3 n}{2}\rfloor-1} \binom{n}{\frac{n}{2}-q, \frac{n}{2}-q, q, q} \pi_1^{n-2q}\pi_3^{2q} \frac{f_n(q)}{2\pi_1} \frac{2^{2q}}{(2q+1)\binom{2q}{q}} \\
&\geq \frac{2^{\pi_3 n}}{2\pi_1 \cdot \binom{\pi_3 n}{\frac{\pi_3 n}{2}}(\pi_3 n+1)} \sum_{q=1}^{\lfloor\frac{\pi_3 n}{2}\rfloor-1} \binom{n}{\frac{n}{2}-q, \frac{n}{2}-q, q, q} \pi_1^{n-2q}\pi_3^{2q} f_n(q) \\
&= \Theta(n^{1.5}) \sum_{q=1}^{\lfloor\frac{\pi_3 n}{2}\rfloor-1} \binom{n}{\frac{n}{2}-q, \frac{n}{2}-q, q, q} \pi_1^{n-2q}\pi_3^{2q} \\
&= \exp(-\Theta(n))
\end{aligned}$$

by Stirling's approximation (Proposition 5.1) and Lemma 5.6; we evaluated $q=\frac{\pi_3 n}{2}$ for $\frac{2^{2q}}{\binom{2q}{q}(2q+1)}$. Hence, by the squeeze theorem, $A_n = \exp(-\Theta(n))$.

Now consider the second summation of Equation (5.45). Notice that $f_n(q) = -\mathcal{O}(n^2)$ (and $f_n(q) = -\Omega(n)$) along the domain $q \in [\frac{n}{6}, \frac{n}{2}]$. Therefore

$$\begin{aligned}
B_n &= -\frac{\binom{n}{\frac{n}{2}}}{2^n} \sum_{q=\frac{n}{6}}^{\frac{n}{2}} \left(\binom{\frac{n}{2}}{q}(2\pi_1)^{\frac{n}{2}-q}(2\pi_3)^q\right)^2 \frac{f_n(q)}{2\pi_1} \frac{2^{2q}}{(2q+1)\binom{2q}{q}} \\
&= \Theta\left(\frac{1}{\sqrt{n}}\right)\Theta\left(\frac{1}{\sqrt{n}}\right)\mathcal{O}(n^2) \sum_{q=\frac{n}{6}}^{\frac{n}{2}} \left(\binom{\frac{n}{2}}{q}(2\pi_1)^{\frac{n}{2}-q}(2\pi_3)^q\right)^2 \\
&= \exp(-\Theta(n))
\end{aligned}$$

by Hoeffding's inequality (Proposition 5.3).

We henceforth focus on the third summation of Equation (5.45), which may be split up using the definition of $f_n(q)$ from Equation (5.44) as

$$\begin{aligned}
C_n &= \frac{\binom{n}{\frac{n}{2}}}{2^n} \sum_{q=\lfloor\frac{\pi_3 n}{2}\rfloor}^{\frac{n}{2}} \left(\binom{\frac{n}{2}}{q}(2\pi_1)^{\frac{n}{2}-q}(2\pi_3)^q\right)^2 \frac{f_n(q)}{2\pi_1} \frac{2^{2q}}{(2q+1)\binom{2q}{q}} \\
&= D_n + E_n + F_n \qquad (5.46)
\end{aligned}$$

where we define

$$D_n = \frac{\binom{n}{\frac{n}{2}}}{2\pi_1 \cdot 2^n} \sum_{q=\lfloor \frac{\pi_3 n}{2} \rfloor}^{\frac{n}{2}} \left(\binom{\frac{n}{2}}{q} (2\pi_1)^{\frac{n}{2}-q} (2\pi_3)^q \right)^2 \frac{2^{2q}}{(2q+1)\binom{2q}{q}} (q - \pi_3 n)^2,$$

$$E_n = \frac{\binom{n}{\frac{n}{2}}(-\pi_1 n + \pi_1 - 2\pi_3)}{2\pi_1 \cdot 2^n} \sum_{q=\lfloor \frac{\pi_3 n}{2} \rfloor}^{\frac{n}{2}} \left(\binom{\frac{n}{2}}{q} (2\pi_1)^{\frac{n}{2}-q} (2\pi_3)^q \right)^2 \frac{2^{2q}}{(2q+1)\binom{2q}{q}} (q - \pi_3 n),$$

$$F_n = \frac{\binom{n}{\frac{n}{2}}(-\pi_1 n)(\pi_1 - 2\pi_3)}{2\pi_1 \cdot 2^n} \sum_{q=\lfloor \frac{\pi_3 n}{2} \rfloor}^{\frac{n}{2}} \left(\binom{\frac{n}{2}}{q} (2\pi_1)^{\frac{n}{2}-q} (2\pi_3)^q \right)^2 \frac{2^{2q}}{(2q+1)\binom{2q}{q}}.$$

Consider the first summation of Equation (5.46). We get

$$\begin{aligned} D_n &= \Theta\left(\frac{1}{\sqrt{n}}\right) \sqrt{2\pi_1\pi_3 n} \sum_{q=\lfloor \frac{\pi_3 n}{2} \rfloor}^{\frac{n}{2}} \left(\binom{\frac{n}{2}}{q} (2\pi_1)^{\frac{n}{2}-q} (2\pi_3)^q \right)^2 \frac{2^{2q}\sqrt{2\pi_1\pi_3 n}}{(2q+1)\binom{2q}{q}} \left(\frac{q - \pi_3 n}{\sqrt{2\pi_1\pi_3 n}}\right)^2 \\ &= \Theta(1) \cdot \pm\mathcal{O}\left(\frac{1}{\sqrt{n}}\right) \qquad (5.47) \\ &= \pm\mathcal{O}\left(\frac{1}{\sqrt{n}}\right) \end{aligned}$$

by Stirling's approximation (Proposition 5.1) and the following lemma, proved in Section 5.3.2. We make the change of variables $\frac{n}{2} \mapsto n$, $q \mapsto k$, and $2\pi_3 \mapsto p$ to apply the lemma to Equation (5.47).

Lemma 5.8. *Let $p \in (0, \frac{2}{3})$ and $S_n \sim Bin(n,p)$. Then*

$$\left| \sum_{k=\lfloor \frac{np}{2} \rfloor}^{n} \left(\frac{k - np}{\sqrt{np(1-p)}}\right)^2 \frac{2^{2k}\sqrt{np(1-p)}}{(2k+1)\binom{2k}{k}} \Pr(S_n = k)^2 \right| = \mathcal{O}\left(\frac{1}{\sqrt{n}}\right).$$

Now consider the second summation of Equation (5.46). We have

$$\begin{aligned}
E_n &= \frac{-\Theta(n)\binom{n}{\frac{n}{2}}}{2^n} \sum_{q=\lfloor \frac{\pi_3 n}{2} \rfloor}^{\frac{n}{2}} \left(\binom{\frac{n}{2}}{q}(2\pi_1)^{\frac{n}{2}-q}(2\pi_3)^q\right)^2 \frac{2^{2q}\sqrt{2\pi_1\pi_3 n}}{(2q+1)\binom{2q}{q}} \frac{(q-\pi_3 n)}{\sqrt{2\pi_1\pi_3 n}} \\
&= -\Theta(\sqrt{n}) \cdot \pm\mathcal{O}\left(\frac{1}{n}\right) \\
&= \pm\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)
\end{aligned}$$

by Stirling's approximation (Proposition 5.1) and the following lemma, proved in Section 5.3.2 with the same change of variables as with Lemma 5.8.

Lemma 5.9. *Let $p \in (0, \frac{2}{3})$ and $S_n \sim Bin(n,p)$. Then*

$$\left| \sum_{k=\lfloor \frac{np}{2} \rfloor}^{n} \left(\frac{k-np}{\sqrt{np(1-p)}}\right) \frac{2^{2k}\sqrt{np(1-p)}}{(2k+1)\binom{2k}{k}} \Pr(S_n = k)^2 \right| = \mathcal{O}\left(\frac{1}{n}\right).$$

Finally, consider the third summation of Equation (5.46). We in-part undo the transformation of Proposition 5.2 to get

$$\begin{aligned}
F_n &= \frac{-\Theta(n)\binom{n}{\frac{n}{2}}}{2^n} \sum_{q=\lfloor \frac{\pi_3 n}{2} \rfloor}^{\frac{n}{2}} \left(\binom{\frac{n}{2}}{q}(2\pi_1)^{\frac{n}{2}-q}(2\pi_3)^q\right)^2 \frac{2^{2q}}{(2q+1)\binom{2q}{q}} \\
&= -\Theta(n) \sum_{q=\lfloor \frac{\pi_3 n}{2} \rfloor}^{\frac{n}{6}-1} \binom{n}{\frac{n}{2}-q, \frac{n}{2}-q, q, q} \pi_1^{n-2q}\pi_3^{2q} \frac{2^{2q}}{(2q+1)\binom{2q}{q}} \\
&\qquad - \Theta(\sqrt{n}) \sum_{q=\frac{n}{6}}^{\frac{n}{2}} \left(\binom{\frac{n}{2}}{q}(2\pi_1)^{\frac{n}{2}-q}(2\pi_3)^q\right)^2 \frac{2^{2q}}{(2q+1)\binom{2q}{q}} && (5.48) \\
&= -\Theta(n)\Theta\left(\frac{1}{n^{1.5}}\right) - \Theta(\sqrt{n})\Theta\left(\frac{1}{\sqrt{n}}\right)\exp(-\Theta(n)) && (5.49) \\
&= -\Theta\left(\frac{1}{\sqrt{n}}\right).
\end{aligned}$$

We get Equation (5.48) by Stirling's approximation (Proposition 5.1), and Equation (5.49) by Lemma 5.6 and Hoeffding's inequality (Proposition 5.3).

This concludes the proof of Lemma 5.5. □

Lemma 5.10.

$$\left|\pi_1 n \sum_{q=1}^{\lfloor \frac{n}{6} \rfloor -1} \binom{n-1}{\frac{n-1}{2}-q, \frac{n-1}{2}-q, q, q} \pi_1^{n-1-2q} \pi_3^{2q} \frac{2^{2q}}{\binom{2q}{q}} \left(\frac{1}{2} + \frac{\pi_3\left(-\frac{n-1}{2}+q-1\right)}{\pi_1(2q+1)}\right)\right| = \mathcal{O}\left(\frac{1}{\sqrt{n}}\right).$$

Proof. We begin by realizing that the objective is equal to

$$\begin{aligned}
&- 2\pi_1 \sum_{q=1}^{\lfloor \frac{n}{6} \rfloor -1} \binom{n-1}{\frac{n-1}{2}-q, \frac{n-1}{2}-q, q, q} \pi_1^{n-1-2q} \pi_3^{2q} \\
&\quad \times \frac{(-\frac{n-1}{2}+q)2^{2q}}{\binom{2q}{q}} \left(\frac{1}{2} + \frac{\pi_3\left(-\frac{n-1}{2}+q-1\right)}{\pi_1(2q+1)}\right) \\
&+ 2\pi_1 \sum_{q=1}^{\lfloor \frac{n}{6} \rfloor -1} \binom{n-1}{\frac{n-1}{2}-q, \frac{n-1}{2}-q, q, q} \pi_1^{n-1-2q} \pi_3^{2q} \\
&\quad \times \frac{(q+\frac{1}{2})2^{2q}}{\binom{2q}{q}} \left(\frac{1}{2} + \frac{\pi_3\left(-\frac{n-1}{2}+q-1\right)}{\pi_1(2q+1)}\right).
\end{aligned} \tag{5.50}$$

The first summation of Equation (5.50) is $\pm\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$ directly by Lemma 5.5, noting that $n-1$ is even. We simplify the second summation of Equation (5.50) similarly to the proof of Lemma 5.5. For the duration of the proof, we make the change of variables $n \mapsto n-1$ to simplify the notation and present the proof similarly to that lemma. Note that

$$\begin{aligned}
&\pi_1(2q+1)\left(\frac{1}{2} + \frac{\pi_3\left(-\frac{n}{2}+q-1\right)}{\pi_1(2q+1)}\right) \\
&= (q - \pi_3 n + \pi_1 - 2\pi_3).
\end{aligned}$$

We are left in the second summation of Equation (5.50) with

$$\begin{aligned}
&\sum_{q=1}^{\frac{n}{6}-1} \binom{n}{\frac{n}{2}-q, \frac{n}{2}-q, q, q} \pi_1^{n-2q} \pi_3^{2q} (q - \pi_3 n + \pi_1 - 2\pi_3) \frac{2^{2q}}{\binom{2q}{q}} \\
&= A_n + B_n + C_n + (\pi_1 - 2\pi_3) D_n
\end{aligned} \tag{5.51}$$

where we define

$$A_n = \sum_{q=1}^{\lfloor \frac{\pi_3 n}{2} \rfloor - 1} \binom{n}{\frac{n}{2}-q, \frac{n}{2}-q, q, q} \pi_1^{n-2q} \pi_3^{2q} (q - \pi_3 n) \frac{2^{2q}}{\binom{2q}{q}},$$

$$B_n = -\frac{\binom{n}{\frac{n}{2}}}{2^n} \sum_{q=\frac{n}{6}}^{\frac{n}{2}} \left(\binom{\frac{n}{2}}{q} (2\pi_1)^{\frac{n}{2}-q} (2\pi_3)^q \right)^2 (q - \pi_3 n) \frac{2^{2q}}{\binom{2q}{q}},$$

$$C_n = \frac{\binom{n}{\frac{n}{2}}}{2^n} \sum_{q=\lfloor \frac{\pi_3 n}{2} \rfloor}^{\frac{n}{2}} \left(\binom{\frac{n}{2}}{q} (2\pi_1)^{\frac{n}{2}-q} (2\pi_3)^q \right)^2 (q - \pi_3 n) \frac{2^{2q}}{\binom{2q}{q}},$$

$$D_n = \sum_{q=1}^{\frac{n}{6}-1} \binom{n}{\frac{n}{2}-q, \frac{n}{2}-q, 2q} \pi_1^{n-2q} (2\pi_3)^{2q},$$

making use of Proposition 5.2, while adding and subtracting the same terms over the regions $q \in [\frac{n}{6}, \frac{n}{2}]$ to make B_n, C_n, and D_n. It is clear that both $A_n, B_n = \pm \exp(-\Theta(n))$ by following similar steps as in Lemma 5.5. Furthermore, D_n is equivalent to the probability of a two-way tie with three alternatives. This is $\Theta\left(\frac{1}{\sqrt{n}}\right)$ by Corollary 5.1. Finally, consider the third summation of Equation (5.51). We have by Stirling's approximation

$$\begin{aligned}
C_n &= \Theta\left(\frac{1}{\sqrt{n}}\right) \sum_{q=\lfloor \frac{\pi_3 n}{2} \rfloor}^{\frac{n}{2}} \left(\binom{\frac{n}{2}}{q} (2\pi_1)^{\frac{n}{2}-q} (2\pi_3)^q \right)^2 \frac{(q - \pi_3 n)}{\sqrt{2\pi_1 \pi_3 n}} (2q+1) \frac{2^{2q}\sqrt{2\pi_1\pi_3 n}}{(2q+1)\binom{2q}{q}} \\
&= \Theta(1) \sum_{q=\lfloor \frac{\pi_3 n}{2} \rfloor}^{\frac{n}{2}} \left(\binom{\frac{n}{2}}{q} (2\pi_1)^{\frac{n}{2}-q} (2\pi_3)^q \right)^2 \frac{(q - \pi_3 n)}{\sqrt{2\pi_1 \pi_3 n}} \frac{q}{\sqrt{2\pi_1 \pi_3 n}} \frac{2^{2q}\sqrt{2\pi_1\pi_3 n}}{(2q+1)\binom{2q}{q}} \\
&\quad + \Theta\left(\frac{1}{\sqrt{n}}\right) \sum_{q=\lfloor \frac{\pi_3 n}{2} \rfloor}^{\frac{n}{2}} \left(\binom{\frac{n}{2}}{q} (2\pi_1)^{\frac{n}{2}-q} (2\pi_3)^q \right)^2 \frac{(q - \pi_3 n)}{\sqrt{2\pi_1 \pi_3 n}} \frac{2^{2q}\sqrt{2\pi_1\pi_3 n}}{(2q+1)\binom{2q}{q}} \\
&= E_n + F_n + G_n
\end{aligned}$$

where we define

$$E_n = \Theta(1) \sum_{q=\lfloor \frac{\pi_3 n}{2} \rfloor}^{\frac{n}{2}} \left(\binom{\frac{n}{2}}{q} (2\pi_1)^{\frac{n}{2}-q} (2\pi_3)^q \right)^2 \left(\frac{q - \pi_3 n}{\sqrt{2\pi_1 \pi_3 n}} \right)^2 \frac{2^{2q}\sqrt{2\pi_1\pi_3 n}}{(2q+1)\binom{2q}{q}},$$

$$F_n = \Theta(\sqrt{n}) \sum_{q=\lfloor \frac{\pi_3 n}{2} \rfloor}^{\frac{n}{2}} \left(\binom{\frac{n}{2}}{q} (2\pi_1)^{\frac{n}{2}-q} (2\pi_3)^q \right)^2 \frac{(q - \pi_3 n)}{\sqrt{2\pi_1 \pi_3 n}} \frac{2^{2q}\sqrt{2\pi_1\pi_3 n}}{(2q+1)\binom{2q}{q}},$$

$$G_n = \Theta\left(\frac{1}{\sqrt{n}}\right) \sum_{q=\lfloor \frac{\pi_3 n}{2} \rfloor}^{\frac{n}{2}} \left(\binom{\frac{n}{2}}{q} (2\pi_1)^{\frac{n}{2}-q} (2\pi_3)^q \right)^2 \frac{(q - \pi_3 n)}{\sqrt{2\pi_1 \pi_3 n}} \frac{2^{2q}\sqrt{2\pi_1\pi_3 n}}{(2q+1)\binom{2q}{q}}.$$

We have that $|E_n| = \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$ by Lemma 5.8, $|F_n| = \Theta(\sqrt{n})\mathcal{O}\left(\frac{1}{n}\right) = \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$ by Lemma 5.9, and $|G_n| = \Theta\left(\frac{1}{\sqrt{n}}\right)\mathcal{O}\left(\frac{1}{n}\right) = \mathcal{O}\left(\frac{1}{n^{1.5}}\right)$ by Lemma 5.9. This concludes the proof of Lemma 5.10. □

5.3.2 Expected Collision Entropy

This section describes the asymptotic rate that certain sequences of summations in Lemmas 5.4 and 5.5 converge to zero, such as this objective equation from Lemma 5.9:

$$\sum_{k=\lfloor \frac{np}{2} \rfloor}^{n} \left(\frac{k - np}{\sqrt{np(1-p)}} \right) \frac{2^{2k}\sqrt{np(1-p)}}{(2k+1)\binom{2k}{k}} \Pr(S_n = k)^2 \tag{5.52}$$

where $p \in (0, \frac{2}{3})$ and $S_n \sim Bin(n,p)$. [8] Intuitively, this equation seems similar to the standardized expectation of a binomial random variable, which is clearly zero. However, there are two complications: the fact that we are squaring the binomial likelihood function and the presence of the value

$$g_{n,k} = \frac{2^{2k}\sqrt{np(1-p)}}{(2k+1)\binom{2k}{k}} \tag{5.53}$$

[8] We name this section "Expected Collision Entropy" for its relationship to Réyni entropy (see e.g., Fehr and Berens (2014)). This is defined for binomial random variables as $-\ln \sum_{k=1}^{n} \left(\binom{n}{k} p^k (1-p)^{n-k}\right)^2$ which details the negative log likelihood of the two random variables being equal. This is "expected" because we're multiplying each collision likelihood by the standardized value $\frac{k-np}{\sqrt{np(1-p)}}$.

in the summation. While $|g_{n,k}| = \Theta(1)$ by Lemma 5.13 (detailed below), it cannot be factored out of the summation through standard techniques because $\frac{k-np}{\sqrt{np(1-p)}}$ takes on both positive and negative values throughout the summation. One intuitively nice method, hypothetically, could partition the summation region at $k = np$, factor out $g_{n,k}$ for each part, and then add the two components back together. However, this method is specious; it would yield too imprecise of an asymptotic bound. Hence, different techniques must be used.

Our methods therefore include replacing the binomial probability with a discrete Gaussian form $\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}$, using triangle inequality, and then applying following theorem to asymptotically bound parts of the objective summations:

Theorem 5.2 (Petrov (1975), Chapter VII.1)**.** *Let $S_n \sim Bin(n,p)$. Then*

$$\sup_{k\in[0,n]}\left|\Pr(S_n = k) - \frac{1}{\sqrt{2\pi np(1-p)}}e^{-\frac{1}{2}\left(\frac{k-np}{\sqrt{np(1-p)}}\right)^2}\right| = \mathcal{O}\left(\frac{1}{n}\right).$$

For the rest of the objective, we make use of properties of $g_{n,k}$ and a change of variables to yield the desired claims. These concepts are described technically in the lemma proofs.

This section is presented in three parts. First, we use different notation in this section than the prior lemmas in order to generalize these claims beyond our specific use-case. Section 5.3.2.1 describes what change of variables are necessary to apply this section's lemmas from the notation used in Lemmas 5.4 and 5.5. Section 5.3.2.2 then lists and proves the three applicable lemmas: 5.7, 5.8, and 5.9. This makes their proofs significantly more complicated. Third, Section 5.3.2.3 proves technical lemmas that are used in the aforementioned lemmas.

5.3.2.1 Preliminaries

Let $p \in (0, \frac{2}{3})$ and $q = 1 - p$. For each $n \in \mathbb{N}$ where $np \in \mathbb{Z}_{\geq 0}$, let $S_n \sim Bin(n,p)$ and define the random variable

$$X_n = \frac{S_n - np}{\sqrt{npq}}. \tag{5.54}$$

The random variable X_n takes on the values

$$x_{n,k} = \frac{k - np}{\sqrt{npq}} \text{ for } 0 \leq k \leq n \tag{5.55}$$

which are evenly spaced out by

$$\Delta_n = \frac{1}{\sqrt{npq}}. \tag{5.56}$$

We have

$$\Pr(S_n = k) = \binom{n}{k} p^k q^{n-k}. \tag{5.57}$$

Finally, we define

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}. \tag{5.58}$$

In order to apply the subsequent lemmas to the claims made throughout the primary theorem, we make the following change of variables:

$$\begin{pmatrix} \frac{n}{2} \\ q \\ 2\pi_3 \\ 2\pi_1 \end{pmatrix} \mapsto \begin{pmatrix} n \\ k \\ p \\ q \end{pmatrix} \tag{5.59}$$

recalling that $2(\pi_1 + \pi_3) = 1$. Hence, we get the variable

$$\frac{k - \pi_3 n}{\sqrt{2\pi_1 \pi_3 n}} \mapsto x_{n,k} = \frac{k - np}{\sqrt{npq}} \tag{5.60}$$

and a new variable definition

$$\frac{2^{2k}\sqrt{2\pi_1\pi_3 n}}{(2k+1)\binom{2k}{k}} \mapsto g_{n,k} = \frac{2^{2k}\sqrt{npq}}{(2k+1)\binom{2k}{k}}. \tag{5.61}$$

5.3.2.2 Proof of Lemmas 5.7, 5.8, and 5.9

Lemma 5.7.

$$\left| \sum_{k=0}^{n} x_{n,k} \Pr(S_n = k)^2 \right| = \mathcal{O}\left(\frac{1}{n}\right).$$

Proof. As described in the introduction to this section, it is clear that

$$\sum_{k=0}^{n} x_{n,k} \Pr(S_n = k) = \frac{1}{\sqrt{npq}} \mathbb{E}[S_n - np] = 0.$$

The challenge with this lemma is the presence of the squared-binomial probability in the objective. Intuitively, we would like to make a symmetry argument that, for any fixed $t > 0$,

$x_{n,np-t} = \frac{-t}{\sqrt{npq}} = -\frac{t}{\sqrt{npq}} = -x_{n,np+t}$ and $\Pr(S_n = np - t) \approx \Pr(S_n = np + t)$. Hence, most terms would cancel out, except for perhaps the tails which occur with exponentially small likelihood by Hoeffding's inequality. This approach does not immediately work because $\Pr(|S_n - np| < t) \xrightarrow{n\to\infty} 0$ for fixed t. Rather, the lemma requires summing up over a range of at least size $\Theta(n)$ around the point $k = np$ (e.g., $[np - t, np + t]$ for $t = \Theta(n)$) whose likelihood of occurrence tends to 1. However, when $t = \Theta(n)$ and $p \neq \frac{1}{2}$, we have $\frac{\Pr(S_n=np+t)}{\Pr(S_n=np-t)} \in \{\exp(\Theta(n)), \exp(-\Theta(n))\}$, so the $x_{n,np-t}$ and $x_{n,np+t}$ terms would not cancel out. [9]

Rather than keeping $\Pr(S_n = k)$ in our summation, which is skewed for $p \neq \frac{1}{2}$, we could replace it with the discretized Gaussian function $f(x_{n,k})\Delta_n = \frac{1}{\sqrt{2\pi npq}} e^{-\frac{x_{n,k}^2}{2}}$, which is symmetric about np. This idea comes from the central limit theorem by which we expect the $\frac{S_n-np}{\sqrt{npq}}$ to converge in distribution to the standard Gaussian. The Berry–Esseen theorem suggests that this convergence rate is $\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$ (see e.g., Durrett (2019)), so, intuitively, the squared-probability should converge at rate $\mathcal{O}\left(\frac{1}{n}\right)$. However, a direct application of Berry–Esseen-like theorems fail since they hold only for cumulative distribution functions. Proving this point-wise for $\Pr(S_n = k)$ at each k and including the value-term $x_{n,k}$ in the summation for our lemma requires nuance.

Hence, we make use of Theorem 5.2 (Petrov, 1975, Chapter VII.1), which bounds the point-wise difference between $\Pr(S_n = k)$ and $f(x_{n,k})\Delta_n$ by the rate $\mathcal{O}\left(\frac{1}{n}\right)$. This lemma's proof proceeds by substituting the binomial probability $\Pr(S_n = k)$ by adding and subtracting $f(x_{n,k})\Delta_n$ to and from the objective. This allows us to bound each term using Theorem 5.2 and several convergence technical lemmas that are described and proved in Section 5.3.2.3.

Notably, we replace the objective with $C_n = \sum_{k=0}^{n} x_{n,k} f(x_{n,k})^2 \Delta_n^2$ where both the value and probability parts of the equation are symmetrical around the center np, plus some additional terms. Still, we run into integral problems by which np may not be an integer. It is easy to show that $|C_n| = \exp(-\Theta(n))$ if np is an integer by symmetry. Demonstrating the desired bound that $|C_n| = \mathcal{O}\left(\frac{1}{n}\right)$ requires a handful of other steps when np is not an integer. We demonstrate both cases in the proof below to build the reader's intuition. The technical details are as follows.

[9] This approach could work by the (local) DeMoivre-Laplace Theorem for $t = \mathcal{O}(\sqrt{n})$ (see e.g., Carlen (2018); Feller (1991)); still, it would not make this proof complete. We would not be able to bound the rate of convergence of the tails specifically enough.

We start off by splitting up the objective into three parts in which we replace $\Pr(S_n = k)$ with $(\Pr(S_n = k) - f(x_{n,k})\Delta_n) + f(x_{n,k})\Delta_n$ at each step:

$$\begin{aligned}
&\sum_{k=0}^{n} x_{n,k} \Pr(S_n = k)^2 \\
&= \sum_{k=0}^{n} x_{n,k} \Pr(S_n = k)\Big(\Pr(S_n = k) - f(x_{n,k})\Delta_n\Big) + \sum_{k=0}^{n} x_{n,k} \Pr(S_n = k) f(x_{n,k})\Delta_n \\
&= A_n + B_n + C_n \qquad (5.62)
\end{aligned}$$

where we define

$$A_n = \sum_{k=0}^{n} x_{n,k} \Pr(S_n = k)\Big(\Pr(S_n = k) - f(x_{n,k})\Delta_n\Big),$$

$$B_n = \sum_{k=0}^{n} x_{n,k} f(x_{n,k})\Delta_n\Big(\Pr(S_n = k) - f(x_{n,k})\Delta_n\Big),$$

$$C_n = \sum_{k=0}^{n} x_{n,k} f(x_{n,k})^2 \Delta_n^2.$$

Consider the first summation of Equation (5.62). We have

$$\begin{aligned}
|A_n| &= \left| \sum_{k=0}^{n} x_{n,k} \Pr(S_n = k) \left(\Pr(S_n = k) - f(x_{n,k})\Delta_n\right) \right| \\
&\leq \sum_{k=0}^{n} |x_{n,k} \Pr(S_n = k)| \cdot |\Pr(S_n = k) - f(x_{n,k})\Delta_n| \\
&\leq \mathcal{O}\left(\frac{1}{n}\right) \sum_{k=0}^{n} |x_{n,k}| \Pr(S_n = k) \qquad (5.63) \\
&= \mathcal{O}\left(\frac{1}{n}\right) \qquad (5.64)
\end{aligned}$$

by triangle inequality. Equation (5.63) follows from Theorem 5.2. Equation (5.64) follows from Lemma 5.11, proved in Section 5.3.2.3.

Lemma 5.11.
$$\sum_{k=0}^{n} |x_{n,k}| \Pr(S_n = k) = \Theta(1).$$

Now, for the second summation of Equation (5.62), we have

$$\begin{aligned}
|B_n| &= \left| \sum_{k=0}^{n} x_{n,k} f(x_{n,k}) \Delta_n \left(\Pr(S_n = k) - f(x_{n,k}) \Delta_n \right) \right| \\
&\leq \sum_{k=0}^{n} |x_{n,k} f(x_{n,k}) \Delta_n| \cdot |\Pr(S_n = k) - f(x_{n,k}) \Delta_n| \\
&\leq \mathcal{O}\left(\frac{1}{n}\right) \sum_{k=0}^{n} |x_{n,k}| f(x_{n,k}) \Delta_n \qquad (5.65) \\
&= \mathcal{O}\left(\frac{1}{n}\right) \qquad (5.66)
\end{aligned}$$

by triangle inequality. Equation (5.65) follows from Theorem 5.2. Equation (5.66) follows from the following lemma.

Lemma 5.12 (Equation 1)**.** *The following equation is* $\Theta(1)$*:*

$$\sum_{k=0}^{n} |x_{n,k}| f(x_{n,k}) \Delta_n.$$

Lemma 5.12 consists of ten equations that we prove are all $\Theta(1)$ in Section 5.3.2.3. Each equation is structured similarly and may be proved in almost an identical manner except for how the proof is initialized. Hence, for convenience and straightforwardness of this section, we pack all ten equations into the same lemma statement.

Finally, consider the third summation of Equation (5.62). We prove that $|C_n| \leq \mathcal{O}\left(\frac{1}{n}\right)$ with the following two cases, depending on whether np is an integer or not. We demonstrate both cases to build the reader's intuition.

<u>**Case 1: np is an integer.**</u> We have

$$
\begin{aligned}
C_n &= \sum_{k=0}^{n} x_{n,k} f(x_{n,k})^2 \Delta_n^2 \\
&= \sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{\left\lceil \frac{3np}{2} \right\rceil} x_{n,k} f(x_{n,k})^2 \Delta_n^2 + \sum_{k=0}^{\left\lfloor \frac{np}{2} \right\rfloor - 1} x_{n,k} f(x_{n,k})^2 \Delta_n^2 + \sum_{k=\left\lceil \frac{3np}{2} \right\rceil + 1}^{n} x_{n,k} f(x_{n,k})^2 \Delta_n^2. \qquad (5.67)
\end{aligned}
$$

The first summation of Equation (5.67) is zero by symmetry since np is assumed to be an integer. The second summation of Equation (5.67) is

$$
-\sum_{k=0}^{\left\lfloor \frac{np}{2} \right\rfloor - 1} \Theta(\sqrt{n}) \exp(-\Theta(n)) \Theta\left(\frac{1}{n}\right) = -\exp(-\Theta(n)),
$$

while the third summation similarly yields $\exp(-\Theta(n))$.

<u>**Case 2: np is not an integer.**</u> Now suppose that np is not an integer and that $np = t_n + b_n$ where $t_n \in \mathbb{N}$ and $b_n \in (0,1)$. Our approach is to split up C_n into four regions: a "positive" region of size npq above np, a "negative" region of size npq below np, and two tails which are clearly exponentially small. We seek to point-wise align the positive and negative regions and have the terms at $k = \lfloor np \rfloor - u$ and $k = \lceil np \rceil + u$, for $u \in [0, \lceil npq \rceil]$ approximately cancel out. We make the appropriate change of variables, which leads to Equation (5.69) below. The final step is to appropriately bound the magnitude of each part of that equation by $\mathcal{O}\left(\frac{1}{n}\right)$ using the Maclaurin–Cauchy integral test from Lemma 5.12. The aforementioned

partition is as follows.

$$
\begin{aligned}
C_n &= \sum_{k=0}^{n} x_{n,k} f(x_{n,k})^2 \Delta_n^2 \\
&= \sum_{k=\lfloor np \rfloor - \lceil npq \rceil}^{\lfloor np \rfloor} x_{n,k} f(x_{n,k})^2 \Delta_n^2 + \sum_{k=\lceil np \rceil}^{\lceil np \rceil + \lceil npq \rceil} x_{n,k} f(x_{n,k})^2 \Delta_n^2 \\
&\quad + \sum_{k=0}^{\lfloor np \rfloor - \lceil npq \rceil - 1} x_{n,k} f(x_{n,k})^2 \Delta_n^2 + \sum_{k=\lceil np \rceil + \lceil npq \rceil + 1}^{n} x_{n,k} f(x_{n,k})^2 \Delta_n^2 \\
&= \sum_{k=\lfloor np \rfloor - \lceil npq \rceil}^{\lfloor np \rfloor} \left(\frac{k - t_n - b_n}{\sqrt{npq}} \right) f\left(\frac{k - t_n - b_n}{\sqrt{npq}} \right)^2 \Delta_n^2 \\
&\quad + \sum_{k=\lceil np \rceil}^{\lceil np \rceil + \lceil npq \rceil} \left(\frac{k - t_n - b_n}{\sqrt{npq}} \right) f\left(\frac{k - t_n - b_n}{\sqrt{npq}} \right)^2 \Delta_n^2 \\
&\quad - \sum_{k=0}^{\lfloor np \rfloor - \lceil npq \rceil - 1} \Theta(n) \exp(-\Theta(n)) \Theta\left(\frac{1}{n} \right) \\
&\quad + \sum_{k=\lceil np \rceil + \lceil npq \rceil + 1}^{n} \Theta(n) \exp(-\Theta(n)) \Theta\left(\frac{1}{n} \right). \qquad (5.68)
\end{aligned}
$$

Notice that our partition is valid: both $np - npq = np(1-q) = np^2 \in (0, n)$ and $np + npq = np(2-p) \in (0, n)$. Next, we make the change of variables $u = \lfloor np \rfloor - k$ in the first line of Equation (5.68) and $u = k - \lceil np \rceil$ in the second line of Equation (5.68). We therefore get

$$
\begin{aligned}
&\sum_{u=0}^{\lceil npq \rceil} \left(\frac{-u - b_n}{\sqrt{npq}} \right) f\left(\frac{-u - b_n}{\sqrt{npq}} \right)^2 \Delta_n^2 \\
&\quad + \sum_{u=0}^{\lceil npq \rceil} \left(\frac{u + 1 - b_n}{\sqrt{npq}} \right) f\left(\frac{u + 1 - b_n}{\sqrt{npq}} \right)^2 \Delta_n^2 \\
&\quad \pm \exp(-\Theta(n)) \\
&= D_n + E_n + F_n + G_n \pm \exp(-\Theta(n)) \qquad (5.69)
\end{aligned}
$$

where we define

$$
D_n = \sum_{u=0}^{\lceil npq \rceil} \left(\frac{u}{\sqrt{npq}} \right) \left(-f\left(\frac{u + b_n}{\sqrt{npq}} \right)^2 + f\left(\frac{u + 1 - b_n}{\sqrt{npq}} \right)^2 \right) \Delta_n^2,
$$

$$E_n = -b_n \sum_{u=0}^{\lceil npq \rceil} f\left(\frac{u+b_n}{\sqrt{npq}}\right)^2 \Delta_n^3,$$

$$F_n = (1-b_n) \sum_{u=0}^{\lceil npq \rceil} f\left(\frac{u+1-b_n}{\sqrt{npq}}\right)^2 \Delta_n^3.$$

We made use of the fact that f is an even function to get D_n and E_n. Consider the first summation of Equation (5.69). We have

$$\begin{aligned}
|D_n| &\leq \sum_{u=0}^{\lceil npq \rceil} \left(\frac{u}{\sqrt{npq}}\right) \left| -f\left(\frac{u}{\sqrt{npq}}\right)^2 + f\left(\frac{u+1}{\sqrt{npq}}\right)^2 \right| \Delta_n^2 \qquad (5.70)\\
&= \sum_{u=0}^{\lceil npq \rceil} \left(\frac{u}{\sqrt{npq}}\right) f\left(\frac{u}{\sqrt{npq}}\right)^2 \Delta_n^2 - \sum_{u=0}^{\lceil npq \rceil} \left(\frac{u+1}{\sqrt{npq}}\right) f\left(\frac{u+1}{\sqrt{npq}}\right)^2 \Delta_n^2\\
&\quad + \sum_{u=0}^{\lceil npq \rceil} \left(\frac{1}{\sqrt{npq}}\right) f\left(\frac{u+1}{\sqrt{npq}}\right)^2 \Delta_n^2\\
&= \pm \exp(-\Theta(n)) + \mathcal{O}\left(\frac{1}{n}\right) \sum_{u=0}^{\lceil npq \rceil} f\left(\frac{u}{\sqrt{npq}}\right)^2 \Delta_n\\
&= \mathcal{O}\left(\frac{1}{n}\right).
\end{aligned}$$

by Lemma 5.12.2. Equation (5.70) comes from the fact that e^{-y^2} is decreasing for $y > 0$, so $f\left(\frac{u+c}{\sqrt{npq}}\right)^2 \leq f\left(\frac{u}{\sqrt{npq}}\right)^2$ for $c \in (0,1)$. Now consider the third summation of Equation (5.69). We have

$$\begin{aligned}
|F_n| &\leq \mathcal{O}\left(\frac{1}{n}\right) \sum_{u=0}^{\lceil npq \rceil} f\left(\frac{u}{\sqrt{npq}}\right)^2 \Delta_n\\
&= \mathcal{O}\left(\frac{1}{n}\right)
\end{aligned}$$

since e^{-y^2} is decreasing for $y > 0$ and by Lemma 5.12.2. It is easy to see that $|G_n| = \mathcal{O}\left(\frac{1}{n}\right)$ by similar reasoning. Collectively, this entails that $|C_n| = \mathcal{O}\left(\frac{1}{n}\right)$.

This concludes the proof of Lemma 5.7. $\square$

Lemma 5.8.

$$\left| \sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{n} x_{n,k}^2 \cdot g_{n,k} \Pr(S_n = k)^2 \right| = \mathcal{O}\left(\frac{1}{\sqrt{n}} \right).$$

Proof. The proof proceeds similar to Lemma 5.7 in that we substitute the binomial probability $\Pr(S_n = k)$ by adding and subtracting the discretized Gaussian function $f(x_{n,k})\Delta_n$ to and from the objective. This allows us to bound each term using Theorem 5.2 and several convergence technical lemmas that are described and proved in Section 5.3.2.3. The final step of this proof is significantly simpler than that in Lemma 5.7 since we only require an asymptotic bound of $\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$. The extra term $g_{n,k}$ does not affect the flow of the proof, as seen below. We start with

$$\begin{aligned}
&\sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{n} x_{n,k}^2 \cdot g_{n,k} \Pr(S_n = k)^2 \\
&= \sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{n} x_{n,k}^2 \cdot g_{n,k} \Pr(S_n = k)\Big(\Pr(S_n = k) - f(x_{n,k})\Delta_n \Big) \\
&\qquad + \sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{n} x_{n,k}^2 \cdot g_{n,k} \Pr(S_n = k) f(x_{n,k})\Delta_n \\
&= A_n + B_n + C_n
\end{aligned} \tag{5.71}$$

where we define

$$A_n = \sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{n} x_{n,k}^2 \cdot g_{n,k} \Pr(S_n = k)\Big(\Pr(S_n = k) - f(x_{n,k})\Delta_n \Big),$$

$$B_n = \sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{n} x_{n,k}^2 \cdot g_{n,k} f(x_{n,k})\Delta_n \Big(\Pr(S_n = k) - f(x_{n,k})\Delta_n \Big),$$

$$C_n = \sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{n} x_{n,k}^2 \cdot g_{n,k} f(x_{n,k})^2 \Delta_n^2.$$

Consider the first summation of Equation (5.71). We have

$$
\begin{aligned}
|A_n| &= \left| \sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{n} x_{n,k}^2 \cdot g_{n,k} \Pr(S_n = k) \left(\Pr(S_n = k) - f(x_{n,k})\Delta_n \right) \right| \\
&\leq \sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{n} \left| x_{n,k}^2 \cdot g_{n,k} \Pr(S_n = k) \right| \cdot \left| \Pr(S_n = k) - f(x_{n,k})\Delta_n \right| \\
&\leq \mathcal{O}\left(\frac{1}{n}\right) \sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{n} x_{n,k}^2 \Pr(S_n = k) && (5.72) \\
&= \mathcal{O}\left(\frac{1}{n}\right) && (5.73)
\end{aligned}
$$

by triangle inequality. Equation (5.72) follows from Theorem 5.2 and since $|g_{n,k}| = \Theta(1)$ by Lemma 5.13, proved in Section 5.3.4.

Lemma 5.13. *Let* $k \in \left[\left\lfloor \frac{np}{2} \right\rfloor, n \right]$. *Then* $|g_{n,k}| = \Theta(1)$.

Equation (5.73) follows from Lemma 5.14, proved in Section 5.3.2.3.

Lemma 5.14.

$$\sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{n} x_{n,k}^2 \Pr(S_n = k) = \Theta(1).$$

Now, for the second summation of Equation (5.71), we have

$$
\begin{aligned}
|B_n| &= \left| \sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{n} x_{n,k}^2 \cdot g_{n,k} f(x_{n,k})\Delta_n \left(\Pr(S_n = k) - f(x_{n,k})\Delta_n \right) \right| \\
&\leq \sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{n} \left| x_{n,k}^2 \cdot g_{n,k} f(x_{n,k})\Delta_n \right| \cdot \left| \Pr(S_n = k) - f(x_{n,k})\Delta_n \right| \\
&\leq \mathcal{O}\left(\frac{1}{n}\right) \sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{n} x_{n,k}^2 f(x_{n,k})\Delta_n && (5.74) \\
&= \mathcal{O}\left(\frac{1}{n}\right) && (5.75)
\end{aligned}
$$

by triangle inequality. Equation (5.74) follows from Theorem 5.2 and since $|g_{n,k}| = \Theta(1)$ by Lemma 5.13. Equation (5.75) follows by Lemma 5.12.3. Finally, consider the third line of

Equation (5.71). We get

$$\begin{aligned}
|C_n| &= \left| \sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{n} x_{n,k}^2 \cdot g_{n,k} f(x_{n,k})^2 \Delta_n^2 \right| \\
&\leq \sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{n} |x_{n,k}^2 \cdot g_{n,k} f(x_{n,k})^2 \Delta_n^2| \\
&\leq \mathcal{O}\left(\frac{1}{\sqrt{n}}\right) \sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{n} x_{n,k}^2 f(x_{n,k})^2 \Delta_n \\
&= \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)
\end{aligned}$$

by triangle inequality, since $|g_{n,k}| = \Theta(1)$ by Lemma 5.13, and by Lemma 5.12.4.

This concludes the proof of Lemma 5.8. □

Lemma 5.9.

$$\left| \sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{n} x_{n,k} \cdot g_{n,k} \Pr(S_n = k)^2 \right| = \mathcal{O}\left(\frac{1}{n}\right).$$

Proof. This proof proceeds in four parts. In the first part, we substitute the binomial probability $\Pr(S_n = k)$ by adding and subtracting the discretized Gaussian function $f(x_{n,k})\Delta_n$ to and from the objective, similar to Lemmas 5.7 and 5.8. We make use of Theorem 5.2 for some parts, as in those lemmas, and are left with $C_n = \sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{n} x_{n,k} \cdot g_{n,k} f(x_{n,k})^2 \Delta_n^2$.

Recall that in Lemma 5.7 we made a symmetry argument to bound $|C_n|$ by $\mathcal{O}\left(\frac{1}{n}\right)$, while in Lemma 5.8 we factored $g_{n,k}$ and Δ_n out of the objective to yield a $\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$ bound. Since $g_{n,k}$ is in this summation and we require an asymptotic bound of $\mathcal{O}\left(\frac{1}{n}\right)$ for this lemma, the techniques of these lemmas used on C_n are no longer valid. In the second step to this proof, we therefore identify meaningful upper- and lower-bounds to C_n in order to apply the squeeze theorem. We do this by exploiting properties of $g_{n,k}$ and identifying upper- and lower-bounds to $g_{n,k}$ that are asymptotically equivalent (see Lemma 5.15 below). The terms composing C_n are both positive and negative on its range $k \in [\left\lfloor \frac{np}{2} \right\rfloor, n]$. We upper-bound C_n by using the upper-bound of $g_{n,k}$ on the positive portion of C_n and lower-bound of $g_{n,k}$ on the negative portion of C_n. The opposite holds to lower-bound C_n. Recall by Lemma 5.13 that $|g_{n,k}| = \Theta(1)$. This bound is remarkably not precise enough to prove Lemma 5.9.

Rather, we require $g_{n,k}$'s bounds to be asymptotically equivalent to attain $\mathcal{O}\left(\frac{1}{n}\right)$ bounds, making use of the stricter Lemma 5.15.

After some simplification, we are left in the third step of the proof with $F_n = \sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{\left\lceil \frac{3np}{2} \right\rceil} x_{n,k}\sqrt{\frac{1}{k+0.5}} f(x_{n,k})^2\Delta_n$. This summation is now symmetrical around np except for the $\mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$ factor in the summation and the possibility that np may not be an integer. To handle the first issue, we make a symmetry argument and pair the terms at $k = np - u$ and $k = np + u$ for $u \in [0, \left\lfloor \frac{np}{2} \right\rfloor]$. This leads to a summation similar to $\sum_{u=0}^{\left\lfloor \frac{np}{2} \right\rfloor} \left(\frac{u}{\sqrt{npq}}\right) f\left(\frac{u}{\sqrt{npq}}\right)^2 \left(\sqrt{\frac{1}{np+u}} - \sqrt{\frac{1}{np-u}}\right)\Delta_n$ (see Equation (5.94) below). We require significant nuance to handle the case where np may not be an integer, described in Step 3 below. All-in-all, this possibility does not affect the convergence rate. Finally, we show in Step 4 that $\left(\sqrt{\frac{1}{np+u}} - \sqrt{\frac{1}{np-u}}\right) = -u\mathcal{O}\left(\frac{1}{n^{1.5}}\right)$, which enables us to prove Lemma 5.9. The technical details are as follows.

Step 1: Substitute the binomial probability. We start off by splitting up the objective into three parts in which we replace $\Pr(S_n = k)$ with $(\Pr(S_n = k) - f(x_{n,k})\Delta_n) + f(x_{n,k})\Delta_n$ at each step:

$$\begin{aligned}
&\sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{n} x_{n,k} \cdot g_{n,k} \Pr(S_n = k)^2 \\
&= \sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{n} x_{n,k} \cdot g_{n,k} \Pr(S_n = k)\Big(\Pr(S_n = k) - f(x_{n,k})\Delta_n\Big) \\
&\quad + \sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{n} x_{n,k} \cdot g_{n,k} \Pr(S_n = k) f(x_{n,k})\Delta_n \\
&= A_n + B_n + C_n
\end{aligned} \tag{5.76}$$

where we define

$$A_n = \sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{n} x_{n,k} \cdot g_{n,k} \Pr(S_n = k)\Big(\Pr(S_n = k) - f(x_{n,k})\Delta_n\Big),$$

$$B_n = \sum_{k=\lfloor \frac{np}{2} \rfloor}^{n} x_{n,k} \cdot g_{n,k} f(x_{n,k}) \Delta_n \Big(\Pr(S_n = k) - f(x_{n,k}) \Delta_n \Big),$$

$$C_n = \sum_{k=\lfloor \frac{np}{2} \rfloor}^{n} x_{n,k} \cdot g_{n,k} f(x_{n,k})^2 \Delta_n^2.$$

Consider the first summation of Equation (5.76). We have

$$\begin{aligned}
|A_n| &= \left| \sum_{k=\lfloor \frac{np}{2} \rfloor}^{n} x_{n,k} \cdot g_{n,k} \Pr(S_n = k) \left(\Pr(S_n = k) - f(x_{n,k}) \Delta_n \right) \right| \\
&\leq \sum_{k=\lfloor \frac{np}{2} \rfloor}^{n} |x_{n,k} \cdot g_{n,k} \Pr(S_n = k)| \cdot |\Pr(S_n = k) - f(x_{n,k}) \Delta_n| \\
&\leq \mathcal{O}\left(\frac{1}{n}\right) \sum_{k=\lfloor \frac{np}{2} \rfloor}^{n} |x_{n,k}| \Pr(S_n = k) \qquad (5.77) \\
&= \mathcal{O}\left(\frac{1}{n}\right) \qquad (5.78)
\end{aligned}$$

by triangle inequality. Equation (5.77) follows from Theorem 5.2 and since $|g_{n,k}| = \Theta(1)$ by Lemma 5.13. Equation (5.78) follows from Lemma 5.11 and Hoeffding's inequality (Proposition 5.3).

Now, for the second summation of Equation (5.76), we have

$$\begin{aligned}
|B_n| &= \left| \sum_{k=\lfloor \frac{np}{2} \rfloor}^{n} x_{n,k} \cdot g_{n,k} f(x_{n,k}) \Delta_n \left(\Pr(S_n = k) - f(x_{n,k}) \Delta_n \right) \right| \\
&\leq \sum_{k=\lfloor \frac{np}{2} \rfloor}^{n} |x_{n,k} \cdot g_{n,k} f(x_{n,k}) \Delta_n| \cdot |\Pr(S_n = k) - f(x_{n,k}) \Delta_n| \\
&\leq \mathcal{O}\left(\frac{1}{n}\right) \sum_{k=\lfloor \frac{np}{2} \rfloor}^{n} |x_{n,k}| f(x_{n,k}) \Delta_n \qquad (5.79) \\
&= \mathcal{O}\left(\frac{1}{n}\right) \qquad (5.80)
\end{aligned}$$

by triangle inequality. Equation (5.79) follows from Theorem 5.2 and since $|g_{n,k}| = \Theta(1)$ by

Lemma 5.13. Equation (5.80) follows from Lemma 5.12.5.

Step 2: Squeeze theorem using properties of $g_{n,k}$. Our next step is to identify meaningful bounds on the third summation of Equation (5.76), C_n, and apply the squeeze theorem. Our upper- and lower-bounds on C_n follow from upper- and lower-bounds on $g_{n,k}$ in the following lemma, described and proved in Section 5.3.3.

Lemma 5.15.

$$\sqrt{\frac{2n}{2n+1}}\sqrt{\frac{2}{\pi(2n+1)}} \leq \frac{2^{2n}}{(2n+1)\binom{2n}{n}} \leq \sqrt{\frac{2}{\pi(2n+1)}}.$$

Recall that $g_{n,k} = \frac{2^{2k}\sqrt{npq}}{(2k+1)\binom{2k}{k}}$, so by Lemma 5.15 we have

$$\sqrt{\frac{2k}{2k+1}}\sqrt{\frac{2npq}{\pi(2k+1)}} \leq g_{n,k} \leq \sqrt{\frac{2npq}{\pi(2k+1)}}. \tag{5.81}$$

Notice that the terms composing C_n are both positive and negative on its range $k \in [\left\lfloor \frac{np}{2} \right\rfloor, n]$. We upper-bound C_n by using the upper-bound of $g_{n,k}$ on the positive portion of C_n and lower-bound of $g_{n,k}$ on the negative portion of C_n. The opposite holds to lower-bound C_n. This procedure partitions the range of C_n into two parts: $k \in [\left\lfloor \frac{np}{2} \right\rfloor, \lfloor np \rfloor]$ and $k \in [\lceil np \rceil, n]$, each of which is $\pm\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$. This bound is not tight enough to prove Lemma 5.9. We therefore want to use a symmetry argument to have the terms at $k = np - u$ and $k = np + u$ for $u \in [0, \left\lfloor \frac{np}{2} \right\rfloor]$ approximately cancel out, like in the proof of Lemma 5.7, to yield a tighter bound. Lemma 5.15's bounds which are asymptotically equivalent (i.e., $\sqrt{\frac{2k}{2k+1}}\sqrt{\frac{2npq}{\pi(2k+1)}} \sim \sqrt{\frac{2npq}{\pi(2k+1)}}$; see Lemma 5.16 below) enables us to do this. This step concludes by bounding $|C_n| \leq \mathcal{O}\left(\frac{1}{n}\right) + |F_n|$ where F_n is a summation that covers the full range $k \in [\left\lfloor \frac{np}{2} \right\rfloor, n]$ and includes an $\mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$ factor in the objective.

From the third summation of Equation (5.76), we get

$$\begin{aligned}
C_n &= \sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{n} x_{n,k} \cdot g_{n,k} f(x_{n,k})^2 \Delta_n^2 \\
&= \sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{\lfloor np \rfloor} x_{n,k} \cdot g_{n,k} f(x_{n,k})^2 \Delta_n^2 + \sum_{k=\lceil np \rceil}^{n} x_{n,k} \cdot g_{n,k} f(x_{n,k})^2 \Delta_n^2 \\
&\leq \sqrt{\frac{\lfloor np \rfloor}{\lfloor np \rfloor + 1}} \sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{\lfloor np \rfloor} x_{n,k} \sqrt{\frac{2npq}{\pi(2k+1)}} f(x_{n,k})^2 \Delta_n^2 \\
&\quad + \sum_{k=\lceil np \rceil}^{n} x_{n,k} \sqrt{\frac{2npq}{\pi(2k+1)}} f(x_{n,k})^2 \Delta_n^2 \qquad (5.82)
\end{aligned}$$

where the lower-bound on $g_{n,k}$ from Equation (5.81) is applied to the negative portion of the summation, where $k \leq \lfloor np \rfloor$, and the upper-bound on $g_{n,k}$ is applied to the positive portion of the summation, where $k \geq \lceil np \rceil$. Note that $\sqrt{\frac{2k}{2k+1}}$ is increasing in k, by the following lemma, so $k = \left\lfloor \frac{np}{2} \right\rfloor$ was inputted to minimize this value over the domain $k \in [\left\lfloor \frac{np}{2} \right\rfloor, \lfloor np \rfloor]$.

Lemma 5.16. *For any constant $t > 0$, $\sqrt{\frac{tn}{tn+1}} = 1 - \mathcal{O}\left(\frac{1}{n}\right)$.*

Lemma 5.16 is proved in Section 5.3.4. By this lemma, Equation (5.82) is equivalent to

$$\begin{aligned}
&\sqrt{\frac{2}{\pi}} \left(1 - \mathcal{O}\left(\frac{1}{n}\right)\right) \sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{\lfloor np \rfloor} x_{n,k} \sqrt{\frac{1}{2k+1}} f(x_{n,k})^2 \Delta_n + \sqrt{\frac{2}{\pi}} \sum_{k=\lceil np \rceil}^{n} x_{n,k} \sqrt{\frac{1}{2k+1}} f(x_{n,k})^2 \Delta_n \\
&= -\mathcal{O}\left(\frac{1}{n}\right) \sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{\lfloor np \rfloor} x_{n,k} \sqrt{\frac{1}{2k+1}} f(x_{n,k})^2 \Delta_n + \sqrt{\frac{2}{\pi}} \sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{n} x_{n,k} \sqrt{\frac{1}{2k+1}} f(x_{n,k})^2 \Delta_n \qquad (5.83)
\end{aligned}$$

We repeat this process to get a lower-bound on C_n:

$$\begin{aligned} C_n &= \sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{n} x_{n,k} \cdot g_{n,k} f(x_{n,k})^2 \Delta_n^2 \\ &= \sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{\lfloor np \rfloor} x_{n,k} \cdot g_{n,k} f(x_{n,k})^2 \Delta_n^2 + \sum_{k=\lceil np \rceil}^{n} x_{n,k} \cdot g_{n,k} f(x_{n,k})^2 \Delta_n^2 \\ &\geq \sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{\lfloor np \rfloor} x_{n,k} \sqrt{\frac{2npq}{\pi(2k+1)}} f(x_{n,k})^2 \Delta_n^2 \\ &\qquad + \sqrt{\frac{2\lceil np \rceil}{2\lceil np \rceil + 1}} \sum_{k=\lceil np \rceil}^{n} x_{n,k} \sqrt{\frac{2npq}{\pi(2k+1)}} f(x_{n,k})^2 \Delta_n^2 \end{aligned} \tag{5.84}$$

where the upper-bound on $g_{n,k}$ from Equation (5.81) is applied to the negative portion of the summation, where $k \leq \lfloor np \rfloor$. Likewise, the lower-bound on $g_{n,k}$ from Equation (5.81) is applied to the positive portion of the summation, where $k \geq \lceil np \rceil$, with $k = \lceil np \rceil$ which is set at $\arg\min_{k \in [\lceil np \rceil, n]} \sqrt{\frac{2k}{2k+1}}$. By Lemma 5.16, Equation (5.84) is then equivalent to

$$\begin{aligned} &\sqrt{\frac{2}{\pi}} \sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{\lfloor np \rfloor} x_{n,k} \sqrt{\frac{1}{2k+1}} f(x_{n,k})^2 \Delta_n + \sqrt{\frac{2}{\pi}} \left(1 - \mathcal{O}\left(\frac{1}{n}\right)\right) \sum_{k=\lceil np \rceil}^{n} x_{n,k} \sqrt{\frac{1}{2k+1}} f(x_{n,k})^2 \Delta_n \\ &= \sqrt{\frac{2}{\pi}} \sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{n} x_{n,k} \sqrt{\frac{1}{2k+1}} f(x_{n,k})^2 \Delta_n - \mathcal{O}\left(\frac{1}{n}\right) \sum_{k=\lceil np \rceil}^{n} x_{n,k} \sqrt{\frac{1}{2k+1}} f(x_{n,k})^2 \Delta_n. \end{aligned} \tag{5.85}$$

To assist the flow of the proof and reduce redundancy, we use a technical variant of the squeeze theorem. We have shown

$$\text{Equation (5.83)} \leq C_n \leq \text{Equation (5.85)}.$$

Rather than prove Equations (5.83) and (5.85) have the same asymptotic bounds, separately, we combine the equations as

$$\begin{aligned} |C_n| &\leq \max\left\{ \Big| \text{Equation (5.83)} \Big|, \Big| \text{Equation (5.85)} \Big| \right\} \\ &\leq \Big| \text{Equation (5.83)} \Big| + \Big| \text{Equation (5.85)} \Big| \end{aligned}$$

by triangle inequality. We continue the proof with

$$
\begin{aligned}
|C_n| &= \left| \sum_{k=\lfloor \frac{np}{2} \rfloor}^{n} x_{n,k} \cdot g_{n,k} f(x_{n,k})^2 \Delta_n^2 \right| \\
&\leq \max \left\{ \Big| \text{Equation (5.83)} \Big|, \Big| \text{Equation (5.85)} \Big| \right\} \\
&\leq \Big| \text{Equation (5.83)} \Big| + \Big| \text{Equation (5.85)} \Big| \\
&= \left| -\mathcal{O}\left(\frac{1}{n}\right) \sum_{k=\lfloor \frac{np}{2} \rfloor}^{\lfloor np \rfloor} x_{n,k} \sqrt{\frac{1}{2k+1}} f(x_{n,k})^2 \Delta_n + \sqrt{\frac{2}{\pi}} \sum_{k=\lfloor \frac{np}{2} \rfloor}^{n} x_{n,k} \sqrt{\frac{1}{2k+1}} f(x_{n,k})^2 \Delta_n \right| \\
&\qquad + \left| \sqrt{\frac{2}{\pi}} \sum_{k=\lfloor \frac{np}{2} \rfloor}^{n} x_{n,k} \sqrt{\frac{1}{2k+1}} f(x_{n,k})^2 \Delta_n - \mathcal{O}\left(\frac{1}{n}\right) \sum_{k=\lceil np \rceil}^{n} x_{n,k} \sqrt{\frac{1}{2k+1}} f(x_{n,k})^2 \Delta_n \right| \\
&\leq |D_n| + |E_n| + 2\sqrt{\frac{2}{\pi}} \cdot |F_n| \qquad (5.86)
\end{aligned}
$$

by triangle inequality, where we define

$$D_n = \mathcal{O}\left(\frac{1}{n}\right) \sum_{k=\lfloor \frac{np}{2} \rfloor}^{\lfloor np \rfloor} x_{n,k} \sqrt{\frac{1}{2k+1}} f(x_{n,k})^2 \Delta_n,$$

$$E_n = \mathcal{O}\left(\frac{1}{n}\right) \sum_{k=\lceil np \rceil}^{n} x_{n,k} \sqrt{\frac{1}{2k+1}} f(x_{n,k})^2 \Delta_n,$$

$$F_n = \sum_{k=\lfloor \frac{np}{2} \rfloor}^{n} x_{n,k} \sqrt{\frac{1}{2k+1}} f(x_{n,k})^2 \Delta_n.$$

Consider the first summation in Equation (5.86). We have

$$\begin{aligned}
|D_n| &= \mathcal{O}\left(\frac{1}{n}\right)\left|\sum_{k=\left\lfloor \frac{np}{2}\right\rfloor}^{\lfloor np\rfloor} x_{n,k}\sqrt{\frac{1}{2k+1}}f(x_{n,k})^2\Delta_n\right| \\
&\leq \mathcal{O}\left(\frac{1}{n^{1.5}}\right)\sum_{k=\left\lfloor \frac{np}{2}\right\rfloor}^{\lfloor np\rfloor} |x_{n,k}|f(x_{n,k})^2\Delta_n \\
&\leq \mathcal{O}\left(\frac{1}{n}\right)\sum_{k=\left\lfloor \frac{np}{2}\right\rfloor}^{\lfloor np\rfloor} f(x_{n,k})^2\Delta_n \\
&= \mathcal{O}\left(\frac{1}{n}\right)
\end{aligned}$$

which follows by Lemma 5.12.6. Identical reasoning follows to upper bound the second summation in Equation (5.86):

$$\begin{aligned}
|E_n| &= \mathcal{O}\left(\frac{1}{n}\right)\left|\sum_{k=\lceil np\rceil}^{n} x_{n,k}\sqrt{\frac{1}{2k+1}}f(x_{n,k})^2\Delta_n\right| \\
&\leq \mathcal{O}\left(\frac{1}{n}\right)
\end{aligned}$$

by Lemma 5.12.7.

Step 3: Handle np may not be an integer. Until this point in the proof, we have demonstrated that the magnitude of the objective is bounded by $\mathcal{O}\left(\frac{1}{n}\right)+|C_n|$ and that $|C_n|\leq \mathcal{O}\left(\frac{1}{n}\right)+|F_n|$ where

$$F_n = \sum_{k=\left\lfloor \frac{np}{2}\right\rfloor}^{n} x_{n,k}\sqrt{\frac{1}{2k+1}}f(x_{n,k})^2\Delta_n.$$

Our aim is to bound $|F_n|\leq \mathcal{O}\left(\frac{1}{n}\right)$. To accomplish this, in this step, we pair the terms at $k=np-u$ and $k=np+u$ for $u\in[0,\left\lfloor\frac{np}{2}\right\rfloor]$, using a change of variables, to yield a summation similar to

$$\sum_{u=0}^{\left\lfloor \frac{np}{2}\right\rfloor}\left(\frac{u}{\sqrt{npq}}\right)f\left(\frac{u}{\sqrt{npq}}\right)^2\left(\sqrt{\frac{1}{np+u}}-\sqrt{\frac{1}{np-u}}\right)\Delta_n$$

(see Equation (5.94) below). We first show that the upper-tail is exponentially small. We then handle the nuance by which np may not be an integer. This possibility does not affect the convergence rate, nor the intuition behind this change-of-variables. The reader may skip from Equation (5.88) to Equation (5.94) without losing the flow of the proof. The step concludes by bounding $|F_n| \leq \mathcal{O}\left(\frac{1}{n}\right) + |L_n|$ where L_n is a summation that covers $u \in [0, \left\lfloor \frac{np}{2} \right\rfloor]$ and includes a factor similar to $\left(\sqrt{\frac{1}{np+u}} - \sqrt{\frac{1}{np-u}}\right)$ in the objective. We proceed as follows.

$$
\begin{aligned}
|F_n| &= \left| \sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{n} x_{n,k} \sqrt{\frac{1}{k+0.5}} f(x_{n,k})^2 \Delta_n \right| \\
&\leq \left| \sum_{k=\left\lceil \frac{3np}{2} \right\rceil + 1}^{n} x_{n,k} \sqrt{\frac{1}{k+0.5}} f(x_{n,k})^2 \Delta_n \right| + \left| \sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{\left\lceil \frac{3np}{2} \right\rceil} x_{n,k} \sqrt{\frac{1}{k+0.5}} f(x_{n,k})^2 \Delta_n \right| \qquad (5.87)
\end{aligned}
$$

by triangle inequality. Notice that for the first summation of Equation (5.87),

$$
\begin{aligned}
&\left| \sum_{k=\left\lceil \frac{3np}{2} \right\rceil + 1}^{n} x_{n,k} \sqrt{\frac{1}{k+0.5}} f(x_{n,k})^2 \Delta_n \right| \\
&\leq \sum_{k=\left\lceil \frac{3np}{2} \right\rceil + 1}^{n} |x_{n,k}| \sqrt{\frac{1}{k+0.5}} f(x_{n,k})^2 \Delta_n \\
&= \Theta(n)\Theta(\sqrt{n})\Theta\left(\frac{1}{\sqrt{n}}\right) \exp(-\Theta(n))\Theta\left(\frac{1}{\sqrt{n}}\right) \\
&= \exp(-\Theta(n))
\end{aligned}
$$

by triangle inequality. Hence, we focus on the range $k \in \left[\left\lfloor \frac{np}{2} \right\rfloor, \left\lceil \frac{3np}{2} \right\rceil\right]$ in the second summation of Equation (5.87):

$$
\begin{aligned}
&\sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{\left\lceil \frac{3np}{2} \right\rceil} x_{n,k} \sqrt{\frac{1}{k+0.5}} f(x_{n,k})^2 \Delta_n \\
&= \sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{\lfloor np \rfloor} \left(\frac{k-np}{\sqrt{npq}}\right) \sqrt{\frac{1}{k+0.5}} f\left(\frac{k-np}{\sqrt{npq}}\right)^2 \Delta_n \\
&\quad + \sum_{k=\lceil np \rceil}^{\left\lceil \frac{3np}{2} \right\rceil} \left(\frac{k-np}{\sqrt{npq}}\right) \sqrt{\frac{1}{k+0.5}} f\left(\frac{k-np}{\sqrt{npq}}\right)^2 \Delta_n. \qquad (5.88)
\end{aligned}
$$

For the first line of Equation (5.88) we make the change of variables $u = \lfloor np \rfloor - k$, which yields

$$\sum_{u=0}^{\lfloor np\rfloor-\left\lfloor\frac{np}{2}\right\rfloor}\left(\frac{\lfloor np\rfloor-u-np}{\sqrt{npq}}\right)\sqrt{\frac{1}{\lfloor np\rfloor-u+0.5}}\ f\left(\frac{\lfloor np\rfloor-u-np}{\sqrt{npq}}\right)^2\Delta_n. \tag{5.89}$$

Suppose that $np = t_n + b_n$ where $t_n \in \mathbb{N}$ and $b_n \in [0, 1)$. Then Equation (5.89) is

$$-\sum_{u=0}^{\lfloor np\rfloor-\left\lfloor\frac{np}{2}\right\rfloor}\left(\frac{u+b_n}{\sqrt{npq}}\right)\sqrt{\frac{1}{t_n-u+0.5}}\ f\left(\frac{u+b_n}{\sqrt{npq}}\right)^2\Delta_n, \tag{5.90}$$

making use of the fact that f is an even function. For the second line of Equation (5.88) we make the change of variables $u = k - \lceil np\rceil$, which yields

$$\begin{aligned}
&\sum_{k=0}^{\left\lceil\frac{3np}{2}\right\rceil-\lceil np\rceil}\left(\frac{u+\lceil np\rceil-np}{\sqrt{npq}}\right)\sqrt{\frac{1}{\lceil np\rceil+u+0.5}}\ f\left(\frac{u+\lceil np\rceil-np}{\sqrt{npq}}\right)^2\Delta_n\\
&=\sum_{k=0}^{\left\lceil\frac{3np}{2}\right\rceil-\lceil np\rceil}\left(\frac{u+1-b_n}{\sqrt{npq}}\right)\sqrt{\frac{1}{t_n+u+1.5}}\ f\left(\frac{u+1-b_n}{\sqrt{npq}}\right)^2\Delta_n.
\end{aligned} \tag{5.91}$$

Let

$$\tau_n = \min\left\{\lfloor np\rfloor-\left\lfloor\frac{np}{2}\right\rfloor, \left\lceil\frac{3np}{2}\right\rceil-\lceil np\rceil\right\}$$

which is near $\frac{np}{2}$ (and is exact, if np is an integer). Putting together Equations (5.90) and (5.91) yields

$$G_n + H_n + I_n \pm \exp(-\Theta(n)) \tag{5.92}$$

where we define

$$G_n = \Delta_n^2(1-b_n)\sum_{u=0}^{\tau_n}\frac{1}{\sqrt{t_n+u+1.5}}f\left(\frac{u+1-b_n}{\sqrt{npq}}\right)^2,$$

$$H_n = -\Delta_n^2 b_n\sum_{u=0}^{\tau_n}\frac{1}{\sqrt{t_n-u+0.5}}f\left(\frac{u+b_n}{\sqrt{npq}}\right)^2,$$

$$I_n = \Delta_n^2 \sum_{u=0}^{\tau_n} \frac{u}{\sqrt{t_n + u + 1.5}} f\left(\frac{u+1-b_n}{\sqrt{npq}}\right)^2 - \frac{u}{\sqrt{t_n - u + 0.5}} f\left(\frac{u+b_n}{\sqrt{npq}}\right)^2.$$

Note that the exponentially small term in Equation (5.92) arises since there may be terms in-between τ_n and either $\lfloor np \rfloor - \lfloor \frac{np}{2} \rfloor$ or $\lceil \frac{3np}{2} \rceil - \lceil np \rceil$. Recall that these terms are near $\frac{np}{2}$. Plugging in $u = \Theta(n)$ for either Equations (5.90) or (5.91) yields $-\exp(-\Theta(n))$ and $\exp(-\Theta(n))$ respectively.

The proof continues by bounding $|G_n|$, $|H_n|$, and $|I_n|$ by $\mathcal{O}\left(\frac{1}{n}\right)$ each and respectively. Consider the first summation of Equation (5.92). Since $t_n = \Theta(n)$ by definition, we have

$$\begin{aligned}
|G_n| &= \mathcal{O}\left(\frac{1}{n}\right) \left| \sum_{u=0}^{\tau_n} f\left(\frac{u+1-b_n}{\sqrt{npq}}\right)^2 \Delta_n \right| \\
&\leq \mathcal{O}\left(\frac{1}{n}\right) \sum_{u=0}^{\tau_n} f\left(\frac{u}{\sqrt{npq}}\right)^2 \Delta_n \\
&= \mathcal{O}\left(\frac{1}{n}\right)
\end{aligned}$$

by triangle inequality and Lemma 5.12.8, making use of the fact that e^{-y^2} is monotone decreasing for $y \geq 0$. A similar argument holds for H_n. Now consider the third summation of Equation (5.92). We get

$$I_n = J_n + K_n + L_n \tag{5.93}$$

where we define

$$J_n = \Delta_n^2 \sum_{u=0}^{\tau_n} \frac{u}{\sqrt{t_n + u + 1.5}} \left(f\left(\frac{u+1-b_n}{\sqrt{npq}}\right)^2 - f\left(\frac{u}{\sqrt{npq}}\right)^2 \right),$$

$$K_n = -\Delta_n^2 \sum_{u=0}^{\tau_n} \frac{u}{\sqrt{t_n - u + 0.5}} \left(f\left(\frac{u+b_n}{\sqrt{npq}}\right)^2 - f\left(\frac{u}{\sqrt{npq}}\right)^2 \right),$$

$$L_n = \Delta_n^2 \sum_{u=0}^{\tau_n} u f\left(\frac{u}{\sqrt{npq}}\right)^2 \left(\frac{1}{\sqrt{t_n + u + 1.5}} - \frac{1}{\sqrt{t_n - u + 0.5}} \right).$$

Consider the first summation of Equation (5.93). We have that $|J_n|$ is equivalent to

$$\begin{aligned}
&\mathcal{O}\left(\frac{1}{n^{1.5}}\right)\left|\sum_{u=0}^{\tau_n} u\left(f\left(\frac{u+1-b_n}{\sqrt{npq}}\right)^2 - f\left(\frac{u}{\sqrt{npq}}\right)^2\right)\right| \\
&\leq \mathcal{O}\left(\frac{1}{n^{1.5}}\right)\sum_{u=0}^{\tau_n} u\left(f\left(\frac{u}{\sqrt{npq}}\right)^2 - f\left(\frac{u+1}{\sqrt{npq}}\right)^2\right) \\
&= \mathcal{O}\left(\frac{1}{n^{1.5}}\right)\sum_{u=0}^{\tau_n}\left(uf\left(\frac{u}{\sqrt{npq}}\right)^2 - (u+1)f\left(\frac{u+1}{\sqrt{npq}}\right)^2\right) + \mathcal{O}\left(\frac{1}{n}\right)\sum_{u=0}^{\tau_n} f\left(\frac{u+1}{\sqrt{npq}}\right)^2\Delta_n \\
&= \mathcal{O}\left(\frac{1}{n}\right)\sum_{u=0}^{\tau_n} f\left(\frac{u}{\sqrt{npq}}\right)^2\Delta_n - \mathcal{O}\left(\frac{1}{n^{1.5}}\right) \\
&= \mathcal{O}\left(\frac{1}{n}\right).
\end{aligned}$$

where the second line is by triangle inequality and since e^{-y^2} is decreasing for $y > 0$; the last line is by Lemma 5.12.8. A similar argument holds for K_n.

Step 4: Handle $\frac{1}{\sqrt{k}}$ using paired terms. Now consider the third summation of Equation (5.93):

$$L_n = \sum_{u=0}^{\tau_n}\left(\frac{u}{\sqrt{npq}}\right)f\left(\frac{u}{\sqrt{npq}}\right)^2\left(\frac{1}{\sqrt{t_n+u+1.5}} - \frac{1}{\sqrt{t_n-u+0.5}}\right)\Delta_n. \tag{5.94}$$

We next simplify the internal difference in this summation. Let $a = t_n + u + 1.5$ and $b = t_n - u + 0.5$. we get:

$$\begin{aligned}
\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}} &= \frac{\sqrt{b}-\sqrt{a}}{\sqrt{ab}}\cdot\frac{\sqrt{b}+\sqrt{a}}{\sqrt{b}+\sqrt{a}} \\
&= \frac{b-a}{b\sqrt{a}+a\sqrt{b}} \\
&= \frac{-(2u+1)}{(t_n-u+0.5)\sqrt{(t_n+u+1.5)} + (t_n+u+1.5)\sqrt{(t_n-u+0.5)}} \\
&= -(2u+1)\cdot\mathcal{O}\left(\frac{1}{n^{1.5}}\right).
\end{aligned} \tag{5.95}$$

Returning to Equation (5.94), which is upper-bounded by zero, we get

$$
\begin{aligned}
&-\mathcal{O}\left(\frac{1}{n^{1.5}}\right)\sum_{u=0}^{\tau_n}\left(\frac{u(2u+1)}{\sqrt{npq}}\right) f\left(\frac{u}{\sqrt{npq}}\right)^2 \Delta_n \\
&= -\mathcal{O}\left(\frac{1}{n}\right)\sum_{u=0}^{\tau_n}\left(\frac{u}{\sqrt{npq}}\right)^2 f\left(\frac{u}{\sqrt{npq}}\right)^2 \Delta_n - \mathcal{O}\left(\frac{1}{n^{1.5}}\right)\sum_{u=0}^{\tau_n}\left(\frac{u}{\sqrt{npq}}\right) f\left(\frac{u}{\sqrt{npq}}\right)^2 \Delta_n \\
&= -\mathcal{O}\left(\frac{1}{n}\right)
\end{aligned}
$$

by Lemmas 5.12.9 and 5.12.10. Hence, we get that $|L_n| \leq \mathcal{O}\left(\frac{1}{n}\right)$.

This concludes the proof of Lemma 5.9. □

5.3.2.3 Technical Lemmas

This subsection describes technical lemmas about the convergence of certain sequences of summations. These are used to support the lemmas in Section 5.3.2.2.

Lemma 5.11.

$$\sum_{k=0}^{n} |x_{n,k}| \Pr(S_n = k) = \Theta(1).$$

Proof. The lemma is implied by the following:

$$\lim_{n\to\infty} \sum_{k=0}^{n} |x_{n,k}| \Pr(S_n = k) = \frac{2}{\sqrt{2\pi}}.$$

We do not assume that np is an integer. Rather, suppose $np = t_n + b_n$ where $t_n \in \mathbb{N}$ and

$b_n \in [0, 1)$. The objective equation is then equal to

$$\begin{aligned}
&\sum_{k=0}^{\lfloor np \rfloor} |x_{n,k}| \Pr(S_n = k) + \sum_{k=\lceil np \rceil}^{n} |x_{n,k}| \Pr(S_n = k) \\
&= -\sum_{k=0}^{\lfloor np \rfloor} \frac{k - t_n - b_n}{\sqrt{npq}} \Pr(S_n = k) + \sum_{k=\lceil np \rceil}^{n} \frac{k - t_n - b_n}{\sqrt{npq}} \Pr(S_n = k) \\
&= \frac{b_n}{\sqrt{npq}} \sum_{k=0}^{\lfloor np \rfloor} \Pr(S_n = k) - \frac{b_n}{\sqrt{npq}} \sum_{k=\lceil np \rceil}^{n} \Pr(S_n = k) \\
&\quad + \sum_{k=0}^{\left\lfloor \frac{np}{2} \right\rfloor - 1} \left| \frac{k - t_n}{\sqrt{npq}} \right| \Pr(S_n = k) + \sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{n} \left| \frac{k - t_n}{\sqrt{npq}} \right| \Pr(S_n = k) \\
&= \pm \mathcal{O}\left(\frac{1}{\sqrt{n}} \right) + \sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{n} \left| \frac{k - t_n}{\sqrt{npq}} \right| \Pr(S_n = k) \qquad (5.96)
\end{aligned}$$

where we partitioned the lower domain of $k \in [0, \frac{\lfloor np \rfloor}{2})$ and realized that it is exponentially small by Hoeffding's inequality (Proposition 5.3). Next, we change the remaining summation into a more convenient form.

$$\begin{aligned}
&\sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{n} \left| \frac{k - t_n}{\sqrt{npq}} \right| \Pr(S_n = k) \\
&= \frac{1}{\sqrt{npq}} \left(-\sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{\lfloor np \rfloor} (k - t_n) \Pr(S_n = k) + \sum_{k=\lceil np \rceil}^{n} (k - t_n) \Pr(S_n = k) \right) \\
&= \frac{1}{\sqrt{npq}} \left(\sum_{k=\lfloor np \rfloor}^{\lfloor np \rfloor + \left\lfloor \frac{np}{2} \right\rfloor} (k - t_n) \Pr(S_n = k) + \sum_{k=\lceil np \rceil}^{n} (k - t_n) \Pr(S_n = k) \right) \\
&= \frac{1}{\sqrt{npq}} \left(-\sum_{k=\lfloor np \rfloor + \left\lfloor \frac{np}{2} \right\rfloor + 1}^{n} (k - t_n) \Pr(S_n = k) + 2 \sum_{k=\lceil np \rceil}^{n} (k - t_n) \Pr(S_n = k) \right) \\
&= \frac{2}{\sqrt{npq}} \sum_{k=\lfloor np \rfloor}^{n} (k - t_n) \Pr(S_n = k) - \exp(-\Theta(n)) \qquad (5.97)
\end{aligned}$$

by Hoeffding's inequality (Proposition 5.3). Let $T = \sum_{k=\lfloor np \rfloor}^{n} k\binom{n}{k}p^k q^{n-k}$. Next, we have

$$\begin{aligned}
T &= np \sum_{k=\lfloor np \rfloor}^{n} \binom{n-1}{k-1} p^{k-1} q^{n-k} \\
&= \frac{np}{q} \sum_{k=\lfloor np \rfloor - 1}^{n-1} \binom{n-1}{k} p^{k} q^{n-k} \\
&= \frac{np}{q} \sum_{k=\lfloor np \rfloor - 1}^{n-1} \binom{n}{k} p^{k} q^{n-k} \left(1 - \frac{k}{n}\right) && (5.98) \\
&= \frac{np}{q} \sum_{k=\lfloor np \rfloor}^{n} \binom{n}{k} p^{k} q^{n-k} \left(1 - \frac{k}{n}\right) \\
&+ \frac{np}{q} \left(\binom{n}{\lfloor np \rfloor - 1} p^{\lfloor np \rfloor - 1} q^{n-(\lfloor np \rfloor - 1)} \left(1 - \frac{(\lfloor np \rfloor - 1)}{n}\right) - \binom{n}{n} p^n q^{n-n} \left(1 - \frac{n}{n}\right) \right) && (5.99)
\end{aligned}$$

where in Equation (5.98) we used the substitution

$$\binom{n-1}{k} = \frac{(n-1)!}{k!(n-1-k)!} = \frac{n!}{k!(n-k)!} \cdot \frac{n-k}{n}.$$

Notice in Equation (5.99) that

$$\begin{aligned}
&\binom{n}{\lfloor np \rfloor - 1} p^{\lfloor np \rfloor - 1} q^{n-(\lfloor np \rfloor - 1)} \left(1 - \frac{(\lfloor np \rfloor - 1)}{n}\right) \\
&= \binom{n}{\lfloor np \rfloor} \frac{\lfloor np \rfloor}{n - \lfloor np \rfloor + 1} p^{\lfloor np \rfloor - 1} q^{n - \lfloor np \rfloor + 1} \left(\frac{n - \lfloor np \rfloor + 1}{n}\right) \\
&= q \left(\frac{\lfloor np \rfloor}{np}\right) \cdot \binom{n}{\lfloor np \rfloor} p^{\lfloor np \rfloor} q^{n - \lfloor np \rfloor} \\
&= \frac{q}{\sqrt{2\pi npq}} \left(1 \pm \mathcal{O}\left(\frac{1}{n}\right)\right)
\end{aligned}$$

by Lemma 5.17, proved in Section 5.3.4.

Lemma 5.17.

$$\binom{n}{\lfloor np \rfloor} p^{\lfloor np \rfloor} q^{n - \lfloor np \rfloor} = \frac{1}{\sqrt{2\pi npq}} \left(1 \pm \mathcal{O}\left(\frac{1}{n}\right)\right).$$

This gets us

$$T = \frac{np}{q}\left(\frac{1}{2} - \frac{T}{n}\right) + \sqrt{\frac{np}{2\pi q}} \left(1 \pm \mathcal{O}\left(\frac{1}{n}\right)\right)$$

using the fact that $\sum_{k=\lfloor np \rfloor}^{n} \binom{n}{k} p^k q^{n-k} = \frac{1}{2}$. Hence,

$$T = \frac{np}{2} + \left(1 \pm \mathcal{O}\left(\frac{1}{n}\right)\right)$$

so that our objective from Equation (5.97) becomes

$$\begin{aligned} &\frac{2}{\sqrt{npq}} \left(T - \frac{\lfloor np \rfloor}{2}\right) - \exp(-\Theta(n)) \\ &= \frac{2}{\sqrt{2\pi}} \left(1 \pm \mathcal{O}\left(\frac{1}{n}\right)\right) \xrightarrow{n\to\infty} \frac{2}{\sqrt{2\pi}} \end{aligned}$$

as claimed. This concludes the proof of Lemma 5.11. □

The following lemma consists of ten equations that we prove are all $\Theta(1)$. Each equation is structured similarly and may be proved in almost an identical manner. Hence, for convenience and straightforwardness of this section, we pack all ten equations into the same lemma statement.

Lemma 5.12. *Let*

$$\tau_n = \min\left\{\lfloor np \rfloor - \left\lfloor \frac{np}{2} \right\rfloor, \left\lceil \frac{3np}{2} \right\rceil - \lceil np \rceil\right\}$$

which is near $\frac{np}{2}$ (and is exact, if np is an integer). The following equations are each $\Theta(1)$:

1. $$\sum_{k=0}^{n} |x_{n,k}| f(x_{n,k}) \Delta_n.$$

2. $$\sum_{u=0}^{\lceil npq \rceil} f\left(\frac{u}{\sqrt{npq}}\right)^2 \Delta_n$$

3. $$\sum_{k=\lfloor \frac{np}{2} \rfloor}^{n} x_{n,k}^2 f(x_{n,k}) \Delta_n$$

4. $$\sum_{k=\lfloor \frac{np}{2} \rfloor}^{n} x_{n,k}^2 f(x_{n,k})^2 \Delta_n$$

5.

$$\sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{n} |x_{n,k}| f(x_{n,k}) \Delta_n.$$

6.

$$\sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{\lfloor np \rfloor} f(x_{n,k})^2 \Delta_n$$

7.

$$\sum_{k=\lceil np \rceil}^{n} f(x_{n,k})^2 \Delta_n$$

8.

$$\sum_{u=0}^{\tau_n} f\left(\frac{u}{\sqrt{npq}}\right)^2 \Delta_n$$

9.

$$\sum_{u=0}^{\tau_n} \left(\frac{u}{\sqrt{npq}}\right) f\left(\frac{u}{\sqrt{npq}}\right)^2 \Delta_n$$

10.

$$\sum_{u=0}^{\tau_n} \left(\frac{u}{\sqrt{npq}}\right)^2 f\left(\frac{u}{\sqrt{npq}}\right)^2 \Delta_n$$

Proof. Each of these equations is proved using similar methods. For conciseness, in this proof, we will demonstrate only the proofs of Equations 5, which is in $x_{n,k}$-format, and 11, which is in u-format. These equations have the largest terms in the objective summation among the $x_{n,k}$- and u-format equations, respectively. Therefore, proving that both Equations 5 and 11 are $\Theta(1)$ entails the same for the remainder of the equations. Our method is summarized as follows.

It is clear that each of these summations are non-negative and concentrated around the mean $k = np$ or $u = 0$ (depending on the format). For each equation and large enough $|k - np|$ or u that are $\Omega(\sqrt{n})$, the term is decreasing in $|k - np|$ or u. Hence, we make use of the Maclaurin–Cauchy integral test for convergence. For smaller $|k - np|$ or u terms that are $\mathcal{O}(\sqrt{n})$, we demonstrate convergence using the definition of the Riemann integral. We make use of the error function $erf(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy$ in these proofs; $erf(x) \in (0, 1)$ for $x > 0$. Then $erfc(x) \equiv 1 - erf(x)$ is the complementary error function.

<u>**Step 1: Demonstrate convergence for Equation** 5.</u> We do not assume that np is an integer. Rather, suppose that $np = t_n + b_n$ where $t_n \in \mathbb{N}$ and $b_n \in [0, 1)$. We partition the objective equation into four regions, as follows:

$$\begin{aligned}
&\sum_{k=\left\lfloor \frac{np}{2} \right\rfloor}^{n} x_{n,k}^2 f(x_{n,k})^2 \Delta_n \\
&= \sum_{k \in \left[\left\lfloor \frac{np}{2} \right\rfloor, n\right] \setminus [\lfloor np \rfloor - \lfloor npq \rfloor, \lceil np \rceil + \lceil npq \rceil]} x_{n,k}^2 f(x_{n,k})^2 \Delta_n + \sum_{k=\lfloor np \rfloor - \left\lfloor \sqrt{npq} \right\rfloor}^{\lceil np \rceil + \left\lceil \sqrt{npq} \right\rceil} x_{n,k}^2 f(x_{n,k})^2 \Delta_n \\
&\quad + \sum_{k=\lceil np \rceil + \left\lceil \sqrt{npq} \right\rceil}^{\lceil np \rceil + \lceil npq \rceil} x_{n,k}^2 f(x_{n,k})^2 \Delta_n + \sum_{k=\lfloor np \rfloor - \lfloor npq \rfloor}^{\lfloor np \rfloor - \left\lfloor \sqrt{npq} \right\rfloor} x_{n,k}^2 f(x_{n,k})^2 \Delta_n
\end{aligned} \tag{5.100}$$

The first summation of Equation (5.100) is

$$\Theta(n)\Theta(n)\exp(-\Theta(n))\Theta\left(\frac{1}{\sqrt{n}}\right) = \exp(-\Theta(n)).$$

The second summation of Equation (5.100) converges to

$$\frac{1}{2\pi}\int_{-1}^{1} y^2 e^{-y^2} dy = \frac{\sqrt{\pi}e \cdot erf(1) - 2}{4\pi e} = \Theta(1)$$

by definition of the Riemann integral. The third summation of Equation (5.100) is equivalent to

$$\begin{aligned}
&\sum_{k=\left\lceil \sqrt{npq} \right\rceil}^{\lceil npq \rceil} \left(\frac{k + \lceil np \rceil - np}{\sqrt{npq}}\right)^2 f\left(\frac{k + \lceil np \rceil - np}{\sqrt{npq}}\right)^2 \Delta_n \\
&= \sum_{R=1}^{\left\lceil \sqrt{npq} \right\rceil - 1} \sum_{r=0}^{\left\lceil \sqrt{npq} \right\rceil - 1} \left(\frac{R\left\lceil \sqrt{npq} \right\rceil + r + 1 - b_n}{\sqrt{npq}}\right)^2 f\left(\frac{R\left\lceil \sqrt{npq} \right\rceil + r + 1 - b_n}{\sqrt{npq}}\right)^2 \Delta_n
\end{aligned}$$

which is at most

$$\sum_{R=1}^{\left\lceil \sqrt{npq} \right\rceil - 1} R^2 f(R)^2,$$

where we plugged in $r = -1 + b_n$ since $y^2e^{-y^2}$ is monotone decreasing along $y \geq 1$. This is taken $\lceil\sqrt{npq}\rceil$ times and cancels out with Δ_n. Furthermore, we used the fact that $\frac{R\lceil\sqrt{npq}\rceil}{\sqrt{npq}} \geq R$. By the integral test for convergence, the third summation of Equation (5.100) converges because

$$\frac{1}{2\pi}\int_1^{\infty} y^2e^{-y^2}dy = \frac{e\sqrt{\pi}\cdot erfc(1) - 2}{4e} = \Theta(1)$$

converges. The fourth summation of (5.100) follows by similar reasoning. Hence, Equation 5 converges; i.e., is $\Theta(1)$ as claimed.

Step 2: Demonstrate convergence for Equation 11. The proof follows almost identically to that of Equation 5 of this lemma. Recall that $\tau_n \approx \frac{np}{2}$. We partition the objective into three regions:

$$\begin{aligned}
&\sum_{u=0}^{\tau_n}\left(\frac{u}{\sqrt{npq}}\right)^2 f\left(\frac{u}{\sqrt{npq}}\right)^2\Delta_n \\
&= \sum_{u=0}^{\lceil\sqrt{npq}\rceil-1}\left(\frac{u}{\sqrt{npq}}\right)^2 f\left(\frac{u}{\sqrt{npq}}\right)^2\Delta_n + \sum_{u=\lceil npq\rceil}^{\tau_n}\left(\frac{u}{\sqrt{npq}}\right)^2 f\left(\frac{u}{\sqrt{npq}}\right)^2\Delta_n \\
&\quad + \sum_{R=1}^{\lceil\sqrt{npq}\rceil-1}\sum_{r=0}^{\lceil\sqrt{npq}\rceil-1}\left(\frac{R\lceil\sqrt{npq}\rceil + r}{\sqrt{npq}}\right)^2 f\left(\frac{R\lceil\sqrt{npq}\rceil + r}{\sqrt{npq}}\right)^2\Delta_n \qquad (5.101)
\end{aligned}$$

The first summation of Equation (5.101) converges to

$$\frac{1}{2\pi}\int_0^{1} y^2e^{-y^2}dy = \frac{e\sqrt{\pi}\cdot erf(1) - 2}{4e} = \Theta(1)$$

by definition of the Riemann integral. The second summation of Equation (5.101) is

$$\Theta(n)\Theta(n)\exp(-\Theta(n))\Theta\left(\frac{1}{\sqrt{n}}\right) = \exp(-\Theta(n)).$$

The third summation of Equation (5.101) is at most

$$\sum_{R=1}^{\lceil\sqrt{npq}\rceil-1} R^2f(R^2) \qquad (5.102)$$

where we plugged in $r = 0$ since $y^2e^{-y^2}$ is monotone decreasing along $y \geq 1$. This is taken $\lceil\sqrt{npq}\rceil$ times and cancels out with Δ_n. Furthermore, we used the fact that $\frac{R\lceil\sqrt{npq}\rceil}{\sqrt{npq}} \geq R$. By the integral test for convergence, the third summation of Equation (5.102) converges because

$$\frac{1}{2\pi}\int_1^\infty y^2e^{-y^2}dy = \frac{e\sqrt{\pi}\cdot erfc(1) - 2}{4e} = \Theta(1)$$

converges. Hence, Equation 11 converges (i.e., is $\Theta(1)$), as claimed. This concludes the proof of Lemma 5.12. □

Lemma 5.14.

$$\sum_{k=\lfloor\frac{np}{2}\rfloor}^{n} x_{n,k}^2 \Pr(S_n = k) = \Theta(1).$$

Proof. The lemma is implied by the following:

$$\lim_{n\to\infty}\sum_{k=\lfloor\frac{np}{2}\rfloor}^{n} x_{n,k}^2 \Pr(S_n = k) = 1.$$

Let $X_n = \frac{S_n - np}{\sqrt{npq}}$ for $S_n \sim Bin(n,p)$. We have that

$$\sum_{k=\lfloor\frac{np}{2}\rfloor}^{n} x_{n,k}^2 \Pr(S_n = k) = \mathbb{E}[X_n^2] - \exp(-\Theta(n))$$

by Hoeffding's inequality (Proposition 5.3). We know that

$$\mathbb{E}[S_n^2] = n^2p^2 + npq.$$

This leads us to the conclusion that

$$\begin{aligned}
\mathbb{E}[X_n^2] &= \frac{1}{npq}\mathbb{E}[S_n^2 - 2S_nnp + n^2p^2] \\
&= \frac{1}{npq}\left(\mathbb{E}[S_n^2] - 2np\mathbb{E}[S_n] + n^2p^2\right) \\
&= \frac{1}{npq}\left((n^2p^2 + npq) - 2np(np) + n^2p^2\right) \\
&= 1.
\end{aligned}$$

This concludes the proof of Lemma 5.14. □

5.3.3 Stirling, Wallis, and Central Binomial Coefficients

Stirling's approximation for the factorial is as follows.

Proposition 5.1 (Stirling's approximation)**.**

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

More precisely, $\forall n \geq 1$*,*

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)e^{\frac{1}{12n+1}} < n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)e^{\frac{1}{12n}}.$$

Plugging in Stirling's approximation for the central binomial coefficient can demonstrate the asymptotic growth:

$$\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{n\pi}}. \tag{5.103}$$

The error of this approximation is known to be $\mathcal{O}(\frac{1}{n})$ (Luke, 1969); see Dutka (1991). For completeness and usefulness in our main theorem, we demonstrate one proof for this asymptotic growth in the following lemma. This argument is transposed from lecture notes by Galvin (2018) and uses the Wallis product for π (Wallis, 1656):

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{4n^2}{4n^2-1} = \prod_{n=1}^{\infty}\left(\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\right). \tag{5.104}$$

Lemma 5.15.

$$\sqrt{\frac{2n}{2n+1}}\sqrt{\frac{2}{\pi(2n+1)}} \leq \frac{2^{2n}}{(2n+1)\binom{2n}{n}} \leq \sqrt{\frac{2}{\pi(2n+1)}}.$$

Proof. For each $n \geq 0$, define $S_n = \int_0^{\pi/2} \sin^n x dx$. We have

$$S_0 = \frac{\pi}{2}, \quad S_1 = \int_0^{\pi/2} \sin x \; dx = 1,$$

and for $n \geq 2$ we get from integration by parts (taking $u = \sin^{n-1} x$ and $dv = \sin x dx$, so that $du = (n-1)\sin^{n-2} x \cos x dx$ and $v = -\cos x$) that

$$\begin{aligned}
S_n &= (\sin^{n-1} x)(-cos x)|_{x=0}^{\pi/2} - \int_0^{\pi/2} -(n-1)\cos x \sin^{n-2} x \cos x dx \\
&= (n-1)\int_0^{\pi/2} \cos^2 x sin^{n-2} x dx \\
&= (n-1)\int_0^{\pi/2} (1-\sin^2 x)\sin^{n-2} x dx \\
&= (n-1)S_{n-2} - (n-1)S_n.
\end{aligned}$$

This leads to the recurrence relation:

$$S_n = \frac{n-1}{n} S_{n-2} \quad \text{for } n \geq 2.$$

Iterating the recurrence relation until the initial conditions are reached, we get that

$$S_{2n} = \left(\frac{2n-1}{2n}\right)\left(\frac{2n-3}{2n-2}\right)\cdots\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\frac{\pi}{2}$$

and

$$S_{2n+1} = \left(\frac{2n}{2n+1}\right)\left(\frac{2n-2}{2n-1}\right)\cdots\left(\frac{4}{5}\right)\left(\frac{2}{3}\right)1.$$

Taking the ratio of these two identities and rearranging gets us that $\frac{\pi}{2}$ is equivalent to

$$\left(\frac{2}{1}\right)\left(\frac{2}{3}\right)\left(\frac{4}{3}\right)\left(\frac{4}{5}\right)\cdots\left(\frac{2n}{2n-1}\right)\left(\frac{2n}{2n+1}\right)\frac{S_{2n}}{S_{2n+1}}.$$

For ease of notation, define

$$\mathcal{W}_n = \left(\frac{2}{1}\right)\left(\frac{2}{3}\right)\left(\frac{4}{3}\right)\left(\frac{4}{5}\right)\cdots\left(\frac{2n}{2n-1}\right)\left(\frac{2n}{2n+1}\right)$$

as the first n terms of Wallis' product, so that $\frac{\pi}{2} = \mathcal{W}_n \frac{S_{2n}}{S_{2n+1}}$. Now, since $0 \leq \sin x \leq 1$ on $[0, \pi/2]$, we have also

$$0 \leq \sin^{2n+1} x \leq \sin^{2x} x \leq \sin^{2n-1} x,$$

and so, integrating and using the recurrence relation, we get

$$0 \leq S_{2n+1} \leq S_{2n} \leq S_{2n-1} = \frac{2n+1}{2n} S_{2n+1}$$

and so

$$1 \le \frac{S_{2n}}{S_{2n+1}} \le 1 + \frac{1}{2n}.$$

Hence, $1 \le \frac{\pi}{2\mathcal{W}_n} \le 1 + \frac{1}{2n}$; equivalently, $\frac{2}{\pi} \ge \mathcal{W}_n \ge \frac{2(2n)}{\pi(2n+1)}$. Wallis' formula can now be used to estimate the central binomial coefficient:

$$\begin{aligned}
\binom{2n}{n} &= \frac{(2n)(2n-1)(2n-2)\dots(3)(2)(1)}{(n)(n-1)\dots(2)(1)\cdot(n)(n-1)\dots(2)(1)} \\
&= 2^n \frac{(2n)(2n-1)(2n-2)\dots(3)(2)(1)}{(n)(n-1)\dots(2)(1)} \\
&= 2^{2n} \frac{(2n)(2n-1)(2n-2)\dots(3)(2)(1)}{(2n)(2n-2)\dots(4)(2)} \\
&= \frac{2^{2n}}{\sqrt{2n+1}} \sqrt{\frac{(2n+1)(2n-1)^2(2n-3)^2\dots(3)^2(1)}{(2n)^2(2n-2)^2\dots(4)^2(2)^2}} \\
&= \frac{2^{2n}}{\sqrt{\mathcal{W}_n(2n+1)}}.
\end{aligned}$$

Therefore:

$$\begin{aligned}
\sqrt{\frac{2n}{2n+1}}\sqrt{\frac{2}{\pi(2n+1)}} \le \frac{2^{2n}}{(2n+1)\binom{2n}{n}} &= \sqrt{\frac{\mathcal{W}_n}{2n+1}} \\
&\le \sqrt{\frac{2}{\pi(2n+1)}}.
\end{aligned}$$

This concludes the proof of Lemma 5.15. □

5.3.4 Technical Lemmas

Proposition 5.2. *Let* $q \in \left[1, \frac{n}{6} - 1\right]$. *Then*

$$\binom{n}{\frac{n}{2}-q, \frac{n}{2}-q, q, q} \pi_1^{n-2q}\pi_3^{2q} = \frac{\binom{n}{\frac{n}{2}}}{2^n}\left(\binom{\frac{n}{2}}{q}(2\pi_1)^{\frac{n}{2}-q}(2\pi_3)^q\right)^2.$$

Proof. First, we have:

$$\begin{aligned}\binom{n}{\frac{n}{2}-q, \frac{n}{2}-q, q, q} &= \frac{n!}{(\frac{n}{2}-q)!^2 q!^2} \\ &= \frac{n!}{(\frac{n}{2})!^2} \frac{(\frac{n}{2})!^2}{(\frac{n}{2}-q)!^2 q!^2} \\ &= \binom{n}{\frac{n}{2}} \binom{\frac{n}{2}}{q}^2.\end{aligned}$$

Second, we note:

$$\pi_1^{n-2q} \pi_3^{2q} = \frac{1}{2^n} \left((2\pi_1)^{\frac{n}{2}-q} (2\pi_3)^q \right)^2.$$

Proposition 5.2 follows by combining these identities. ∎

Proposition 5.3 (Hoeffding's Inequality)**.** *Let $p \in (0,1)$ and $q = 1 - p$; let $a, b \in \mathbb{R}$ such that $0 \leq a < b \leq 1$. If $p \notin [a, b]$ then*

$$\sum_{k=\lfloor an \rfloor}^{\lceil bn \rceil} \left(\binom{n}{k} p^{n-k} q^k \right)^2 = \exp(-\Theta(n)).$$

Proof. Consider first the case where $p < a$. Then

$$\begin{aligned} 0 &\leq \sum_{k=\lfloor an \rfloor}^{\lceil bn \rceil} \left(\binom{n}{k} p^{n-k} q^k \right)^2 \\ &\leq \sum_{k=\lfloor an \rfloor}^{\lceil bn \rceil} \left(\binom{n}{k} p^{n-k} q^k \right) \\ &= \Pr\Big(S_n - pn \geq \lfloor an \rfloor - pn\Big) - \Pr\Big(S_n - pn \geq \lceil bn \rceil - pn + 1\Big) \\ &\leq \exp(-\Theta(n)) \end{aligned}$$

by Hoeffding's inequality, where $S_n \sim Bin(n, p)$. Proposition 5.3 follows because the case where $p > b$ is similar. ∎

The following identities are used in Lemma 5.1 to simplify several conditional expectation equations. This lemma makes use of technical statements presented in Lemma 5.18, which are proved later in this section.

Lemma 5.3. *For $q \geq 1$, the following identities hold:*

1.

$$\sum_{\beta=q+1}^{2q} \binom{2q}{\beta} \frac{1}{2^{2q}} = \frac{1}{2} - \frac{1}{2^{2q}} \binom{2q-1}{q-1}$$

2.

$$\sum_{\beta=q+1}^{2q} \beta \binom{2q}{\beta} \frac{1}{2^{2q}} = \frac{q}{2}$$

3.

$$\sum_{\beta=0}^{q} \binom{2q+1}{\beta} \frac{1}{2^{2q+1}} = \frac{1}{2}$$

4.

$$\sum_{\beta=0}^{q} \beta \binom{2q+1}{\beta} \frac{1}{2^{2q+1}} = \left(\frac{2q+1}{4}\right) - \frac{2q+1}{2^{2q+1}} \binom{2q-1}{q-1}.$$

5.

$$\sum_{\beta=q+1}^{2q+1} \binom{2q+1}{\beta} \frac{1}{2^{2q+1}} = \frac{1}{2}$$

6.

$$\sum_{\beta=q+1}^{2q+1} \beta \binom{2q+1}{\beta} \frac{1}{2^{2q+1}} = \left(\frac{2q+1}{4}\right) + \frac{2q+1}{2^{2q+1}} \binom{2q-1}{q-1}.$$

7.

$$\sum_{\beta=0}^{q-1} \binom{2q}{\beta} \frac{1}{2^{2q}} = \frac{1}{2} - \frac{1}{2^{2q+1}} \binom{2q}{q}$$

8.

$$\sum_{\beta=0}^{q-1} \beta \binom{2q}{\beta} \frac{1}{2^{2q}} = \frac{q}{2} - \frac{q}{2^{2q}} \binom{2q}{q}.$$

Proof. We take these equations one at a time.

Equation 1.

$$\begin{aligned}
\sum_{\beta=q+1}^{2q} \binom{2q}{\beta} \frac{1}{2^{2q}} &= \sum_{\beta=0}^{2q} \binom{2q}{\beta} \frac{1}{2^{2q}} - \sum_{\beta=0}^{q-1} \binom{2q}{\beta} \frac{1}{2^{2q}} - \frac{1}{2^{2q}} \binom{2q}{q} \\
&= 1 - \left[\frac{1}{2} - \frac{1}{2^{2q}} \binom{2q-1}{q-1} \right] - \frac{1}{2^{2q}} \binom{2q}{q} \\
&= \frac{1}{2} + \frac{1}{2^{2q}} \left[\binom{2q-1}{q-1} - \binom{2q}{q} \right] \\
&= \frac{1}{2} - \frac{1}{2^{2q}} \binom{2q-1}{q-1}
\end{aligned}$$

where the third row is by (Lemma 5.18, Equation 2) and the last row follows from Pascal's rule.

Equation 2.

$$\begin{aligned}
\sum_{\beta=q+1}^{2q} \beta \binom{2q}{\beta} \frac{1}{2^{2q}} &= \sum_{\beta=0}^{2q} \beta \binom{2q}{\beta} \frac{1}{2^{2q}} - \sum_{\beta=0}^{q-1} \beta \binom{2q}{\beta} \frac{1}{2^{2q}} - \frac{q}{2^{2q}} \binom{2q}{q} \\
&= q - \left[\frac{q}{2} - \frac{2q}{2^{2q}} \binom{2q-1}{q-1} \right] - \frac{q}{2^{2q}} \binom{2q}{q} \\
&= \frac{q}{2} + \frac{q}{2^{2q}} \left[2\binom{2q-1}{q-1} - \binom{2q}{q} \right] \\
&= \frac{q}{2}
\end{aligned}$$

where the third row is by (Lemma 5.18, Equation 3) and the last row follows from Pascal's rule.

Equation 3. By symmetry, we have that

$$\sum_{\beta=0}^{q} \binom{2q+1}{\beta} \frac{1}{2^{2q+1}} = \frac{2^{2q}}{2^{2q+1}} = \frac{1}{2}.$$

Equation 4. We recall Claim 5.1 from Chapter 3:

Claim 5.1. *For any $u \in \mathbb{N}$ and any $t \in [0, u]$, we have*

$$\sum_{v=t}^{u} \binom{u}{v}(u-2v) = -t\binom{u}{t}.$$

As a result,

$$\begin{aligned}\sum_{v=0}^{t} \binom{u}{v}(u-2v) &= \sum_{v=0}^{u} \binom{u}{v}(u-2v) - \sum_{v=0}^{t-1} \binom{u}{v}(u-2v) \\ &= (t+1)\binom{u}{(t+1)}\end{aligned}$$

which implies

$$\begin{aligned}u\sum_{v=0}^{t} \binom{u}{v} - 2\sum_{v=0}^{t} v\binom{u}{v} &= (t+1)\binom{u}{(t+1)} \\ \Rightarrow \sum_{v=0}^{t} v\binom{u}{v} &= \frac{u}{2}\sum_{v=0}^{t} \binom{u}{v} - \frac{(t+1)}{2}\binom{u}{(t+1)}\end{aligned}$$

Substituting $u \leftarrow (2q+1)$ and $t \leftarrow (q)$ into Equation 4, we get

$$\begin{aligned}\sum_{\beta=0}^{q} \beta\binom{2q+1}{\beta}\frac{1}{2^{2q+1}} &= \frac{1}{2^{2q+1}}\left(\frac{(2q+1)}{2}\sum_{\beta=0}^{q}\binom{2q+1}{\beta} - \frac{q+1}{2}\binom{2q+1}{q+1}\right) \\ &= \frac{(2q+1)}{2^{2q+2}}\sum_{\beta=0}^{q}\binom{2q+1}{\beta} - \frac{q+1}{2^{2q+2}}\binom{2q+1}{q+1} \\ &= \left(\frac{2q+1}{4}\right) - \frac{2q+1}{2^{2q+1}}\binom{2q-1}{q-1}\end{aligned}$$

where the second row comes from applying Claim 5.1, the third row is by simplification, and the fourth row is by applying Equation 3 and simplification of the binomial.

Equation 5. Proof by symmetry.

Equation 6. Recall from Equation 4 of Lemma 5.3 that

$$\sum_{\beta=0}^{q} \beta\binom{2q+1}{\beta}\frac{1}{2^{2q+1}} = \left(\frac{2q+1}{4}\right) - \frac{2q+1}{2^{2q+1}}\binom{2q-1}{q-1}.$$

The equation follows since

$$\sum_{\beta=0}^{2q+1} \beta \binom{2q+1}{\beta} \frac{1}{2^{2q+1}} = \left(\frac{2q+1}{2}\right)$$

by definition of the expectation of a binomial random variable.

<u>**Equation 7.**</u> Recall Equation 1 from Lemma 5.3:

$$\sum_{\beta=q+1}^{2q} \binom{2q}{\beta} \frac{1}{2^{2q}} = \frac{1}{2} - \frac{1}{2^{2q}} \binom{2q-1}{q-1}.$$

The equation follows by recognizing that

$$\sum_{\beta=0}^{2q} \binom{2q}{\beta} \frac{1}{2^{2q}} = 1.$$

<u>**Equation 8.**</u> Recall Equation 2 from Lemma 5.3:

$$\sum_{\beta=q+1}^{2q} \beta \binom{2q}{\beta} \frac{1}{2^{2q}} = \frac{q}{2}.$$

The equation follows by recognizing that

$$\sum_{\beta=0}^{2q} \beta \binom{2q}{\beta} \frac{1}{2^{2q}} = q.$$

This concludes the proof of Lemma 5.3. □

The following lemma applies Theorem 2.3 ((Xia, 2021a, Theorem 1)) to prove that the likelihood an $(n, 4)$-PMV fits into a set describing a two-way tie such that there are a specific number of agents with each ranking R_j (recall Definition 2.8). This additional constraint reduces the likelihood from $\Theta\left(\frac{1}{\sqrt{n}}\right)$, by Corollary 5.1, to $\Theta\left(\frac{1}{n}\right)$. This holds as long as $\pi_3 n$ is contained in the summation region; the likelihood is exponentially small otherwise.

Lemma 5.6. *Fix* $a, b \in (0, \frac{1}{6})$, $a < b$. *Then*

$$\sum_{q=\lfloor an \rfloor}^{\frac{n}{6}-1} \binom{n}{\frac{n}{2}-q, \frac{n}{2}-q, q, q} \pi_1^{n-2q} \pi_3^{2q} = \begin{cases} \Theta\left(\frac{1}{n}\right), & \pi_3 \geq a \\ \exp(-\Theta(n)), & \textit{otherwise} \end{cases}$$

and

$$\sum_{q=1}^{\lceil bn \rceil} \binom{n}{\frac{n}{2}-q, \frac{n}{2}-q, q, q} \pi_1^{n-2q} \pi_3^{2q} = \begin{cases} \Theta\left(\frac{1}{n}\right), & \pi_3 \leq b \\ \exp(-\Theta(n)), & \textit{otherwise.} \end{cases}$$

Proof. To prove the lemma, we may assume without loss of generality that $an, bn \in \mathbb{Z}_{\geq 0}$ are integers. This follows because $|x - \lfloor x \rfloor| \leq 1 = o(n)$ for any $x \in \mathbb{R}$, so Xia (2021a)'s theorems are indifferent to the distinction between x and $\lfloor x \rfloor$. The same holds for x and $\lceil x \rceil$.

Consider n random variables $Q_1, \ldots, Q_n$, such that $Q_i \in \{R_1, \ldots, R_4\}$, which are distributed identically and independently according to π by Assumption 5.1. Let $\vec{X}_\pi$ denote the corresponding $(n, 4)$-PMV to $Q_1, \ldots, Q_n$ according to Definition 2.8; we have $\mu = 4$. Let us define the sets:

$$\begin{aligned} \mathcal{T}^a &= \left\{ \left(\frac{n}{2} - q, \frac{n}{2} - q, q, q \right) : q \in \left[an, \frac{n}{6} - 1 \right] \right\} \\ \mathcal{T}^b &= \left\{ \left(\frac{n}{2} - q, \frac{n}{2} - q, q, q \right) : q \in [1, bn] \right\}. \end{aligned}$$

Then we have

$$\begin{aligned} \sum_{q=an}^{\frac{n}{6}-1} \binom{n}{\frac{n}{2}-q, \frac{n}{2}-q, q, q} \pi_1^{n-2q} \pi_3^{2q} &= \Pr(\vec{X}_\pi \in \mathcal{T}^a) \\ \sum_{q=1}^{bn} \binom{n}{\frac{n}{2}-q, \frac{n}{2}-q, q, q} \pi_1^{n-2q} \pi_3^{2q} &= \Pr(\vec{X}_\pi \in \mathcal{T}^b) \end{aligned}$$

which are instances of the PMV-in-polyhedron problem. Specifically, notice that

$$\mathcal{T}^a = \left\{ \vec{x} \in \mathbb{R}^4 \ : \ \mathbf{A}^a \vec{x} \leq \vec{b}^a \right\} \tag{5.105}$$

where

$$\mathbf{A}^a = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ -\frac{1}{6} & -\frac{1}{6} & \frac{5}{6} & -\frac{1}{6} \\ a & a & -1+a & a \end{pmatrix}, \quad \vec{b}^a = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

and

$$\mathcal{T}^b = \left\{ \vec{x} \in \mathbb{R}^4 \ : \ \mathbf{A}^b \vec{x} \leq \vec{b}^b \right\} \tag{5.106}$$

where

$$\mathbf{A}^b = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 \\ -b & -b & 1-b & -b \end{pmatrix}, \quad \vec{b}^b = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}.$$

For any $\pi_3 \in (0, \frac{1}{6})$ we will demonstrate in Step 1 below that $[\mathcal{T}^a]_n^{\mathbb{Z}} \neq 0$ and $[\mathcal{T}^a]_n^{\mathbb{Z}} \neq b$; hence, the zero case of the above theorem does not apply. Next, in Step 2, we will demonstrate that $\pi \in \mathcal{T}^a \iff \pi_3 \geq a$ and $\pi \in \mathcal{T}^b \iff \pi_3 \leq b$; hence, the polynomial and exponential cases of the theorem apply when the respective conditions hold. In Step 3, we will finally demonstrate that $dim([\mathcal{T}^a]_{\leq 0}) = dim([\mathcal{T}^b]_{\leq 0}) = 2$, so that the polynomial power is $\frac{2-4}{2} = -1$ for each polyhedron.

Step 1: Zero case does not apply. It is easy to see that $\vec{t^q} \in \mathcal{T}^a$ for $q = an$ and that $\vec{t^q} \in \mathcal{T}^b$ for $q = bn$. This holds because $a < \frac{1}{6}$ and $b > 0$ and implies that $[\mathcal{T}^a]_n^{\mathbb{Z}} \neq \emptyset$ and $[\mathcal{T}^b]_n^{\mathbb{Z}} \neq \emptyset$. Hence, the zero case of Theorem 2.3 does not apply.

Step 2: Differentiate polynomial and exponential cases. The next condition of Theorem 2.3 is a comparison between $[\mathcal{T}^a]_{\leq 0}$ or $[\mathcal{T}^b]_{\leq 0}$ and $\Pi = \{\pi^n\}$ using Assumption 5.1.

Consider the (fractional) vote profile πn and the last row of $\mathbf{A}^a$. For $\mathcal{T}^a$, we have

$$(a, a, -1+a, a) \cdot \pi n = (-\pi_3 + a)n \leq 0$$

if and only if $\pi_3 \geq a$. It is easy to see that $\vec{v} \cdot \pi n \leq 0$ for any other row-vector $\vec{v} \in \mathbf{A}^a$. This holds by our Assumption 5.1 on π that $\pi_1 = \pi_2 > 2\pi_3 = 2\pi_4 > 0$, which already necessitates that $\pi_3 \in (0, \frac{1}{6})$.

Likewise, in the last row of $\mathbf{A}^b$ for the case of $\mathcal{T}^b$, we have

$$(-b, -b, 1-b, -b) \cdot \pi n = (\pi_3 - b)n \leq 0$$

if and only if $\pi_3 \leq b$. Similarly, $\vec{v} \cdot \pi n \leq 0$ for any other row-vector $\vec{v} \in \mathbf{A}^b$. Therefore, the polynomial cases of Theorem 2.3 apply to $\Pr(\vec{X}_\pi \in \mathcal{T}^a)$ and $\Pr(\vec{X}_\pi \in \mathcal{T}^b)$ when the lemma's respective conditions hold; otherwise the exponential case applies.

Step 3: Determine dimension of characteristic cones. Following the proof of Theorem 1 in Xia (2021a), we start with the following definition.

Definition 5.2 (Equation (2) on page 99 of Schrijver (1998))**.** *For any matrix* $\mathbf{A}$ *that defines a polyhedron* $\mathcal{H}$*, let* $\mathbf{A}^=$ *denote the* implicit equalities*, which is the maximal set of rows of* $\mathbf{A}$ *such that for all* $\vec{x} \in \mathcal{H}_{\leq 0}$*, we have* $\mathbf{A}^= \cdot (\vec{x})^T = (\vec{0})^T$*. Let* $\mathbf{A}^+$ *denote the remaining rows of* $\mathbf{A}$*.*

By Equation (9) on page 99 of Schrijver (1998) we know that $dim([\mathcal{T}^a]_{\leq 0}) = \mu - rank([\mathcal{A}^a]^=)$ and $dim([\mathcal{T}^b]_{\leq 0}) = \mu - rank([\mathcal{A}^b]^=)$. From Equations (5.105) and (5.106) we can deduce that

$$[\mathbf{A^a}]^= = [\mathbf{A^b}]^= = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

which has rank 2. Hence, the polynomial powers when we apply Theorem 2.3 are

$$\frac{(\mu - rank([\mathcal{A}^a]^=)) - \mu}{2} = \frac{(\mu - rank([\mathcal{A}^b]^=)) - \mu}{2} = -1.$$

This concludes the proof of Lemma 5.6. $\square$

Lemma 5.13. *Let* $k \in \left[\left\lfloor \frac{np}{2} \right\rfloor, n\right]$*. Then*

$$\left| \frac{2^{2k}\sqrt{npq}}{(2k+1)\binom{2k}{k}} \right| = \Theta(1).$$

Proof. By Lemma 5.15 we get that

$$\sqrt{\frac{2}{3}}\sqrt{\frac{2}{\pi(2k+1)}} \le \sqrt{\frac{2k}{2k+1}}\sqrt{\frac{2}{\pi(2k+1)}}$$
$$\le \frac{2^{2k}}{(2k+1)\binom{2k}{k}}$$
$$\le \sqrt{\frac{2}{\pi(2k+1)}}.$$

Lemma 5.13 follows since $k = \Theta(n)$ by assumption and there is an extra $\Theta(\sqrt{n})$ term in the lemma's objective. $\square$

Lemma 5.16. *For any constant $t > 0$, $\sqrt{\frac{tn}{tn+1}} = 1 - \mathcal{O}\left(\frac{1}{n}\right)$.*

Proof. The lemma is implied by the following:

$$\lim_{n\to\infty} n\left(\sqrt{\frac{tn}{tn+1}} - 1\right) = -\frac{1}{2t}.$$

Fix $\epsilon > 0$ and define $N = \frac{1}{2t^2\epsilon}$. Then $\forall n > N$,

$$\begin{aligned}
&\left|n\left(\sqrt{\frac{tn}{tn+1}} - 1\right) + \frac{1}{2t}\right| \\
&= \left|n\left(\frac{\sqrt{tn} - \sqrt{tn+1}}{\sqrt{tn+1}}\right)\left(\frac{\sqrt{tn} + \sqrt{tn+1}}{\sqrt{tn} + \sqrt{tn+1}}\right) + \frac{1}{2t}\right| \\
&= \left|\frac{-n}{\sqrt{tn+1}\left(\sqrt{tn} + \sqrt{tn+1}\right)} + \frac{1}{2t}\right| \\
&= \frac{1}{2t}\left|\frac{-tn + 1 + \sqrt{(tn)(tn+1)}}{tn + 1 + \sqrt{(tn)(tn+1)}}\right| \\
&\le \frac{1}{2t}\left|\frac{-tn + 1 + tn + 1}{2(tn)}\right| \\
&= \frac{1}{2t^2 n} \\
&< \frac{1}{2t^2 N} \\
&= \epsilon.
\end{aligned}$$

Lemma 5.16 follows by definition of the limit. $\square$

The following lemma is adapted from the proof of the local DeMoivre-Laplace theorem, demonstrated in lecture notes by Carlen (2018).

Lemma 5.17.

$$\binom{n}{\lfloor np \rfloor} p^{\lfloor np \rfloor} q^{n-\lfloor np \rfloor} = \frac{1}{\sqrt{2\pi npq}} \left(1 \pm \mathcal{O}\left(\frac{1}{n}\right)\right).$$

Proof. A more precise version of Stirling's formula for all $n \geq 1$ is

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n+1}} \leq n! \leq \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}.$$

Taking logarithms, it follows that

$$\left|\log n! - \frac{1}{2}\log(2\pi n) - n \log n + n\right| \leq \frac{1}{12n}.$$

For $n \in \mathbb{N}$ and $k \in [0, n]$ an integer, we compute

$$\begin{aligned}
\log\binom{n}{k} &= \log n! - \log k! - \log(n-k)! \\
&\approx -\frac{1}{2}\log(2\pi) + \left(n + \frac{1}{2}\right)\log n - \left(k + \frac{1}{2}\right)\log k - \left(n - k + \frac{1}{2}\right)\log(n-k) \\
&= \frac{1}{2}\log\left(\frac{1}{2\pi n}\right) - \left(k + \frac{1}{2}\right)\log\left(\frac{k}{n}\right) - \left(n - k + \frac{1}{2}\right)\log\left(\frac{n-k}{n}\right)
\end{aligned}$$

where we have used

$$\left(n + \frac{1}{2}\right) = \left(k + \frac{1}{2}\right) + \left(n - k + \frac{1}{2}\right) - \frac{1}{2}$$

to obtain the last line. Therefore

$$\begin{aligned}
&\log\left(\binom{n}{k} p^k q^{n-k}\right) \\
&\approx -\frac{1}{2}\log(2\pi npq) - \left(k + \frac{1}{2}\right)\log\left(\frac{k}{np}\right) - \left(n - k + \frac{1}{2}\right)\log\left(\frac{n-k}{nq}\right).
\end{aligned} \tag{5.107}$$

Note that the error made in Equation (5.107) is no greater than

$$\frac{1}{12}\left(\frac{1}{n} + \frac{1}{k} + \frac{1}{n-k}\right) = \mathcal{O}\left(\frac{1}{n}\right)$$

in magnitude. We do not assume that np is an integer. Rather, suppose $np = t_n + b_n$ where $t_n \in \mathbb{N}$ and $b_n \in (0, 1)$. Plugging in $k = \lfloor np \rfloor = t_n$ into Equation (5.107) yields

$$\begin{aligned} & -\frac{1}{2}\log(2\pi npq) + \left(\lfloor np \rfloor + \frac{1}{2}\right)\log\left(\frac{np}{\lfloor np \rfloor}\right) + \left(n - \lfloor np \rfloor + \frac{1}{2}\right)\log\left(\frac{nq}{n - \lfloor np \rfloor}\right) \\ & = -\frac{1}{2}\log(2\pi npq) + \left(t_n + \frac{1}{2}\right)\log\left(1 + \frac{b_n}{t_n}\right) + \left(n - t_n + \frac{1}{2}\right)\log\left(1 - \frac{b_n}{n - t_n}\right). \quad (5.108) \end{aligned}$$

We apply the Taylor expansion for the natural logarithm, which is

$$\log(1+t) = t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \mathcal{O}(t^4)$$

and converges for $|t| < 1$. This is an alternating sequence, meaning that

$$\left|\log(1+t) - t + \frac{1}{2}t^2\right| \leq \frac{1}{3}|t|^3.$$

Hence, for $t = \pm\mathcal{O}\left(\frac{1}{n}\right)$ from Equation (5.108), the error in approximating the logarithm is $\mathcal{O}\left(\frac{1}{n^3}\right)$. Through this approximation, we get

$$\begin{aligned} & -\frac{1}{2}\log(2\pi npq) + \left(t_n + \frac{1}{2}\right)\left(\frac{b_n}{t_n} - \frac{b_n^2}{2t_n^2}\right) + \left(n - t_n + \frac{1}{2}\right)\left(-\frac{b_n}{n - t_n} + \frac{b_n^2}{2(n - t_n)^2}\right) \\ & = -\frac{1}{2}\log(2\pi npq) \pm \mathcal{O}\left(\frac{1}{n}\right). \quad (5.109) \end{aligned}$$

Lemma 5.17's statement follows by noticing that $e^{\pm\mathcal{O}\left(\frac{1}{n}\right)} = \left(1 \pm \mathcal{O}\left(\frac{1}{n}\right)\right)$ by the Maclaurin series of the exponential. □

The following identities are used to prove Lemma 5.3.

Lemma 5.18. *For $q \geq 2$, the following identities hold:*

1.
$$\sum_{\beta=0}^{q-1} \binom{2q-1}{\beta} \frac{1}{2^{2q-1}} = \frac{1}{2}$$

2.
$$\sum_{\beta=0}^{q-1} \beta \binom{2q-1}{\beta} \frac{1}{2^{2q-1}} = \left(\frac{2q-1}{4}\right) - \frac{q}{2^{2q}} \binom{2q-1}{q-1}$$

3.

$$\sum_{\beta=q+1}^{2q} \binom{2q}{\beta} \frac{1}{2^{2q}} = \frac{1}{2} - \frac{1}{2^{2q}} \binom{2q-1}{q-1}.$$

Proof. We take these equations one at a time.

Equation 1. By symmetry, we have that

$$\sum_{\beta=0}^{q-1} \binom{2q-1}{\beta} \frac{1}{2^{2q-1}} = \frac{2^{2q-2}}{2^{2q-1}} = \frac{1}{2}.$$

Equation 2.

$$\begin{aligned}
\sum_{\beta=0}^{q-1} \binom{2q}{\beta} \frac{1}{2^{2q}} &= \frac{1}{2^{2q}} \left(\sum_{\beta=1}^{q-1} \binom{2q-1}{\beta} + \sum_{\beta=1}^{q-1} \binom{2q-1}{\beta-1} + 1 \right) \\
&= \frac{1}{2^{2q}} \left(\sum_{\beta=0}^{q-1} \binom{2q-1}{\beta} + \sum_{\beta=0}^{q-2} \binom{2q-1}{\beta} \right) \\
&= \frac{1}{2^{2q}} \left(2 \sum_{\beta=0}^{q-1} \binom{2q-1}{\beta} - \binom{2q-1}{q-1} \right) \\
&= \frac{1}{2} - \frac{1}{2^{2q}} \binom{2q-1}{q-1}
\end{aligned}$$

where the second row holds by Pascal's rule, the third row is by changing the second summation's base, the fourth row is by simplification, and the fifth row follows from applying Equation 1.

Equation 3.

$$\begin{aligned}
\sum_{\beta=0}^{q-1} \beta \binom{2q}{\beta} \frac{1}{2^{2q}} &= \frac{1}{2^{2q}} \sum_{\beta=1}^{q-1} (2q) \binom{2q-1}{\beta-1} \\
&= \frac{q}{2^{2q-1}} \sum_{\beta=0}^{q-2} \binom{2q-1}{\beta} \\
&= \frac{q}{2^{2q-1}} \left(\sum_{\beta=0}^{q-1} \binom{2q-1}{\beta} - \binom{2q-1}{q-1} \right) \\
&= \frac{q}{2} - \frac{2q}{2^{2q}} \binom{2q-1}{q-1}
\end{aligned}$$

where the second row holds since $b\binom{a}{b} = a\binom{a-1}{b-1}$ for any $a, b \in \mathbb{Z}_{\geq 0}$ and $0 < b \leq a$, the third row is by changing the summation's base, the fourth row is by simplification, and the fifth row is by applying Equation 1.

This concludes the proof of Lemma 5.18. $\square$

CHAPTER 6
CONCLUSION AND FUTURE WORK

Iterative voting (IV) is a naturalistic model for strategic behavior when agents have the opportunity to negotiate their votes prior to finalizing the collective decision. Agents play an extended game by updating their votes over time with myopic reasoning; they do not work to convince others of their point of view or modify others' preferences. Prior work has studied the convergence, equilibrium, and some performance properties of IV by varying the social choice rule used and the models of agent behavior and information schemes. Our work in this thesis advanced these aspects through new settings and techniques.

In Chapter 3, we studied the effect strategic behavior has on the quality of electoral outcomes. Our results naturally extend those of Brânzei et al. (2013) by differentiating the rank-based utility vector $\vec{u}$ from the iterative positional scoring rule $f_{\vec{s}}$. When $\vec{u} = \vec{s}$, prior work found IV's performance to be "very good" for plurality ($ADPoA = 1$), "not bad" for veto ($DPoA = \Omega(m)$ for $m \geq 4$), and "very bad" for Borda ($DPoA = \Omega(n)$). In contrast, we proved that iterative plurality has a $\Theta(n)$ adversarial loss in the worst case when $\vec{u} \neq \vec{s}$ (Theorem 3.1). By distributing agents' preferences according to the impartial culture (IC), we overcame this negative result and obtained a constant order average improvement in social welfare, regardless of the order of agents' best response steps (Theorem 3.2).

In Chapter 4, we explored how convergence of iterative plurality over a single issue, as found by Meir et al. (2010), Meir et al. (2014), and Meir (2015), extends to multiple referenda as agents have limited access to information. We found that for binary issues, the existence of cycles hinges on the interdependence of issues in agents' preference rankings. Specifically, once an agent j takes a local dominance improvement (LDI) step on an issue i, they only subsequently revert their vote if their preference for i changes. This occurs in the event that the set of possible winning alternatives, among other issues that affect j's preference for i, changes. Agents don't have this interdependence if their preferences are $\mathcal{O}$-legal – i.e., if

Portions of this chapter have previously appeared as:

Kavner, J., & Xia, L. (2021). Strategic behavior is bliss: iterative voting improves social welfare. In *Advances in neural information processing systems* (Vol. 34, pp. 19021–19032). Curran Associates, Inc.

Kavner, J., Meir, R., Rossi, F., & Xia, L. (2023, August). Convergence of multi-issue iterative voting under uncertainty. In *Proceedings of the 32nd international joint conference on artificial intelligence* (pp. 2783-2791). ©2023 IJCAI.

Kavner, J., & Xia, L. (2024). *Average-case analysis of iterative voting.* arXiv. https://arxiv.org/abs/2402.08144.

preferences for each issue is independent of later issues in an order $\mathcal{O}$, conditioned on the outcomes of earlier issues in the order. Agent preferences over individual issues then change only finite times, so LDI dynamics converge (Theorem 4.1).

We also found that as uncertainty increases over issues other than the one agents are changing, fewer preference rankings admit LDI steps, eliminating cycles (Theorem 4.2). This result assumes agents have *alternating uncertainty* – i.e., agents may gather more information about the issue they're changing their vote over, thus reducing their uncertainty about that issue, prior to making the change. Finally, convergence does not extend to multi-alternative issues since LDI dynamics may cycle if agents only have partial order preference information (Corollary 4.1). Our experiments confirmed that convergence is practically guaranteed with uncertainty, despite its possibility, and suggests IV improves agents' social welfare over truthful outcomes.

Putting these results together with experimental evidence of IV to date leads us to mixed conclusions about IV's effect on social welfare. Our theoretical result of Theorem 3.2 and experiments in Section 4.6 complement findings that IV improves welfare (e.g., see Reijngoud and Endriss (2012) and Grandi et al. (2013) about the single-issue setting and Bowman et al. (2014) and Grandi et al. (2022) about the multi-issue setting). Tsang and Larson (2016) shows a similar gain in social welfare when agents have single-peaked preferences, are embedded on a social network, and make their manipulations based on estimates of their neighbors' reports. Put together, IV seems to provide a benefit that serves as an additional defense of strategic manipulation to those discussed by Dowding and Hees (2008). Still, it contrasts certain lab experiments by Meir et al. (2020) and simulations by Koolyk et al. (2017). The variability in performance across experiments are due, in part, to differences in model parameters being tested: the setting, social choice function, model of agent information and behavior, and whether the experiment is based on human decisions or synthetic vote profiles sampled from some distribution. In particular, our average-case analysis in Chapter 3 was specific to IC. Although IC has been widely used in social choice (e.g., in the likelihood of ties of elections (Gillett, 1977, 1980; Marchant, 2001)), it is understood to be an unrealistic assumption (Lehtinen & Kuorikoski, 2007; Spielman & Teng, 2009). [10]

Theorem 3.2 took the first step at understanding IV beyond the worst-case analysis

[10] Empirically, this is in terms of fitting real-world election data. Theoretically, IC maximizes the likelihood of Condorcet's paradox among three-alternative elections (Regenwetter, 2006; Tsetlin et al., 2003; Van Deemen, 2014).

and toward more realistic preference distributions. Theorem 5.1 takes the next step by studying IV's performance for a wider class of agents' preference distributions. In Chapter 5, we demonstrated a threshold for which IV improves or degrades expected welfare over the truthful vote. We contributed several novel binomial and multinomial lemmas that may be useful for future study of IV and applied Xia (2021a)'s theorems to expectations of random functions, rather than the likelihood of events. Furthermore, we continued Chapter 3's representation of agents' preferences as a Bayesian network to gain further insight in behavioral social choice.

Our work may be interpreted within the smoothed analysis framework put forth by Xia (2020) and Xia (2021a). Namely, Xia expressed the smoothed likelihood of an event as the supremum (and infimum) expectation of an indicator function, representing the worst- (and best-) average-case analysis where input distributions are sampled from a set $\Pi \subseteq \Delta(\mathcal{L}(\mathcal{A}))$. A comparable "smoothed additive dynamic price of anarchy" notion would study $\sup_{\vec{\pi}\in\Pi^n}$ (and $\inf_{\vec{\pi}\in\Pi^n}$) $\text{EADPoA}(f,\vec{u},\vec{\pi})$. Our work provides insights into these values if Π contains the uniform distribution or any distribution π that follows Assumption 5.1.

There are a number of future directions for IV research, both theoretically and empirically. First, while we discussed extensions of the EADPoA of iterative plurality to other preference distributions in Proposition 5.2, our work is foremost limited in its real-world applicability by our assumptions: $m = 3$ alternatives and Assumption 5.1, which notably restricts the support in π over preference rankings. Relaxing these assumptions to account for all identical and identically distributed preferences in $\Delta(\mathcal{L}(\mathcal{A}))$ is an interesting direction for future work. We expect that extending $m > 3$ will be the most involved since, in order to apply our methods of partitioning EADPoA by the potential winning sets, the set $\mathcal{T}^{1,q}$ in Lemma 5.1 would need to be adapted to suggest

$$|\{j : t_j = 1\}| = |\{j : t_j = 2\}| > |\{j : t_j = \ell\}|,\ \forall \ell \geq 3; \tag{6.1}$$

likewise for $\mathcal{T}^{2,q}$. This would significantly complicate our present analysis, though our technical lemmas in Sections 5.3.1 – 5.3.4 may assist this future direction.

A second theoretical direction would study IV quality according to other behavioral procedures besides myopic best responses. While only a few behavioral and information schemes guarantee convergence (Meir, 2017), this assumption limits IV's real-world applicability to situations that assert best response agent behavior. Furthermore, our results may be sensitive to even a few agents playing other strategies, as exemplified by the study of *trembling hand equilibrium* (Obraztsova, Rabinovich, Elkind, Polukarov, & Jennings, 2016). The welfare of voting games without guaranteed convergence may instead be characterized by the worst-case ratio between the game's truthful outcome and stationary distribution over any cycle – known as the *price of sinking* (Goemans, Mirrokni, & Vetta, 2005). Bounding the welfare in each cycle could extend the DPoA, left for future work.

A third avenue of future work is testing the empirical significance of our theoretical results, as with the experiments by Zou et al. (2015), Tal et al. (2015), and Meir et al. (2020). Understanding to what extent strategic behavior actually affects electoral outcome quality would help mechanism designers elicit more authentic preferences. This could be tested, for example, by fixing peoples' preferences to align with Assumption 5.1 and varying π_1 across the dichotomy threshold: either $<$, $=$, or > 0.4. It is still uncertain how well the iterative plurality protocol models real-world strategic behavior. While we assume myopic best responses in this thesis, peoples' actual behavior through an IV procedure may yield different quality results, even while fixing their preferences. It may further be fruitful to test whether researchers could provide recommendations by suggesting, prescriptively, that people implement best responses in order to help them make better decisions (Xia, 2017). Artificial intelligence-powered recommendations should (i) generally yield better social outcomes, (ii) be perceived as fair, both from a procedural and distributional perspective and in comparison to human-made decisions, and (iii) be desirable by the relevant stakeholders.

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