The Axiomatic Approach to Ranking Systems

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The Axiomatic Approach to Ranking Systems

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Abstract

The ranking of agents based on other agents' input is fundamental to multi-agent systems. Moreover, it has become a central ingredient of a variety of Internet sites, where perhaps the most famous examples are Google's PageRank algorithm and eBay's reputation system.

The setting of ranking systems introduces a new social choice model. The novel feature of this setting is that the set of agents and the set of alternatives **coincide**. Therefore, in such setting one may need to consider the transitive effects of voting. For example, if agent a reports on the importance of (i.e. votes for) agent b then this may influence the credibility of a report by b on the importance of agent c; these indirect effects should be considered when we wish to aggregate the information provided by the agents into a social ranking.

A natural interpretation/application of this setting is the ranking of Internet pages. In this case, the set of agents represents the set of Internet pages, and the links from a page p to a set of pages Q can be viewed as a two-level ranking where agents in Q are preferred by agent(page) p to the agents(pages) which are not in Q. The problem of finding an appropriate social ranking in this case is in fact the problem of (global) page ranking. Particular approaches for obtaining a useful page ranking have been implemented by search engines such as Google.

Due to Arrow-like impossibility results and inspiration from the page ranking setting above, we will limit ourselves to the discussion of ranking systems where agents have dichotomous preferences. In these settings agents have only two levels of preferences: either they vote for some agent, or they do not.

The theory of social choice consists of two complementary axiomatic perspectives:

- The descriptive perspective: given a particular rule r for the aggregation of individual rankings into a social ranking, find a set of axioms that are sound and complete for r. That is, find a set of requirements that r satisfies; moreover, every social aggregation rule that satisfies these requirements should coincide with r. A result showing such an axiomatization is termed a *representation theorem* and it captures the exact essence of (and assumptions behind) the use of the particular rule.

- The normative perspective: devise a set of requirements that a social aggregation rule should satisfy, and try to find whether there is a social aggregation rule that satisfies these requirements.

In this thesis, we apply both these approaches the the ranking systems setting. We begin by applying the descriptive perspective and providing a representation theorem for the well-known PageRank algorithm, which is the basis of Google's search technology. This theorem shows a set of five axioms which are uniquely imply an idealized version of the PageRank ranking system.

In the normative perspective, we begin by defining two important properties of ranking systems: Transitivity and Ranked Independence of Irrelevant Alternatives. We prove an impossibility result for satisfying both of these properties together, but show that when the transitivity axiom is weakened, both can be satisfied by an interesting ranking system. We formally define this recursive-in-degree ranking system and provide an efficient algorithm for its computation.

Still in the normative approach to ranking systems, we tackle the issue of incentives. We consider the case where a self-interested agent may try and manipulate its outgoing votes in order to improve its position in the ranking. We prove a full classification of the existence of incentive compatible ranking systems under four very basic axioms, each with a weak and a strong version. As this classification indicates that no reasonable ranking system can be fully incentive compatible, we expand our discussion to quantifying the level of incentive compatibility of ranking systems. In that setting we prove positive as well as negative results

Finally, we present a variation of ranking systems where a personalized ranking is generated for every participant in the system. We adapt the transitivity, IIA, and incentive compatibility axioms from the general ranking systems setting and prove a surprisingly positive result — a representation theorem for the systems which satisfy all of these axioms. We further show that all of the axioms are required for this proof, while relaxing any axiom leads to new personalized ranking systems.

Abbreviations and Notations

Symbol	Meaning	Page
AIIA	Arrow's Independence of Irrelevant Alternatives	42
AV	Approval Voting	32
$\delta_G^F(v)$	Deviation magnitude of v in G under ranking system F	64
$d_G(v_1, v_2)$	Length of shortest path from v_1 to v_2 in G	73
$\mathbf{Del}(G, v)$	Delete vertex v with in and out-degree of 1 from graph G	17
$\mathbf{Delete}(G, v)$	Strong deletion operator	19
$\mathbf{Duplicate}(G, v, m)$	Duplication operator	20
F_D	The distance personalized ranking system	73
$\mathbb{G}_1$	Set of all directed graphs with out-degree of 1	34
$\mathbb{G}_B$	Set of all bipartite directed graphs	36
$\mathbb{G}_{SC}$	Set of all strongly connected directed graphs	36
$\mathbb{G}_V$	Set of all directed graphs over vertex set V	8
IIA	Independence of Irrelevant Alternatives	29
$L(A)$	Set of all linear orderings on A	7
$\mathcal{M}_{both}$	Sybil & Out-degree manipulation	76
$\mathcal{M}_{out}$	Out-degree manipulation	75
$\mathcal{M}_{sybil}$	Sybil manipulation	76
$n_G(v_1, v_2)$	Number of shortest paths from v_1 to v_2 in G	85
$\mathcal{P}$	Set of all comparison profiles.	32
$P_G(v)$	Predecessor set of vertex v in graph G	10
$\text{Path}(v)$	Set of all almost-simple directed paths to vertex v	37
PPR	Personalized PageRank	73
PR	PageRank	11
PRS	Personalized Ranking System	72
$r_G^F(v)$	Rank of agent v in graph G as ranked by system F.	51
RID	Recursive In-degree	37
RIIA	Ranked Independence of Irrelevant Alternatives	32
$S_G(v)$	Successor set of vertex v in graph G	10
$\mathbf{SelfEdge}(G, v)$	Graph G with self edge added to v	12
$\mathbf{SelfEdge}^{-1}(G, v)$	Graph G with self edge removed from v	12
$\text{value}_r(v)$	Value function from recursive in-degree	37
$\text{vp}_r(v_1, \ldots, v_m)$	Value of path $v_1, \ldots, v_m$ (recursive in-degree)	37
wlog	without loss of generality	39

Chapter 1

Introduction

The ranking of agents based on other agents' input is fundamental to multi-agent systems (see e.g. Resnick et al. (2000)). Moreover, it has become a central ingredient of a variety of Internet sites, where perhaps the most famous examples are Google's PageRank algorithm (Page et al., 1998) and eBay's reputation system (Resnick and Zeckhauser, 2001).

In the classical theory of social choice, as manifested by Arrow(1963), a set of agents/voters is called to rank a set of alternatives. Given the agents' input, i.e. the agents' individual rankings, a social ranking of the alternatives is generated. The theory studies desired properties of the aggregation of agents' rankings into a social ranking. In particular, Arrow's celebrated impossibility theorem(Arrow, 1963) shows that there is no aggregation rule that satisfies some minimal requirements, while by relaxing any of these requirements appropriate social aggregation rules can be defined.

The classical theory of social choice lay the foundations to large part of the rigorous work on the design and analysis of social interactions. Indeed, the most classical results in the theory of mechanism design (e.g. the Gibbard (1973); Satterthwaite (1975) theorems) are applications of the theory of social choice. While economic mechanism design had become an extensive line of study in computer science (see e.g. Nisan and Ronen (1999)) and electronic commerce (see e.g. Lehmann et al. (1999); Parkes (2001); Conitzer et al. (2003)), our work introduces another connection between algorithms and Internet technologies to the mathematical theory of social choice.

The setting of ranking systems introduces a new social choice model. The novel feature of this setting is that the set of agents and the set of alternatives **coincide**. Therefore, in such setting one may need to consider the transitive effects of voting. For example, if agent a reports on the importance of (i.e. votes for) agent b then this may influence the credibility of a report by b on the importance of agent c; these indirect effects should be considered when we wish to aggregate the information provided by the agents into a social ranking.

A natural interpretation/application of this setting is the ranking of Internet pages. In this case, the set of agents represents the set of Internet pages, and the links from a page p to a set of pages Q can be viewed as a two-level ranking where agents in Q are preferred by agent(page) p to the agents(pages) which are not in Q. The problem of finding an appropriate social ranking in this case is in fact the problem of (global) page ranking. Particular approaches for obtaining a useful page ranking have been implemented by search engines such as Google (Page et al., 1998).

Due to Arrow-like impossibility results and inspiration from the page ranking setting above, we will limit ourselves to the discussion of ranking systems where agents have dichotomous preferences (see Bogomolnaia et al. (2005) for a discussion in the social choice setting). In these settings agents have only two levels of preferences: either they vote for some agent, or they do not.

There has been some previous axiomatic work on the case where the agents and alternatives coincide (Rubinstein and Kasher, 1998; Samet and Schmeidler, 1998), where the result of the system is also dichotomous, specifying a subset of the agents that is *qualified* in some sense or has some property J. Our approach differs in the fact the we generate a general ranking of the agents, and our axioms specify criteria on this ranking that could not be easily formalized in the dichotomous output setting.

Another relevant line of research is the ranking of players in tournaments (Rubinstein, 1980; Slutzki and Volij, 2005). Although the mathematical model of tournaments overlaps our model of ranking systems, the two models differ in interpretation. While in our model we consider a link from agent a to b as a *vote* from a to b that is under the full control of a, the tournament setting considers this as an indication of a win (in a sports match, or in a pairwise election) of b over a, which is of course out of the control of a.

The theory of social choice consists of two complementary axiomatic perspectives:

- The descriptive perspective: given a particular rule r for the aggregation of individual rankings into a social ranking, find a set of axioms that are sound and complete for r. That is, find a set of requirements that r satisfies; moreover, every social aggregation rule that satisfies these requirements should coincide with r. A result showing such an axiomatization is termed a *representation theorem* and it captures the exact essence of (and assumptions behind) the use of the particular rule.

- The normative perspective: devise a set of requirements that a social aggregation rule should satisfy, and try to find whether there is a social aggregation rule that satisfies these requirements.

Many efforts have been invested in the descriptive approach in the framework of the classical theory of social choice. In that setting, representation theorems have been presented to major voting rules such as the majority rule (May (1952), see Moulin (1991) for an overview).

An excellent example for the normative perspective is Arrow's impossibility theorem mentioned above. Borodin et al. (2005) have compared various known ranking systems both experimentally and by various mathematical criteria. However, they have not proven any representation theorems or impossibility results. Tennenholtz (2004) has presented some preliminary results for ranking systems where the set of voters and the set of alternatives coincide. However, the axioms presented in that work consist of several very strong requirements which naturally lead to an impossibility result.

In this thesis, we apply both these approaches to the ranking systems setting. We begin by applying the descriptive perspective and providing a representation theorem for the well-known PageRank algorithm, which is the basis of Google's search technology (Brin and Page, 1998). This theorem shows a set of five axioms which are uniquely imply an idealized version of the PageRank ranking system. This theorem is presented in Chapter 2.

In the normative perspective, we begin by defining two important properties of ranking systems: Transitivity and Ranked Independence of Irrelevant Alternatives. We prove an impossibility result for satisfying both of these properties together, but show that when the transitivity axiom is weakened, both can be satisfied by an interesting ranking system. We formally define this recursive-in-degree ranking system and provide an efficient algorithm for its computation. These results are presented in Chapter 3.

Still in the normative approach to ranking systems, we tackle the issue of incentives. We consider the case where a self-interested agent may try and manipulate its outgoing votes in order to improve its position in the ranking. We prove a full classification of the existence of incentive compatible ranking systems under four very basic axioms, each with a weak and a strong version. As this classification indicates that no reasonable ranking system can be fully incentive compatible, we expand our discussion to quantifying the level of incentive compatibility of ranking systems. In that setting we prove positive as well as negative results. These results are presented in Chapter 4.

Finally, we present a variation of ranking systems where a personalized ranking is generated for every participant in the system. We adapt the transitivity, IIA, and incentive compatibility axioms from the general ranking systems setting and prove a surprisingly positive result — a representation theorem for the systems which satisfy all of these axioms. We further show that all of the axioms are required for this proof, while relaxing any axiom leads to new personalized ranking systems. These results are presented in Chapter 5.

In Chapter 6, we provide concluding remarks for the entire thesis.

1.1 Ranking Systems

Before describing our results regarding ranking systems, we must first formally define what we mean by the words "ranking system" in terms of graphs and linear orderings:

Definition 1.1: Let A be some set. A relation $R \subseteq A \times A$ is called a *linear ordering* on A if it is reflexive, transitive, antisymmetric, and complete. Let $L(A)$ denote the set of all linear orderings on A.

Notation: Let $\preceq$ be a linear ordering, then $\simeq$ is the equality predicate of $\preceq$, and $\prec$ is the strict order induced by $\preceq$. Formally, $a \simeq b$ if and only if $a \preceq b$ and $b \preceq a$; and $a \prec b$ if and only if $a \preceq b$ but not $b \preceq a$.

Given the above we can define what a ranking system is:

Definition 1.2: Let $\mathbb{G}_V$ be the set of all directed graphs $G = (V, E)$ with no parallel edges, but possibly with self-loops. A *ranking system* F is a functional that for every finite vertex set V maps graphs $G \in \mathbb{G}_V$ to an ordering $\preceq^F_G \in L(V)$. If F is a partial function then it is called a *partial ranking system*, otherwise it is called a *general ranking system*.

One can view this setting as a variation/extension of the classical theory of social choice as modeled by Arrow (1963). The ranking systems setting differs in two main properties. First, in this setting we assume that the set of voters and the set of alternatives coincide, and second, we allow agents only two levels of preference over the alternatives, as opposed to Arrow's setting where agents could rank alternatives arbitrarily.

Chapter 2

The PageRank Axioms

2.1 Introduction

An important set of ranking systems are page ranking systems. It is well known that page ranking is fundamental for search technology, as well as for other applications. A major problem therefore is the study of the rationale of using a particular page ranking algorithm. What are the properties of a particular page ranking algorithm that characterize and differentiate it from other page ranking algorithms? In order to address this challenge we adapt the axiomatic approach, adopted in the mathematical theory of social choice, into the context of page ranking.

If we treat the Internet as a graph, where the nodes/pages are agents, and the links originating from node/page p define the preferences of the corresponding agent (i.e. a page that p links to is preferable to a page that p does not link to) then the page ranking problem becomes the problem of aggregating individual rankings into a global (social) ranking.

In this chapter we address the above challenge by introducing a representation theorem for PageRank. Needless to say that PageRank (Page et al., 1998) is the most famous page ranking procedure. In particular, PageRank is the basis for Google's search technology[1] (Brin and Page, 1998). If we treat the Internet as a strongly connected graph, where the nodes are the pages and the edges are links between pages, then PageRank can be defined as the limit probability distribution reached in a random walk on that graph. Roughly speaking, page p_1 will be ranked higher than page p_2 if the probability of reaching p_1 is greater than the probability of reaching p_2. We will show several simple properties (called axioms) one may require a page ranking algorithm to satisfy and prove that the PageRank algorithm does satisfy these axioms. Then, we prove our main result: any page ranking algorithm that does satisfy these axioms **must** coincide with PageRank!

[1] In fact, ranking based on similar ideas can be found in other contexts as well. See Pinski and Narin (1976) for the use of PageRank-like procedure in the comparison of journals' impact.

The only previous work that we are familiar with which deals with a related axiomatization is a recent work on the axiomatization of citation indexes by Palacios-Huerta and Volij (2004). This work deals however with the case of numeric inputs (e.g. the inputs are not only graphs, as in page ranking, but include also numeric measures for the number of citations by each node, and by each node for each other node), and (most importantly) the axioms considered are numeric as well (e.g. when defining the axioms we are allowed for computations such as division or matrix multiplication). Our aim is quite different: we are after ordinal, graph-theoretic requirements that will provide sound and complete axiomatization for PageRank. This creates a most significant challenge: while the PageRank algorithm is numeric and is based on the computation of eigenvectors, we are after simple graph-theoretic properties that will fully characterize the related ranking procedure.

In the next section we define some preliminaries, including the PageRank ranking system. In Section 2.3 we introduce five axioms one may require to hold for any page ranking procedure, and claim that PageRank does satisfy these axioms. In Section 2.4 we show some useful properties implied by the axioms. In Section 2.5 we use these properties for proving that any page ranking procedure that does satisfy the axioms should coincide with PageRank. Further discussion of the approach taken in this chapter is presented in Section 2.6.

2.2 Page Ranking

The current practice of the ranking of Internet pages is based on the idea of computing the limit stationary probability distribution of a random walk on the Internet graph, where the nodes are pages, and the edges are links among the pages. In order for the result of that process will be well defined, we restrict our attention to strongly connected graphs:

Definition 2.1: A directed graph $G = (V, E)$ is called *strongly connected* if for all vertices $v_1, v_2 \in V$ there exists a path from v_1 to v_2 in E.

In order to define the PageRank ranking system, we first recall the following standard definitions:

Definition 2.2: Let $G = (V, E)$ be a directed graph, and let $v \in V$ be a vertex in G. Then: The *successor set* of v is $S_G(v) = \{u|(v, u) \in E\}$, and the *predecessor set* of v is $P_G(v) = \{u|(u, v) \in E\}$.

We now define the PageRank matrix which is the matrix which captures the random walk created by the PageRank procedure. Namely, in this process we start in a random page, and iteratively move to one of the pages that are linked to by the current page, assigning equal probabilities to each such page.

Definition 2.3: Let $G = (V, E)$ be a directed graph, and assume $V =$

$\{v_1, v_2, \ldots, v_n\}$. the *PageRank Matrix* A_G (of dimension $n \times n$) is defined as:

$$[A_G]_{i,j} = \begin{cases} 1/|S_G(v_j)| & (v_j, v_i) \in E \\ 0 & \text{Otherwise.} \end{cases}$$

The PageRank procedure will rank pages according to the stationary probability distribution obtained in the limit of the above random walk; this is formally defined as follows:

Definition 2.4: Let $G = (V, E)$ be some strongly connected graph, and assume $V = \{v_1, v_2, \ldots, v_n\}$. Let $\mathbf{r}$ be the unique solution of the system $A_G \cdot \mathbf{r} = \mathbf{r}$ where $r_1 = 1$. The *PageRank* $PR_G(v_i)$ of a vertex $v_i \in V$ is defined as $PR_G(v_i) = r_i$. The *PageRank ranking system* is a ranking system that for the vertex set V maps G to $\preceq_G^{PR}$, where $\preceq_G^{PR}$ is defined as: for all $v_i, v_j \in V$: $v_i \preceq_G^{PR} v_j$ if and only if $PR_G(v_i) \leq PR_G(v_j)$.

The above defines a powerful heuristic for the ranking of Internet pages, as adopted by search engines (Page et al., 1998). This is however a particular numeric procedure, and our aim is to treat it from an axiomatic social choice perspective, providing graph-theoretic, ordinal representation theorem for PageRank.

2.3 The Axioms

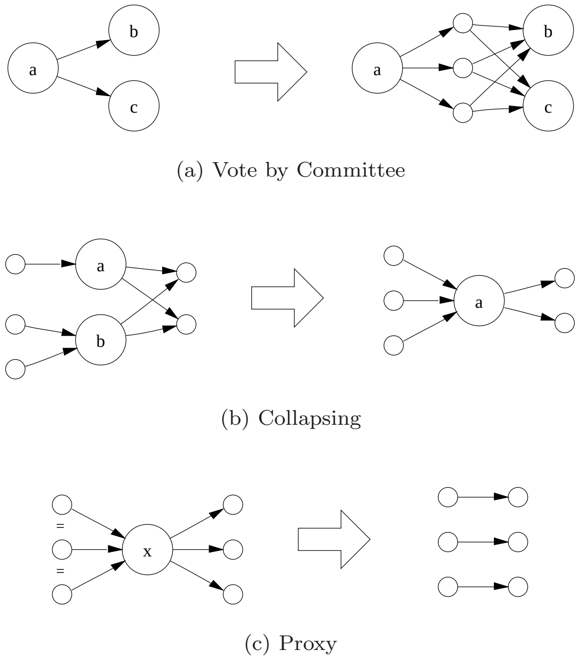
From the perspective of the theory of social choice, each page in the Internet graph is viewed as an agent, where this agent prefers the pages (i.e. agents) it links to upon pages it does not link to. The problem of finding a social aggregation rule will become therefore the problem of page ranking. The idea is to search for simple axioms, i.e. requirements we wish the page ranking system to satisfy. Most of these requirements will have the following structure: page a is preferable to page b when the graph is G if and only if a is preferable to b when the graph is G'. Our aim is to search for a small set of axioms that can be shown to be satisfied by PageRank. The axioms need to be simple graph-theoretic, ordinal properties, which do not refer to numeric computations.

In explaining some of the axioms we will refer to Figure 2.1. For simplicity, while the axioms are stated as "if and only if" statements, we will sometimes emphasize in the intuitive explanation of an axiom only one of the directions (in all cases similar intuitions hold for the other direction).

The first axiom is straightforward:

Axiom 2.5: (Isomorphism) A ranking system F satisfies *isomorphism* if for every isomorphism function $\varphi : V_1 \mapsto V_2$, and two isomorphic graphs $G \in \mathbb{G}_{V_1}, \varphi(G) \in \mathbb{G}_{V_2}$: $\preceq_{\varphi(G)}^F = \varphi(\preceq_G^F)$.

The isomorphism axiom tells us that the ranking procedure should be independent of the names we choose for the vertices.



(a) Vote by Committee

(b) Collapsing

(c) Proxy

Figure 2.1: Sketch of several axioms

The second axiom is also quite intuitive. It tells us that if a is ranked at least as high as b if the graph is G, where in G a does not link to itself, then a should be ranked higher than b if all that we add to G is a link from a to itself. Moreover, the relative ranking of other vertices in the new graph should remain as before. Formally, we have the following notation and axiom:[2]

Notation: Let $G = (V, E) \in \mathbb{G}_V$ be a graph s.t. $(v, v) \notin E$. Let $G' = (V, E \cup \{(v, v)\})$. Let us denote $\mathbf{SelfEdge}(G, v) = G'$ and $\mathbf{SelfEdge}^{-1}(G', v) = G$. Note that $\mathbf{SelfEdge}^{-1}(G', v)$ is well defined.

Axiom 2.6: (Self edge) Let F be a ranking system. F satisfies the *self edge* axiom if for every vertex set V and for every vertex $v \in V$ and for every graph $G = (V, E) \in \mathbb{G}_V$ s.t. $(v, v) \notin E$, and for every $v_1, v_2 \in V \setminus \{v\}$: Let $G' = \mathbf{SelfEdge}(G, v)$. If $v_1 \preceq^F_G v$ then $v \not\preceq^F_{G'} v_1$; and $v_1 \preceq^F_G v_2$ iff $v_1 \preceq^F_{G'} v_2$.

The following third axiom (titled *Vote by committee*) captures the following idea, which is illustrated in Figure 2.1(a). If page a links to pages b and c, then

[2]One may claim that this axiom makes no sense if we do not allow self loops. This is however only a simple technical issue. If we do not allow self loops then the axiom should be replaced by a new one, where the addition of self-loop to a is replaced by the addition of a new page, a', where a links to a' and where a' links only to a. Our results will remain similar.

the relative ranking of all pages should be the same as in the case where the direct links from a to b and c are replaced by links from a to a new set of pages, which link (only) to b and c. The idea here is that the amount of importance a provides to b and c by linking to them, should not change due to the fact that a assigns its power through a committee of (new) representatives, all of which behave as a. More generally, and more formally, we have the following:

Axiom 2.7: (Vote by committee) Let F be a ranking system. F satisfies *vote by committee* if for every vertex set V, for every vertex $v \in V$, for every graph $G = (V, E) \in \mathbb{G}_V$, for every $v_1, v_2 \in V$, and for every $m \in \mathbb{N}$: Let $G' = (V \cup \{u_1, u_2, \ldots, u_m\}, E \setminus \{(v, x)|x \in S_G(v)\} \cup \{(v, u_i)|i = 1, \ldots, m\} \cup \{(u_i, x)|x \in S_G(v), i = 1, \ldots, m\})$, where $\{u_1, u_2, \ldots, u_m\} \cap V = \emptyset$. Then, $v_1 \preceq^F_G v_2$ iff $v_1 \preceq^F_{G'} v_2$.

The 4th axiom, termed *collapsing* is illustrated in Figure 2.1(b). The idea of this axiom is that if there is a pair of pages, say a and b, where both a and b link to the same set of pages, but the sets of pages that link to a and b are disjoint, then if we collapse a and b into a singleton, say a, where all links to b become now links to a, then the relative ranking of all pages, excluding a and b of course, should remain as before. The intuition here is that if there are two voters (i.e. pages), a and b, who vote similarly (i.e. have the same outgoing links), and the power of each one of them stems from the fact a set of other voters have voted for him, where the sets of voters for a and for b are disjoint, then if all voters for a and b would vote only for a (dropping b) then a should provide the same importance to other agents as a and b did together. This of course relies on having a and b voting for the same individuals. As a result, the following axiom is quite intuitive:

Axiom 2.8: (collapsing) Let F be a ranking system. F satisfies *collapsing* if for every vertex set V, for every $v, v' \in V$, for every $v_1, v_2 \in V \setminus \{v, v'\}$, and for every graph $G = (V, E) \in \mathbb{G}_V$ for which $S_G(v) = S_G(v')$, $P_G(v) \cap P_G(v') = \emptyset$, and $[P_G(v) \cup P_G(v')] \cap \{v, v'\} = \emptyset$: Let $G' = (V \setminus \{v'\}, E \setminus \{(v', x)|x \in S_G(v')\} \setminus \{(x, v')|x \in P_G(v')\} \cup \{(x, v)|x \in P_G(v')\})$. Then, $v_1 \preceq^F_G v_2$ iff $v_1 \preceq^F_{G'} v_2$.

The last axiom we introduce, termed the *proxy* axiom, is illustrated in Figure 2.1(c). Roughly speaking, this axiom tells us that if there is a set of k pages, all having the same importance, which link to a, where a itself links to k pages, then if we drop a and connect directly, and in a 1-1 fashion, the pages which linked to a to the pages that a linked to, then the relative ranking of all pages (excluding a) should remain the same. This axiom captures equal distribution of importance. The importance of a is received from k pages, all with the same power, and is split among k pages; alternatively, the pages that link to a could pass directly the importance to pages that a link to, without using a as a proxy for distribution. More formally, and more generally, we have the following:

Axiom 2.9: (proxy) Let F be a ranking system. F satisfies *proxy* if for every vertex set V, for every vertex $v \in V$, for every $v_1, v_2 \in V \setminus \{v\}$, and for every graph $G = (V, E) \in \mathbb{G}_V$ for which $|P_G(v)| = |S_G(v)|$, for all $p \in P_G(v)$: $S_G(p) = \{v\}$, and for all $p, p' \in P_G(v)$: $p \simeq_G^F p'$: Assume $P_G(v) = \{p_1, p_2, \ldots, p_m\}$ and $S_G(v) = \{s_1, s_2, \ldots, s_m\}$. Let $G' = (V \setminus \{v\}, E \setminus \{(x, v), (v, x) | x \in V\} \cup \{(p_i, s_i) | i \in \{1, \ldots, m\}\})$. Then, $v_1 \preceq_G^F v_2$ iff $v_1 \preceq_{G'}^F v_2$.

2.3.1 Soundness

Although we have provided some intuitive explanation for the axioms, one may argue that particular axiom(s) are not that reasonable. As it turns out however, all the above axioms are satisfied by the PageRank procedure. The proof of the basic soundness proposition is provided below. In Section 2.5 we show that the above axioms are not only satisfied by PageRank, but also completely and uniquely characterize the PageRank procedure.

Proposition 2.1: The PageRank ranking system PR satisfies isomorphism, self edge, vote by committee, collapsing, and proxy.

Proof: The isomorphism axiom is satisfied directly from the definition by the assumption that $V = \{v_1, v_2, \ldots, v_n\}$.

For the vote by committee axiom, let $V = \{v_1, v_2, \ldots, v_n\}$ be a vertex set, let $G = (V, E) \in \mathbb{G}_V$ be a graph, and let $v_s, v_t \in V$ be vertices and let $m \in \mathbb{N}$ be a natural number. Assume $v_s \preceq_G^{PR} v_t$.

Let $G' = (V \cup \{v_{n+1}, v_{n+2}, \ldots, v_{n+m}\}, E \setminus \{(v_1, x) | x \in S_G(v_1)\} \cup \{(v_1, v_{n+j}) | j = 1, \ldots, m\} \cup \{(v_{n+j}, x) | x \in S_G(v_1), j = 1, \ldots, m\})$. Let $\mathbf{r}$ be the solution of $A_G \cdot \mathbf{r} = \mathbf{r}$, where $r_1 = 1$. Let $\mathbf{r}'$ be the following vector:

$$\mathbf{r}' = \begin{pmatrix} r_1 \\ \vdots \\ r_n \\ r_1/m \\ \vdots \\ r_1/m \end{pmatrix}$$

We will now prove that $A_{G'}\mathbf{r}' = \mathbf{r}'$. Note that by definition of G', the matrix $A_{G'}$ is

$$A_{G'} = \begin{pmatrix} 0 & a_{1,2} & \cdots & a_{1,n} & a_{1,1} & \cdots & a_{1,1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n,2} & \cdots & a_{n,n} & a_{n,1} & \cdots & a_{n,1} \\ 1/m & & & & & & \\ \vdots & & & 0 & & & \\ 1/m & & & & & & \end{pmatrix}$$

If we multiply, we get: for $i \in \{1, \ldots n\}$:

$$[A_{G'} r']_i \quad = \quad \sum_{j=2}^{n} a_{i,j} r_j + m a_{i,1} \cdot r_1/m = \sum_{j=1}^{n} a_{i,j} r_j = r_i,$$

and for $i \in \{n+1, \ldots n+m\}$, $[A_{G'}r']_i = 1/m \cdot r_1$, as required. Also $r'_1 = r_1 = 1$, so $PR_{G'}(v_j) = r'_j$ for all $j \in \{1, \ldots, n+m\}$. Now, $PR_{G'}(v_s) = r'_s = r_s = PR_G(v_s) \leq PR_G(v_t) = r_t = r'_t = PR_{G'}(v_t)$, as required.

For the collapsing axiom, let $V = \{v_1, v_2, \ldots, v_n\}$, and let $G = (V, E) \in \mathbb{G}_V$. Assume $S(v_n) = S(v_{n-1})$ and $P(v_n) \cap P(v_{n-1}) = \emptyset$. Let $v_k, v_l \in V$ be vertices $(k, l < n-1)$. Assume $v_k \preceq_G^{PR} v_l$.

Let $G' = (V \setminus \{v_n\}, E \setminus \{(v_n, x) | x \in S_G(v_n)\} \setminus \{(x, v_n) | x \in P_G(v_n)\} \cup \{(x, v_{n-1}) | x \in P_G(v_n)\})$. Let $\mathbf{r}$ be the solution of $A_G \cdot \mathbf{r} = \mathbf{r}$, where $r_1 = 1$. Let $\mathbf{r}'$ be the following vector:

$$\mathbf{r}' = \begin{pmatrix} r_1 \\ \vdots \\ r_{n-2} \\ r_{n-1} + r_n \end{pmatrix}$$

We will now prove that $A_{G'}\mathbf{r}' = \mathbf{r}'$. Note that by definition of G', the matrix $A_{G'}$ is

$$A_{G'} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-2,1} & a_{n-2,2} & \cdots & a_{n-2,n-1} \\ a_{n-1,1} + a_{n,1} & a_{n-1,2} + a_{n,2} & \cdots & 0 \end{pmatrix}$$

If we multiply, we get for $i \in \{1, \ldots n-2\}$:

$$[A_{G'}r']_i \quad = \quad a_{i,n-1}(r_n + r_{n-1}) + \sum_{j=1}^{n-2} a_{i,j} r_j = a_{i,n-1} r_n + a_{i,n-1} r_{n-1} + \sum_{j=1}^{n-2} a_{i,j} r_j$$

Note that $a_{i,n} = a_{i,n-1} = \frac{1}{|S(v_n)|}$, so

$$\begin{aligned} [A_{G'}r']_i &= \sum_{j=1}^{n-2} a_{i,j} r_j + a_{i,n-1} r_{n-1} + a_{i,n} r_n = \sum_{j=1}^{n} a_{i,j} r_j = r_i. \\ [A_{G'}r']_{n-1} &= \sum_{j=1}^{n-2} (a_{n-1,j} + a_{n,j}) r_j = \sum_{j=1}^{n-2} a_{n-1,j} r_j + \sum_{j=1}^{n-2} a_{n,j} r_j \end{aligned}$$

Note that $a_{n-1,n-1} = a_{n-1,n} = a_{n,n-1} = a_{n,n} = 0$, so

$$[A_{G'}r']_{n-1} \quad = \quad \sum_{j=1}^{n} a_{n-1,j} r_j + \sum_{j=1}^{n} a_{n,j} r_j = r_{n-1} + r_n$$

So, we get $A_{G'}\mathbf{r}' = \mathbf{r}'$ as required. Also $r'_1 = r_1 = 1$, so $PR_{G'}(v_j) = r'_j$ for all $j \in \{1, \ldots, n-1\}$. Now, $PR_{G'}(v_k) = r'_k = r_k = PR_G(v_k) \leq PR_G(v_l) = r_l = r'_l = PR_{G'}(v_l)$, as required.

For the proxy axiom, let $V = \{v_1, v_2, \ldots, v_n\}$, and let $G = (V, E) \in \mathbb{G}_V$. Assume $P(v_n) = \{v_1, v_2, \ldots, v_m\}$, $v_1 \simeq v_2 \simeq \cdots \simeq v_m$, and $S(v_n) = \{v_{t+1}, v_{t+2}, \ldots, v_{t+m}\}$, where $t \in \{0, \ldots, m\}$. Let $v_k, v_l \in V$ be vertices $(k, l < n)$. Assume $v_k \preceq_G^{PR} v_l$.

Let $G' = (V \setminus \{v_n\}, E \setminus \{(x, v_n), (v_n, x) | x \in V\} \cup \{(v_i, v_{t+i}) | i \in \{1, \ldots, m\}\})$. Let $\mathbf{r}$ be the solution of $A_G \cdot \mathbf{r} = \mathbf{r}$, where $r_1 = 1$. Since $v_1 \simeq v_2 \simeq \cdots \simeq v_m$, we have $r_1 = r_2 = \cdots = r_m$, and note that because $P_G(v_n) = \{v_1, v_2, \ldots, v_m\}$ and $S(v_i) = \{v_n\}$ for all $i \in \{1, \ldots, m\}$:

$$r_n = \sum_{i=1}^{n} a_{n,i} r_i = r_1 + r_2 + \cdots + r_m = m r_1 = m.$$

Let $\mathbf{r}' = \mathbf{r}_{-n}$. By definition of G', the matrix $A_{G'}$ is

$$A_{G'} = \begin{pmatrix}
0 & 0 & \cdots & 0 & a_{1,m+1} & a_{1,m+2} & \cdots & a_{1,n-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & a_{t,m+1} & a_{t,m+2} & \cdots & a_{t,n-1} \\
1 & 0 & \cdots & 0 & a_{t+1,m+1} & a_{t+1,m+2} & \cdots & a_{t+1,n-1} \\
0 & 1 & \cdots & 0 & a_{t+2,m+1} & a_{t+2,m+2} & \cdots & a_{t+2,n-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & a_{t+m,m+1} & a_{t+m,m+2} & \cdots & a_{t+m,n-1} \\
0 & 0 & \cdots & 0 & a_{t+m+1,m+1} & a_{t+m+1,m+2} & \cdots & a_{t+m+1,n-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & a_{n-1,m+1} & a_{n-1,m+2} & \cdots & a_{n-1,n-1}
\end{pmatrix}$$

We multiply can now multiply, and since $a_{i,n} = 0$ for all $i \in \{1, \ldots t, t+m+1, \ldots, n-1\}$ (because $S(v_n) = \{t+1, \ldots, t+m\}$) and $a_{i,j} = 0$ for all $i \in \{1, \ldots, n-1\}$ and $j \in \{1, \ldots, m\}$ (because $S(v_j) = \{v_n\}$), we get for $i \in \{1, \ldots t, t+m+1, \ldots, n-1\}$:

$$[A_{G'} r']_i = \sum_{j=m+1}^{n-1} a_{i,j} r_j = \sum_{j=1}^{n} a_{i,j} r_j = r_i$$

and for $i \in \{t+1, \ldots, t+m\}$:

$$\begin{aligned}
[A_{G'} r']_i &= \sum_{j=m+1}^{n-1} a_{i,j} r_j + r_{i-t} = \sum_{j=1}^{n-1} a_{i,j} r_j + 1 = \sum_{j=1}^{n-1} a_{i,j} r_j + \frac{1}{m} r_n = \\
&= \sum_{j=1}^{n-1} a_{i,j} r_j + a_{i,n} r_n = \sum_{j=1}^{n} a_{i,j} r_j = r_i
\end{aligned}$$

So, we get $A_{G'} \mathbf{r}' = r'$ as required. Also $r'_1 = r_1 = 1$, so $PR_{G'}(v_j) = r'_j$ for all $j \in \{1, \ldots, n-1\}$. Now, $PR_{G'}(v_k) = r'_k = r_k = PR_G(v_k) \leq PR_G(v_l) = r_l = r'_l = PR_{G'}(v_l)$, as required.

For the self edge axiom, let $V = \{v_1, v_2, \ldots, v_n\}$, and let $G = (V, E) \in \mathbb{G}_V$. Assume $(v_1, v_1) \notin E$. Let $\mathbf{r}$ be the solution of $A_G \cdot \mathbf{r} = \mathbf{r}$, where $r_1 = 1$. Let $G' = (V, E \cup \{(v_1, v_1)\})$ and let $m = |S_G(v_1)|$. Let $\mathbf{r}'$ be the following vector:

$$\mathbf{r}' = \begin{pmatrix} r_1 \\ \frac{m}{m+1} r_2 \\ \vdots \\ \frac{m}{m+1} r_n \end{pmatrix}$$

We will now prove that $A_{G'}\mathbf{r}' = \mathbf{r}'$. Note that by definition of G', the matrix $A_{G'}$ is

$$A_{G'} = \begin{pmatrix} \frac{1}{m+1} & a_{1,2} & \cdots & a_{1,n} \\ \frac{m}{m+1} a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{m}{m+1} a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix}$$

If we multiply, we get: for $i \in \{2, \ldots n\}$:

$$\begin{aligned} [A_{G'}r']_1 &= \frac{1}{m+1} r_1 + \sum_{j=2}^{n} a_{1,j} \frac{m}{m+1} r_j = \frac{1}{m+1} r_1 + \frac{m}{m+1} \sum_{j=2}^{n} a_{1,j} r_j = \\ &= \frac{1}{m+1} r_1 + \frac{m}{m+1} \sum_{j=1}^{n} a_{1,j} r_j = \frac{1}{m+1} r_1 + \frac{m}{m+1} r_1 = r_1 \\ [A_{G'}r']_i &= \frac{m}{m+1} a_{i,1} r_1 + \sum_{j=2}^{n} a_{i,j} \frac{m}{m+1} r_j = \frac{m}{m+1} \sum_{j=1}^{n} a_{i,j} r_j = \frac{m}{m+1} r_i \end{aligned}$$

So, we get $A_{G'}\mathbf{r}' = \mathbf{r}'$ as required. Also $r'_1 = r_1 = 1$, so $PR_{G'}(v_j) = r'_j$ for all $j \in \{1, \ldots, n-1\}$.

Assume $v_2 \preceq_G^{PR} v_1$. Then, $PR_{G'}(v_2) = r'_2 < r_2 = PR_G(v_2) \leq PR_G(v_1) = r_1 = r'_1 = PR_{G'}(v_1)$, as required.

Now assume $v_2 \preceq_G^{PR} v_3$. Then, $PR_{G'}(v_2) = r'_2 = r_2 = PR_G(v_2) \leq PR_G(v_3) = r_3 = r'_3 = PR_{G'}(v_3)$, as required. $\square$

2.4 Several Useful Properties

In this section we prove three technical properties which are implied by our axioms. As a result, these three properties are satisfied by the PageRank ranking system. The purpose of presenting them is rather technical: they will be used in the next section, when we show that the PageRank ranking system is the only one that satisfies our axioms.

Notation: Let V be a vertex set and let $v \in V$ be a vertex. Let $G = (V, E) \in \mathbb{G}_V$ be a graph where $S(v) = \{s\}$, $P(v) = \{p\}$, and $(s, p) \notin E$. We will use $\mathbf{Del}(G, v)$ to denote the graph $G' = (V', E')$ defined by:

$$\begin{aligned} V' &= V \setminus \{v\} \\ E' &= E \setminus \{(p, v), (v, s)\} \cup \{(p, s)\}. \end{aligned}$$

The $\mathbf{Del}(\cdot,\cdot)$ operator simply removes a vertex from the graph that has an in-degree and out-degree of 1, replacing it by an edge from its predecessor to its successor. The following lemma says that when our axioms are satisfied then this operator does not change the relative ranking of all (remaining) pages.

Definition 2.10: Let F be a ranking system. F has the *weak deletion* property if for every vertex set V, for every vertex $v \in V$ and for all vertices $v_1, v_2 \in V \setminus \{v\}$, and for every graph $G = (V, E) \in \mathbb{G}_V$ s.t. $S(v) = \{s\}$, $P(v) = \{p\}$, and $(s, p) \notin E$: Let $G' = \mathbf{Del}(G, v)$. Then, $v_1 \preceq^F_G v_2$ iff $v_1 \preceq^F_{G'} v_2$.

Lemma 2.2: Let F be a ranking system that satisfies isomorphism, vote by committee and proxy. Then, F has the weak deletion property.

Proof: Let V be a vertex set, let $v \in V; v_1, v_2 \in V \setminus \{v\}$ be vertices and let $G = (V, E) \in \mathbb{G}_V$ be a graph s.t. $S(v) = \{s\}$, $P(v) = \{p\}$, and $(s, p) \notin E$. Assume $v_1 \preceq^F_G v_2$. Let $s_0 = v$ and $S(p) = \{s_0, s_1, s_2, \ldots, s_m\}$.

- Let $G_1 = (V_1, E_1)$, where

$$\begin{aligned} V_1 &= V \cup \{p'\} \\ E_1 &= E \setminus \{(p, s_i) | i = 0, \ldots, m\} \cup \{p, p'\} \cup \\ &\quad \cup \{(p', s_i) | i = 0, \ldots, m\}. \end{aligned}$$

  By the vote by committee axiom with parameter 1, $v_1 \preceq^F_{G_1} v_2$.

- Let $G_2 = (V_2, E_2)$, where

$$\begin{aligned} V_2 &= V_1 \cup \{u_i | i = 0, \ldots, m\} \\ E_2 &= E_1 \setminus \{(p, p')\} \cup \\ &\quad \cup \{(p, u_i), (u_i, p') | i = 0, \ldots, m\}. \end{aligned}$$

  By the vote by committee axiom with parameter $m + 1$, $v_1 \preceq^F_{G_2} v_2$.

- Let $G_3 = (V_3, E_3)$, where

$$\begin{aligned} V_3 &= V_2 \setminus \{p'\} \\ E_3 &= E_2 \setminus \{(u_i, p'), (p', s_i) | i = 0, \ldots, m\}. \\ &\quad \cup \{(u_i, s_i) | i = 0, \ldots, m\}. \end{aligned}$$

  By the isomorphism axiom, $u_i \simeq_{G_2} u_j$ for all $i, j \in \{0, \ldots, m\}$. By the proxy axiom, $v_1 \preceq^F_{G_3} v_2$.

- Let $G_4 = (V_4, E_4)$, where

$$\begin{aligned} V_4 &= V_3 \setminus \{v\} \\ E_4 &= E_3 \setminus \{(u_0, v), (v, s)\} \cup \{(u_0, s)\}. \end{aligned}$$

  By the vote by committee axiom with parameter 1, $v_1 \preceq^F_{G_4} v_2$.

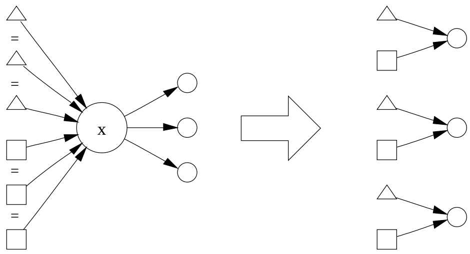


Figure 2.2: Sketch of $\mathbf{Delete}(G, x)$

- Let $G' = \mathbf{Del}(G, v)$. By the vote by committee, isomorphism, and proxy axioms, as between G and G_3 above, $v_1 \preceq^F_{G'} v_2 \Leftrightarrow v_1 \preceq^F_{G_4} v_2$. Thus, $v_1 \preceq^F_{G'} v_2$ as required.

$\square$

We now move to a second deletion property satisfied by the axioms.
Notation: Let V be a vertex set and let $v \in V$ be a vertex. Let $G = (V, E) \in \mathbb{G}_V$ be a graph where $S(v) = \{s_1, s_2, \ldots, s_t\}$ and $P(v) = \{p^i_j | j = 1, \ldots, t; i = 0, \ldots, m\}$, and $S(p^i_j) = \{v\}$ for all $j \in \{1, \ldots t\}$ and $i \in \{0, \ldots, m\}$. We will use $\mathbf{Delete}(G, v, \{(s_1, \{p^i_1 | i = 0, \ldots m\}), \ldots, (s_t, \{p^i_t | i = 0, \ldots m\})\})$ to denote the graph $G' = (V', E')$ defined by:

$$\begin{aligned} V' &= V \setminus \{v\} \\ E' &= E \setminus \{(p^i_j, v), (v, s_j) | i = 0, \ldots, m; j = 1, \ldots, t\} \cup \\ &\quad \cup \{(p^i_j, s_j) | i = 0, \ldots, m; j = 1, \ldots, t\}. \end{aligned}$$

When the grouping of the predecessors is trivial or understood from context, we will sloppily use $\mathbf{Delete}(G, v)$.

A sketch of the Delete operator can be found in Figure 2.2. In this figure we see that node x which links to three other nodes, and has two sets of three predecessors, where the nodes in each such set are of the same importance. The Delete operator will drop x and connect exactly one element from each of the predecessor sets to exactly one node in the successor set. The following lemma says that when our axioms are satisfied then this operator does not change the relative ranking of all (remaining) pages.

Definition 2.11: Let F be a ranking system. F has the *strong deletion* property if for every vertex set V, for every vertex $v \in V$, for all $v_1, v_2 \in V \setminus \{v\}$, and for every graph $G = (V, E) \in \mathbb{G}_V$ s.t. $S(v) = \{s_1, s_2, \ldots, s_t\}$, $P(v) = \{p^i_j | j = 1, \ldots, t; i = 0, \ldots, m\}$, $S(p^i_j) = \{v\}$ for all $j \in \{1, \ldots t\}$ and $i \in \{0, \ldots, m\}$, and $p^i_j \simeq^F_G p^i_k$ for all $i \in \{0, \ldots, m\}$ and $j, k \in \{1, \ldots t\}$: Let $G' = \mathbf{Delete}(G, v, \{(s_1, \{p^i_1 | i = 0, \ldots m\}), \ldots (s_t, \{p^i_t | i = 0, \ldots m\})\})$. Then, $v_1 \preceq^F_G v_2$ iff $v_1 \preceq^F_{G'} v_2$.

Lemma 2.3: Let F be a ranking system that satisfies collapsing and proxy. Then, F has the strong deletion property.

Proof: Let V be a vertex set, let $v \in V; v_1, v_2 \in V \setminus \{v\}$ be vertices and let $G = (V, E) \in \mathbb{G}_V$ be a graph s.t. $S(v) = \{s_1, s_2, \dots, s_t\}$, $P(v) = \{p_j^i | j = 1, \dots, t; i = 0, \dots, m\}$, $S(p_j^i) = \{v\}$ for all $j \in \{1, \dots t\}$ and $i \in \{0, \dots, m\}$, and $p_j^i = p_j^k$ for all $j \in \{1, \dots t\}$ and $i, k \in \{0, \dots, m\}$. Assume $v_1 \preceq_G^F v_2$. Denote $u^0 = v$.

- Let $G_1 = (V_1, E_1)$, where

$$\begin{aligned} V_1 &= V \cup \{u^i | i = 1, \dots, m\} \\ E_1 &= E \setminus \{(p_j^i, v) | i = 1, \dots, m; j = 1, \dots, t\} \cup \\ &\quad \cup \{(p_j^i, u^i), (u^i, s_j) | i = 1, \dots, m; j = 1, \dots, t\} \end{aligned}$$

  By the collapsing axiom applied in the reverse direction a total of m times for $\{(u^{i-1}, u^i) | i = 1, \dots, m\}$, $v_1 \preceq_{G_1}^F v_2$.

- Let $G_2 = (V_2, E_2)$, where

$$\begin{aligned} V_2 &= V_1 \setminus \{u^i | i = 0, \dots, m\} \\ E_2 &= E_1 \setminus \{(p_j^i, u^i), (u^i, s_j) | i = 0, \dots, m; j = 1, \dots, t\} \cup \\ &\quad \cup \{(p_j^i, s_j) | i = 0, \dots, m; j = 1, \dots, t\}. \end{aligned}$$

  By the proxy axiom applied a total of $m + 1$ times for $\{u^i | i = 0, \dots, m\}$, $v_1 \preceq_{G_2}^F v_2$.

Note that G_2 is exactly $G' = \mathbf{Delete}(G, v, \{(s_1, \{p_1^i | i = 0, \dots m\}), \dots (s_t, \{p_t^i | i = 0, \dots m\})\})$, so $v_1 \preceq_{G'}^F v_2$ as required.

$\square$

We conclude with a third property which is also satisfied by the axioms.

Notation: Let V be a vertex set and let $G = (V, E) \in \mathbb{G}_V$ be a graph. Let $S(v) = \{s_1^0, s_2^0, \dots, s_t^0\}$. We will use $\mathbf{Duplicate}(G, v, m)$ to denote the graph $G' = (V', E')$ defined by:

$$\begin{aligned} V' &= V \cup \{s_j^i | i = 1, \dots, m-1; j = 1, \dots t\} \\ E' &= E \cup \{(v, s_j^i) | i = 1, \dots, m-1; j = 1, \dots t\} \cup \\ &\quad \cup \{(s_j^i, u) | i = 1, \dots, m-1; j = 1, \dots t; u \in S_G(s_j^0)\}. \end{aligned}$$

A sketch of the Duplicate operator can be found in Figure 2.3. In this figure we see that a links to two nodes, each of which has its own successor set. Then, each node in the successor set of a is duplicated by a factor of three, i.e. for each node a' in the successor set of a we add two new nodes to the successor set of a, each of which with the same successor set as a'. The following lemma

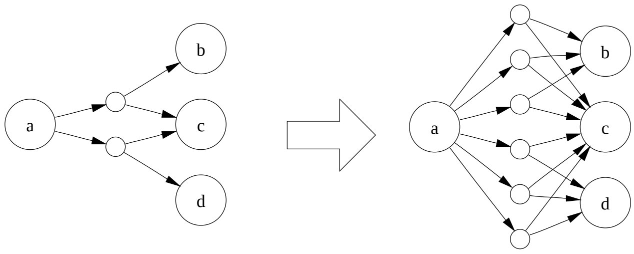


Figure 2.3: Sketch of $\textbf{Duplicate}(G, a, 3)$

says that when our axioms are satisfied then this operator does not change the relative ranking of the pages, excluding the ones which have been duplicated. The proof appears in the Appendix.

Definition 2.12: Let F be a ranking system. F has the *edge duplication* property if for every vertex set V, for all vertices $v, v_1, v_2 \in V$, for every $m \in \mathbb{N}$, and for every graph $G = (V, E) \in \mathbb{G}_V$: Let $S(v) = \{s_1^0, s_2^0, \ldots, s_t^0\}$, and let $G' = \textbf{Duplicate}(G, v, m)$. Then, $v_1 \preceq_G^F v_2$ iff $v_1 \preceq_{G'}^F v_2$.

Lemma 2.4: Let F be a ranking system that satisfies isomorphism, vote by committee, collapsing, and proxy. Then, F has the edge duplication property.

Proof: Let V be a vertex set, let $v, v_1, v_2 \in V$ be vertices, and let $m' \in \mathbb{N}$ be a natural number. Assume $m' > 1$ (otherwise $G' = G$), and let $m = m' - 1$. Let $G = (V, E) \in \mathbb{G}_V$ be a graph. Assume $v_1 \preceq_G^F v_2$, and let $S(v) = \{s_1^0, s_2^0, \ldots, s_t^0\}$.

- Let $G_1 = (V_1, E_1)$, where

$$\begin{aligned} V_1 &= V \cup \{u_j^i | i = 0, \ldots, m; j = 1, \ldots t\} \\ E_1 &= E \setminus \{(v, x) | x \in S_G(v)\} \cup \{(v, u_j^i) | i = 0, \ldots, m; j = 1, \ldots t\} \cup \\ &\quad \cup \{(u_j^i, x) | x \in S_G(v), i = 0, \ldots, m; j = 1, \ldots t\}. \end{aligned}$$

  By the vote by committee axiom with parameter $(m+1)t$, $v_1 \preceq_{G_1}^F v_2$.

- Let $G_2 = (V_2, E_2)$, where

$$\begin{aligned} V_2 &= V_1 \cup \{w_j^i | i = 0, \ldots, m; j = 1, \ldots t\} \\ E_2 &= E_1 \setminus \{(v, u_j^i) | i = 0, \ldots, m; j = 1, \ldots t\} \cup \\ &\quad \cup \{(v, w_j^i), (w_j^i, u_j^i) | i = 0, \ldots, m; j = 1, \ldots t\} \end{aligned}$$

  By the vote by committee axiom (applied $(m-1)t$ times) with parameter 1, $v_1 \preceq_{G_2}^F v_2$.

- Let $G_3 = (V_3, E_3)$, where

$$\begin{array}{rcl} V_3 & = & V_2 \setminus \{u^i_j | i = 0, \ldots, m; j = 2, \ldots, t\} \\ E_3 & = & E_2 \setminus \{(u^i_j, x | x \in S_G(v); i = 0, \ldots, m; j = 2, \ldots, t\} \setminus \\ & & \setminus \{(w^i_j, u^i_j) | i = 0, \ldots, m; j = 2, \ldots, t\} \cup \\ & & \cup \{(w^i_j, u^i_1) | i = 0, \ldots, m; j = 2, \ldots, t\}. \end{array}$$

By the collapsing axiom applied a total of $(m+1)(t-1)$ times for $\{(u^i_{j-1}, u^i_j) | j = 2, \ldots, t; i = 0, \ldots m\}$, $v_1 \preceq^F_{G_3} v_2$.

- Let $G_4 = (V_4, E_4)$, where

$$\begin{array}{rcl} V_4 & = & V_3 \setminus \{u^i_1 | i = 0, \ldots, m\} \\ E_4 & = & E_3 \setminus \{(u^i_1, x) | i = 0, \ldots, m; x \in S_G(v)\} \setminus \\ & & \setminus \{(w^i_j, u^i_1) | i = 0, \ldots, m; j = 1, \ldots, t\} \cup \\ & & \cup \{(w^i_j, s^0_j) | i = 0, \ldots, m; j = 1, \ldots, t\}. \end{array}$$

By the isomorphism axiom, $w^i_j \simeq w^i_k$ for all $i \in \{0, \ldots, m\}$ and $j, k \in \{1, \ldots, t\}$. By the proxy axiom (applied a total of $m+1$ times for $\{u^i_1 | i = 0, \ldots m\}$), $v_1 \preceq^F_{G_4} v_2$.

- Let $G_5 = (V_5, E_5)$, where

$$\begin{array}{rcl} V_5 & = & V_4 \cup \{s^i_j | i = 1, \ldots, m; j = 1, \ldots, t\} \\ E_5 & = & E_4 \setminus \{(w^i_j, s^0_j) | i = 1, \ldots, m; j = 1, \ldots, t\} \cup \\ & & \cup \{(w^i_j, s^i_j) | i = 1, \ldots, m; j = 1, \ldots, t\} \cup \\ & & \cup \{(s^i_j, x) | x \in S(s^0_j); i = 1, \ldots, m\}. \end{array}$$

By the collapsing axiom applied in the reverse direction a total of $m \cdot t$ times for $\{(s^{i-1}_j, s^i_j) | i = 1, \ldots, m; j = 1, \ldots, t\}$, $v_1 \preceq^F_{G_5} v_2$.

- Let $G_6 = (V_6, E_6)$, where

$$\begin{array}{rcl} V_6 & = & V_5 \setminus \{w^i_j | i = 0, \ldots, m; j = 1, \ldots t\} \\ E_6 & = & E_5 \setminus \{(v, w^i_j), (w^i_j, s^i_j) | i = 0, \ldots, m; j = 1, \ldots t\} \cup \\ & & \cup \{(v, s^i_j) | i = 0, \ldots, m; j = 1, \ldots t\}. \end{array}$$

By the vote by committee axiom applied in the reverse direction a total of $(m+1) \cdot t$ times for $\{w^i_j | i = 0, \ldots, m; j = 1, \ldots t\}$, $v_1 \preceq^F_{G_6} v_2$.

Note that G_6 is exactly $\mathbf{Duplicate}(G, v, m+1) = \mathbf{Duplicate}(G, v, m') = G'$, so $v_1 \preceq^F_{G'} v_2$ as required.

$\square$

2.5 Completeness

We are now ready to show that that our axioms fully characterize the PageRank ranking system. We can prove:

Theorem 2.5: A ranking system F satisfies isomorphism, self edge, vote by committee, collapsing, and proxy if and only if F is the PageRank ranking system.

Given Proposition 2.1, it is enough to prove the following:

Proposition 2.6: Let F_1 and F_2 be a ranking systems that have the weak deletion, strong deletion, and edge duplication properties, and satisfy the self edge and isomorphism axioms. Then, F_1 and F_2 are the same ranking system (notation: $F_1 \equiv F_2$).

We shall now describe a sketch of the proof. The basic idea of the proof is to begin with a graph $G = (V, E)$ and two arbitrary vertices a and b in V, and manipulate G by applying **Del**$(\cdot,\cdot)$, **Delete**$(\cdot,\cdot,\cdot)$, **Duplicate**$(\cdot,\cdot,\cdot)$, and **SelfEdge**$(\cdot,\cdot)$ to achieve a new graph G_n for which F_1 and F_2 rank a and b the same as in G (Formally $a \preceq^F_{G_n} b \Leftrightarrow a \preceq^F_G b$ for $F \in \{F_1, F_2\}$). Afterwards, G_n is further manipulated to generate $G_{n+\delta}$ for which $a \simeq^F_{G_{n+\delta}} b$, but $a \preceq^F_{G_n} b \Rightarrow b \not\preceq^F_{G_{n+\delta}} a$ for $F \in \{F_1, F_2\}$ or vice versa (with a and b replaced). So, we conclude that $a \preceq^{F_1}_{G_n} b \Leftrightarrow a \preceq^{F_2}_{G_n} b$, and thus $a \preceq^{F_1}_G b \Leftrightarrow a \preceq^{F_2}_G b$.

The steps required to generate G_n from G, and then $G_{n+\delta}$ from G_n may be described algorithmically. These steps are illustrated in Figure 2.4:

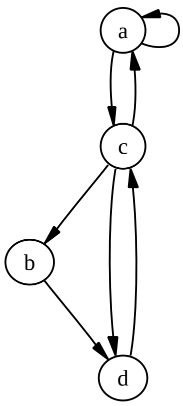
1. Add a new vertex on every edge on the initial graph (Figure 2.4b), thus splitting each original edge into two new edges. These vertices do not change the relative ranking of a and b due to the weak deletion property.

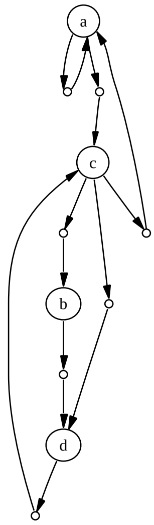
2. If no original vertices exist in the graph except a and b, go to step 8. Otherwise, select an original vertex $x \notin \{a, b\}$ (in Figure 2.4 we start by selecting c).

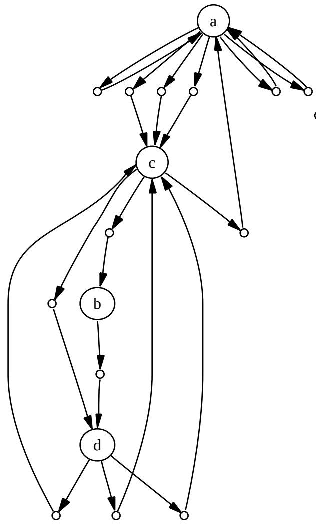
3. Remove all vertices that are both predecessors and successors of x and all edges connected to these vertices. All of these are new vertices, which have an in-degree and out-degree of 1.

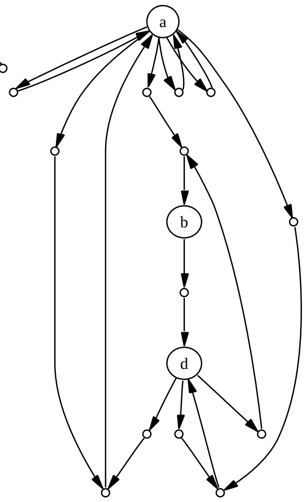
   Basically, this step removes all self-edges of x (with an added vertex on them). These deletions do not change the relative ranking of a and b due to the weak deletion property and the self edge axiom.

4. Duplicate all predecessors of predecessors of x by x's out-degree. This does not change the relative ranking of a and b due to the duplication property (Figure 2.4c).







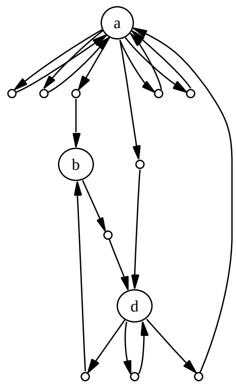


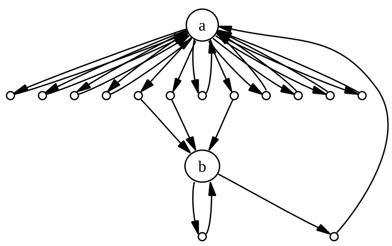
(a) Initial graph

(b) After adding vertices

(c) After duplication of c's predecessors

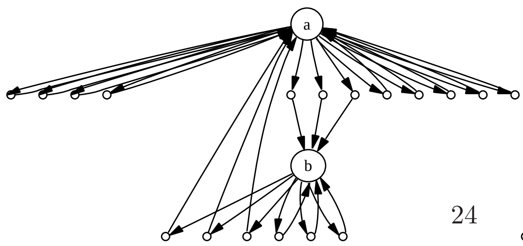
(d) After deletion of c

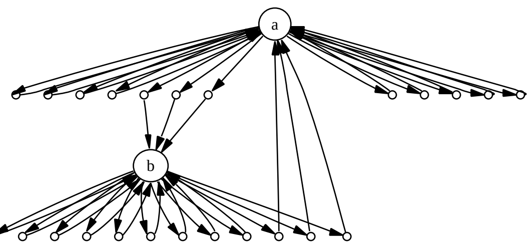




(e) After deletion of c's intermediate successors

(f) After deletion of d





(g) After duplication of b

(h) Final isomorphic graph

Conclusion: $a \not\preceq b$.

Figure 2.4: Example run of the completeness algorithm

Note that all the vertices we duplicate are original ones (possibly a or b, but not x), so to add additional in-between vertices before x, making the in-degree of x a multiple of its out degree, split into groups of isomorphic, and thus equally ranked, vertices.

5. Delete x using $\mathbf{Delete}(G, x)$ (Figure 2.4d).

6. Delete the successors of x (new vertices) to retain the state of one new vertex between each pair of original vertices (Figure 2.4e). These deletions do not change the relative ranking of a and b due to the strong deletion property.

7. Go to step 2 (Figure 2.4f illustrates the second iteration, where d is selected).

8. Now, a and b are the only original vertices remaining in the graph, and the graph could be defined by the number of vertices (with edges) between a and b, between b and a, between a and a, and between b and b.

9. Duplicate a by the number of edges with vertices from b to a and vice versa, thus equalizing the number of edges with vertices from a to b the number from b to a (Figure 2.4g). This relative ranking between a and b is retained due to the duplication property.

10. Now, add self edges (with vertices) to the vertex $v \in \{a, b\}$ with fewer self-edges (with vertices), until the number of self edges is equal between a and b (Figure 2.4h). Let $v' = \{a, b\} \setminus \{v\}$. By the self edge axiom and the weak deletion property, if $v' \preceq^F v$ before adding the self edges, then now $v \not\preceq^F v'$ for $F \in \{F_1, F_2\}$.

11. By the isomorphism axiom, in this graph, $a \simeq b$, therefore in the graph after step 9, $v' \preceq^F v$ for $F \in \{F_1, F_2\}$. But as the relative ranking of a and b did not change until step 10, $v' \preceq^F_G v$ for $F \in \{F_1, F_2\}$, and thus $a \preceq^{F_1}_G b \Leftrightarrow a \preceq^{F_2}_G b$.

□

We shall now present the complete and general proof in full detail.

Proof: Let V be a vertex set and let $G = (V, E) \in \mathbb{G}_V$ be some graph. If $|V| = 1$, then there exists only one ordering on V, so trivially $\preceq^{F_1}_G \equiv \preceq^{F_2}_G$. Assume $V = \{v_1, v_2, \ldots, v_n\}$. We will show that $v_1 \preceq^{F_1}_G v_2 \Leftrightarrow v_1 \preceq^{F_2}_G v_2$. Without loss of generality we can show only one direction. Let $F \in \{F_1, F_2\}$.

Let $G_2 = (V_2, E_2)$ be the following graph (G with a vertex added on every edge):

$$\begin{aligned} V_2 &= V \cup \{u_{i,j} | (v_i, v_j) \in E\} \\ E_2 &= \{(v_i, u_{i,j}), (u_{i,j}, v_j) | (v_i, v_j) \in E\}. \end{aligned}$$

Note that

$$G = \mathbf{Del}(\mathbf{Del}(\cdots \mathbf{Del}(G_2, u_1) \cdots, u_{|E|-1}), u_{|E|})$$

where $\{u_1, \dots, u_{|E|}\} = \{u_{i,j} | (v_i, v_j) \in E\}$ and that G_2 satisfies the conditions of weak deletion property for the vertices $\{u_{i,j} | (v_i, v_j) \in E\}$, thus $v_1 \preceq_G^F v_2 \Leftrightarrow v_1 \preceq_{G_2}^F v_2$.

For all strongly connected directed graphs G' such that for all $v \in V$ and for all $v' \in P_{G'}(v) \cup S_{G'}(v)$ s.t. $|S_{G'}(v')| = |P_{G'}(v')| = 1$, let us denote for all $v \in V$: $S_G^2(v) = \{v' \in V : x \in S_{G'}(v), S_{G'}(x) = \{v'\}\}$ and $P_G^2(v) = \{v' \in V : x \in P_{G'}(v), P_{G'}(x) = \{v'\}\}$.

For, $i = 3, \dots, n$, we recursively define G_i as follows: Let $\{q_1, q_2, \dots, q_m\} = S_{G_{i-1}}(v_i) \cap P_{G_{i-1}}(v_i)$. Let G'_{i-1} be the graph

$$G'_{i-1} = \mathbf{SelfEdge}^{-1}(\mathbf{Del}(\cdots \mathbf{SelfEdge}^{-1}(\mathbf{Del}(G_{i-1}, q_1), v_i) \cdots, q_m), v_i).$$

Now, let $P^2_{G'_{i-1}}(v) = \{p_1, \dots, p_k\}$. and let $S_{G'_{i-1}}(v_i) = \{s_1, s_2, \dots, s_l\}$. Let G''_{i-1} be defined as:

$$G''_{i-1} = \mathbf{Duplicate}(\cdots \mathbf{Duplicate}(G'_{i-1}, p_1, l) \cdots, p_k, l)$$

Let $\{p_j^i | i = 1, \dots, l\} = S_{G''_{i-1}}(p_j)$ be the duplicated successors of p_j for $j = 1 \dots k$. Now let $G_i = (V_i, E_i)$ be defined as:

$$\begin{aligned} G'''_{i-1} &= \mathbf{Delete}(G'_{i-1}, v_i, \{(s_1, \{p_j^1 | j = 1, \dots, k\}), \dots, (s_l, \{p_j^l | j = 1, \dots, k\})\}) \\ G_i &= \mathbf{Delete}(\cdots \mathbf{Delete}(\mathbf{Delete}(G'''_{i-1}, s_1), s_2) \cdots, s_l). \end{aligned}$$

By the edge duplication and strong deletion properties and the self edge axiom, $v_1 \preceq_{G_i}^F v_2$ for all $i \in \{2, \dots, n\}$.

We will now prove that for all $i \in \{2, \dots, n\}$ and for all $v \in V_i \setminus V$: $|P_{G_i}(v)| = |S_{G_i}(v)| = 1$ and $P_{G_i}(v) \cup S_{G_i}(v) \subseteq V$ and for all $v \in V$: $(P_{G_i}(v) \cup S_{G_i}(v)) \cap V = \emptyset$. Proof by induction: G_2 trivially satisfies both requirements. Now assume that for all $v \in V_i \setminus V$: $|P_{G_i}(v)| = |S_{G_i}(v)| = 1$ and $P_{G_i}(v) \cup S_{G_i}(v) \subseteq V$ and for all $v \in V$: $(P_{G_i}(v) \cup S_{G_i}(v)) \cap V = \emptyset$. Clearly, G'_i satisfies the conditions, because we only removed elements from V_i, and not changed the predecessors or successors of any $v \in V \setminus V_i$. Also, all edges added between vertices in V were removed. The $\mathbf{Duplicate}(\cdot, \cdot, \cdot)$ operation adds vertices with in-degree 1 and out-degree equal to the out degree of the successors of v, which is also 1. So, the new vertices added in G''_i satisfy the conditions. Furthermore, no edges were added between elements of V. Thus, G''_i satisfies the conditions. In G_{i+1}, we removed v and all its successors. The predecessors of v in G''_i keep their out-degree 1, and point to elements of $S^2_{G''_i}(v)$, and thus still meet the requirements. Other elements of $V''_i \setminus V$ have not changed their edges, and thus still meet the requirements. Still, no edges were added between elements of V. Therefore, for all $v \in V_{i+1} \setminus V$: $|P_{G_{i+1}}(v)| = |S_{G_{i+1}}(v)| = 1$ and $P_{G_{i+1}}(v) \cup S_{G_{i+1}}(v) \subseteq V$ and for all $v \in V$: $(P_{G_{i+1}}(v) \cup S_{G_{i+1}}(v)) \cap V = \emptyset$.

Specifically, this is true for $G_n = (V_n, E_n)$. Furthermore, $V_n \cap V = \{v_1, v_2\}$. Thus, G_n could be described as:

$$\begin{aligned} V_n &= \{v_1, v_2\} \cup \{v_{jk}^i | j, k \in \{1, 2\}; i = 1, \dots, n_{jk}\} \\ E_n &= \{(v_j, v_{jk}^i), (v_{jk}^i, v_k) | j, k \in \{1, 2\}; i = 1, \dots, n_{jk}\}. \end{aligned}$$

The only parameters which affect the structure of G_n are n_{jk} $(j, k \in \{1, 2\})$, so we can denote $G_n = \mathbf{G}[n_{11}, n_{12}, n_{21}, n_{22}]$. Now, let

$$\begin{aligned} G'_n &= \mathbf{Duplicate}(\mathbf{Duplicate}(G_n, v_1, n_{21}), v_2, n_{12}) \\ &= \mathbf{G}[n_{21}n_{11}, n_{21}n_{12}, n_{12}n_{21}, n_{12}n_{22}]. \end{aligned}$$

By the edge duplication property, $v_1 \preceq^F_G v_2 \Leftrightarrow v_1 \preceq^F_{G'_n} v_2$.

Consider the following 3 cases:

- If $n_{21}n_{11} = n_{12}n_{22}$, then the graph is isomorphic to itself, replacing v_1 with v_2 and v^i_{jk} with v^i_{kj}. In this case, by the isomorphism axiom, $v_1 \simeq^F_{G'_n} v_2$ and thus $v_1 \simeq^F_G v_2$, and therefore $v_1 \preceq^F_G v_2$ for $F \in \{F_1, F_2\}$.
- If $n_{21}n_{11} > n_{12}n_{22}$, let $\delta = n_{21}n_{11} - n_{12}n_{22} > 0$. Now we define for $i = n+1, \dots n+\delta$:

  $$\begin{aligned} G'_i &= \mathbf{SelfEdge}(G_{i-1}, v_2) \\ G_i &= \mathbf{G}[n_{21}n_{11}, n_{21}n_{12}, n_{12}n_{21}, n_{12}n_{22} + i - n]. \end{aligned}$$

  Note that $G'_i = \mathbf{Del}(G_i, v_{22}^{n_{12}n_{22}+i-n})$. Thus, by the self-edge axiom and the weak deletion property, $v_1 \preceq^F_G v_2 \Rightarrow v_2 \not\preceq^F_{G_{n+\delta}} v_1$. Now, note that $G_{n+\delta} = \mathbf{G}[n_{21}n_{11}, n_{12}n_{21}, n_{12}n_{21}, n_{21}n_{11}]$, thus as before, by isomorphism, $v_1 \simeq^F_{G_{n+\delta}} v_2$. Therefore we conclude that $v_1 \not\preceq^F_G v_2$ for $F \in \{F_1, F_2\}$.
- If $n_{21}n_{11} < n_{12}n_{22}$, we can similarly conclude that $v_2 \not\preceq^F_G v_1$, and therefore $v_1 \preceq^F_G v_2$ for $F \in \{F_1, F_2\}$.

We have shown that for every vertex set V, for all $G = (V, E) \in \mathbb{G}_V$, and for every $v_1, v_2 \in V$: $v_1 \preceq^{F_1}_G v_2 \Leftrightarrow v_1 \preceq^{F_2}_G v_2$. Thus, $F_1 \equiv F_2$, concluding the proof of the proposition. $\square$

2.6 Discussion

Representation theorems are the formal mathematical tool for the justification of decision and choice rules. We have already mentioned the formal theory of social choice, but representation theorems also lay mathematical foundations for other branches of decision and choice theory. For example, the crowning achievement of the theory of (single-agent) choice is Savage's representation theorem (1954), which provides sound and complete axiomatization for the expected utility maximization decision criterion. Here also one looks for ordinal requirements, which do not refer to numeric computations, under which an agent can be viewed as an expected utility maximizer. This is similar to our work, where we considered only graph-theoretic ordinal axioms to justify the numeric computations done by PageRank.

Although PageRank is probably the most popular page ranking procedure, it may be interesting to attempt and provide axiomatization for other page ranking procedures, such as Hubs and Authorities Kleinberg (1999). Once such axiomatization is found the different axiomatic systems can be compared as a basis for rigorous evaluation.

We believe that the problem of ranking of Internet pages is indeed a fundamental problem. We see the fact that this central problem is a new type of social choice problem as especially intriguing. In order to provide mathematical foundations to page ranking systems we therefore need to search for basic representation theorems that will provide ordinal, graph theoretic axiomatizations for basic heuristics and approaches for page ranking. Representation theorems isolate the "essence" of particular ranking systems, and provide means for the evaluation (and potentially comparison) of such systems. In this chapter we initiated work on this topic by introducing such representation theorem for PageRank. We hope that others will join us in exploring the connections between page ranking algorithms and the mathematical theory of social choice.

Chapter 3

The Normative Approach

3.1 Introduction

In this chapter we provide an extensive normative study of ranking systems. We introduce two fundamental axioms. One of these axioms captures the transitive effects of voting in ranking systems, and the other adapts Arrow's well-known independence of irrelevant alternatives(IIA) axiom to the context of ranking systems. Surprisingly, we find that no general ranking system can simultaneously satisfy these two axioms! We further show that our impossibility result holds under various restrictions on the class of ranking problems considered. On the other hand, we show a positive result for the case when the transitivity axiom is relaxed. This new ranking system is practical and useful. Finally, we use our IIA axiom to present a positive result in the form of a representation theorem for the well-known approval voting ranking system, which ranks the agents based on the number of votes received. This axiomatization shows that when ignoring transitive effects, there is only one ranking system that satisfies our IIA axiom.

This chapter is structured as follows: Sections 3.2 and 3.3 introduce our axioms of Transitivity and Ranked Independence of Irrelevant Alternatives respectively. Our main impossibility result is presented in Section 3.4, and further strengthened in Section 3.5. Our main positive result, in the form of a ranking system satisfying a weaker version of transitivity is given in Section 3.6. Finally, an axiomatization for the Approval Voting ranking system in presented in Section 3.7.

3.2 Transitivity

A basic property one would assume of ranking systems is that if an agent a's voters are ranked higher than those of agent b, then agent a should be ranked higher than agent b. This notion is formally captured below:

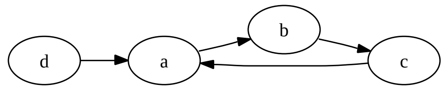


Figure 3.1: Example of Transitivity

Definition 3.1: Let F be a ranking system. We say that F satisfies *strong transitivity* if for all graphs $G = (V, E)$ and for all vertices $v_1, v_2 \in V$: Assume there is a 1-1 mapping $f : P(v_1) \mapsto P(v_2)$ s.t. for all $v \in P(v_1)$: $v \preceq f(v)$. Further assume that either f is not onto or for some $v \in P(v_1)$: $v \prec f(v)$. Then, $v_1 \prec v_2$.

Example 3.1: Consider the graph G in Figure 3.1 and any ranking system F that satisfies strong transitivity. F must rank vertex d below all other vertices, as it has no predecessors, unlike all other vertices. If we assume that $a \preceq_G^F b$, then by strong transitivity we must conclude that $b \preceq_G^F c$ as well. But then we must conclude that $b \prec_G^F a$ (as b's predecessor a is ranked lower than a's predecessor c, and a has an additional predecessor d), which leads to a contradiction. Given $b \prec_G^F a$, again by transitivity, we must conclude that $c \prec_G^F b$, so the only ranking for the graph G that satisfies strong transitivity is $d \prec_G^F c \prec_G^F b \prec_G^F a$.

Tennenholtz (2004) has suggested an algorithm that defines a ranking system that satisfies strong transitivity by iteratively refining an ordering of the vertices.

Note that the PageRank ranking system defined in Section 2.2 above does not satisfy strong transitivity. This is due to the fact that PageRank reduces the weight of links (or votes) from nodes which have a higher out-degree. Thus, assuming Yahoo! and Microsoft are equally ranked, a link from Yahoo! means less than a link from Microsoft, because Yahoo! links to more external pages than does Microsoft. Noting this fact, we can weaken the definition of transitivity to require that the predecessors of the compared agents have an equal out-degree:

Definition 3.2: Let F be a ranking system. We say that F satisfies *weak transitivity* if for all graphs $G = (V, E)$ and for all vertices $v_1, v_2 \in V$: Assume there is a 1-1 mapping $f : P(v_1) \mapsto P(v_2)$ s.t. for all $v \in P(v_1)$: $v \preceq f(v)$ and $|S(v)| = |S(f(v))|$. Further assume that either f is not onto or for some $v \in P(v_1)$: $v \prec f(v)$. Then, $v_1 \prec v_2$.

Indeed, our idealized version of the PageRank ranking system satisfies this weakened version of transitivity. Furthermore, the result in the example above does not change when we consider weak transitivity in place of strong transitivity.

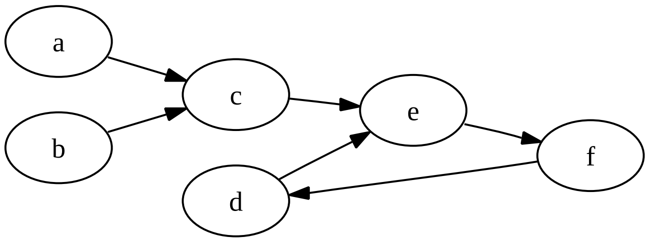


Figure 3.2: Example of Ranked Independence of Irrelevant Alternatives

3.3 Ranked Independence of Irrelevant Alternatives

A standard assumption in social choice settings is that a pair of agents' relative rank should only depend on (some property of) their immediate predecessors. Such axioms are usually called independence of irrelevant alternatives(IIA) axioms.

In our setting, we require the relative ranking of two agents must only depend on the pairwise comparisons of the ranks of their predecessors, and not on their identity or cardinal value. Our IIA axiom, called *ranked* IIA, differs from the one suggested by Arrow (1963) in the fact that we do not consider the identity of the voters, but rather their relative rank.

Example 3.2: Consider the graph in Figure 3.2. Furthermore, assume a ranking system F has ranked the vertices of this graph as following: $a \simeq b \prec c \simeq d \prec e \simeq f$. Now look at the comparison between c and d. c's predecessors, a and b, are both ranked equally, and both ranked lower than d's predecessor f. This is also true when considering e and f – e's predecessors c and d are both ranked equally, and both ranked lower than f's predecessor e. Therefore, if we agree with ranked IIA, the relation between c and d, and the relation between e and f must be the same, which indeed it is – both $c \simeq d$ and $e \simeq f$. However, this same situation also occurs when comparing c and f (c's predecessors a and b are equally ranked and ranked lower than f's predecessor e), but in this case $c \prec f$. So, we can conclude that the ranking system F which produced these rankings does not satisfy ranked IIA.

To formally define this condition, one must consider all possibilities of comparing two nodes in a graph based only on ordinal comparisons of their predecessors. We call these possibilities comparison profiles:

Definition 3.3: A *comparison profile* is a pair $\langle \mathbf{a}, \mathbf{b} \rangle$ where $\mathbf{a} = (a_1, \ldots, a_n)$,

$\mathbf{b} = (b_1, \ldots, b_m)$, $a_1, \ldots, a_n, b_1, \ldots, b_m \in \mathbb{N}$, $a_1 \leq a_2 \leq \cdots \leq a_n$, and $b_1 \leq b_2 \leq \cdots \leq b_m$. Let $\mathcal{P}$ be the set of all such profiles.

A ranking system F, a graph $G = (V, E)$, and a pair of vertices $v_1, v_2 \in V$ are said to *satisfy* such a comparison profile $\langle \mathbf{a}, \mathbf{b} \rangle$ if there exist 1-1 mappings $f_1 : P(v_1) \mapsto \{1 \ldots n\}$ and $f_2 : P(v_2) \mapsto \{1 \ldots m\}$ such that given $f : (\{1\} \times P(v_1)) \cup (\{2\} \times P(v_2)) \mapsto \mathbb{N}$ defined as:

$$\begin{aligned} f(1, v) &= a_{f_1(v)} \\ f(2, u) &= b_{f_2(u)}, \end{aligned}$$

$f(i, x) \leq f(j, y) \Leftrightarrow x \preceq_G^F y$ for all $(i, x), (j, y) \in (\{1\} \times P(v_1)) \cup (\{2\} \times P(v_2))$.

Example 3.2 (cont.): In the example considered above, all of the pairs (c, d), (c, f), and (e, f) satisfy the comparison profile $\langle (1, 1), (2) \rangle$.

We now require that for every such profile the ranking system ranks the nodes consistently:

Definition 3.4: Let F be a ranking system. We say that F satisfies *ranked independence of irrelevant alternatives (RIIA)* if there exists a mapping $f : \mathcal{P} \mapsto \{0, 1\}$ such that for every graph $G = (V, E)$ and for every pair of vertices $v_1, v_2 \in V$ and for every comparison profile $p \in \mathcal{P}$ that v_1 and v_2 satisfy, $v_1 \preceq_G^F v_2 \Leftrightarrow f(p) = 1$.

As RIIA is an independence property, the ranking system $F_=$, that ranks all agents equally, satisfies RIIA. A more interesting ranking system that satisfies RIIA is the approval voting ranking system, defined below.

Definition 3.5: The *approval voting ranking system AV* is the ranking system defined by:

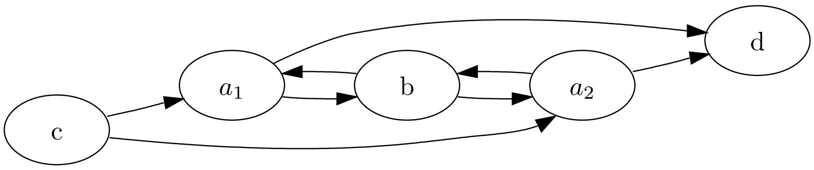
$$v_1 \preceq_G^{AV} v_2 \Leftrightarrow |P(v_1)| \leq |P(v_2)|$$

A full axiomatization of the approval voting ranking system is given in section 3.7. Another ranking system satisfying RIIA will be presented in section 3.6.

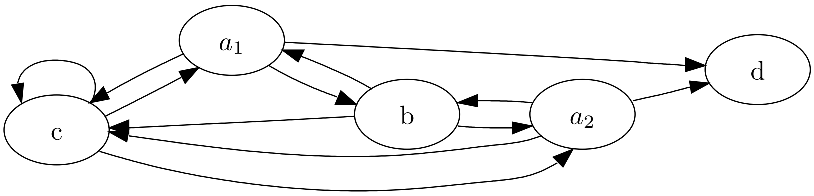
3.4 Impossibility

Our main result illustrates the impossibility of satisfying (weak) transitivity and RIIA simultaneously.

Theorem 3.1: There is no general ranking system that satisfies weak transitivity and RIIA.



(a) Graph G_1



(b) Graph G_2

Figure 3.3: Graphs for the proof of Theorem 3.1

Proof: Assume for contradiction that there exists a ranking system F that satisfies weak transitivity and RIIA. Consider first the graph G_1 in Figure 3.3(a). First, note that a_1 and a_2 satisfy some comparison profile $p_a = ((x,y),(x,y))$ because they have identical predecessors. Thus, by RIIA, $a_1 \preceq^F_{G_1} a_2 \Leftrightarrow a_2 \preceq^F_{G_1} a_1$, and therefore $a_1 \simeq^F_{G_1} a_2$. By weak transitivity, it is easy to see that $c \prec^F_{G_1} a_1$ and $c \prec^F_{G_1} b$. If we assume $b \preceq^F_{G_1} a_1$, then by weak transitivity, $a_1 \prec^F_{G_1} b$ which contradicts our assumption. So we conclude that $c \prec^F_{G_1} a_1 \prec^F_{G_1} b$.

Now consider the graph G_2 in Figure 3.3(b). Again, by RIIA, $a_1 \simeq^F_{G_2} a_2$. By weak transitivity, it is easy to see that $a_1 \prec^F_{G_2} c$ and $b \prec^F_{G_2} c$. If we assume $a_1 \preceq^F_{G_2} b$, then by weak transitivity, $b \prec^F_{G_2} a_1$ which contradicts our assumption. So we conclude that $b \prec^F_{G_2} a_1 \prec^F_{G_2} c$.

Consider the comparison profile $p = ((1,3),(2,2))$. Given F, a_1 and b satisfy p in G_1 (because $c \prec^F_{G_1} a_1 \simeq^F_{G_1} a_2 \prec^F_{G_1} b$) and in G_2 (because $b \prec^F_{G_2} a_1 \simeq^F_{G_2} a_2 \prec^F_{G_2} c$). Thus, by RIIA, $a_1 \preceq^F_{G_1} b \Leftrightarrow a_1 \preceq^F_{G_2} b$, which is a contradiction to the fact that $a_1 \prec^F_{G_1} b$ but $b \prec^F_{G_2} a_1$. $\square$

This result is quite a surprise, as it means that every reasonable definition of a ranking system must either consider cardinal values for nodes and/or edges (like Page et al. (1998)), or operate ordinally on a global scale (like the axiomatization presented in Chapter 2).

3.5 Relaxing Generality

A hidden assumption in our impossibility result is the fact that we considered only general ranking systems. In this section we analyze several special classes of graphs that relate to common ranking scenarios.

3.5.1 Small Graphs

A natural limitation on a preference graph is a cap on the number of vertices (agents) that participate in the ranking. Indeed, when there are three or less agents involved in the ranking, strong transitivity and RIIA can be simultaneously satisfied. An appropriate ranking algorithm for this case is the one suggested in Tennenholtz (2004).

However, when there are four or more agents, strong transitivity and RIIA cannot be simultaneously satisfied (the proof is similar to that of Theorem 3.1, but with vertex d removed in both graphs). When five or more agents are involved, even weak transitivity and RIIA cannot be simultaneously satisfied, as implied by the proof of Theorem 3.1.

3.5.2 Single Vote Setting

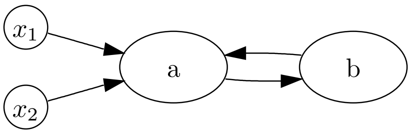
Another natural limitation on the domain of graphs that we might be interested in is the restriction of each agent(vertex) to exactly one vote(successor). For example, in the voting paradigm this could be viewed as a setting where every agent votes for exactly one agent. The following proposition shows that even in this simple setting weak transitivity and RIIA cannot be simultaneously satisfied.

Proposition 3.2: Let $\mathbb{G}_1$ be the set of all graphs $G = (V, E)$ such that $|S(v)| = 1$ for all $v \in V$. There is no partial ranking system over $\mathbb{G}_1$ that satisfies weak transitivity and RIIA.

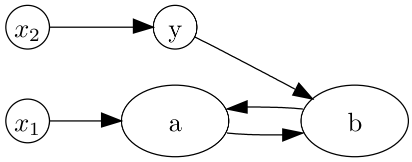
Proof: Assume for contradiction that there is a partial ranking system F over $\mathbb{G}_1$ that satisfies weak transitivity and RIIA. Let $f : \mathcal{P} \mapsto \{0, 1\}$ be the mapping from the definition of RIIA for F.

Let $G_1 \in \mathbb{G}_1$ be the graph in Figure 3.4a. By weak transitivity, $x_1 \simeq^F_{G_1} x_2 \prec^F_{G_1} b \prec^F_{G_1} a$. (a, b) satisfies the comparison profile $\langle(1, 1, 2), (3)\rangle$, so we must have $f\langle(1, 1, 2), (3)\rangle = 0$. Now let $G_2 \in \mathbb{G}_1$ be the graph in Figure 3.4b. By weak transitivity $x_1 \simeq^F_{G_2} x_2 \prec^F_{G_2} y \prec^F_{G_2} a \prec^F_{G_2} b$. (b, a) satisfies the comparison profile $\langle(2, 3), (1, 4)\rangle$, so we must have $f\langle(2, 3), (1, 4)\rangle = 0$.

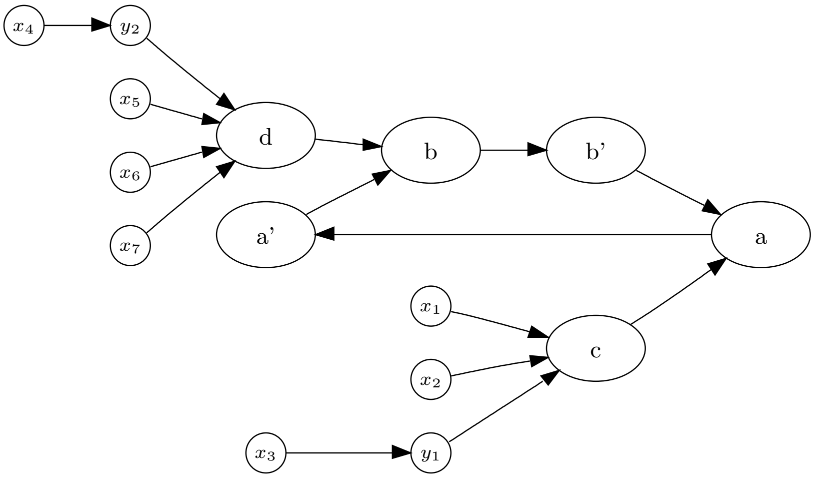
Let $G_3 \in \mathbb{G}_1$ be the graph in Figure 3.4c. By weak transitivity it is easy to see that $x_1 \simeq^F_{G_3} \cdots \simeq^F_{G_3} x_7 \prec^F_{G_3} y_1 \simeq^F_{G_3} y_2 \prec^F_{G_3} c \prec^F_{G_3} d$. Furthermore, by weak transitivity we conclude that $a \prec^F_{G_3} b$ and $a' \prec^F_{G_3} b'$ from $c \prec^F_{G_3} d$; and $y_1 \prec^F_{G_3} b$ from $x_3 \prec^F_{G_3} d$. Now consider the vertex pair (c, b'). We have shown that $x_1 \simeq^F_{G_3} x_2 \prec^F_{G_3} y_1 \prec^F_{G_3} b$. So, (c, b') satisfies the comparison profile



(a) Graph G_1



(b) Graph G_2



(c) Graph G_3

Figure 3.4: Graphs for the proof of proposition 3.2

$\langle(1,1,2),(3)\rangle$, thus by RIIA $b' \prec^F_{G_3} c$. Now consider the vertex pair (b,a). We have already shown that $a' \prec^F_{G_3} b' \prec^F_{G_3} c \prec^F_{G_3} d$. So, (a,b) satisfies the comparison profile $\langle(2,3),(1,4)\rangle$, thus by RIIA $b \prec^F_{G_3} a$. However, we have already shown that $a \prec^F_{G_3} b$ – a contradiction. Thus, the ranking system F cannot exist. □

3.5.3 Bipartite Setting

In the world of reputation systems (Resnick et al., 2000), we frequently observe a distinction between two types of agents such that each type of agent only ranks agents of the other type. For example buyers only interact with sellers and vice versa. This type of limitation is captured by requiring the preference graphs to be bipartite, as defined below.

Definition 3.6: A graph $G = (V, E)$ is called *bipartite* if there exist V_1, V_2

such that $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, and $E \subseteq (V_1 \times V_2) \cup (V_2 \times V_1)$. Let $\mathbb{G}_B$ be the set of all bipartite graphs.

Our impossibility result extends to the limited domain of bipartite graphs.

Proposition 3.3: There is no partial ranking system over $\mathbb{G}_B \cap \mathbb{G}_1$ that satisfies weak transitivity and RIIA.

Proof: The proof is exactly the same as for $\mathbb{G}_1$, considering that all graphs in Figure 3.4 are bipartite. □

3.5.4 Strongly Connected Graphs

The well-known PageRank ranking system is (ideally) defined on the set of strongly connected graphs. That is, the set of graphs where there exists a directed path between any two vertices.

Let us denote the set of all strongly connected graphs by $\mathbb{G}_{SC}$. The following proposition extends our impossibility result to strongly connected graphs.

Proposition 3.4: There is no partial ranking system over $\mathbb{G}_{SC}$ that satisfies weak transitivity and RIIA.

Proof: The proof is similar to the proof of Theorem 3.1, but with an additional vertex e in both graphs that has edges to and from all other vertices. □

3.6 Relaxing Transitivity

Our impossibility result becomes a possibility result when we relax the transitivity requirement. Instead of comparing only vertices with similar out-degree as in the weak transitivity axiom above, we weaken the requirement for strict preference to hold only in the case where the matching predecessors of one agent are preferred to the *all* predecessors of the other.

Definition 3.7: Let F be a ranking system. We say that F satisfies *strong quasi-transitivity* if for all graphs $G = (V, E)$ and for all vertices $v_1, v_2 \in V$: Assume there is a 1-1 mapping $f : P(v_1) \mapsto P(v_2)$ s.t. for all $v \in P(v_1)$:$v \preceq f(v)$. Then, $v_1 \preceq v_2$. And, if $P(v_1) \neq \emptyset$ and for all $v \in P(v_1)$: $v \prec f(v)$, then $v_1 \prec v_2$.

When we only require strong *quasi*-transitivity and RIIA, we find an interesting family of ranking systems that rank the agents according to their in-degree, breaking ties by comparing the ranks of the strongest predecessors. These recursive in-degree systems work by assigning a rational value for every vertex, that is based on the following idea: rank first based on the in-degree. If there is a tie, rank based on the strongest predecessor's value, and so on. Loops are

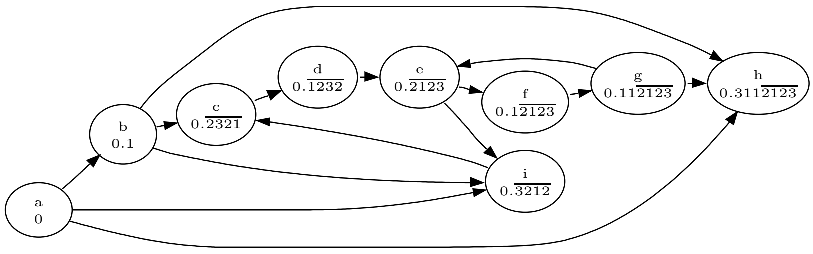


Figure 3.5: Values assigned by the recursive in-degree algorithm

ranked as periodical rational numbers in base $(n+1)$ with a period the length of the loop, in the case that continuing on the loop is the maximally ranked option.

The recursive in-degree systems differ in the way different in-degrees are compared. Any monotone increasing mapping of the in-degrees could be used for the initial ranking. To show these systems are well-defined and that the values can be calculated we define these systems algorithmically as follows:

Definition 3.8: Let $r : \mathbb{N} \mapsto \mathbb{N}$ be a monotone nondecreasing function such that $r(i) \leq i$ for all $i \in \mathbb{N}$. The *recursive in-degree ranking system with rank function r* is defined as follows: Given a graph $G = (V, E)$,

$$v_1 \preceq_G^{RID_r} v_2 \Leftrightarrow \text{value}_r(v_1) \leq \text{value}_r(v_2),$$

where value is defined as:

$$\text{value}_r(v) \quad = \max_{\mathbf{a} \in \text{Path}(v)} \text{vp}_r(\mathbf{a}), \tag{3.1}$$

where the maximum is over the set of almost-simple reverse paths to v:

$$\begin{aligned} \text{Path}(v) \quad = \{ \quad & (v = a_1, a_2, \ldots, a_m)| \\ & (a_m, \ldots, a_1) \text{ is a path in } G \ \wedge (a_{m-1}, \ldots, a_1) \text{ is simple}\} \end{aligned}$$

and valuation the function $\text{vp}_r : V^* \mapsto \mathbb{Q}$ is defined as:

$$\text{vp}_r(a_1, a_2, \ldots, a_m) = \frac{1}{n+1} \left[\begin{array}{l} r(|P(a_1)|) + \\ \begin{cases} 0 & m = 1 \\ \text{vp}_r(a_2, \ldots, a_m, a_2) & a_1 = a_m \wedge m > 1 \\ \text{vp}_r(a_2, \ldots, a_m) & \text{Otherwise.} \end{cases} \end{array} \right] \tag{3.2}$$

Note that $\text{vp}_r(a_1, a_2, \ldots, a_m)$ is infinitely recursive in the case when $a_1 =$

$a_m \wedge m > 1$. For computation sake we can redefine this case finitely as:

$$\begin{aligned}
\mathrm{vp}_r(a_1, \ldots, a_m, a_1) &= \sum_{i=0}^{\infty} \frac{1}{(n+1)^{mi}} \sum_{j=1}^{m} \frac{r(|P(a_j)|)}{(n+1)^j} = \\
&= \frac{(n+1)^m}{(n+1)^m - 1} \mathrm{vp}_r(a_1, \ldots, a_m).
\end{aligned}$$

Example 3.3: An example of the values assigned for a particular graph when r is the identity function is given in Figure 3.5. As $n = 9$, the values are decimal. Note that the loop (c, d, e, i) generates a periodical decimal $\mathrm{value}_r(c) = \mathrm{vp}_r(c, i, e, d, c) = 0.\overline{2321}$ by the infinite recursion in (3.2).

The recursive in-degree system satisfies an interesting fixed point property that can be used to facilitate its efficient computation:

Proposition 3.5: Let $r : \mathbb{N} \mapsto \mathbb{N}$ be a monotone nondecreasing function such that $r(i) \leq i$ for all $i \in \mathbb{N}$ and define $r(0) = 0$. The value function for the recursive in-degree ranking system satisfies:

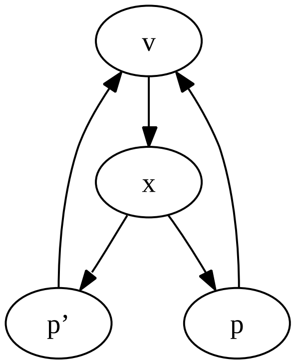
$$\mathrm{value}_r(v) = \begin{cases} \frac{1}{n+1}\left[r(|P(v)|) + \max_{p \in P(v)} \mathrm{value}_r(p)\right] & P(v) \neq \emptyset \\ 0 & \text{Otherwise} \end{cases} \tag{3.3}$$

Proof: Denote $\mathrm{Path}'(p, v)$ as the set of almost-simple directed paths to p which do not pass through v unless immediately looping back to p:

$$\begin{aligned}
\mathrm{Path}'(p, v) = \{ \quad & (p = a_1, a_2, \ldots, a_m) | \\
& (a_m, \ldots, a_1) \text{ is a path in } G \ \wedge (a_{m-1}, \ldots, a_1) \text{ is simple} \wedge \\
& \forall i \in \{1, \ldots, m-2, m\} : a_i \neq v \wedge \\
& a_{m-1} = v \Leftrightarrow a_m = p \}.
\end{aligned}$$

Let $v \in V$ be some vertex. Then,

$$\begin{aligned}
\mathrm{value}_r(v) &= \max_{\mathbf{a} \in \mathrm{Path}(v)} \mathrm{vp}_r(\mathbf{a}) = \\
&= \frac{1}{n+1} \left[\begin{array}{l} r(|P(v)|) + \max_{(v=a_1,\ldots,a_m) \in \mathrm{Path}(v)} \\ \begin{cases} \mathrm{vp}_r(a_2, \ldots, a_m, a_2) & a_1 = a_m \wedge m > 1 \\ \mathrm{vp}_r(a_2, \ldots, a_m) & \text{Otherwise.} \end{cases} \end{array} \right] = \quad (3.4) \\
&= \frac{1}{n+1} \left[r(|P(v)|) + \max_{p \in P(v)} \max_{\mathbf{a} \in \mathrm{Path}'(p,v)} \mathrm{vp}_r(\mathbf{a}) \right] = \quad (3.5) \\
&= \frac{1}{n+1} \left[r(|P(v)|) + \max_{p \in P(v)} \max_{\mathbf{a} \in \mathrm{Path}(p)} \mathrm{vp}_r(\mathbf{a}) \right] = \\
&= \frac{1}{n+1} \left[r(|P(v)|) + \max_{p \in P(v)} \mathrm{value}_r(p) \right].
\end{aligned}$$



$$\begin{aligned}\mathbf{a} &= (p, x, v, p', x)\\ \mathbf{b} &= (p, x, v, p)\\ \mathbf{c} &= (p', x, v, p')\end{aligned}$$

Figure 3.6: Example paths from the proof of Proposition 3.6.

Note that (3.4) is equal to zero 0 if $P(v) = \emptyset$, as required. To show that the equality (3.5) holds, assume for contradiction that there exists $p \in P(v)$ and $\mathbf{a} \in \text{Path}(p)$ such that

$$\text{vp}_r(\mathbf{a}) > \max_{p' \in P(v)} \max_{\mathbf{a}' \in \text{Path}'(p',v)} \text{vp}_r(\mathbf{a}'). \tag{3.6}$$

From $\mathbf{a} \in \text{Path}(p) \setminus \text{Path}'(p, v)$, we know that $a_i = v$ for some $i \in \{1, \dots, m\}$. Assume wlog that i is minimal. Let $\mathbf{b}$ denote the path $(p = a_1, a_2, \dots, a_i, p)$ and let $\mathbf{c}$ denote the path $(p' = a_{i+1}, \dots, a_m, a_{j+1}, \dots, a_{i+1})$ if $a_m = a_j$ for some $j < i$ or $(p' = a_{i+1}, \dots, a_m)$ otherwise. An example of such paths is given in Figure 3.6. Note that $\mathbf{b} \in \text{Path}'(p, v)$ and $\mathbf{c} \in \text{Path}'(p', v)$, where $p, p' \in P(v)$. Now, note that

$$\text{vp}_r(\mathbf{a}) \quad = \quad \frac{(n+1)^j - 1}{(n+1)^j}\text{vp}_r(\mathbf{b}) + \frac{1}{(n+1)^j}\text{vp}_r(\mathbf{c}),$$

and thus $\text{vp}_r(\mathbf{a})$ must be between $\text{vp}_r(\mathbf{b})$ and $\text{vp}_r(\mathbf{c})$, in contradiction to assumption (3.6). □

We shall now show this ranking system does in fact satisfy RIIA and our weakened version of transitivity.

Proposition 3.6: Let $r : \mathbb{N} \mapsto \mathbb{N}$ be a monotone nondecreasing function such that $r(i) \leq i$ for all $i \in \mathbb{N}$ and define $r(0) = 0$. The recursive in-degree ranking system with rank function r satisfies strong quasi-transitivity and RIIA.

Proof: The fixed point result in Proposition 3.5 further implies $0 \leq \text{value}_r(v) < 1$, and thus vertices are ordered first by $r(|P(v)|)$ and then by $\max_{p \in P(v)} \text{value}_r(p)$. Therefore, every comparison profile $\langle \mathbf{a}, \mathbf{b} \rangle$ where $\mathbf{a} = (a_1, \dots, a_k)$, $\mathbf{b} = (b_1, \dots, b_l)$ is ranked as follows:

$$f\langle \mathbf{a}, \mathbf{b} \rangle = 1 \quad \Leftrightarrow \quad (k = 0) \vee (r(k) < r(l)) \vee [(r(k) = r(l)) \wedge (a_k \leq b_l)] \,.$$

Algorithm 1 Efficient algorithm for recursive in-degree

1. Initialize $\text{value}_r(v) \leftarrow \frac{1}{n+1} r(|P(v)|)$ for all $v \in V$, where $r(0)$ is assumed to be 0.

2. Let V' be the set of vertices with incoming edges.

3. Iterate $|V|$ times:

   (a) For every vertex $v \in V'$:

      i. Update $\text{value}_r(v) \leftarrow \frac{1}{n+1}\left[r(|P(v)|) + \max_{p \in P(v)} \text{value}_r(p)\right]$.

4. Sort V' by $\text{value}_r(\cdot)$.

5. Output all vertices in $V \setminus V'$ as weakest, followed by the vertices in V' sorted by $\text{value}_r(\cdot)$ in ascending order.

This ranking of profiles trivially yields strong quasi-transitivity as required. □

We now provide an equivalent recursive definition for value:

$$\text{value}_r(v) = \text{pv}_r((), v) \tag{3.7}$$

$$\text{pv}_r(\mathbf{a}, v) = \begin{cases} (v) & P(v) = \emptyset \\ \left(v, \max_{p \in P(v)} \text{pv}_r(\mathbf{a}, v, p)\right) & v \notin \mathbf{a} \\ (a_k, \ldots, a_m, v) & \mathbf{a} = (a_1, \ldots, a_k = v, \ldots, a_m), \end{cases} \tag{3.8}$$

where the maximum on the paths is taken over $\text{vp}_r(\text{pv}_r(\mathbf{a}, v, p))$.

In Algorithm 1, we present an efficient algorithm for ranking all vertices in a graph simultaneously by recursive-in-degree. Algorithm 1 works in $O(|V| \cdot |E|)$ time. A simple heuristic for improving the efficiency of the algorithm for practical purposes is to reduce the number of iterations, like in other fixed point algorithms such as PageRank. We shall now prove the correctness and complexity of this algorithm.

Proposition 3.7: Algorithm 1 outputs vertices in V in the order of $\preceq^{RID}$ as defined in Definition 3.8 and works in $O(|V| \cdot |E|)$ time.

Proof: Let us first denote

$$\begin{aligned} \text{vp}'_r(a_1, a_2, \ldots, a_m, \ldots) &= \frac{1}{n+1}[r(|P(a_1)| + \text{vp}'_r(a_2, \ldots, a_m, \ldots)] \\ \text{vp}'_r() &= 0. \end{aligned}$$

Note that for all $v \in V$ and for all $a_1, \ldots, a_m \in \text{Path}(v)$: If $a_1, \ldots, a_m$ is simple, $\text{vp}'_r(a_1, \ldots, a_m) = \text{vp}_r(a_1, \ldots, a_m)$. Otherwise if $a_n = a_i$, then $\text{vp}_r(a_1 \ldots, a_m) = \text{vp}'_r(a_1, \ldots a_m, a_{i+1}, \ldots, a_m, \ldots)$. Let $\mathbb{P}(v)$ be the set of all reverse paths to v in

G, simple or otherwise. We then have for all $v \in V$:

$$\text{value}_r(v) \quad = \quad \max_{p \in \text{Path}(v)} \text{vp}_r(p) = \max_{p \in \mathbb{P}(v)} \text{vp}'_r(p),$$

because the first loop in $p \in \mathbb{P}(v)$ can be replaced with the one maximizing $\text{vp}_r(\cdot)$, thus increasing value.

The iteration in step 3 of the algorithm calculates for all v:

$$\frac{1}{n+1}\left[r_0 + \max_{p_1 \in P(v)}\left[\cdots \frac{1}{n+1}\left[r_{|V|-1} + \max_{p_{|V|} \in P(p_{|V|-1})} \frac{1}{n+1} r_{|V|}\right]\cdots\right]\right],$$

where $r_i = r(|P(p_i)|)$ and $p_0 = v$. This value is equal to

$$\begin{aligned} & \max_{p_1 \in P(v)} \max_{p_2 \in P(p_1)} \cdots \max_{p_{|V|} \in P(p_{|V|}-1)} \sum_{i=0}^{|V|} \frac{r_i}{(n+1)^{i+1}} = \\ = \quad & \max_{(p_1,\ldots,p_{|V|+1}) \in \mathbb{P}_{|V|}(v)} \sum_{i=1}^{|V|+1} \frac{r_i}{(n+1)^i} = \\ = \quad & \max_{p \in \mathbb{P}_{|V|+1}(v)} \text{vp}'_r(v), \end{aligned} \tag{3.9}$$

where $\mathbb{P}_m(v)$ is the set of all reverse paths of length $\leq m$ to v, simple or otherwise.As there are only $|V|$ vertices, any two vertices that differ in the value assigned by the value function from (3.1) must also differ the value (3.9) calculated by the algorithm and in the same direction.

We shall now prove the time complexity of the algorithm, by tracing each step. Steps 1 and 2 take $O(|V|)$ time. The iteration in step 3 is repeated $|V|$ times, and for every vertex in V' performs $O(|P(v)|)$ calculations, so each iteration takes $O(|E|)$ time and thus the total time is $O(|V| \cdot |E|)$. Step 4 takes $O(|V'| \log |V'|) \leq O(|V| \log |E|) \leq O(|V| \cdot |E|)$. Finally, the output step 5 takes $O(|V|)$ time. As every step takes no more than $O(|V| \cdot |E|)$ time, so does the entire algorithm. $\square$

3.7 Axiomatization of Approval Voting

In Sections 3.4 and 3.5 we have seen mostly negative results which arise when trying to accommodate (weak) transitivity and RIIA. We have shown that although each of the axioms can be satisfied separately, there exists no general ranking system that satisfies both axioms.

Tennenholtz (2004) has previously shown a non-trivial ranking system that satisfies (weak) transitivity, and in the previous section we have seen such a system for RIIA. However, we have not provided a representation theorem for our new system.

In this section we provide a representation theorem for a ranking system that satisfies RIIA but not weak transitivity — the approval voting ranking system. This system ranks the agents based on the number of votes each agent received, with no regard to the rank of the voters. The axiomatization we provide in this section shows the power of RIIA, as it shows that there exists only one (interesting) ranking system that satisfies it without introducing transitive effects.

In order to specify our axiomatization, we recall several classical definitions from the theory of social choice.

The strong positive response axiom essentially means that if an agent receives additional votes, its rank must improve:

Definition 3.9: Let F be a ranking system. F satisfies *strong positive response* if for all graphs $G = (V, E)$ and for all $(v_1, v_2) \in (V \times V) \setminus E$, and for all $v_3 \in V$: Let $G' = (V, E \cup (v_1, v_2))$. If $v_3 \preceq^F_G v_2$, then $v_3 \prec^F_{G'} v_2$.

The anonymity and neutrality axioms mean that the names of the voters and alternatives respectively do not matter for the ranking:

Definition 3.10: A ranking system F satisfies *anonymity* if for all $G = (V, E)$, for all permutations $\pi : V \mapsto V$, and for all $v_1, v_2 \in V$: Let $E' = \{(\pi(v_1), v_2) | (v_1, v_2) \in E\}$. Then, $v_1 \preceq^F_{(V,E)} v_2 \Leftrightarrow v_1 \preceq^F_{(V,E')} v_2$.

Definition 3.11: A ranking system F satisfies *neutrality* if for all $G = (V, E)$, for all permutations $\pi : V \mapsto V$, and for all $v_1, v_2 \in V$: Let $E' = \{(v_1, \pi(v_2)) | (v_1, v_2) \in E\}$. Then, $v_1 \preceq^F_{(V,E)} v_2 \Leftrightarrow v_1 \preceq^F_{(V,E')} v_2$.

Arrow's classical Independence of Irrelevant Alternatives axiom requires that the relative rank of two agents be dependant only on the set of agents that preferred one over the other.

Definition 3.12: A ranking system F satisfies *Arrow's Independence of Irrelevant Alternatives (AIIA)* if for all $G = (V, E)$, for all $G' = (V, E')$, and for all $v_1, v_2 \in V$: Let $P_G(v_1) \setminus P_G(v_2) = P_{G'}(v_1) \setminus P_{G'}(v_2)$ and $P_G(v_2) \setminus P_G(v_1) = P_{G'}(v_2) \setminus P_{G'}(v_1)$. Then, $v_1 \preceq^F_G v_2 \Leftrightarrow v_1 \preceq^F_{G'} v_2$.

Our representation theorem states that together with positive response and RIIA, any one of the three independence conditions above (anonymity, neutrality, and AIIA) are essential and sufficient for a ranking system being AV[1]. In addition, we show that as in the classical social choice setting when only considering two-level preferences, positive response, anonymity, neutrality, and AIIA are an essential and sufficient representation of approval voting. This result extends the well known axiomatization of the majority rule due to May (1952):

Proposition 3.8 (May's Theorem): A social welfare functional over two alternatives is a majority social welfare functional if and only if it satisfies anonymity, neutrality, and positive response.

[1]In fact, an even weaker condition of *decoupling,* that in essence allows us to permute the graph structure while keeping the edges' names is sufficient in this case.

We can now formally state our theorem:

Theorem 3.9: Let F be a general ranking system. Then, the following statements are equivalent:

1. F is the approval voting ranking system ($F = AV$)
2. F satisfies positive response, anonymity, neutrality, and AIIA
3. F satisfies positive response, RIIA, and either one of anonymity, neutrality, and AIIA

Proof: It is easy to see that AV satisfies positive response, RIIA, anonymity, neutrality, and AIIA. It remains to show that (2) and (3) entail (1) above.

To prove (2) entails (1), assume that F satisfies positive response, anonymity, neutrality, and AIIA. Let $G = (V, E)$ be some graph and let $v_1, v_2 \in V$ be some agents. By AIIA, the relative ranking of v_1 and v_2 depends only on the sets $P_G(v_1) \setminus P_G(v_2)$ and $P_G(v_2) \setminus P_G(v_1)$. We have now narrowed our consideration to a set of agents with preferences over two alternatives, so we can apply Proposition 3.8 to complete our proof.

To prove (3) entails (1), assume that F satisfies positive response, RIIA and either anonymity or neutrality or AIIA. As F satisfies RIIA we can limit our discussion to comparison profiles. Let $f : \mathcal{P} \mapsto \{0, 1\}$ be the function from the definition of RIIA. We will use the notation $\mathbf{a} \preceq \mathbf{b}$ to mean $f\langle \mathbf{a}, \mathbf{b}\rangle = 1$, $\mathbf{a} \prec \mathbf{b}$ to mean $f\langle \mathbf{b}, \mathbf{a}\rangle = 0$, and $\mathbf{a} \simeq \mathbf{b}$ to mean $\mathbf{a} \preceq \mathbf{b}$ and $\mathbf{b} \preceq a$.

By the definition of RIIA, it is easy to see that $\mathbf{a} \simeq \mathbf{a}$ for all $\mathbf{a}$. By positive response it is also easy to see that $(\underbrace{1, 1, \ldots, 1}_{n}) \preceq (\underbrace{1, 1, \ldots, 1}_{m})$ iff $n \leq m$. Let $P = \langle (a_1, \ldots, a_n), (b_1, \ldots, b_m)\rangle$ be a comparison profile. Let $G = (V, E)$ be the following graph (an example of such graph for the profile $\langle (1, 3, 3), (2, 4)\rangle$ is in Figure 3.7):

$$
\begin{aligned}
V &= \{x_1, \ldots, x_{\max\{a_n, b_m\}}\} \cup \\
&\quad \cup \{v_1, \ldots, v_n, v'{}_1, \ldots, v'_n, v\} \cup \\
&\quad \cup \{u_1, \ldots, u_m, u'_1, \ldots, u'_m, u\} \\
E &= \{(x_i, v_j) | i \leq a_j\} \cup \{(x_i, u_j) | i \leq b_j\} \cup \\
&\quad \cup \{(v_i, v) | i = 1, \ldots, n\} \cup \{(u_i, u) | i = 1, \ldots, m\}.
\end{aligned}
$$

It is easy to see that in the graph G, v and u satisfy the profile P. Let π be the following permutation:

$$
\pi(x) = \begin{cases} v'_i & x = v_i \\ v_i & x = v'_i \\ u'_i & x = u_i \\ u_i & x = u'_i \\ x & \text{Otherwise.} \end{cases}
$$

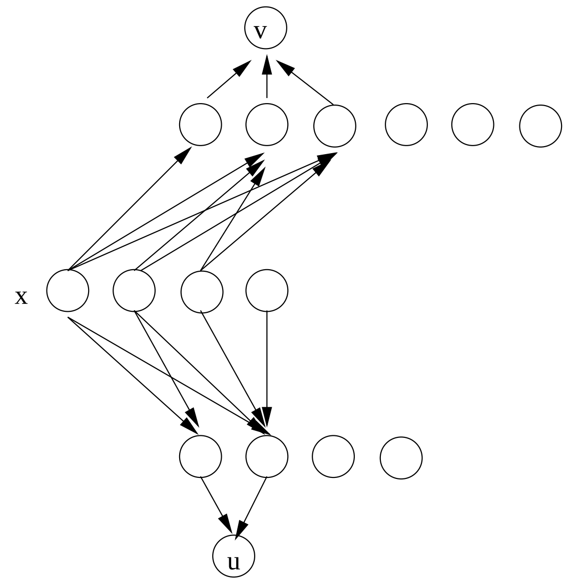


Figure 3.7: Example graph G for the profile $\langle(1,3,3),(2,4)\rangle$

The remainder of the proof depends on which additional axiom F satisfies:

- If F satisfies anonymity, let $E' = \{(\pi(x), y) | (x, y) \in E\}$. Note that in the graph (V, E') v and u satisfy the profile $\langle(\underbrace{1,1,\ldots,1}_{n}),(\underbrace{1,1,\ldots,1}_{m})\rangle$, and thus $v \preceq^F_{(V,E')} u \Leftrightarrow n \le m$. By anonymity, $u \preceq^F_{(V,E)} v \Leftrightarrow u \preceq^F_{(V,E')} v$, thus proving that $f(P) = 1 \Leftrightarrow n \le m$ for an arbitrary comparison profile P, and thus $F = AV$.
- If F satisfies neutrality, let $E' = \{(x, \pi(y)) | (x, y) \in E\}$. Note that in the graph (V, E') v and u satisfy the profile $\langle(\underbrace{1,1,\ldots,1}_{n}),(\underbrace{1,1,\ldots,1}_{m})\rangle$, and thus $v \preceq^F_{(V,E')} u \Leftrightarrow n \le m$. By neutrality, $u \preceq^F_{(V,E)} v \Leftrightarrow u \preceq^F_{(V,E')} v$, again showing that $f(P) = 1 \Leftrightarrow n \le m$ for an arbitrary comparison profile P, and thus $F = AV$.
- If F satisfies AIIA, let $E' = \{(x, \pi(y)) | (x, y) \in E\}$ as before. So, also $v \preceq^F_{(V,E')} u \Leftrightarrow n \le m$. Note that $P_G(v) = P_{(V,E')}(v)$ and $P_G(u) = P_{(V,E')}(u)$, so by AIIA, $u \preceq^F_{(V,E)} v \Leftrightarrow u \preceq^F_{(V,E')} v$, and thus as before, $F = AV$.

$\square$

Chapter 4

Incentive Compatible Ranking Systems

4.1 Introduction

Ranking systems to not exist in empty space. Many ranking systems systems settings involve self-interested agents who try and manipulate the ranking system in order to improve their own position in the resulting ranking. It is of great importance to design ranking systems that are resistant to such manipulations, and to study the conditions for their existence.

The issue of incentives has been extensively studied in the classical social choice literature. The Gibbard–Satterthwaite theorem (Gibbard, 1973; Satterthwaite, 1975) shows that in the classical social welfare setting, it is impossible to aggregate the rankings in a strategy-proof fashion under some basic conditions. The incentives of the candidates themselves were considered in the context of elections (Dutta et al., 2001), where a related impossibility result is presented. Another notion of incentives was considered in the case where a single agent may create duplicates of itself (Cheng and Friedman, 2005). Furthermore, the computation of equilibria in the more abstract context of ranking games was also discussed (Brandt et al., 2006).

In this chapter we present our research on the issue of incentives in ranking systems. We define two notions of incentive compatibility, where the agent is concerned with its expected position in the ranking under *affine* or *general* utility functions.

We then consider some very basic properties of ranking systems, which are satisfied by almost all known ranking systems, and prove that these properties cannot be all satisfied by an incentive compatible ranking system. This finding is far from trivial, as different ranking systems may require different manipulations by an agent in order to increase its rank in different situations. Furthermore, we show that when we assume only a subset of the basic properties, some artificial incentive compatible ranking systems can be constructed.

Together, these results form a complete characterization of incentive compatible ranking systems under these basic properties.

Our results expose some surprising and illuminating effects of some basic properties one may require a ranking system to satisfy on the existence of incentive compatible ranking systems.

Next, we consider *non imposing* ranking systems, i.e. systems in which any strict ordering of the agents is feasible. We show that there are no fully incentive compatible general non imposing ranking systems, and provide a full axiomatization of a non-imposing incentive compatible ranking system for the setting with exactly three agents. We then briefly discuss the strong fairness axiom of isomorphism.

As full incentive compatibility is shown to be practically impossible, we proceed to define three notions of limited incentive compatibility. We use these notions to quantify the incentive compatibility of known ranking systems and to prove general bounds. Specifically, we quantify the incentive compatibility of the Approval Voting and PageRank ranking systems and prove a significant lower bound on the incentive compatibility of any ranking system satisfying the basic *strong monotonicity* property, which is satisfied by almost all practical ranking systems. When non-imposition is considered, we show a ranking system that is incentive compatible up to a deviation by one agent by at most one rank. This sets a tight bound as no such fully incentive compatible ranking system exists.

This chapter is structured as follows: In Section 4.2 we define some basic properties of ranking systems. In Section 4.3 we introduce our two notions of incentive compatibility. We then show a strong possibility result in Section 4.4, when we do not assume the *minimal fairness* property. In Section 4.5 we provide a full classification of the existence of incentive compatible ranking systems when we do assume minimal fairness. Section 4.6 provides some illuminating lessons learned from this classification. In section 4.7 we define the non-imposition property and show that although no general incentive compatible non-imposing ranking system exists, one does exist for exactly three agents. In Section 4.8 we introduce the isomorphism property and briefly discuss to the classification of incentive compatibility under isomorphism. In Section 4.9 we define our weaker notions of incentive compatibility, quantify the incentive compatibility some existing and new ranking systems, and prove some upper and lower bounds on incentive compatibility.

4.2 Basic Properties of Ranking Systems

As this chapter deals with incentives, we find it best to assume that self edges are not allowed in the input to the ranking systems.

In order to classify the incentive compatibility features of ranking systems, we must first define the criteria for the classification. We define some very basic properties that are satisfied by almost all known ranking systems. Most properties have two versions – one weak and one strong, both satisfied by almost all known ranking systems.

First of all, we define the notion of a trivial ranking system, which ranks any two vertices the same way in all graphs.

Definition 4.1: A ranking system F is called *trivial* if for all vertices v_1, v_2 and for all graphs G, G' which include these vertices: $v_1 \preceq_G^F v_2 \Leftrightarrow v_1 \preceq_{G'}^F v_2$. A ranking system F is called *nontrivial* if it is not trivial.

A ranking system F is called *infinitely nontrivial* if there exist vertices v_1, v_2 such that for all $N \in \mathbb{N}$ there exists $n > N$ and graphs $G = (V, E)$ and $G' = (V', E')$ s.t. $|V| = |V'| = n$, $v_1 \preceq_G^F v_2$, but $v_2 \prec_{G'}^F v_1$.

A basic requirement from a ranking system is that when there are no votes (or all votes) in the system, all agents must be ranked equally. We call this requirement minimal fairness[1].

Definition 4.2: A ranking system F is *minimally fair* if for every graph $G = (V, \emptyset)$ with no edges, and for every $v_1, v_2 \in V$: $v_1 \simeq_G^F v_2$. It further satisfies *strong minimal fairness* if for every graph $G_\top = (V, V \times V \setminus \{(v,v)|v \in V\})$ with all edges and for every $v_1, v_2 \in V$: $v_1 \simeq_{G_\top}^F v_2$.

Another basic requirement from a ranking system is that as agents gain additional votes, their rank must improve, or at least not worsen. Surprisingly, this vague notion can be formalized in (at least) two distinct ways: the monotonicity property considers the situation where one agent has a superset of the votes another has *in the same graph*, while the positive response[2] property considers the addition of a vote for an agent *between graphs*. This distinction is important because, as we will see, the two properties are neither equivalent, nor imply each other.

Definition 4.3: Let F be a ranking system. F satisfies *weak positive response* if for all graphs $G = (V, E)$ and for all $(v_1, v_2) \in (V \times V) \setminus \{(v,v)|v \in V\} \setminus E$, and for all $v_3 \in V \setminus \{v_2\}$: Let $G' = (V, E \cup (v_1, v_2))$. Then, $v_3 \preceq_G^F v_2$ implies $v_3 \preceq_{G'}^F v_2$ and $v_3 \prec_G^F v_2$ implies $v_3 \prec_{G'}^F v_2$. F furthermore satisfies *strong positive response* if $v_3 \preceq_G^F v_2$ implies $v_3 \prec_{G'}^F v_2$.

Definition 4.4: A ranking system F satisfies *weak monotonicity* if for all $G = (V, E)$ and for all $v_1, v_2 \in V$: If $P(v_1) \subseteq P(v_2)$ then $v_1 \preceq_G^F v_2$. F furthermore satisfies *strong monotonicity* if $P(v_1) \subsetneq P(v_2)$ additionally implies $v_1 \prec_G^F v_2$.

Example 4.1: Consider the graphs G_1 and G_2 in Figure 4.1. Assume a ranking system F ranks $a \simeq_{G_1}^F d$ in graph G_1. Then, if F satisfies weak positive

[1] A stronger notion of fairness, the isomorphism property, will be considered in Section 4.8.

[2] We have previously defined strong positive response in the context of the axiomatization of Approval Voting in Section 3.7 on page 41. We expand this definition below.

(a) G_1 (b) G_2

Figure 4.1: Example graphs for the basic properties of ranking systems

response, it must also rank $a \preceq^F_{G_2} d$ in G_2. If F satisfies the strong positive response, then it must strictly rank $a \prec^F_{G_2} d$ in G_2. However, if we do not assume $a \preceq^F_{G_1} d$, F may rank a and d arbitrarily in G_2.

Now consider the graph G_1, and note that $P(a) = \{c\} \subsetneq \{c, d\} = P(b)$. This is the requirement of the weak (and strong) monotonicity property, and thus any ranking system F that satisfies weak monotonicity must rank $a \preceq^F_{G_1} b$, and if it satisfies strong monotonicity, it must strictly rank $a \prec^F_{G_1} b$.

Note that the weak monotonicity property implies minimal fairness. This is due to the fact that when no votes are cast, all vertices have exactly the same predecessor sets and thus must be ranked equally.

Yet another simple requirement from a ranking system is that it does not behave arbitrarily differently when two sets of agents with their respective votes are considered one set.

Definition 4.5: Let F be a ranking system and let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs s.t. $V_1 \cap V_2 = \emptyset$ and let $v_1, v_2 \in V_1$ be two vertices. Let $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. F satisfies the *weak union condition* if $v_1 \preceq^F_{G_1} v_2 \Leftrightarrow v_1 \preceq^F_{G_1 \cup G_2} v_2$. Let $G' = (V_1 \cup V_2, E_1 \cup E_2 \cup E)$, where $E \subseteq V_1 \times V_2$ is in an arbitrary set of edges from V_1 to V_2. F satisfies the *strong union condition* if $v_1 \preceq^F_{G_1} v_2 \Leftrightarrow v_1 \preceq^F_{G'} v_2$.

Surprisingly, we will see that even the weak union condition has great significance towards the existence of a ranking system or lack thereof. One reason for this effect, is that a ranking system satisfying this condition cannot behave differently depending on the size of the graph.

4.2.1 Satisfiability

As we have mentioned above, these properties are very basic and, with the exception of the strong union condition, all the properties above are satisfied by almost all known ranking systems such as the PageRank ranking system (with a damping factor) and the authority ranking by the Hubs&Authorities algorithm (Kleinberg, 1999). These ranking systems *do not* satisfy the strong union condition, as in both systems outgoing links outside an agent's strongly connected component may affect ranks inside the strongly connected component, either by dividing the importance (in PageRank) or by affecting the hubbiness score in Hubs&Authorities.

Furthermore, the simple *approval voting* ranking system (see definition 3.5 on page 32) satisfies all the strong properties mentioned above including the strong union condition.

Fact 4.1: The approval voting ranking system AV satisfies strong minimal fairness, strong monotonicity, strong positive response, the strong union condition, and infinite nontriviality.

These facts lead us to believe that the properties defined above (perhaps with the exception of the strong union condition), should all be satisfied by any reasonable ranking system, at least in their weak form. We will soon show that this is not possible when requiring incentive compatibility.

4.3 Incentive Compatibility

Ranking systems do not exist in empty space. The results given by ranking systems frequently have implications for the agents being ranked, which are the same agents that are involved in the ranking. Therefore, the incentives of these agents should in many cases be taken into consideration.

In our approach, we require that our ranking system will not rank agents better for stating untrue preferences, but we assume that the agents are interested only in their own ranking (and not, say, in the ranking of those they prefer).

We assume that for strict rankings (with no ties), for every agent count n, there exists a utility function $u_n : \mathbb{N} \mapsto \mathbb{R}$ that maps an agent's *rank* (i.e. the number of agents ranked below it) to a utility value for being ranked that way. We assume u_n is nondecreasing, that is every agent weakly prefers to be ranked higher.

This utility function can be extended to the case of ties, by treating these as a uniform randomization over the matching strict orders. Thus the utility of an agent with k agents strictly below it and m agents tied is

$$E[u_n] = u_n^*(k, m) = \frac{1}{m} \sum_{i=k}^{k+m-1} u_n(i).$$

We can now define the utility of a ranking for an agent as follows:

Definition 4.6: The *utility* $u_G^F(v)$ of a vertex v in graph $G = (V, E)$ under the ranking system F and utility function u is defined as

$$\begin{aligned} u_G^F(v) &= u^*_{|V|}(|\{v' : v' \prec v\}|, |\{v' : v' \simeq v\}|) = \\ &= \frac{1}{|\{v' : v' \simeq v\}|} \sum_{i=|\{v':v' \prec v\}|}^{|\{v':v' \preceq v\}|-1} u_n(i). \end{aligned}$$

This definition allows us to define a preference relation over rankings for each agent. Using this preference relation, we can now define the general notion of incentive compatibility as immunity of utility to manipulation of outgoing edges[3]:

Definition 4.7: Let F be a ranking system. F is called *incentive compatible* under utility function u if for all graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ s.t. for some $v \in V$, and for all $v' \in V \setminus \{v\}, v'' \in V : (v', v'') \in E_1 \Leftrightarrow (v', v'') \in E_2$: $u_{G_1}^F(v) = u_{G_2}^F(v)$.

A strong notion of incentive compatibility is compatibility under *any* utility function:

Definition 4.8: Let F be a ranking system. F satisfies *strong incentive compatibility* if for any nondecreasing utility function $u : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{R}$, F is incentive compatible under u.

A simple utility function one may consider is the identity function $u_n(k) \equiv k$. This basic utility function means that any change in rank has the same significance. The utility of a ranking with k weaker agents and m equal agents under this function is:

$$u_n^*(k, m) = \frac{1}{m} \sum_{i=k}^{k+m-1} u_n(i) = k + \frac{m-1}{2}.$$

It turns out that the preference relation over rankings produced by the identity utility function is the same as the one produced by any affine utility function $u(k) = a \cdot k + b$, as $u_n^*(k, m)$ in this case is simply $a \cdot (k + \frac{m-1}{2}) + b$. Therefore, it is interesting to look at incentive compatibility under an affine utility function u:

Definition 4.9: Let F be a ranking system and let. F is called *weakly incentive compatible* if for every utility function $u : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{R}$ such that $u_n(k) = a \cdot k + b$ for some constants $a, b \in \mathbb{R}$: F is incentive compatible under u.

[3] A more general discussion of manipulations in personalized ranking systems is available in Section 5.3.1.

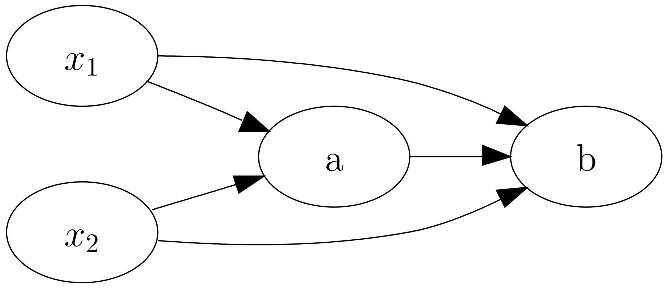


Figure 4.2: Example graph for ranking system F

Notation: In order to prevent ambiguity, in the remainder of this chapter we will use $r_G^F(v)$ ("rank") to denote $u_G^F(v)$ under the utility function $u_n(k) = k+\frac{1}{2}$. So that

$$r_G^F(v) = |\{v' : v' \prec v\}| + \frac{1}{2} |\{v' : v' \simeq v\}| .$$

Note that due to the fact that all affine ranking functions give the same ordering over $u^*(k, m)$, we can, wlog, consider only $u_n(k) = k + \frac{1}{2}$ when proving weak incentive compatibility or lack thereof.

Interestingly, we will see in the remainder of this chapter that these incentive compatibility properties are very hard to satisfy, and no common nontrivial ranking system satisfies them. In particular, the PageRank, Hubs&Authorities, and Approval Voting ranking systems mentioned above are not weakly incentive compatible.

Example 4.2: One may think that under positive response, impossibility of weak incentive compatibility is a direct result of an alleged dominant strategy not to vote for any agent.

However, this is not true, as sometimes the best response does involve voting for some agent. Consider the ranking system F defined by:

$$v_1 \preceq_G^F v_2 \Leftrightarrow |P(v_1)| + \tfrac{1}{3}|S(v_1)| \leq |P(v_2)| + \tfrac{1}{3}|S(v_2)|.$$

This ranking system satisfies strong positive response, but is not weakly incentive compatible. For example, in the graph depicted in Figure 4.2, the agent a can improve its rank either by not voting for b, or by voting for both x_1 and x_2. The maximal increase in a's rank is achieved by doing both.

Note that under this ranking system, agents do not have a dominant strategy that maximizes their rank, and thus there is no general dominant deviation that demonstrates lack of incentive compatibility.

4.4 Possibility without Minimal Fairness

To begin our classification of the existence of incentive compatible ranking systems, we first consider ranking systems which do not satisfy minimal fairness. We have already seen that minimal fairness is implied by weak monotonicity, so we cannot hope to satisfy weak monotonicity without minimal fairness. As it turns out, the strong versions of all the remaining properties considered above can, in fact, be satisfied simultaneously.

Proposition 4.2: There exists a ranking system F_1 that satisfies strong incentive compatibility, strong positive response, infinite nontriviality, and the strong union condition.

Proof: Assume a lexicographic order $<$ over vertex names, and assume three consecutive vertices $v_1 < v_2 < v_3$. Then, F_1 is defined as follows (let $G = (V, E)$ be some graph):

$$\begin{aligned} v \preceq_G^{F_1} u \quad \Leftrightarrow \quad & [v \leq u \wedge (v \neq v_2 \vee u \neq v_3)] \vee \\ & [v = v_2 \wedge u = v_3 \wedge (v_1, v_2) \notin E] \vee \\ & [v = v_3 \wedge u = v_2 \wedge (v_1, v_2) \in E]. \end{aligned}$$

That is, vertices are ranked strictly according to their lexicographic order, except when $(v_1, v_2) \in E$, whereas the ranking of v_2 and v_3 is reversed.

F_1 is infinitely nontrivial because graphs with the vertices v_1, v_2, v_3 are ranked differently depending on the existence of the edge (v_1, v_2), and these exist for any $|V| \geq 3$.

F_1 satisfies strong incentive compatibility because the only vertex that can make any change in the ranking is v_1 and it cannot ever change its own position in the ranking at all.

F_1 satisfies strong positive response because the ordering of the vertices remains unchanged by anything but the (v_1, v_2) edge, and is always strict. The addition of the (v_1, v_2) edge only increases the relative rank of v_2 as required.

Assume for contradiction that F_1 does not satisfy the strong union condition. Then, there exist two disjoint graphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ and an edge set $E \subseteq V_1 \times V_2$ such that the ranking $\preceq_G^{F_1}$ of graph $G = (V_1 \cup V_2, E_1 \cup E_2 \cup E)$ is inconsistent with $\preceq_{G_1}^{F_1}$. First note that the only inconsistency that may arise is with the ranking of v_2 compared to v_3. Therefore, $\{v_2, v_3\} \subseteq V_1$. Furthermore, for the ranking to be inconsistent $(v_1, v_2) \notin E_1$ and $(v_1, v_2) \in E_1 \cup E_2 \cup E$ (the opposite is impossible due to inclusion). Furthermore, $v_2 \in V_1 \Rightarrow v_2 \notin V_2 \Rightarrow (v_1, v_2) \notin V_1 \times V_2 \Rightarrow (v_1, v_2) \notin E$. Thus we conclude that $(v_1, v_2) \in E_2$, and thus $v_2 \in V_2$, in contradiction to the fact that $v_2 \in V_1$. $\square$

4.5 Full Classification under Minimal Fairness

We are now ready to state our main results:

Theorem 4.3: There exist weakly incentive compatible, infinitely nontrivial, minimally fair ranking systems F_2, F_3, F_4 that satisfy weak monotonicity; weak positive response; and the weak union condition respectively. However, there is no weakly incentive compatible, nontrivial, minimally fair ranking system that satisfies any two of those three properties.

Theorem 4.4: There is no weakly incentive compatible, nontrivial, minimally fair ranking system that satisfies either one the four properties: strong monotonicity, strong positive response, the strong union condition and strong incentive compatibility.

The proof of these two theorems is split into ten different cases that must be considered – three possibility proofs for F_2, F_3. and F_4, three impossibility results with pairs of weak properties, and four impossibility results with each of the strong properties. We will now prove each of these cases.

4.5.1 Possibility Proofs

Proposition 4.5: There exists a weakly incentive compatible ranking system F_2 that satisfies minimal fairness, weak positive response, and infinite nontriviality.

Proof: Let v_1, v_2, v_3 be some vertices and let $G = (V, E)$ be some graph, then F_2 is defined as follows:

$$\begin{aligned} v \preceq u \quad \Leftrightarrow \quad & [v \neq v_3 \wedge u \neq v_2] \vee v = u \vee \\ & (v_1, v_3) \notin E \vee v_2 \notin V. \end{aligned}$$

That is, F_2 ranks all vertices equally, except when the edge (v_1, v_3) exists. Then, F_2 ranks $v_2 \prec v \simeq u \prec v_3$ for all $v, u \in V \setminus \{v_2, v_3\}$.

F_2 satisfies minimal fairness because when no edges exist, the clause $(v_1, v_3) \notin E$ always matches, and thus all vertices are ranked equally, as required. F_2 satisfies infinite nontriviality, because for all $|V| \geq 3$ there exists a graph which includes the vertices v_1, v_2, v_3 and the edge (v_1, v_3), which is ranked nontrivially.

F_2 satisfies weak positive response because the only edge addition that changes the ranks of the vertices in the graph (the addition of (v_1, v_3)) indeed doesn't weaken the target vertex v_3.

F_2 is weakly incentive compatible because only v_1 can affect the ranking of the vertices in the graph (by voting for v_3 or not), but $r(v_1)$ is always $\frac{|V|}{2}$. $\square$

Proposition 4.6: There exists a weakly incentive compatible ranking system F_3 that satisfies minimal fairness, the weak union condition, and infinite nontriviality.
Proof: Let v_1, v_2, v_3 be some vertices and let $G = (V, E)$ be some graph, then F_3 is defined as follows:

$$\begin{aligned} v \preceq u \quad \Leftrightarrow \quad & [v \neq v_3 \wedge u \neq v_2] \vee v = u \vee \\ & \{(v_1, v_2), (v_1, v_3)\} \not\subseteq E. \end{aligned}$$

That is, F_3 ranks all vertices equally, except when the edges $(v_1, v_2), (v_1, v_3)$ exist. Then, F_3 ranks $v_2 \prec v \simeq u \prec v_3$ for all $v, u \in V \setminus \{v_2, v_3\}$.

F_3 satisfies minimal fairness because when no edges exist, the clause $\{(v_1, v_2), (v_1, v_3)\} \not\subseteq E$ always matches, as required. F_3 satisfies infinite nontriviality, because for all $|V| \geq 3$ there exists a graph which includes the vertices v_1, v_2, v_3 and the edges $\{(v_1, v_2), (v_1, v_3)\}$, which is ranked nontrivially.

To prove F_3 satisfies the weak union condition, let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be some graphs such that $V_1 \cap V_2 = \emptyset$, and let $G = G_1 \cup G_2$. If $\{(v_1, v_2), (v_1, v_3)\} \not\subseteq E_1 \cup E_2$ then by the definition of F_3, it must rank all vertices in all graphs G_1, G_2, G equally, as required. Otherwise, for all $v, u \in (V_1 \cup V_2) \setminus \{v_2, v_3\}$: $v_2 \prec_G^{F_3} v \simeq_G^{F_3} u \prec_G^{F_3} v_3$. Assume wlog that $(v_1, v_2) \in E_1$ and thus $v_1, v_2 \in V_1$. But then also $(v_1, v_3) \in E_1$ and thus also $v_3 \in V_1$. By the definition of F_3, for all $v, u \in V_1 \setminus \{v_2, v_3\}$: $v_2 \prec_{G_1}^{F_3} v \simeq_{G_1}^{F_3} u \prec_{G_1}^{F_3} v_3$. As $v_1, v_2, v_3 \notin G_2$, trivially for all $v, u \in V_2$: $v \simeq_{G_2}^{F_3} u$, as required.

F_3 is weakly incentive compatible because only v_1 (if at all) can affect the ranking of the vertices in the graph (by voting for v_2 and v_3 or not), but $r(v_1)$ is always $\frac{|V|}{2}$. □

Proposition 4.7: There exists a weakly incentive compatible ranking system F_4 that satisfies minimal fairness, weak monotonicity, and infinite nontriviality.
Proof: The ranking system F_4 ranks all vertices equally, except for graphs $G = (V, E)$ for which $|V| \geq 7$, where $V = \{w, s, m_0, \ldots, m_{n-1}\}$, and for all $i \in \{0, \ldots, n-1\}$: $(m_i, s) \in E$, $(m_i, w) \notin E$, and for all $j \in \{0, \ldots, n-1\}$: $(m_i, m_j) \in E$ if and only if $j = (i+1) \bmod n$ or $j = (i+2) \bmod n$. Figure 4.3 includes an example graph that satisfies these conditions. In such graphs, F_4 ranks $w \prec_G^{F_4} m_1 \simeq_G^{F_4} \cdots \simeq_G^{F_4} m_n \prec_G^{F_4} s$.

F_4 is minimally fair by definition, as when there are no edges, all vertices are ranked equally. F_4 satisfies infinite nontriviality because such nontrivially ranked graphs G exist for all $|V| \geq 7$.

F_4 satisfies weak monotonicity because in the graphs that it doesn't rank all vertices equally we see that $P(w) \not\supseteq P(m_i) \not\supseteq P(s)$ for all $i \in \{0, \ldots n-1\}$, which is consistent with the ordering F_4 specifies.

To prove F_4 is weakly incentive compatible, we let G_1, G_2 be two graphs that differ only in the outgoing edges of a single vertex v, and show that $r_{G_1}^{F_4}(v) = r_{G_2}^{F_4}(v)$. Because all graphs in which not all vertices are ranked equally are of

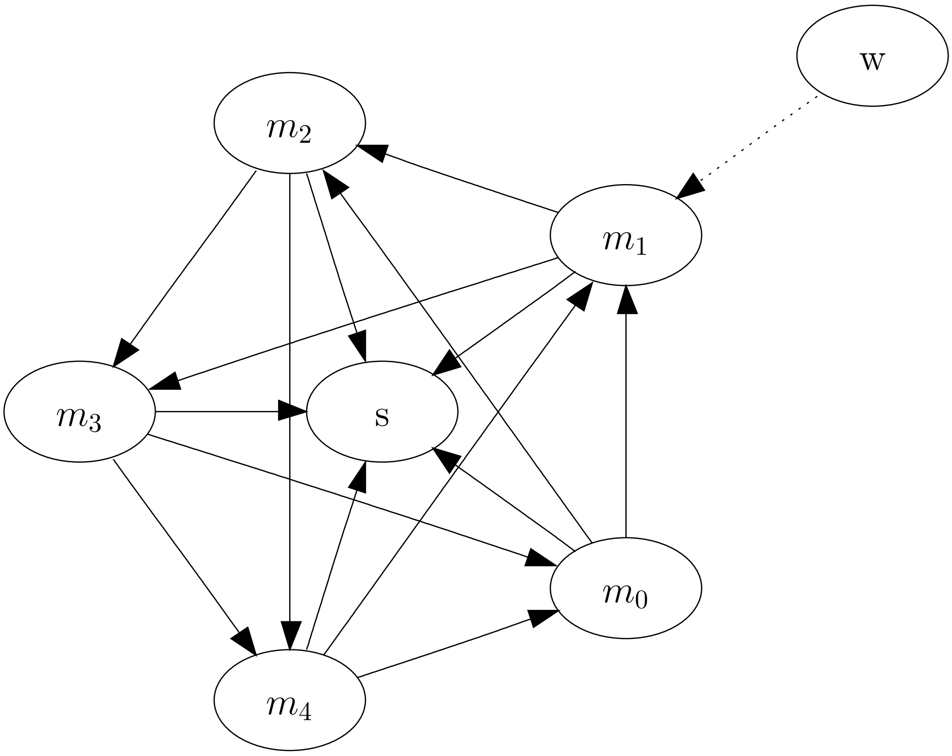


Figure 4.3: Nontrivially ranked graph for F_4

the form defined above, at least one of the graphs G_1, G_2 must have this form. Let us assume wlog that this graph is G_1, and mark the vertices of this graph as defined above.

Now consider two cases:

1. If $v = w$ or $v = s$, then by the definition of F_4, $\preceq_{G_1}^{F_4} \equiv \preceq_{G_2}^{F_4}$, thus trivially, $r_{G_1}^{F_4}(v) = r_{G_2}^{F_4}(v)$, as required.

2. If $v = m_i$ for some $i \in \{0, \ldots, n-1\}$, then first note that $r_{G_1}^{F_4}(v) = \frac{|V|}{2}$. If G_2 is not of the form defined above then all its vertices are ranked equally and specifically $r_{G_2}^{F_4} = \frac{|V|}{2}$, as required. Otherwise, G_2 is of the form defined above. Let w' and s' be the w and s vertices for G_2 in the form defined above. By the definition, $2 \leq |P_{G_1}(v)| \leq 4$, while $|P_{G_2}(w')| \leq 1$ and $|P_{G_2}(s')| \geq 5$. Therefore, $v \notin \{w', s'\}$. By the definition of F_4, $r_{G_2}^{F_4}(v) = \frac{|V|}{2}$, as required.

$\square$

4.5.2 Impossibility proofs with pairs of weak properties

We prove the impossibility results with pairs of weak properties, by assuming existence of a ranking system and analyzing the minimal graph in which the ranking system does not rank all agents equally. This is done in the following lemma:

Lemma 4.8: Let F be a weakly incentive compatible minimally fair nontrivial ranking system. Then, there exists a graph $G = (V, E)$ and vertices $v_\perp, v_\top, v \in V$ such that:

1. For all graphs $G' = (V', E')$ where $|E'| < |E|$ or $|E'| = |E|$ and $|V'| < |V|$, $v_1 \simeq^F_{G'} v_2$ for all $v_1, v_2 \in V'$.
2. $r^F_G(v) = \frac{|V|}{2}$
3. $v_\perp \prec^F_G v \prec^F_G v_\top$
4. For all $v' \in V$: $v_\perp \preceq^F_G v' \preceq^F_G v_\top$.
5. $S(v) \neq \emptyset$ and for all $v' \in V$ such that $S(v') \neq \emptyset$: $v' \simeq^F_G v$.

Proof: Let $G = (V, E)$ be a minimal (in edges, then vertices) graph such that there exist v_1, v_2 where $v_1 \prec^F_G v_2$. Such a graph exists because F is nontrivial. This graph immediately satisfies condition 1. Let $v_\perp, v_\top$ be vertices such that for all $v' \in V$: $v_\perp \preceq^F_G v' \preceq^F_G v_\top$ (such vertices exist because $\preceq$ is an ordering). Note that these vertices satisfy condition 4.

$E \neq \emptyset$ because minimal fairness will force $v_1 \simeq v_2$. Let $(v, v') \in E$ be some edge. From minimallity, $r^F_{(V, E \setminus \{(v,v')\})}(v) = \frac{|V|}{2}$. From weak incentive compatibility, $r^F_G(v) = \frac{|V|}{2}$, satisfying condition 2. Therefore,

$$\begin{aligned}
\tfrac{1}{2}\left|\{v'|v' \prec v\}\right| + \tfrac{1}{2}\left|\{v'|v' \preceq v\}\right| &= \tfrac{1}{2}|V| \\
\left|\{v'|v' \prec v\}\right| + \left|\{v'|v' \preceq v\}\right| &= \left|\{v'|v' \preceq v\}\right| + \\
&\quad + \left|\{v'|v' \succ v\}\right| \\
\left|\{v'|v' \prec v\}\right| &= \left|\{v'|v' \succ v\}\right|.
\end{aligned}$$

From the assumption that $v_1 \prec^F_G v_2$: $v_\perp \preceq^F_G v_1 \prec^F_G v_2 \preceq^F_G v_\top$. Therefore, $v_\perp \prec v$ or $v \prec v_\top$. But as $|\{v'|v' \prec v\}| = |\{v'|v' \succ v\}|$, and at least one is nonempty, both $v_\perp \prec v \prec v_\top$, satisfying condition 3.

Condition 5 is satisfied by noting that for all v' such that $S(v) \neq \emptyset$, $r^F_G(v') = \frac{|V|}{2} = r^F_G(v)$, and thus $v' \simeq^F_G v$. $\square$

Now we can prove the impossibility results for any pair of weak properties:

Proposition 4.9: There exists no weakly incentive compatible nontrivial ranking system that satisfies the weak monotonicity and weak positive response conditions.

Proof: Assume for contradiction a ranking system F that satisfies the conditions. First note that F is minimally fair, because in a graph with no edges, all vertices have exactly the same predecessor set. Thus, the conditions of Lemma 4.8 are satisfied, so we can let $G = (V, E)$ and $v, v_\bot, v_\top \in V$ be the graph and the vertices from the lemma.

Now, let $(v_1, v_2) \in E$ be some edge. Let $G' = (V, E \setminus \{(v_1, v_2)\})$. By condition 1, $v_2 \simeq^F_{G'} v_\top$. By weak positive response, $v_\top \preceq^F_G v_2$. Since this is true for all $v_2 \in V$ with $P(v_2) = \emptyset$, and $v_\bot \prec^F_G v \prec^F_G v_\top$, we conclude that $P_G(v_\bot) = P_G(v) = \emptyset$. Now, by weak monotonicity $v_\bot \simeq^F_G v$, in contradiction to the fact that $v_\bot \prec^F_G v$. □

Proposition 4.10: There exists no weakly incentive compatible nontrivial ranking system that satisfies the weak monotonicity and weak union conditions.
Proof: Assume for contradiction a ranking system F that satisfies the conditions. First note that F is minimally fair, because in a graph with no edges, all vertices have exactly the same predecessor set. Thus, the conditions of Lemma 4.8 are satisfied, so we can let $G = (V, E)$ and $v, v_\bot, v_\top \in V$ be the graph and the vertices from the lemma.

Now let $G' = (V \cup \{x\}, E)$ be a graph with an additional vertex $x \notin V$. By the weak union condition, $v_\bot \prec^F_{G'} v$. By weak monotonicity, $x \preceq^F_{G'} v_\bot$. Therefore, by the weak union condition, $r^F_{G'}(v) = r^F_G(v) + 1 = \frac{|V|}{2} + 1$. Let $G'' = (V \cup \{x\}, E \setminus \{(v', v) | v' \in V\})$. By condition 1 and the fact that $S_{G'}(v) \neq \emptyset$, $r^F_{G''}(v) = \frac{|V|+1}{2}$. From weak incentive compatibility, $r^F_{G''}(v) = r^F_{G'}(v)$, which is a contradiction. □

Proposition 4.11: There exists no weakly incentive compatible nontrivial minimally fair ranking system that satisfies the weak union and weak positive response conditions.
Proof: Assume for contradiction a ranking system F that satisfies the conditions. As the conditions of Lemma 4.8 are satisfied, let $G = (V, E)$ and $v, v_\bot, v_\top \in V$ be the graph and the vertices from the lemma. Now let $G_1 = (V \setminus \{v_\bot\}, E)$ and let $G_2 = (\{v_\bot\}, \emptyset)$. From conditions 3 and 5, $S(v_\bot) = \emptyset$. If $P_G(v_\bot) \neq \emptyset$, then by condition 1 in the graph $G' = (V, E \setminus \{(x, v_\bot)\})$ where $x \in P_G(v_\bot)$, $v_\top \preceq^F_{G'} v_\bot$. But then by weak positive response $v_\top \preceq^F_G v_\bot$ in contradiction to condition 3.

Therefore, $P_G(v_\bot) = S_G(v_\bot) = \emptyset$. Thus, G_1 and G_2 satisfy the conditions of the weak union condition with regard to G. Therefore, $v \prec^F_G v_\top \Rightarrow v \prec^F_{G_1} v_\top$, in contradiction to condition 1, because the edge set is the same and $|V_1| < |V|$. □

4.5.3 Impossibility proofs with the strong properties

Proposition 4.12: There exists no weakly incentive compatible minimally fair ranking system that satisfies strong positive response.

Proof: Assume for contradiction a ranking system F that satisfies the conditions. Assume a graph G with two vertices $V = \{v_1, v_2\}$ and no edges. By minimal fairness, $v_1 \simeq_G^F v_2$. Now assume a graph $G' = (V, \{(v_1, v_2)\})$ with an added edge between v_1 and v_2. By strong positive response, $v_1 \prec_G^F v_2$. However, by weak incentive compatibility, $1 = r_G^F(v_1) = r_{G'}^F(v_1) = \frac{1}{2}$, which is a contradiction. $\square$

Proposition 4.13: There exists no weakly incentive compatible ranking system that satisfies strong monotonicity.

Proof: Assume for contradiction a ranking system F that satisfies the conditions. Assume a graph G with two vertices $V = \{v_1, v_2\}$ and no edges. As $P_G(v_1) = P_G(v_2)$, by strong monotonicity, $v_1 \simeq_G^F v_2$. Now assume a graph $G' = (V, \{(v_1, v_2)\})$ with an added edge between v_1 and v_2. As $P_{G'}(v_1) \subsetneq P_{G'}(v_2)$, $v_1 \prec_G^F v_2$. However, by weak incentive compatibility, $1 = r_G^F(v_1) = r_{G'}^F(v_1) = \frac{1}{2}$, which is a contradiction. $\square$

Proposition 4.14: There exists no nontrivial strongly incentive compatible minimally fair ranking system..

Proof: We will prove that for any $G = (V, E)$, and for any $v_1, v_2 \in V$: $v_1 \preceq_G^F v_2$. We will use the incentive function $u_n(k) = n^k$, which gives a different value for each $u_n^*(k, m)$. The proof is by induction on $|E|$.

Induction Base: Assume $E = \emptyset$, and let $v_1, v_2 \in V$ be vertices. By minimal fairness, $v_1 \preceq v_2$.

Inductive Step: Assume correctness for $|E| \leq n$ and prove for $|E| = n + 1$. Assume for contradiction that for some $v_1, v_2 \in V$: $v_2 \prec v_1$. Let $v \in V$ be a vertex such that $S(v) \neq \emptyset$ (such a vertex exists because $|E| > 0$). Note that $|\{x \in V | v \simeq_G^F x\}| < |V|$, because otherwise $v_1 \preceq_G^F x \preceq_G^F v_2$. Let $E' = E \setminus \{(v, x) | x \in V\}$ and $G' = (V, E')$. By the assumption of induction, $|\{x \in V | v \simeq_{G'}^F x\}| = |V|$. Thus, $|\{x \in V | v \prec_{G'}^F x\}| = 0$. By strong incentive compatibility, $0 \leq |\{x \in V | v \prec_G^F x\}| \leq |\{x \in V | v \prec_{G'}^F x\}| = 0$, thus $|V| = |\{x \in V | v \simeq_{G'}^F x\}| \leq |\{x \in V | v \simeq_G^F x\}| < |V|$ which yields a contradiction. $\square$

Proposition 4.15: There exists no weakly incentive compatible nontrivial minimally fair ranking system that satisfies the strong union condition.

Proof: Assume for contradiction a ranking system F that satisfies the conditions. As the conditions of Lemma 4.8 are satisfied, let $G = (V, E)$ and $v, v_\perp, v_\top \in V$ be the graph and the vertices from the lemma. Now let

$G_1 = (V \setminus \{v_\top\}, E \setminus \{(v', v_\top) \in E | v' \in V\})$ and let $G_2 = (\{v_\top\}, \emptyset)$. From conditions 3 and 5, $S(v_\top) = \emptyset$ and thus G_1 and G_2 satisfy the conditions of the strong union condition with regard to G. Therefore, $v_\perp \prec^F_G v \Rightarrow v_\perp \prec^F_{G_1} v$, in contradiction to condition 1, because $|E_1| \leq |E|$ and $|V_1| < |V|$. $\square$

4.6 Some Illuminating Lessons

Theorems 4.3 and 4.4 teach us some surprising lessons about the implications of various versions of the basic properties.

4.6.1 Strong incentive compatibility is different than weak incentive compatibility

We have seen in Proposition 4.14 that, as one would expect, strong incentive compatibility is impossible when assuming minimal fairness. However, it turns out that when we slightly weaken the requirement of incentive compatibility to cover only the *expected* rank of the agent, Proposition 4.7 shows us this is possible. This means that the level of incentive compatibility has an effect on the existence of ranking systems.

4.6.2 Positive Response is not the same as Monotonicity

The Positive response and Monotonicity properties seem, at a glance, to be very similar, as they both informally require that the more votes an agent has, the higher it is ranked. However, looking more deeply, we see that the Positive Response properties require this behavior to be manifested across graphs, while the Monotonicity properties require that the effect be seen within a single graph.

This leads to interesting facts, such as not being able to nontrivially satisfy both Weak Monotonicity and Weak Positive response with incentive compatibility (Proposition 4.9), while each of the properties could be satisfied separately (Propositions 4.7 and 4.2). Furthermore, Strong Monotonicity cannot be satisfied at all (Proposition 4.13) with weak incentive compatibility, while Strong Positive Response *can* be satisfied even with strong incentive compatibility (Proposition 4.2).

4.6.3 The Weak Union property matters

Recall that the weak union property requires that when two disjoint graphs are put together, the subgraphs must still be ranked as before.

This property might seem trivial, but the impossibility results in Theorem 4.3 imply that this property has a part in inducing impossibility. The reason for this is twofold:

- The combination of two graphs adds more options for the agents in both subgraphs to vote for, which in order to preserve incentive compatibility, must all preserve the agent's relative rank in the combined graph.

- The weak union property further implies that the ranking system must not rely on the number of vertices in the graph, and moreover, that the minimal nontrivially ranked graph for a given ranking system must be connected.

4.7 Non-imposing Ranking Systems

An important extension of nontriviality is non-imposition. Non-imposing ranking systems are not only nontrivial, but can accommodate any strict order on the vertices.

Definition 4.10: Let F be a ranking system, F satisfies *non imposition* if for all V and for all strict linear orderings $L \in L(V)$: there exists some $G \in \mathbb{G}_V$ such that $F(G) \equiv L$.

We will now show that non-imposition cannot be satisfied when requiring incentive compatibility.

Fact 4.16: There exists no non-imposing incentive compatible ranking system.
Proof: Assume the vertex set $V = \{v_1, v_2\}$. There are two potential edges in this graph $e_1 = (v_1, v_2)$ and $e_2 = (v_2, v_1)$. Let $G = (V, E)$ be a graph s.t. $v_1 \prec_G v_2$ and let G' be a graph s.t. $v_2 \prec_{G'} v_1$. As $r_{G'}(v_1) \neq r_G(v_1)$ and $r_G(v_2) \neq r_{G'}(v_2)$, from incentive compatibility, the symmetric difference $E \oplus E' = (E \cup E') \setminus (E \cap E') = \{e_1, e_2\}$. Let $E'' = E \oplus \{e_1\} = E' \oplus \{e_2\}$. From incentive compatibility $r_{G''}(v_1) = r_G(v_1) = \frac{1}{2} = r_{G'}(v_2) = r_{G''}(v_2)$, but this cannot be as if $v_1 \simeq_{G''} v_2$, $r_{G''}(v_1) = r_{G''}(v_1) = 1$. □

4.7.1 A Fully Incentive Compatible Non-imposing Ranking System for 3 Agents

We have just shown that there exists no general incentive compatible non-imposing ranking system. However, if we limit our domain we may find that there exist such ranking systems. In this section, we will provide a full axiomatization for non-imposing incentive compatible ranking systems when there are exactly three agents.

Definition 4.11: A ranking system is called *three-plurality* if for every graph $G = (V, E)$ such that $|V| = 3$ there exists an ordering v_0, v_1, v_2 of the vertices in

		$v_0 \to v_1$	$v_0 \to v_2$
$v_2 \to v_0$	$v_1 \to v_2$	$\simeq$	$v_1 \prec v_0 \prec v_2$
	$v_1 \to v_0$	$v_2 \prec v_1 \prec v_0$	$v_1 \prec v_2 \prec v_0$
$v_2 \to v_1$	$v_1 \to v_2$	$v_0 \prec v_2 \prec v_1$	$v_0 \prec v_1 \prec v_2$
	$v_1 \to v_0$	$v_2 \prec v_0 \prec v_1$	$\simeq$

Figure 4.4: Schematic representation of the three-plurality ranking systems

V such that F ranks $u \preceq v \Leftrightarrow f(u) \leq f(v)$, where $f(v)$ is one of the following:

$$\begin{aligned} f_1(v_i) &= I\left[(v_{i-1}, v_i) \in E\right] + I\left[(v_{i+1}, v_{i-1}) \notin E\right] \\ f_2(v_i) &= I\left[(v_{i-1}, v_i) \in E \wedge (v_{i-1}, v_{i+1}) \notin E\right] + \\ &\quad + I\left[(v_{i+1}, v_i) \in E \vee (v_{i+1}, v_{i-1}) \notin E\right], \end{aligned}$$

where all the indices are calculated modulo 3, and I is the indicator function.

There are exactly four three-plurality ranking systems for graphs with $V = \{v_0, v_1, v_2\}$. These ranking systems all implement plurality voting when each agent must vote, as illustrated in Figure 4.4, and differ in the interpretation of the cases where agents cast no votes or both votes.

Theorem 4.17: Let F be a ranking system over the set of graphs with 3 vertices. F is three-plurality iff it satisfies all of the following criteria: incentive compatibility, non-imposition, weak positive response, and minimal fairness.

Furthermore, these conditions are independent.

Proof: We must first show that any three-plurality ranking system F satisfies these four criteria. Incentive compatibility and non-imposition can easily be deduced from Figure 4.4. To show that F satisfies weak positive response, note that any added edge (v_i, v_j) may only increase $f(v_j)$ and decrease $f(v_k)$ for $k \neq j$, thus satisfying weak positive response. Minimal fairness is also satisfied by noticing the symmetry in the definitions of f_1, f_2.

Now we need to prove that any ranking system F satisfying the four criteria is three-plurality. By non-imposition, there exist graphs G_1, G_2, G_3 such that: $v_0 \prec^F_{G_1} v_1 \prec^F_{G_1} v_2$, $v_2 \prec^F_{G_2} v_0 \prec^F_{G_2} v_1$, and $v_1 \prec^F_{G_3} v_2 \prec^F_{G_3} v_0$. The set of allowable strategies for agent v_i for $i \in \{0, 1, 2\}$ is $\{s^i_1, s^i_2, s^i_3, s^i_4\} = \wp(V \setminus \{v_i\})$. We can use strategy vectors of the form (s^0_i, s^1_j, s^2_k) to represent the graph $(V, s^0_i \cup s^1_j \cup s^2_k)$.

Let $\mathbf{s_1}, \mathbf{s_2}, \mathbf{s_3}$ be the strategy vectors representing G_1, G_2, G_3 respectively. By incentive compatibility, $\mathbf{s_1}$ and $\mathbf{s_2}$ differ by the strategies of at least 2 agents. Assume that $s^0_1 \neq s^0_2 \wedge s^1_1 \neq s^1_2 \wedge s^2_1 \neq s^2_2$. By IC, in the graph (s^0_2, s^1_1, s^2_1): $r(v_0) = 0.5$ and in the graph (s^0_2, s^1_1, s^2_2): $r(v_1) = 2.5$. As these two graphs differ only in the outgoing edges of v_2, its rank must be equal, thus must be $r(v_2) = 1.5$ in both. Therefore, in both (s^0_2, s^1_1, s^2_1) and (s^0_2, s^1_1, s^2_2), F ranks $v_0 \prec v_2 \prec v_1$. Again from IC, graph (s^0_2, s^1_2, s^2_1) must be ranked $v_2 \prec v_0 \prec v_1$

and (s_1^0, s_1^1, s_2^2) must be ranked $v_0 \prec v_1 \prec v_2$. We can now let $G_2 = (s_2^0, s_2^1, s_1^2)$, and thus differ from G_1 by the strategies of only two agents.

It is easy to see we can always choose G_2 such that G_1 and G_2 only differ in the strategies of v_0 and v_1. Similarly, G_3 can be chosen such that G_1 and G_3 differ only by the strategies of v_1 and v_2. Assume now that $s_3^1 \neq s_2^1$. By IC, in graph (s_1^0, s_3^1, s_1^2): $r(v_2) = r(v_1) = 1.5$ and thus $v_0 \simeq v_1 \simeq v_2$. Now, in graph (s_1^0, s_2^1, s_1^2): $r(v_0) = r(v_1) = 1.5$ and thus $v_0 \simeq v_1 \simeq v_2$. Now, in graph (s_1^0, s_2^1, s_3^2): $r(v_1) = 0.5$ and $r(v_2) = 1.5$, so $v_1 \prec v_2 \prec v_0$. We now let $G_3 = (s_1^0, s_2^1, s_3^2)$ and thus now every pair of graphs from G_1, G_2, G_3 differ by strategies of two agents. After renaming strategies, we get a structure isomorphic to the one described in Figure 4.4, but without any mapping between the names of the strategies and actual edge selection by the agents.

We will first show that the additional strategies of the agents simply reflect these existing strategies. In (s_3^0, s_1^1, s_1^2), by IC, $r(v_0) = 1.5$. So assume that F ranks $v_2 \prec v_0 \prec v_1$. However, in that case in (s_3^0, s_2^1, s_1^2), $r(v_1) = r(v_0) = 2.5$, which is impossible. However, in the two remaining cases it is easy to see that s_3^0 reflects s_1^0 or s_2^0. The same is true for all other agents. Therefore, we only need to map the four strategies for each agent to one of the two options for that agent.

Note that agent v_2 is strengthened when agent v_0 switches from s_1^0 to s_2^0 and agent v_1 is weakened. Assume $S(v_0) = \{v_1\}$ maps to s_2^0, then by weak positive response, $S(v_0) = \{v_1, v_2\}$ and $S(v_0) = \emptyset$ must also map to s_2^0, and furthermore then $S(v_0) = \{v_2\}$ must also map to s_2^0, in contradiction to the fact that s_1^0 must be playable (by non-imposition). Similarly, in all cases where $|S(v)| = 1$, $S(v)$ maps to the relevant strategy in Figure 4.4.

By minimal fairness, when $E = \emptyset$, the strategy profile must be (s_1^0, s_1^1, s_1^2) or (s_2^0, s_2^1, s_2^2), thus if $S(v_0) = \emptyset$ maps to a strategy s_i^0, then $S(v_1) = \emptyset$ and $S(v_2) = \emptyset$ must map to strategies s_i^1 and s_i^2 respectively. The same goes for $S(v_0) = \{v_1, v_2\}$ — if it maps to a strategy s_i^0, then $S(v_1) = \{v_0, v_2\}$ and $S(v_2) = \{v_0, v_1\}$ must map to strategies s_i^1 and s_i^2 respectively.

So, we are left with four mapping options:

- $S(v_0) = \emptyset$ maps to s_2^0 and $S(v_0) = \{v_1, v_2\}$ maps to s_1^0 .
- $S(v_0) = \emptyset$ maps to s_1^0 and $S(v_0) = \{v_1, v_2\}$ maps to s_2^0 .
- $S(v_0) = \emptyset$ and $S(v_0) = \{v_1, v_2\}$ both map to s_2^0 .
- $S(v_0) = \emptyset$ and $S(v_0) = \{v_1, v_2\}$ both map to s_1^0 .

These mapping options exactly correspond to the four three-plurality ranking systems — The first two correspond to f_1, and the second two to f_2. Of each pair, the first corresponds to the ordering v_0, v_1, v_2 and the second corresponds to v_0, v_2, v_1.

We have shown any ranking system satisfying the four conditions must be three-plurality.

To show that the conditions are independent we must show different ranking systems satisfying all conditions except one:

- Incentive compatibility — The approval voting ranking system satisfies all aforementioned conditions except incentive compatibility.
- Non-imposition — The trivial ranking system that always ranks all vertices equally satisfies IC, weak positive response and minimal fairness.
- Weak positive response — We can swap the meanings of s_1^i and s_2^i for all agents and get a ranking system satisfying all conditions except weak positive response.
- Minimal fairness — If we do not assume minimal fairness, we can assign the strategies for $S(v) = \emptyset$ and $S(v) = V \setminus \{v\}$ differently for each agent v.

□

4.8 Isomorphism

Most of the ranking systems we have seen up to now in the possibility proofs take advantage of the names of the vertices to determine the ranking. A natural requirement from a ranking system is that the names assigned to the vertices will not take part in determining the ranking. This is formalized by the isomorphism property.

Definition 4.12: A ranking system F satisfies *isomorphism* if for every isomorphism function $\varphi : V_1 \mapsto V_2$, and two isomorphic graphs $G \in \mathbb{G}_{V_1}, \varphi(G) \in \mathbb{G}_{V_2}$: $\preceq^F_{\varphi(G)} = \varphi(\preceq^F_G)$.

It turns out that the ranking system F_4 from the possibility proof for weak incentive compatibility and weak monotonicity (Proposition 4.7) satisfies isomorphism as well, and thus there exists a weakly incentive compatible ranking system satisfying isomorphism and weak monotonicity. The existence of weakly incentive compatible ranking systems satisfying isomorphism in conjunction with either the weak union property or the weak positive response is an open question.

4.9 Quantifying Incentive Compatibility

We have seen that there are no incentive compatible ranking systems satisfying all of the basic properties we have outlined above. Therefore, it is essential to weaken this requirement of incentive compatibility. This weakening would allow ranking systems that permit manipulations up to a specific magnitude or by a specific number of agents.

In order to define these limited manipulations we must first define the magnitude of an agent's best deviation:

Definition 4.13: Let F be ranking system and let $G = (V, E)$ be a graph for which F is defined. The *deviation magnitude* $\delta_G^F(v)$ *of* v *in* G *under ranking system* F is defined as $\max\{r^F_{(V,E')}(v) - r^F_G(v) \Big| F(V, E') \text{ is defined}, \forall v' \in V \setminus \{v\}, v'' \in V : (v', v'') \in E \Leftrightarrow (v', v'') \in E'\}$. That is, the maximum rank difference v can obtain for itself by changing its outgoing vertices in G under F.

We can now define three different quantifications of the level of incentive compatibility of a ranking system:

Definition 4.14: Let F be a ranking system. F is called *k-worst case incentive compatible* over a set of graphs $\mathbb{G}$ if for all graphs $G \in \mathbb{G}$ and for all $v \in V$: $\delta_G^F(v) \leq k$. We say that the *worst case incentive compatibility* of F is k if it is k-worst case incentive compatible, but not $(k-\varepsilon)$-worst case incentive compatible for all $\varepsilon > 0$.

Definition 4.15: Let F be a ranking system. F is called *k-mean incentive compatible* over a set of graphs $\mathbb{G}$ if for all graphs $G \in \mathbb{G}$: $\sum_{v \in V} \delta_G^F(v)/|V| \leq k$. We say that the *mean incentive compatibility* of F is k if it is k-mean incentive compatible, but not $(k - \varepsilon)$-mean incentive compatible for all $\varepsilon > 0$.

Definition 4.16: Let F be a ranking system. F is called *k-agent incentive compatible* for a set of graphs $\mathbb{G}$ if for all graphs $G \in \mathbb{G}$: $|\{v \in V | \delta_G^F(v) > 0\}| \leq k$. We say that the *agent incentive compatibility* of F is k if it is k-agent incentive compatible, but not $(k - 1)$-agent incentive compatible.

Notation: In this section we will use the term *fully incentive compatible* in place of *weakly incentive compatible* for contrast with these weaker notions of incentive compatibility.

Note that when k is zero, all of these definitions coincide with full incentive compatibility.

Of the basic properties we defined above, we have shown that weak positive response, weak monotonicity and minimal fairness could each be satisfied by a fully incentive compatible ranking system. This leads us to concentrate on the levels of incentive compatibility attainable under strong monotonicity and non-imposition. In these fundamental cases, full incentive compatibility cannot be obtained, and thus it is interesting to try and obtain a more limited degree of incentive compatibility. In the sequel we show tight bounds for the levels of incentive compatibility under these two conditions.

4.9.1 Incentive Compatibility Under Strong Monotonicity

When we study the incentive compatibility of ranking systems satisfying strong monotonicity, it is helpful to keep in mind that this property is satisfied by almost all practical ranking systems, including Approval Voting, PageRank, and

Hubs&Authorities. Specifically, we are going to quantify the incentive compatibility of the Approval Voting and PageRank ranking systems, when the out-degree of each vertex is limited to some constant k.

First, we are going to prove a general negative result about ranking systems that satisfy strong monotonicity.

Theorem 4.18: There exists no strongly monotone ranking system that is $(\frac{k}{2}-\varepsilon)$-worst case incentive compatible on the set of graphs with max out-degree k for all $\varepsilon > 0$. Furthermore, there exists no strongly minimally fair strongly monotone ranking system that is $(\frac{k}{2}-\varepsilon)$-mean incentive compatible on the set of graphs with max out-degree k for all $\varepsilon > 0$.

Proof: Assume a strongly monotone ranking system F and assume a graph $G = (V, E)$ with $k+1$ vertices $V = \{v_0, v_1, \ldots, v_k\}$ and edges $E = \{(v_0, v_1), (v_0, v_2), \ldots, (v_0, v_k)\}$. Assume a strongly monotone l-worst case incentive compatible ranking system F. By strong monotonicity, F ranks

$$v_0 \prec_G^F v_1 \simeq_G^F v_2 \simeq_G^F \cdots \simeq_G^F v_k.$$

This gives $r_G^F(v_0) = \frac{1}{2}$. However, if v_0 changes its votes to $\emptyset$, the rank will become (by strong monotonicity) $v_0 \simeq v_1 \simeq v_2 \simeq \cdots \simeq v_k$, and thus $r_{G'}^F(v_0) = \frac{k+1}{2}$. We have shown a manipulation of magnitude $\frac{k}{2}$, in contradiction to the fact that F is $(\frac{k}{2} - \varepsilon)$-IC, where $\varepsilon > 0$.

Now assume a strongly minimally fair strongly monotone ranking system F'. We will show a graph $G = (V, E)$ in which all agents have a deviation of magnitude $\frac{k}{2}$. The graph is the complete clique with $k + 1$ vertices: $V = \{0, \ldots, k\}$ and $E = V \times V \setminus \{(v, v)|v \in V\}$. Note that this graph has a maximal out-degree of k and F ranks all agents equally (due to minimal fairness). However, if any agent v removes all its outgoing edges to form a graph G', then that agent will be, by strong monotonicity, ranked above all other agents. Thus, $r_G^{F'}(v) = \frac{k+1}{2}$, while $r_{G'}^{F'}(v) = k + \frac{1}{2}$. Thus $\delta_G^{F'}(v) = \frac{k}{2}$ for all $v \in V$. Therefore, F' is not $(\frac{k}{2} - \varepsilon)$-mean incentive compatible for all $\varepsilon > 0$. $\square$

We can now quantify the incentive compatibility of the approval voting ranking system, showing that the aforementioned lower bound is tight.

Proposition 4.19: The approval voting ranking system AV satisfies the following over the set of graphs with max out-degree k:

- The worst case incentive compatibility of AV is $\frac{k}{2}$.
- The mean incentive compatibility of AV is $\frac{k}{2}$.
- The agent incentive compatibility of AV over the set of graphs with n vertices $(n > 1)$ is n.

Proof: First we will prove that AV_k is $\frac{k}{2}$-worst case incentive compatible. Let $G = (V, E), G' = (V, E') \in \mathbb{G}$ be graphs that differ in the outgoing edges from v. Note that $|P_G(v)| = |P_{G'}(v)|$ as neither G nor G' include self-edges. Let $S_{del} = \{u \in S_G(v) \setminus S_{G'}(v)\}$. Note that $|S_{del}| \leq k$. For all $u \in V \setminus S_{del}$: $|P_G(u)| = |\{w|(w,u) \in E\}| \leq |\{w|(w,u) \in E'\}| = |P_{G'}(u)|$, and thus $|P_{G'}(u)| < |P_{G'}(v)| \Rightarrow |P_G(u)| < |P_G(v)|$ and $|P_{G'}(u)| \leq |P_{G'}(v)| \Rightarrow |P_G(u)| \leq |P_G(v)|$. Furthermore, for all $u \in S_{del} : |P_{G'}(u)| = |P_G(u)| + 1$. Let $S_a = \{u \in S_{del} : |P_G(u)| = |P_G(v)|\}$ and $S_b = \{u \in S_{del} : |P_G(u)| + 1 = |P_G(v)|\}$ Now,

$$\begin{aligned} r^{AV}_{G'}(v) - r^{AV}_G(v) &= \tfrac{1}{2}\left|\{v'|v' \prec_{G'} v\}\right| \\ &\quad -\tfrac{1}{2}\left|\{v'|v' \prec_{G} v\}\right| \\ &\quad +\tfrac{1}{2}\left|\{v'|v' \preceq_{G'} v\}\right| \\ &\quad -\tfrac{1}{2}\left|\{v'|v' \preceq_{G} v\}\right| \\ &\leq \tfrac{1}{2}|S_a| + \tfrac{1}{2}|S_b| \leq \tfrac{1}{2}|S_{del}| \leq \tfrac{k}{2}. \end{aligned}$$

The $\frac{k}{2}$-mean incentive compatibility immediately follows, and the n-agent incentive compatibility is trivial. AV_k satisfies strong monotonicity and minimal fairness, and thus it is not $(\frac{k}{2} - \varepsilon)$-mean incentive compatible, and not $(\frac{k}{2} - \varepsilon)$-worst case incentive compatible for all $\varepsilon > 0$.

To show that AV_k is not $(n-1)$-agent incentive compatible over the set of graphs with n vertices ($n > 1$), assume the full loop with n vertices $G = (V, E)$ defined as follows:

$$\begin{aligned} V &= \{0, \ldots, n-1\} \\ E &= \{(i, i+1 \mod n), | i = 0 \ldots n-1\} \end{aligned}$$

Now, by removing all of its edges, each agent can improve its own relative rank by $\frac{1}{2}$, and thus all n agents have a deviation, and thus AV is not $(n-1)$-agent incentive compatible. $\square$

We now shall define the PageRank procedure with damping factor d. Recall the definition of the PageRank matrix (definition 2.3 on page 10) which is the matrix which captures the random walk created by the PageRank procedure. In this process we start in a random page, and iteratively move to one of the pages that are linked to by the current page, assigning equal probabilities to each such page. The damping factor d defines the probability of "teleporting" to a random page at each step of the walk. The PageRank procedure will rank pages according to the stationary probability distribution obtained in the limit of the this random walk. This is formally defined as follows:

Definition 4.17: Let $G = (V, E)$ be some strongly connected graph, and assume $V = \{v_1, v_2, \ldots, v_n\}$. Let $0 < d < 1$ be a *damping factor*. Let $\mathbf{r}$ be the unique solution of the system $(1-d) \cdot A_G \cdot \mathbf{r} + d \cdot \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}^T = \mathbf{r}$ where $\sum r_i = n$. The damped *PageRank* $PR^d_G(v_i)$ of a vertex $v_i \in V$ is defined as $PR^d_G(v_i) = r_i$. The *PageRank ranking system with damping factor d* is a ranking system that for the vertex set V maps G to $\preceq_G^{PR^d}$, where $\preceq_G^{PR^d}$ is defined as: for all $v_i, v_j \in V$: $v_i \preceq_G^{PR^d} v_j$ if and only if $PR_G^d(v_i) \leq PR_G^d(v_j)$.

We will now quantify the incentive compatibility of the PageRank ranking system:

Proposition 4.20: The PageRank ranking system PR^d with damping factor d is not $(\frac{n}{2}-2)$-mean incentive compatible nor $(n-1)$-agent incentive compatible on the set of graphs with n vertices ($n > 2$) and out-degree 1.
Proof: Consider the graph $G = (V, E)$ where $V = \{0, \ldots, n-1\}$ and $E = \{(i, i+1 \mod n) | i = 0, \ldots n-1\}$. In this graph PR^d ranks all agents equally due to symmetry. Let $v \in V$ be some agent. Assume wlog $v = n-1$ and let $G' = (V, E')$ be defined as $E' = E \setminus \{(n-1, 0)\} \cup \{(n-1, n-2)\}$. Applying linear algebra, we conclude that PR^d ranks $0 \prec 1 \prec \cdots \prec n-3 \prec n-1 \prec n-2$ in G' and thus $r_{G'}^{PR^d}(v) = r_{G'}^{PR^d}(n-1) = n-1.5$. However, $r_G^{PR^d}(v) = \frac{n}{2}$, and thus $\delta_G^{PR^d}(v) \geq \frac{n-3}{2}$. This is true for all $v \in V$, so we see that PR^d is not $(\frac{n}{2}-2)$-mean incentive compatible nor $(n-1)$-agent incentive compatible for an arbitrary graph G with n vertices. $\square$

A similar lower bound showing deviations of magnitude $O(n)$ by all agents can be shown for the Hubs&Authorities ranking system as presented by Kleinberg (1999).

4.9.2 Non-imposing Ranking Systems

Recall that non-imposing ranking systems are those that accommodate any strict order on the vertices, and that no such fully incentive compatible ranking systems exist. We will now show a 1-worst case incentive compatible ranking system satisfying non-imposition. This ranking system is also 1-agent incentive compatible, which sets a tight bound.

Theorem 4.21: There exists a ranking system F that satisfies non-imposition, 1-worst case incentive compatibility, $\frac{1}{n}$-mean incentive compatibility on graphs with n vertices, 1-agent incentive compatibility, and weak positive response.
Proof: The ranking system F is defined as follows: Assume a graph $G = (V, E)$ with $V = \{v_1, v_2, \ldots, v_n\}$. For each $v \neq v_1$ we define

$$p(v) = \begin{cases} |P(v) \setminus S(v_1)| + n & v \in S(v_1) \\ |P(v) \cap S(v_1)| & v \notin S(v_1) \end{cases}.$$

Now we define a strict ordering $\preceq^*$ on $V \setminus \{v_1\}$:

$$\begin{aligned} v_i \preceq^* v_j \quad \Leftrightarrow \quad & [p(v_i) < p(v_j)] \vee \\ & \vee [p(v_i) = p(v_j) \wedge i \leq j]. \end{aligned}$$

Given this ordering we can finally define $\preceq_G^F$:

$$\begin{aligned} v_i \preceq_G^F v_j \quad \Leftrightarrow \quad & (i \neq 1 \wedge j \neq 1 \wedge v_i \preceq^* v_j) \vee \\ & \vee (i = 1 \wedge |\{u | u \preceq^* v_j\}| \geq |P(v_1)|) \vee \\ & \vee (j = 1 \wedge |\{u | u \preceq^* v_i\}| < |P(v_1)|). \end{aligned}$$

The weak positive response property is satisfied because addition of an edge (u, v) either weakly increases $p(v)$ if $v \neq v_1$, increasing the relative rank of v, or increases $|P(v_1)|$ if $v = v_1$, and thus again increases the relative rank of v.

To prove F satisfies non-imposition, assume a vertex set $V = \{v_1, \ldots, v_n\}$ and strict ordering $\preceq'$ on V. Let $u_1, u_2, \ldots, u_{n-1}$ be the vertices in $V \setminus \{v_1\}$ ordered according to $\preceq'$ and let $k = |\{v \in V : v \preceq' v_1\}|$. Let $G = (V, E)$ be the graph defined as follows:

$$\begin{aligned} E \quad = \quad & \{(v_1, u_i) | i > \tfrac{n-1}{2}\} \cup \{(u_i, v_1) | i < k\} \cup \\ & \{(u_i, u_j) | \tfrac{n-1}{2} < i < j - 1 + \tfrac{n}{2}\} \cup \\ & \{(u_i, u_j) | i < j - \tfrac{n-1}{2}\}. \end{aligned}$$

First note that for all $u_i \in V \setminus \{v_1\}$:

$$p(u_i) \quad = \quad \begin{cases} i + \left\lfloor \frac{n}{2} \right\rfloor & i > \frac{n-1}{2} \\ i - 1 & \text{Otherwise} \end{cases}$$

Thus, $u_1 \prec^* u_2 \prec^* \cdots \prec^* u_{n-1}$. As $|P(v_1)| = k - 1$, $u_1 \prec_G^F \cdots \prec_G^F u_{k-1} \prec_G^F v_1 \prec_G^F u_k \prec_G^F \cdots \prec_G^F u_{n-1}$, and thus $\preceq_G^F \equiv \preceq'$, as required.

We will now prove the incentive compatibility features of this ranking system. Let $G = (V, E)$ be some graph. Note that both $\preceq^*$ and $\preceq_G^F$ are strict orderings. The deviation magnitude of agent v_1 is 0, as its rank is dependent only on its in-degree, which it cannot manipulate:

$$\begin{aligned} \delta_G^F(v_1) \quad & = \quad \max\{r_{(V,E')}^F(v) - r_G^F(v)\} = \\ & = \quad \max\{(|P_{(V,E')}(v_1)| + \tfrac{1}{2}) - (|P_G(v_1)| + \tfrac{1}{2})\} = \\ & = \quad \max\{|P_G(v_1)| - |P_G(v_1)|\} = 0. \end{aligned}$$

Let $v_i \in V \setminus \{v_1\}$ be an agent. The rank $r_G^F(v_i)$ is:

$$\begin{aligned} r_G^F(v_j) \quad & = \quad \tfrac{1}{2} \left|\{v' : v' \prec_G^F v_i\}\right| + \tfrac{1}{2} \left|\{v' : v' \preceq_G^F v_i\}\right| = \\ & = \quad \left|\{v' : v' \prec_G^F v_i\}\right| + \tfrac{1}{2} = \\ & = \quad \begin{cases} |\{v' : v' \prec^* v_i\}| + 1.5 & |\{v' | v' \preceq^* v_i\}| \geq |P(v_1)| \\ |\{v' : v' \prec^* v_i\}| + \frac{1}{2} & \text{Otherwise.} \end{cases} \end{aligned}$$

Now, $|\{v' : v' \prec^* v_i\}|$ is independent of the outgoing edges of v_i given $S(v_1)$, as $v_i \in S(v_1)$ iff its outgoing edges are used to rank agents $\notin S(v_1)$. Thus, the only manipulation v_i might do is to change $|P(v_1)|$, and thus increase its rank by 1. In order to increase its rank, v_i must decrease $|P(v_1)|$. v_i can do so by at most 1, by removing an edge (v_i, v_1) if it exists. This manipulation can only be done if $|\{v'|v' \preceq^* v_i\}| = |P_G(v_1)|$. As $\preceq^*$ is strict and $0 \leq |P_G(v_1)| \leq n-1$, there exists exactly one agent v_i satisfying this condition.

Thus, for some $v_i \in V$: $\delta_G^F(v_i) \leq 1$, and for all $v_j \in V \setminus \{v_i\}$: $\delta_G^F(v_i) = 0$. So we conclude that F is 1-worst case incentive compatible, $\frac{1}{n}$-mean incentive compatible on graphs with n vertices, and 1-agent incentive compatible. $\square$

Chapter 5

Personalized Ranking Systems

5.1 Introduction

Personalized ranking systems and trust systems are an essential tool for collaboration in a multi-agent environment. In these systems, agents report on their peers' performance, and these reports are aggregated to form a ranking of the agents. In the previous chapters, we have discussed global ranking systems, where all agents see the same ranking. In this chapter, we consider personalized ranking systems, where each agent is provided with her own ranking of the agents.

Examples of personalized ranking systems include the personalized version of PageRank(Haveliwala et al., 2003) and the MoleTrust ranking system (Avesani et al., 2005). Furthermore, trust systems which provide each agent with a set of agents he or she can trust can be viewed as personalized ranking systems which supply a two-level ranking over the agents. Many of these systems can be easily adapted to provide a full ranking of the agents. Examples of trust systems include OpenPGP(Pretty Good Privacy)'s trust system (Callas et al., 1998), the ranking system employed by Advogato (Levien, 2002), and the epinions.com web of trust.

A central challenge in the study of ranking systems, is to provide means and rigorous tools for the evaluation of these systems. This challenge equally applies to both global and personalized ranking systems. A central approach to the evaluation of such systems is the experimental approach. In the general ranking systems setting, this approach was successfully applied to Hubs&Authorities (Kleinberg, 1999) and to various other ranking systems (Borodin et al., 2005). In the trust systems setting, Massa and Avesani (2005) suggest a similar experimental approach.

In the previous chapters we have suggested the axiomatic approach to the evaluation of ranking systems. Cheng and Friedman (2005) discuss a specific *sybilproofness* property, and have proven that a personalized ranking system must be applied in order to satisfy this property. While the axiomatic approach has been extensively applied to the global ranking systems setting, no general attempt has been made to apply such an approach to the context of personalized ranking systems.

In this chapter, we introduce an extensive axiomatic study of the personalized ranking system setting, by adapting axioms that have been applied to global ranking systems earlier in this thesis. We compare several existing personalized ranking systems in the light of these axioms, and provide novel ranking systems that satisfy various sets of axioms. Moreover, we prove a full characterization of the personalized ranking systems satisfying all suggested axioms.

We consider four basic axioms. The first axiom, self confidence, requires that an agent would be ranked at the top of his own personalized rank. The second axiom, transitivity, captures the idea that an agent preferred by more highly trusted agents, should be ranked higher than an agent preferred by less trusted agents. The third axiom, Ranked Independence of Irrelevant Alternatives, requires that under the perspective of any agent, the relative ranking of two other agents would depend only on the pairwise comparisons between the rank of the agents that prefer them. The last axiom, strong incentive compatibility, captures the idea that an agent cannot gain trust by any agent's perspective by manipulating its reported trust preference.

We fully characterize the set of ranking systems satisfying all four axioms, and show ranking systems satisfying every three of the four axioms (but not the fourth).

This chapter is organized as follows. Section 5.2 introduces the setting of personalized ranking systems and discusses some known systems. In section 5.3 we present our axioms, and classify the ranking systems shown according to these axioms. In section 5.4 we provide a full characterization of the ranking systems satisfying all of our axioms, and in section 5.5 we study ranking systems satisfying every three of the four axioms. Section 5.6 presents some concluding remarks and suggestions for future research.

5.2 Personalized Ranking Systems

5.2.1 The Setting

We define a personalized ranking system as a slight variation of a general system:

Definition 5.1: Let $\mathbb{G}_V$ be the set of all directed graphs $G = (V, E)$ with no parallel edges, but possibly with self-loops[1]. A *personalized ranking system(PRS)* F is a functional that for every finite vertex set V and for every *source* $s \in V$ maps every graph $G \in \mathbb{G}_V$ to an ordering $\preceq^F_{G,s} \in L(V)$.

Note that our definition of a personalized ranking system considers only the ordinal ranking of the vertices and does not assign cardinal values to vertices.

[1]Unless otherwise noted, all our results still apply when self loops are not allowed.

Also note that our definition does not assume the existence of a path from s to every vertex. However, in some settings this may be considered a useful assumption. Therefore, we shall use these kind of graphs in all examples and counter-examples, but prove our results for the more general case defined above.

5.2.2 Some personalized ranking systems

We shall now give examples of some known PRSs. A basic ranking system that is at the basis of many trust systems ranks the agents based on the minimal distance of the agents from the source.

Notation: Let $G = (V, E)$ be some directed graph and $v_1, v_2 \in V$ be some vertices, we will use $d_G(v_1, v_2)$ to denote the length of the shortest directed path in G between v_1 and v_2. If no such path exists, $d_G(v_1, v_2) \triangleq \infty$.

Definition 5.2: The *distance PRS* F_D is defined as follows: Given a graph $G = (V, E)$ and a source s, $v_1 \preceq^{F_D}_{G,s} v_2 \Leftrightarrow d_G(s, v_1) \geq d_G(s, v_2)$

Another family of PRSs can be derived from the well-known PageRank ranking system by modifying the so-called teleportation vector in the definition of PageRank (Haveliwala et al., 2003). The Personalized PageRank procedure ranks pages according to the stationary probability distribution obtained in the limit of a random walk with a random teleportation to the source s with probability d. This is formally defined as follows:

Definition 5.3: Let $G = (V, E)$ be some graph, and assume $V = \{s, v_2, \ldots, v_n\}$. Let $\mathbf{r}$ be the unique solution of the system $(1-d) \cdot A_G \cdot \mathbf{r} + d \cdot (1, 0, \ldots, 0)^T = \mathbf{r}$. The *Personalized PageRank with damping factor d* of a vertex $v_i \in V$ is defined as $PPR^d_{G,s}(v_i) = r_i$. The *Personalized PageRank Ranking System with damping factor d* is a PRS that for the vertex set V and source $s \in V$ maps G to $\preceq^{PPR_d}_{G,s}$, where $\preceq^{PPR_d}_{G,s}$ is defined as: for all $v_i, v_j \in V$: $v_i \preceq^{PPR_d}_{G,s} v_j$ if and only if $PPR^d_{G,s}(v_i) \leq PPR^d_{G,s}(v_j)$.

We now suggest a variant of the Personalized PageRank system, which, as we will later show, has more positive properties than Personalized PageRank.

Definition 5.4: Let $G = (V, E)$ be some graph and assume $V = \{s, v_2, \ldots, v_n\}$. Let B_G be the link matrix for G. That is, $[B_G]_{i,j} = 1 \Leftrightarrow (j, i) \in E$. Let $\alpha = 1/n^2$ and let $\mathbf{a}$ be the unique solution of the system $\alpha \cdot B_G \cdot \mathbf{a} + (1, \alpha^n, \ldots, \alpha^n)^T = \mathbf{a}$. The *$\alpha$-Rank* of a vertex $v_i \in V$ is defined as $r_{G,s}(v_i) = a_i$. The *α-Rank PRS* is a PRS that for the vertex set V and source $s \in V$ maps G to $\preceq^{\alpha R}_{G,s}$, where $\preceq^{\alpha R}_{G,s}$ is defined as: for all $v_i, v_j \in V$: $v_i \preceq^{\alpha R}_{G,s} v_j$ if and only if $r_{G,s}(v_i) \leq r_{G,s}(v_j)$.

The α-Rank system ranks the agents based on their distance from s, breaking ties by the summing of the trust values of the predecessors. By selecting $\alpha = 1/n^2$, it is ensured that a slight difference in rank of nodes closer to s will be more significant than a major difference in rank of nodes further from s.

Additional personalized ranking systems are presented in Section 5.5 as part of our axiomatic study.

5.3 Some Axioms

A basic requirement of a PRS is that the source — the agent under whose perspective we define the ranking system — must be ranked strictly at the top of the trust ranking, as each agent implicitly trusts herself. We refer to this property as self confidence.

Definition 5.5: Let F be a PRS. We say that F satisfies *self confidence* if for all graphs $G = (V, E)$, for all sources $s \in V$ and for all vertices $v \in V \setminus \{s\}$: $v \prec^F_{G,s} s$.

Recall the notions of *strong transitivity* and *strong quasi transitivity* (definitions 3.1 on page 30 and 3.7 on page 36), which require that if an agent a's voters are ranked higher than those of agent b, then agent a should be ranked higher than agent b. We adapt these notions to the personalized setting, and introduce a new weaker version of transitivity as follows:

Definition 5.6: Let F be a PRS. We say that F satisfies *quasi transitivity* if for all graphs $G = (V, E)$, for all sources $s \in V$ and for all vertices $v_1, v_2 \in V \setminus \{s\}$: Assume there is a 1-1 mapping $f : P(v_1) \mapsto P(v_2)$ s.t. for all $v \in P(v_1)$:$v \preceq f(v)$. Then, $v_1 \preceq v_2$. F further satisfies *strong quasi transitivity* if when $P(v_1) \neq \emptyset$ and for all $v \in P(v_1)$: $v \prec f(v)$, then $v_1 \prec v_2$. F further satisfies *strong transitivity* if when either f is not onto or for some $v \in P(v_1)$: $v \prec f(v)$, then $v_1 \prec v_2$.

The new notion of quasi transitivity requires that agents with stronger matching predecessors be ranked at least as strong as agents with weaker predecessors without any requirement for strict preference.

Recall the *Ranked IIA* axiom (definition 3.4 on page 32), which intuitively means that the relative ranking of agents must be consistent across all comparisons with the same rank relations. We now adapt this axiom to the setting of PRSs, by requiring this independence for all vertices reachable from from the source, except for the source itself.

Notation: We will use V^G_s to denote the set of vertices that have a directed path from s in a graph G. We will sloppily use V_s when G is understood from context.

Definition 5.7: Let F be a PRS. We say that F satisfies *ranked independence of irrelevant alternatives (RIIA)* if there exists a mapping $f : \mathcal{P} \mapsto \{0, 1\}$ such that for every graph $G = (V, E)$, for every source $s \in V$ and for every pair of vertices $v_1, v_2 \in V^G_s \setminus \{s\}$ and for every comparison profile $p \in \mathcal{P}$ that v_1 and v_2 satisfy, $v_1 \preceq^F_{G,s} v_2 \Leftrightarrow f(p) = 1$.

Notation: We will sloppily use the notation $\mathbf{a} \preccurlyeq \mathbf{b}$ to denote $f\langle \mathbf{a}, \mathbf{b} \rangle = 1$.

5.3.1 Incentive Compatibility

The issue of incentives has been extensively studied in classical social choice (Gibbard, 1973; Satterthwaite, 1975; Dutta et al., 2001). We have further studied the issue of incentives in ranking systems in Chapter 4. As with global ranking systems, agents ranked by personalized ranking systems may wish to manipulate their reported preferences in order to improve their trustworthiness in the eyes of a specific agent. Therefore, the incentives of these agents should in many cases be taken into consideration.

We would like our ranking systems to stand against various types of manipulations. It is important to formally define what a manipulation is, and the types of manipulations we would like to defend against.

Definition 5.8: A *manipulation* is a function $\mathcal{M}$ that maps every graph $G = (V, E) \in \mathbb{G}$ and every vertex $v \in V$ in that graph to a set of graphs $M \subseteq \mathbb{G}$ such that $G \in M$ and $v \in G'$ for all $G' \in M$.

That is, a manipulation defines for every vertex in any graph, what different graphs can that agent cause to be presented to the ranking system as a result of a manipulation.

Our standard for incentive compatibility is strong incentive compatibility, which requires that agents will not improve their rank in the terms of the number of agents ranked above them and the number or agents ranked the same as them[2]:

Definition 5.9: Let F be a PRS. F satisfies *strong incentive compatibility under manipulation* $\mathcal{M}$ if for all true preference graphs $G = (V, E)$, for all sources $s \in V$, for all vertices $v \in V$, and for all manipulations $G' \in \mathcal{M}(G, v)$: $|\{x \in V' | v \prec^F_{G'} x\}| \geq |\{x \in V | v \prec^F_G x\}|$; and if $|\{x \in V' | v \prec^F_{G'} x\}| = |\{x \in V | v \prec^F_G x\}|$ then $|\{x \in V' | v \simeq^F_{G'} x\}| \geq |\{x \in V | v \simeq^F_G x\}|$.

In chapter 4, we considered manipulation by modification of an agent's outgoing links. Such outgoing link manipulation can be defined as:

$$\mathcal{M}_{out}(V, E, v) = \{(V, E') | \forall u \in V \setminus \{v\} : \forall u' \in V : (u, u') \in E \Leftrightarrow (u, u') \in E'\}.$$

The outgoing link manipulation $\mathcal{M}_{out}$ is actually a special kind of manipulation in the sense that the agent can perform the manipulation in both directions.

Definition 5.10: A manipulation $\mathcal{M}$ is called *reversible* if for all $G = (V, E) \in \mathbb{G}$, for all $v \in V$, and for all $G' \in \mathcal{M}(G, v)$: $G \in \mathcal{M}(G', v)$.

Reversible manipulations are important due to the following simple fact:

[2] In chapter 4, we have defined the notion of a utility function $u_n : \mathbb{N} \mapsto \mathbb{R}$ that for every graph size n maps the number of agents ranked below a specific agent in a strict ranking to a utility value, and we assumed such utility functions are nondecreasing. If we further assume that $u_n(i) = u_m(i + n - m)$ for all $0 < i < m < n$, that is, an agent's utility in a strict ranking depends only on the number of agents ranked above it, we can show that our current definition of strong incentive compatibility is equivalent to the one in Chapter 4.

Fact 5.1: Let $\mathcal{M}$ be a reversible manipulation and let F be a PRS . F satisfies strong incentive compatibility under $\mathcal{M}$ if and only if for all graphs $G = (V, E)$, for all sources $s \in V$, for all vertices $v \in V$, and for all manipulations $G' \in \mathcal{M}(G, v)$: $|\{x \in V' | v \prec^F_{G'} x\}| = |\{x \in V | v \prec^F_G x\}|$ and $|\{x \in V' | v \simeq^F_{G'} x\}| = |\{x \in V | v \simeq^F_G x\}|$.

Therefore, in a PRS that is incentive compatible under a reversible manipulation an agent cannot change its rank at all by performing a manipulation.

Another type of manipulation, considered by Cheng and Friedman (2005) is concerned with the generation of fraudulent identities in order to manipulate one's rank. Their setting considered weighted edges, as opposed to our setting where the edges are binary. However, we can adapt their sybil form of manipulation by simply removing these weights.

A sybil manipulation, or sybling strategy is a manipulation in which an agent controlling one vertex v in the graph can create any number of fraudulent identities (or sybils) and freely manipulate the links among these sybils, while maintaining the same set of incoming and outgoing links (possibly duplicated) among the sybil group as a whole.

Thus, we can define the sybil manipulation as:

$$\begin{aligned}
\mathcal{M}_{sybil}(V, E, v) \quad = \quad & \{(V', E')| \\
& V \subseteq V' \wedge \forall u, u' \in V \setminus \{v\} : (u, u') \in E \Leftrightarrow (u, u') \in E' \wedge \\
& P_G(v) \setminus \{v\} = (V \setminus \{v\}) \cap \bigcup_{u \in V' \setminus V \cup \{v\}} P_{G'}(u) \wedge \\
& S_G(v) \setminus \{v\} = (V \setminus \{v\}) \cap \bigcup_{u \in V' \setminus V \cup \{v\}} S_{G'}(u)\}.
\end{aligned}$$

We can also consider the combined manipulation of the two, which is not the same as the simple union of these manipulations:

$$\begin{aligned}
\mathcal{M}_{both}(V, E, v) \quad = \quad & \{(V', E')| \\
& V \subseteq V' \wedge \forall u, u' \in V \setminus \{v\} : (u, u') \in E \Leftrightarrow (u, u') \in E' \wedge \\
& P_G(v) \setminus \{v\} = (V \setminus \{v\}) \cap \bigcup_{u \in V' \setminus V \cup \{v\}} P_{G'}(u).
\end{aligned}$$

It turns out that strong incentive compatibility under both outgoing edge and sybling manipulations is equivalent to strong incentive compatibility under the combined manipulation:

Fact 5.2: Let F be a PRS. F satisfies strong incentive compatibility under $\mathcal{M}_{out}$ and under $\mathcal{M}_{sybil}$ if and only if it satisfies strong incentive compatibility under $\mathcal{M}_{both}$.

Proof: The "if" direction is trivial. For the "only if" direction, let $G = (V, E)$ be a graph and $v \in V$. Consider a manipulation $(V', E') \in \mathcal{M}_{both}(V, E, v)$. Let $U = \{x | \exists u \in V' \setminus V \cup \{v\} : (u, x) \in E'\}$. Let $E'' = E \setminus \{(v, x) | x \in V\} \cup \{(v, x) | x \in$

$U\}$. Now $(V, E'') \in \mathcal{M}_{out}(V, E, v)$ and $(V', E') \in \mathcal{M}_{sybil}(V, E'', v)$, and due to strong incentive compatibility under these manipulations, F also satisfies strong incentive compatibility under manipulation (V', E') and indeed under any manipulation in $\mathcal{M}_{both}$. $\square$

5.3.2 Satisfication

We will now demonstrate the aforementioned axioms by showing which axioms are satisfied by the PRSs mentioned in Section 5.2.2.

Proposition 5.3: The distance PRS F_D satisfies self confidence, ranked IIA, transitivity, and strong incentive compatibility under $\mathcal{M}_{both}$, but does not satisfy strong transitivity.

Proof: Self-confidence is satisfied by definition of F_D. F_D satisfies RIIA, because it ranks every comparison profile in the connected section consistently according to the following rule:

$$(a_1, a_2, \ldots, a_n) \preccurlyeq (b_1, b_2, \ldots, b_m) \Leftrightarrow a_n \leq b_m.$$

That is, any two vertices are compared according to their strongest predecessor. F_D satisfies strong quasi transitivity, because the ranking of the profiles above is consistent with strong quasi transitivity. The unconnected vertices are all equal to each other and weaker than the connected vertices which is also true for their predecessors, and thus strong quasi transitivity is satisfied.

To prove that F_D satisfies strong incentive compatibility, note the fact that an agent x cannot modify the shortest path from s to x by changing its outgoing links or adding sybils since any such shortest path necessarily does not include x or its sybils (except as target). Moreover, x or its sybils cannot change the shortest path to any agent y with $d(s, y) \leq d(s, x)$, because x and its sybils are necessarily not on the shortest path from s to y. Therefore, the amount of agents ranked above x and its sybils and the amount of agents ranked equal to x or its sybils cannot decrease due to x's manipulations.

To prove F_D does not satisfy strong transitivity, consider the graph in Figure 5.1a. In this graph, x and y are ranked the same, even though $P(x) \subsetneq P(y)$, in contradiction to strong transitivity. $\square$

Proposition 5.4: The Personalized PageRank ranking systems satisfy self confidence if and only if the damping factor is set to more than one half[3]. Moreover, Personalized PageRank does not satisfy weak transitivity, ranked IIA or strong incentive compatibility under $\mathcal{M}_{out}$ or $\mathcal{M}_{sybil}$ for any damping factor.

[3] If we do not allow self-loops this bound becomes $(\sqrt{5} - 1)/2 \approx 0.618$.

Proof: To prove the that PPR does not satisfy self-confidence for $d \leq \frac{1}{2}$, consider the graph in Figure 5.1b. For any damping factor d, the PPR will be $PPR(s) = d$ and $PPR(x) = 1 - d$. If $d \leq \frac{1}{2}$ then $PPR(s) \leq PPR(x)$ and thus $s \preceq^{PPR_d} x$, in contradiction to the self confidence axiom.

PPR satisfies self-confidence for $d > \frac{1}{2}$ because then $PPR(s) \geq d > \frac{1}{2}$, while for all $v \in V \setminus \{s\}$, $PPR(v) \leq 1 - d < \frac{1}{2}$.

To prove that PPR does not satisfy strong quasi transitivity and ranked IIA, consider the graph in Figure 5.1c. The PPR of this graph for any damping factor d is as follows: $PPR(s) = d$; $PPR(a) = \frac{d(1-d)}{2}$; $PPR(b) = \frac{d(1-d)^2}{4}$; $PPR(c) = \frac{d(1-d)^2}{2}$. Therefore, the ranking of this graph is: $b \prec c \prec a \prec s$. Quasi transitivity is violated because $b \prec c$ even though $P(b) = P(c) = a$. This also violates ranked IIA because the ranking profile $\langle (1), (1) \rangle$ must be ranked as equal due to trivial comparisons such as a and a.

Strong incentive compatibility under $\mathcal{M}_{out}$ is not satisfied, because in the graph in Figure 5.1c, if any of the b agents b' would have voted for themselves, they would have been ranked $b \prec b' \prec c \prec a \prec s$, which is a strict increase in b' rank.

To show that strong incentive compatibility under $\mathcal{M}_{sybil}$ is not satisfied, consider the graph in Figure 5.1d. Note that $a \simeq b \prec s$ in this graph. Consider the manipulation by a where a sybil a' is added along with the edges $\{(s, a'), (a', a)\}$. In this case, the PageRank value of b would be $\frac{1}{3}(1-d)d$ while the PageRank value of a will be $\frac{(1-d)+1}{3}(1-d)d$. Therefore, $b \prec a \prec s$ in the manipulated graph, and thus strong incentive compatibility is not satisfied. $\square$

It is interesting to note that although Personalized Pagerank does not satisfy strong incentive compatibility under $\mathcal{M}_{sybil}$, a weighted version of Personalized PageRank *is* in fact sybilproof with regard to the weighted definition of sybilproofness presented in Cheng and Friedman (2005).

Strong transitivity is also satisfied by a natural PRS — the α-Rank system:

Proposition 5.5: The α-Rank system satisfies self confidence and strong transitivity, but does not satisfy ranked IIA or strong incentive compatibility under $\mathcal{M}_{out}$ or $\mathcal{M}_{sybil}$.

Proof: To show α-Rank satisfies self confidence, note that by definition $r_{G,s}(s) \geq 1$. Assume for contradiction that $\max_{v \neq s} r_{G,s}(v) \geq 1$. Then,

$$\begin{aligned} r_{G,i}(s) &\leq 1 + \alpha \sum_{v \in V} r_{G,s}(v) \\ &\leq 1 + \alpha \left[(n-1) \max_{v \neq s} r_{G,s}(v) + r_{G,i}(s) \right] \end{aligned}$$

$$\begin{aligned} r_{G,i}(s) &\leq \frac{1}{1-\alpha} + \frac{\alpha}{1-\alpha}(n-1) \max_{v \neq s} r_{G,s}(v) \leq \\ &\leq 2 + \max_{v \neq s} r_{G,s}(v) \end{aligned}$$

$$
\begin{aligned}
\max_{v \neq s} r_{G,s}(v) &\leq \alpha^n + \alpha \sum_{v \in V} r_{G,s}(v) \\
&\leq \alpha^n + \alpha \left[n \cdot \max_{v \neq s} r_{G,s}(v) + 2 \right]
\end{aligned}
$$

$$
\begin{aligned}
\left[1 - \frac{n}{n^2}\right] \max_{v \neq s} r_{G,s}(v) &\leq \frac{2}{n^2} + \frac{1}{n^{2n}} \\
n^2 - n &\leq 2 + 1/n^{2n-2} \\
n^2 - n - 1/n^{2n-2} &\leq 2 \\
2 \leq n(n-1) &< 2
\end{aligned}
$$

To prove α-Rank satisfies strong transitivity, consider two vertices $a, b \in V \setminus \{s\}$ and a function $f : P(a) \mapsto P(b)$ such that $v \preceq f(v)$ for all $v \in P(a)$. Then,

$$
\begin{aligned}
r_{G,s}(a)/\alpha - \alpha^n &= \sum_{v \in P(a)} r_{G,s}(v) \leq \sum_{v \in f(P(a))} r_{G,s}(v) \leq \\
&\leq \sum_{v \in P(b)} r_{G,s}(v) = r_{G,s}(b)/\alpha - \alpha^n,
\end{aligned} \tag{5.1}
$$

which implies $a \preceq b$. If for some $v \in P(a)$: $v \prec f(v)$, or if f is not onto, then the first or the second inequality respectively in (5.1) above is strict, which implies $a \prec b$, as required.

To prove α-Rank does not satisfy strong incentive compatibility under $\mathcal{M}_{out}$, consider the graph in Figure 5.1e. In this graph α-Rank ranks $d \prec b$. However, if d removes the link to b they will be ranked equally and thus reducing the number of agents stronger than d. To prove α-Rank does not satisfy strong incentive compatibility under $\mathcal{M}_{sybil}$, consider again the graph in Figure 5.1e. Agent c is ranked below agent b in this graph. However, she can duplicate herself and add edges (c, c') and (c', c) to be ranked above b thus decreasing the number of agents ranked better than herself.

To prove α-Rank does not satisfy RIIA, consider the graph in Figure 5.1f. It is easy to calculate the following α-Rank values:

$$
\begin{aligned}
r(s) &= 1 \\
r(i) = r(h) &= \alpha + \alpha^{10} \\
r(d) = r(e) &= \alpha^2 + \alpha^{10} + \alpha^{11} \\
r(f) &= 2\alpha^2 + \alpha^3 + \alpha^{10} + 3\alpha^{11} + \alpha^{12} \\
r(g) &= \alpha^2 + \alpha^3 + \alpha^{10} + 2\alpha^{11} + \alpha^{12} \\
r(a) &= 2\alpha^3 + \alpha^{10} + 2\alpha^{11} + 2\alpha^{12} \\
r(b) &= 2\alpha^3 + \alpha^4 + \alpha^{10} + \alpha^{11} + 3\alpha^{12} + \alpha^{13} \\
r(c) &= \alpha^3 + \alpha^4 + \alpha^{10} + \alpha^{11} + 2\alpha^{12} + \alpha^{13}.
\end{aligned}
$$

Therefore, this graph is ranked $c \prec a \prec b \prec d \simeq e \prec g \prec f \prec i \simeq h \prec s$. Note that (a, b) and (a, c) both satisfy the profile $\langle(1, 1), (2)\rangle$, however $a \prec b$ and $c \prec a$ in contradiction to RIIA. $\square$

5.4 A Characterization Theorem

Our main result is a full characterization of the PRSs that satisfy the axioms above. We will see that these systems are the generalized strong count systems. Strong count ranks agents based on their strongest predecessors, breaking ties according to the *number* of equal strongest predecessors the agents have. The function r below determines how such ties are broken. As s is stronger than all other agents, the strongest predecessor of each agent in $V_s \setminus \{s\}$ must be closer to s.

The strong count systems are formally defined as follows:

Definition 5.11: Let $r : \mathbb{N} \mapsto \mathbb{N}$ be a monotone nondecreasing function such that $r(i) \leq i$ for all $i \in \mathbb{N}$. The strong count system SC_r is recursively defined as follows: First of all, $y \simeq y' \prec x \prec s$ for all $x \in V_s \setminus \{s\}$ and $y, y' \in V \setminus V_s$. For $x \in V_s \setminus \{s\}$, denote $P'(x) = P(x) \cap \{y|d(s, y) < d(s, x)\}$, and $P_{max}(x) = \{y|y \in P'(x), \forall z \in P'(x) : z \preceq^{SC_r} y\}$. Now for $a, b \in V_s \setminus \{s\}$:

$$
\begin{aligned}
a \preceq^{SC_r} b \quad \Leftrightarrow \quad & (\exists x \in P_{max}(a), y \in P_{max}(b) : x \prec^{SC_r} y) \vee \\
\vee[\quad & (\forall x \in P_{max}(a), y \in P_{max}(b) : x \simeq^{SC_r} y) \wedge \\
& \wedge((r(|P_{max}(a)|) \leq r(|P_{max}(b)|))].
\end{aligned}
$$

The strong count systems rank based on the strongest predecessor's rank and then break ties based on the number of strongest predecessors. Unconnected vertices are equally ranked at the bottom. Note that for $r \equiv 1$, the Strong Count PRS is exactly the distance system.

Our main result claims that these strong count systems are the only systems that satisfy all aforementioned axioms.

Theorem 5.6: Let F be a PRS. The following three statements are equivalent:

1. F is a strong count system for some r.
2. F satisfies self confidence, strong quasi transitivity, ranked IIA and strong incentive compatibility under $\mathcal{M}_{out}$.
3. F satisfies self confidence, strong quasi transitivity, ranked IIA and strong incentive compatibility under $\mathcal{M}_{both}$.

We begin our proof by showing that the strong count systems do in fact satisfy all these axioms.

Proof: (1 $\Rightarrow$ 3): Let r be a monotone nondecreasing function such that $r(x) \leq x$. SC_r satisfies self confidence by definition.

To show that SC_r satisfies RIIA and strong quasi transitivity on elements of V_s, we will show that it ranks any profile $p = \langle (a_1, \ldots, a_n); (b_1, \ldots, b_m) \rangle$ as follows: Let $c_a = \max\{i \in \mathbb{N} | a_{n-i} = a_{n-i+1} = \cdots = a_n\}$ and $c_b = \max\{i \in \mathbb{N} | b_{m-i} = b_{m-i+1} = \cdots = b_m\}$.

$$\begin{aligned} f(p) = 1 \quad \Leftrightarrow & \quad (a_n < b_m) \vee \\ \vee & \quad [(a_n = b_m) \wedge (r(c_a) \leq r(c_b))] \end{aligned}$$

This almost follows from the recursive definition of SC_r, however it remains to show that $\forall x, y \in V : d(s,x) < d(s,y) \Rightarrow x \prec^{SC} y$. This can be proven by induction on $d(s,y)$. If $y = s$ this is trivial by definition. Otherwise, by the assumption of induction, $\exists x' \in P_{max}(x), y' \in P_{max}(y) : x' \prec^{SC} y'$ and thus by the recursive definition, $x \prec^{SC} y$.

Strong quasi transitivity involving elements in $V \setminus V_s$ and elements either in $V \setminus V_s$ or in $V_s \setminus \{s\}$ is satisfied because for all $x \in V \setminus V_s$ and $y \in V \setminus \{s\}$ we have $x \preceq y$ (by definition) and if $x \prec y$ then $y \in V_s \setminus \{s\}$ and thus there is some $y' \in P(y)$ such that for all $x' \in P(x)$: $x' \preceq y'$.

With regard to the strong incentive compatibility under $\mathcal{M}_{both}$, due to the distance feature proven above, all sybils of v will be strictly weaker than the vertices with smaller distance from s. Furthermore, any other vertices that were stronger than v in the original graph will be stronger than any of v's sybils, due to the fact that the relative rank of two vertices is determined only based on incoming links from vertices closer to s, and more incoming edges cannot decrease an agent's rank. By the same logic, vertices which were equal to v in the original graph, will either be stronger or equal to v in the manipulated graph. □

In order to prove the hard direction of Theorem 5.6 (2 $\Rightarrow$ 1), we will first show that a strong notion of transitivity is implied by the axioms:

Definition 5.12: Let F be a PRS. We say that F satisfies *weak maximum transitivity* if for all graphs $G = (V, E)$, for all sources $s \in V$ and for all vertices $v_1, v_2 \in V_s$: Let m_1, m_2 be the maximally ranked vertices in $P(v_1), P(v_2)$ respectively. Assume $m_1 \prec m_2$. Then, $v_1 \prec v_2$.

Lemma 5.7: Let F be a PRS that satisfies self confidence, strong quasi transitivity, RIIA and strong incentive compatibility. Then, F satisfies weak maximum transitivity.

Proof: In order to show that F satisfies weak maximum transitivity, we will show that for every comparison profile the ranking must be consistent with weak maximum transitivity. Let $p = \langle (a_1, a_2, \ldots, a_k), (b_1, b_2, \ldots, b_l) \rangle$ be a comparison profile where $a_k \neq b_l$. Assume wlog that $b_l < a_k$ and assume for contradiction that $\langle (a_1, a_2, \ldots, a_k) \preceq (b_1, b_2, \ldots, b_l) \rangle$. Consider the graph $G = (V, E)$ defined as follows:

$$\begin{aligned} V &= \{s,a,b\} \cup \{u_i^j | i \in \{1,\ldots,\max(k,l)\}; j \in \{0,\ldots,a_k\}\} \\ E &= \{(u_i^j, u_i^{j-1}) | i \in \{1,\ldots,\max(k,l)\}; j \in \{1,\ldots,a_k\}\} \cup \\ &\cup \{(s, u_i^{b_l}) | i \in \{1,\ldots,\max(k,l)\}\} \cup \\ &\cup \{(u_i^j, a) | a_i = j\} \cup \{(u_i^j, b) | b_i = j\}. \end{aligned}$$

Figure 5.2 contains such a graph for the profile $\langle(1,4),(2,2,3)\rangle$.

Note that by strong quasi transitivity and self confidence, for all i, i', j, j': $u_i^j \preceq u_{i'}^{j'}$ iff $, j \leq j'$. Therefore, we will use u^j to denote any u_i^j. By the construction of G, a and b satisfy p. Thus, from our assumption, $a \preceq b$.

By strong quasi transitivity, $a \succeq u^{b_l}$, and thus from our assumption also $b \succeq u^{b_l}$. Now consider the point of view of agent $u_l^{b_l}$. She can perform a manipulation by not voting for b. This manipulation must not change her relative rank, as it is in $\mathcal{M}_{out}$. As the relative ranks of the u_i^j agents and s are unaffected by this manipulation, it cannot affect the ranks of a and b relative to $u_l^{b_l}$, and thus after the edge $(u_l^{b_l}, b)$ is removed, we still have $b \succeq u_l^{b_l}$. We can repeat this process for all $i = b_l, \ldots, 2$, with the result that in the graph G' for the profile $\langle(a_1, a_2, \ldots, a_k),(b_1)\rangle$, $b \succeq u^{b_2} \succeq u^{b_1}$. However, by strong quasi transitivity, $b \simeq_{G'} u^{b_1-1} \prec_{G'} u^{b_1} \preceq_{G'} b$, which is a contradiction.

□

We can now prove the hard direction of Theorem 5.6.

Proof: (Theorem 5.6: 2 ⇒ 1) Given Lemma 5.7, it remains to look at profiles $\langle(a_1, a_2, \ldots, a_k),(b_1, b_2, \ldots, b_l)\rangle$ where $a_k = b_l$. Denote $M = a_k = b_l$. Let p be such a profile. Denote $x_a = |\{n | a_n = M\}|$ and similarly $x_b = |\{n | b_n = M\}|$. These values denote the number of strongest predecessors a and b have in profile p.

We will now prove by induction on $k + l - x_a - x_b$ that F ranks p the same as it ranks $\langle(\underbrace{1,\ldots,1}_{x_a \text{ times}}),(\underbrace{1,\ldots,1}_{x_b \text{ times}})\rangle$. If $k + l - x_a - x_b = 0$, then $a_1 = a_k = b_1 = b_l$, and thus the requirement is trivially satisfied. Otherwise, we assume correctness for $k + l - x_a - x_b - 1$. Further assume wlog that $a_1 \neq a_k$. Denote $r = a_{k-x_a}$ and $y_a = |\{n | a_n = r\}|$.

We shall now consider two cases:

- If $b_1 = b_l$ or $a_{k-x_a} \neq b_{l-x_b}$. If $b_1 \neq b_l$, then further assume wlog that $a_{k-x_a} > b_{l-x_b}$. Consider the graph $G = (V, E)$ defined as follows:

$$\begin{aligned} V &= \{s,a\} \cup \{b^1,\ldots,b^{y_a}\} \cup \\ &\cup \{u_i^j | i \in \{1,\ldots,\max(k,l)\}; j \in \{0,\ldots,M\}\} \\ E &= \{(u_i^j, u_i^{j-1}) | i \in \{1,\ldots,\max(k,l)\}; j \in \{1,\ldots,M\}\} \cup \\ &\cup \{(s, u_i^M) | i \in \{1,\ldots,\max(k,l)\}\} \cup \{(u_i^j, a) | a_i = j \neq r\} \cup \\ &\cup \{(u_i^j, b^n) | b_i = j, n = 1,\ldots,y_a\} \cup \{(b^n, a) | n = 1,\ldots,y_a\}. \end{aligned}$$

Figure 5.3 contains such a graph for the profile $\langle(1,3,3,4),(1,2,4,4)\rangle$. Note that by strong quasi transitivity and self confidence, for all i,i',j,j': $u_i^j \preceq u_{i'}^{j'}$ iff $,j \leq j'$. Therefore, we will use u^j to denote any u_i^j. Similarly, all b^n are equal to each other, and by weak maximum transitivity (Lemma 5.7), $u^{M-1} \preceq a, b \prec u^M$ (we will similarly use b to denote any b^n). Therefore, a and b satisfy p. Now consider the following manipulation by b^1: Removing the outgoing edge to a. This manipulation is in $\mathcal{M}_{out}$ and thus should not change the relative rank of b^1. Note that b^1's predecessors remain the same and equal to the ones of $b^2,\ldots,b^{y_a}$, and all b^n remain equal. We must now show that for every allowable relative ranking of u^{M-1}, a, and b the manipulation cannot change a and b's relative rank. We will do this by considering all cases:

Ordering	# Vertices equal to b	# Vertices stronger than b
$u^{M-1} \simeq b \prec a$	$y_a + \max(k,l)$	$(M-r)\cdot\max(k,l)+2$
$u^{M-1} \prec b \prec a$	y_a	$(M-r)\cdot\max(k,l)+2$
$u^{M-1} \simeq a \simeq b$	$y_a + \max(k,l)+1$	$(M-r)\cdot\max(k,l)+1$
$u^{M-1} \prec a \simeq b$	y_a+1	$(M-r)\cdot\max(k,l)+1$
$u^{M-1} \simeq a \prec b$	y_a	$(M-r)\cdot\max(k,l)+1$
$u^{M-1} \prec a \prec b$	y_a	$(M-r)\cdot\max(k,l)+1$

We see that any change in the relation between a and b will surely change b's rank in a way that is not strategyproof.

We have shown that profile p must be ranked the same as the profile

$$\langle(a_1,a_2,\ldots,a_{k-x_a-1},a_{k-x_a+1},\ldots,a_k),(b_1,b_2,\ldots,b_l)\rangle,$$

which by the assumption of induction gives us the desired result.

- Otherwise, $a_{k-x_a} = b_{l-x_b}$. Denote $y_b = |\{n|b_n = r\}|$ and assume wlog that $y_b \geq y_a$. Consider the graph $G=(V,E)$ defined as follows:

$$\begin{aligned} V &= \{s,a\}\cup\{b^0,\ldots,b^{y_b}\}\cup \\ &\quad \cup\{u_i^j|i\in\{1,\ldots,\max(k,l)\};j\in\{0,\ldots,M\}\} \\ E &= \{(u_i^j,u_i^{j-1})|i\in\{1,\ldots,\max(k,l)\};j\in\{1,\ldots,M\}\}\cup \\ &\quad \cup\{(s,u_i^M)|i\in\{1,\ldots,\max(k,l)\}\}\cup\{(u_i^j,a)|a_i=j\neq r\}\cup \\ &\quad \cup\{(u_i^j,b^n)|b_i=j\neq r, n=0,\ldots,y\}\cup \\ &\quad \cup\{(b^n,a)|n=1,\ldots,y_a\}\cup\{(b^n,b^m)|n\neq m\in\{0,\ldots,y_b\}\}. \end{aligned}$$

Figure 5.4 contains such a graph for the profile $\langle(1,1,2,2),(1,1,1,2)\rangle$. As before, for all i,i',j,j': $u_i^j \preceq u_{i'}^{j'}$ iff $,j\leq j'$ and we will use u^j to denote any u_i^j. All b^n are equal to each other because if wlog $b^1 \prec b^2$ then b^1's predecessors will be stronger than b^2's predecessors and thus by strong quasi transitivity $b^2 \preceq b^1$. Again, by weak maximum transitivity, $u^{M-1} \preceq a, b \prec u^M$ and we will use b to denote any b^n. Therefore, a and b satisfy p. We can again consider a manipulation by b^1 removing an edge to a, again all b^n remain equal and as before the manipulation cannot change a and b's relative rank, and when again applying the assumption of induction we get the desired result.

By strong quasi transitivity, profiles where all predecessors are equal are ranked $\langle 1\rangle \preceq \langle 1,1\rangle \preceq \cdots$. When considering the result above, we conclude any two vertices should be weakly ranked according to the number of strongest predecessors they have, and by RIIA the tie-breaking rule must be universal.

It remains to show that vertices in $V \setminus V_s$ will be ranked equally and strictly weaker than those in V_s. Let $m \in V_s$ be a minimally ranked vertex in V_s. Consider a manipulation by m adding edges to all vertices in $V \setminus V_s$. By the above proof, all vertices in $V \setminus V_s$ will be equally ranked weaker than m. As m does not worsen its position by performing this manipulation and the internal ranking in V_s does not change we conclude that in any graph all vertices in $V \setminus V_s$ must be ranked strictly weaker than those in V_s.

We can show the vertices in $V \setminus V_s$ are ranked equally by induction on the number of edges between them. If there are no such edges, then by strong quasi transitivity, the requirement is satisfied. Otherwise, consider an edge (v_1, v_2) such that $v_1, v_2 \in V \setminus V_s$. A manipulation by v_1 adding this edge must retain its position and thus all agents in $V \setminus V_s$ must be ranked equally.

We have shown that all vertices must be ranked according to strong count and thus the system must be a strong count system. $\square$

5.5 Relaxing the Axioms

We shall now prove the conditions in Lemma 5.7 (and thus also in Theorem 5.6(2)) are all necessary by showing PRSs that satisfy each three of the four conditions, but do not satisfy weak maximum transitivity. Some of these systems are quite artificial, while others are interesting and useful.

Proposition 5.8: There exists a PRS that satisfies strong quasi transitivity, RIIA and strong incentive compatibility, but not self confidence nor weak maximum transitivity.

Proof: Let F_D^- be the PRS that ranks strictly the opposite of the distance system F_D. That is, $v_1 \preceq_{G,s}^{F_D^-} v_2 \Leftrightarrow v_2 \preceq_{G,s}^{F_D} v_1$. The proof F_D^- satisfies strong quasi transitivity, RIIA and strong incentive compatibility follows the proof of Proposition 5.3, with the following rule for ranking comparison profiles:

$$(a_1, a_2, \ldots, a_n) \preccurlyeq (b_1, b_2, \ldots, b_m) \Leftrightarrow a_1 \leq b_1.$$

F_D^- does not satisfy self confidence, because, by definition s is weaker than all other agents, and does not satisfy weak maximum transitivity because in graph from Figure 5.1a, F_D^- ranks x and y equally even though the strongest predecessor of y, which is x, is stronger than the strongest predecessor of x, which is s. $\square$

This PRS is highly unintuitive, as the most trusted agents are the ones furthest from the source, which is by itself the least trusted.

Relaxing strong quasi transitivity leads to a PRS that is almost trivial:

Proposition 5.9: There exists a PRS that satisfies self confidence, ranked IIA and strong incentive compatibility, but not strong quasi transitivity nor weak maximum transitivity.

Proof: Let F be the PRS which ranks for every $G = (V, E)$, for every source $s \in V$, and for every $v_1, v_2 \in V \setminus \{s\}$: $v_1 \simeq v_2 \prec s$. That is, F ranks s on the top, and all of the other agents equally. F trivially satisfies self confidence, RIIA and strong incentive compatibility, as s is indeed stronger than all other agents and every comparison profile is ranked equally. F does not satisfy strong quasi transitivity or weak maximum transitivity, because in a chain of vertices starting from s all except s will be ranked equally, $\square$

5.5.1 Relaxing Ranked IIA

When Ranked IIA is relaxed, we find a new ranking system that ranks according to the distance from s, breaking ties according to the number of shortest paths from s.

Notation: Let $G = (V, E)$ be some directed graph and $v_1, v_2 \in V$ be some vertices, we will use $n_G(v_1, v_2)$ to denote the number of directed paths of minimum length between v_1 and v_2 in G. We will sloppily use the notations $d(v)$ and $n(v)$ to denote $d_G(s, v)$ and $n_G(s, v)$ respectively.

Definition 5.13: The Path Count PRS F_P is defined as follows: Given a graph $G = (V, E)$ and a source s, for all $v_1, v_2 \in V \setminus \{s\}$:

$$\begin{aligned} v_1 \preceq^{F_P}_{G,s} v_2 \quad \Leftrightarrow \quad & d_G(s, v_1) > d_G(s, v_2) \vee \\ & (d_G(s, v_1) = d_G(s, v_2) \wedge \\ & \wedge n_G(s, v_1) \leq n_G(s, v_2)) \end{aligned}$$

Proposition 5.10: The path count PRS F_P satisfies self confidence, strong quasi transitivity and strong incentive compatibility under $\mathcal{M}_{both}$, but not ranked IIA nor weak maximum transitivity.

Proof: Self confidence is trivial as $d(s) = 0 < d(v)$ for all $v \neq s$.

To prove F_P satisfies quasi transitivity consider a graph $G = (V, E)$, a source $s \in V$ and two vertices $v_1, v_2 \in V \setminus \{s\}$. Assume for contradiction that $v_2 \prec v_1$ and there exists a 1-1 function $f : P(v_1) \mapsto P(v_2)$ such that $v \preceq f(v)$ for all $v \in P(v_1)$. By the definition of F_P: $d(v_1) \leq d(v_2)$, but

$$d(v_1) \quad = \quad \min_{v \in P(v_1)} d(v) + 1 \geq \min_{v \in f(P(v_1))} d(v) + 1 \geq \min_{v \in P(v_2)} d(v) + 1 = d(v_2),$$

and thus $d(v_1) = d(v_2)$. Now,

$$\begin{aligned} n(v_1) &= \sum_{v \in P(v_1) \wedge d(v)+1=d(v_1)} n(v) \leq \\ &\leq \sum_{v \in f(P(v_1)) \wedge d(f^{-1}(v))+1=d(v_1)} n(v) \leq \\ &\leq \sum_{v \in P(v_2) \wedge d(v)+1=d(v_2)} n(v) = n(v_2). \end{aligned}$$

Therefore, $v_1 \preceq v_2$ in contradiction to our assumption.

For strong quasi transitivity, assume now that $v_2 \preceq v_1$, $P(v_1) \neq \emptyset$, and there exists a 1-1 function $f : P(v_1) \mapsto P(v_2)$ such that $v \prec f(v)$ for all $v \in P(v_1)$. As above we find that $d(v_1) = d(v_2)$. Now,

$$\begin{aligned} n(v_1) &= \sum_{v \in P(v_1) \wedge d(v)+1=d(v_1)} n(v) < \\ &< \sum_{v \in f(P(v_1)) \wedge d(f^{-1}(v))+1=d(v_1)} n(v) \leq n(v_2), \end{aligned}$$

which yields $v_1 \prec v_2$ in contradiction to our assumption.

To show F_P satisfies strong incentive compatibility under $\mathcal{M}_{both}$, note that a manipulation by v cannot change $d(v)$ or $d(v') \; \forall v' : d(v') < d(v)$. Moreover, v and its sybils cannot gain any new edges from vertices closer to v or change their internal edges. For this reason, $n(v)$ cannot increase and $n(v')$ cannot decrease for all v' s.t. $d(v') \leq d(v)$. Thus, F_P does indeed satisfy strong incentive compatibility under $\mathcal{M}_{both}$.

To show F_P does not satisfy ranked IIA nor weak maximum transitivity, consider the graph in Figure 5.5. F_P ranks this graph as follows: $a \prec b \prec y \prec z \prec x \prec s$. Consider the profile $\langle (2); (1,1) \rangle$. If we compare x and y we get $(1,1) \prec (2)$, but if we compare a and b we get $(2) \prec (1,1)$, in violation of ranked IIA. Furthermore, the latter comparison is in violation of weak maximum transitivity, as required. □

5.5.2 Relaxing incentive compatibility

When we relax incentive compatibility we find that the familiar Recursive In-degree ranking systems from section 3.6 on page 36 can be easily adapted to the personalized setting as well. The only difference from the previous definition is that we use base $(n+2)$ and assign a maximal strength of $\frac{n+1}{n+2}$ to the source vertex s.

Definition 5.14: Let $r : \mathbb{N} \mapsto \mathbb{N}$ be a monotone nondecreasing function such that $r(i) \leq i$ for all $i \in \mathbb{N}$. The *recursive in-degree PRS with rank function r* is defined as follows: Given a graph $G = (V, E)$ and source s,

$$v_1 \preceq^{RID_r}_{G,s} v_2 \Leftrightarrow \text{value}_{r,s}(v_1) \leq \text{value}_{r,s}(v_2),$$

where value is defined as:

$$\text{value}_{r,s}(v) \quad = \quad \max_{\mathbf{a} \in \text{Path}_s(v)} \text{vp}_{r,s}(\mathbf{a}) \tag{5.2}$$

where the maximum is over the set of almost-simple paths to v not passing through s (but which may start at s):

$$\begin{aligned} \text{Path}_s(v) \quad = \{ \quad & (v = a_1, a_2, \ldots, a_m) | \\ & (a_m, \ldots, a_1) \text{ is a path in } G \ \wedge (a_{m-1}, \ldots, a_1) \text{ is simple} \wedge \\ & \forall i \in \{1 \ldots m-1\} : a_i \neq s\}. \end{aligned}$$

and valuation function $\text{vp} : V^* \mapsto \mathbb{Q}$ is defined as:

$$\text{vp}_{r,s}(a_1, a_2, \ldots, a_m) = \frac{1}{n+2} \left[\begin{array}{l} \begin{cases} n+1 & a_1 = s \\ r(|P(a_1)|) & \text{Otherwise} \end{cases} \\ \quad + \\ \begin{cases} 0 & m = 1 \\ \text{vp}_{r,s}(a_2, \ldots, a_m, a_2) & a_1 = a_m \wedge m > 1 \\ \text{vp}_{r,s}(a_2, \ldots, a_m) & \text{Otherwise.} \end{cases} \end{array} \right] \tag{5.3}$$

Note that $\text{vp}_{r,s}(a_1, a_2, \ldots, a_m)$ is infinitely recursive in the case when $a_1 = a_m \wedge m > 1$. For computation sake we can redefine this case finitely as:

$$\begin{aligned} \text{vp}_{r,s}(a_1, \ldots, a_m, a_1) \quad &= \quad \sum_{i=0}^{\infty} \frac{1}{(n+2)^{mi}} \sum_{j=1}^{m} \frac{r(|P(a_j)|)}{(n+2)^j} = \\ &= \quad \frac{(n+2)^m}{(n+2)^m - 1} \text{vp}_{r,s}(a_1, \ldots, a_m). \end{aligned}$$

Further note that when the r function is constant ($r \equiv 1$), then the recursive in-degree system becomes the distance system on V_s, where the vertices in $V \setminus V_s$ are ranked weaker, and the ordering among them is set according to the length of the longest path (simple or not) leading to the vertex.

Example 5.1: An example of the values assigned for a particular graph when r is the identity function is given in Figure 5.6. As $n = 8$, the trust values are decimal. Note that the loop (b, d) generates a periodical decimal $\text{value}_{r,s}(b) = \text{vp}_{r,s}(b, d) = 0.\overline{32}$ by the infinite recursion in (5.3).

These systems satisfy the axioms as required:

Proposition 5.11: Let $r : \mathbb{N} \mapsto \mathbb{N}$ be a monotone nondecreasing function such that $r(i) \leq i$ for all $i \in \mathbb{N}$ and define $r(0) = 0$. The recursive in-degree ranking system with rank function r satisfies self-confidence, strong quasi-transitivity and RIIA. If r is not constant[4] then the recursive in-degree system further does not satisfy weak maximum transitivity nor strong incentive compatibility under either $\mathcal{M}_{out}$ or $\mathcal{M}_{sybil}$.

Proof: We will prove that in the entire graph (not just V_s) every comparison profile $\langle \mathbf{a}, \mathbf{b} \rangle$ where $\mathbf{a} = (a_1, \ldots, a_k)$, $\mathbf{b} = (b_1, \ldots, b_l)$ is ranked as follows:

$$f\langle \mathbf{a}, \mathbf{b} \rangle = 1 \quad \Leftrightarrow \quad (k = 0) \vee (r(k) < r(l)) \vee [(r(k) = r(l)) \wedge (a_k \le b_l)] \,.$$

Note that this ranking of comparison profiles also implies strong quasi transitivity. To show comparison profiles are ranked as such, we will prove that

$$\text{value}_{r,s}(v) \quad = \quad \begin{cases} 0 & v \neq s \wedge P(v) = \emptyset \\ \frac{n+1}{n+2} & v = s \\ \frac{1}{n+2}\left[r(|P(v)|) + \max_{p \in P(v)} \text{value}_{r,s}(p)\right] & \text{Otherwise} \end{cases} \tag{5.4}$$

and note that $0 \le \text{value}_{r,s}(v) \le \frac{n+1}{n+2}$, and thus vertices other than s are ordered first by $r(|P(v)|)$ and then by $\max_{p \in P(v)} \text{value}_{r,s}(p)$, as required. Moreover, self confidence is satisfied because for all $v \neq s$: $\text{value}_{r,s}(v) < \frac{n+1}{n+2}$.

The two edge cases are trivial, we shall now concentrate on the primary case in (5.4). Let $v \in V \setminus \{s\}$ be some vertex where $P(v) \neq \emptyset$. Denote $\text{Path}'_s(p, v)$ as the set of almost-simple directed paths to p stopping at s which do not pass through v unless immediately looping back to p:

$$\begin{aligned} \text{Path}'_s(p, v) \quad = \{ \quad & (p = a_1, a_2, \ldots, a_m) | \\ & (a_m, \ldots, a_1) \text{ is a path in } G \ \wedge (a_{m-1}, \ldots, a_1) \text{ is simple} \wedge \\ & \forall i \in \{1 \ldots m-1\} : a_i \neq s \wedge \\ & \forall i \in \{1, \ldots, m-2, m\} : a_i \neq v \wedge a_{m-1} = v \Leftrightarrow a_m = p \}. \end{aligned}$$

Now we see that:

$$\begin{aligned} \text{value}_{r,s}(v) \quad &= \quad \max_{\mathbf{a} \in \text{Path}_s(v)} \text{vp}_{r,s}(\mathbf{a}) = \\ &= \quad \frac{1}{n+2} \left[\begin{array}{l} r(|P(v)|) + \max_{(v=a_1,\ldots,a_m) \in \text{Path}_s(v)} \\ \begin{cases} \text{vp}_{r,s}(a_2, \ldots, a_m, a_2) & a_1 = a_m \wedge m > 1 \\ \text{vp}_{r,s}(a_2, \ldots, a_m) & \text{Otherwise.} \end{cases} \end{array} \right] = \qquad (5.5) \\ &= \quad \frac{1}{n+2} \left[r(|P(v)|) + \max_{p \in P(v)} \max_{\mathbf{a} \in \text{Path}'_s(p,v)} \text{vp}_{r,s}(\mathbf{a}) \right] = \qquad (5.6) \\ &= \quad \frac{1}{n+2} \left[r(|P(v)|) + \max_{p \in P(v)} \max_{\mathbf{a} \in \text{Path}_s(p)} \text{vp}_{r,s}(\mathbf{a}) \right] = \\ &= \quad \frac{1}{n+2} \left[r(|P(v)|) + \max_{p \in P(v)} \text{value}_{r,s}(p) \right]. \end{aligned}$$

[4]If r is constant, the system still does not satisfy strong incentive compatibility under either $\mathcal{M}_{out}$ or $\mathcal{M}_{sybil}$, but only if we allow vertices that have no path from s.

To show that the equality (5.6) holds, assume for contradiction that there exists $p \in P(v)$ and $\mathbf{a} \in \text{Path}_s(p)$ such that

$$\text{vp}_{r,s}(\mathbf{a}) > \max_{p' \in P(v)} \max_{\mathbf{a}' \in \text{Path}'_s(p',v)} \text{vp}_{r,s}(\mathbf{a}'). \tag{5.7}$$

From $\mathbf{a} \in \text{Path}_s(p) \setminus \text{Path}'_s(p,v)$, we know that $a_i = v$ for some $i \in \{1, \ldots, m\}$. Assume wlog that i is minimal. Let $\mathbf{b}$ denote the path $(p = a_1, a_2, \ldots, a_i, p)$ and let $\mathbf{c}$ denote the path $(p' = a_{i+1}, \ldots, a_m, a_{j+1}, \ldots, a_{i+1})$ if $a_m = a_j$ for some $j < i$ or $(p' = a_{i+1}, \ldots, a_m)$ otherwise. An example of such paths is given in Figure 5.7. Note that $\mathbf{b} \in \text{Path}'_s(p,v)$ and $\mathbf{c} \in \text{Path}'_s(p',v)$, where $p, p' \in P(v)$. Now, note that

$$\text{vp}_{r,s}(\mathbf{a}) \quad = \quad \frac{(n+2)^j - 1}{(n+2)^j}\text{vp}_{r,s}(\mathbf{b}) + \frac{1}{(n+2)^j}\text{vp}_{r,s}(\mathbf{c}),$$

and thus $\text{vp}_{r,s}(\mathbf{a})$ must be between $\text{vp}_{r,s}(\mathbf{b})$ and $\text{vp}_{r,s}(\mathbf{c})$, in contradiction to assumption (5.7).

We shall now prove that recursive in-degree is not incentive compatible under $\mathcal{M}_{out}$ or $\mathcal{M}_{sybil}$ and does not satisfy weak maximum transitivity. Let $i \in \mathbb{N}$ be the minimum number such that $r(i) > 1$. Consider the graph G in Figure 5.8, where there are i vertices labeled x. This graph is ranked $x \prec t \prec s$, where x refers to all vertices labeled x. Weak maximum transitivity is not satisfied because $x \prec t$ even though $s \succ x$. Let x' be one of the vertices labeled x. It can perform a manipulation in $\mathcal{M}_{out}$ by removing its edge to t, and thus changing the ranking to $x \simeq x' \simeq t \prec s$. It can also perform a manipulation in $\mathcal{M}_{sybil}$ by creating i additional sybils of themselves and create a complete clique thus changing the ranking to $x \prec v \simeq x' \simeq t \prec s$, where v are the new vertices involved from the manipulation. □

For an extensive study of the recursive in-degree system in the context of general ranking systems see Section 3.6.

5.6 Concluding Remarks

We have presented a method for the evaluation of personalized ranking systems by using axioms adapted from general ranking systems, and evaluated existing and new personalized ranking systems according to these axioms. As most existing PRSs do not satisfy these axioms, we have presented several new and practical personalized ranking systems that satisfy subsets, or indeed all, of these axioms. We argue that these new ranking systems have a more solid theoretical basis, and thus may very well be successful in practice. Furthermore, we have proven a representation theorem for the Strong Count ranking systems, which are the only systems that satisfy all axioms.

This study is far from exhaustive. Further research is due in formulating new axioms, and proving representation theorems for the various PRSs suggested in this chapter. An additional avenue for research is modifying the setting in order to accommodate for more elaborate input such as trust/distrust relations or numerical trust ratings, as seen in some existing personalized ranking systems used in practice.

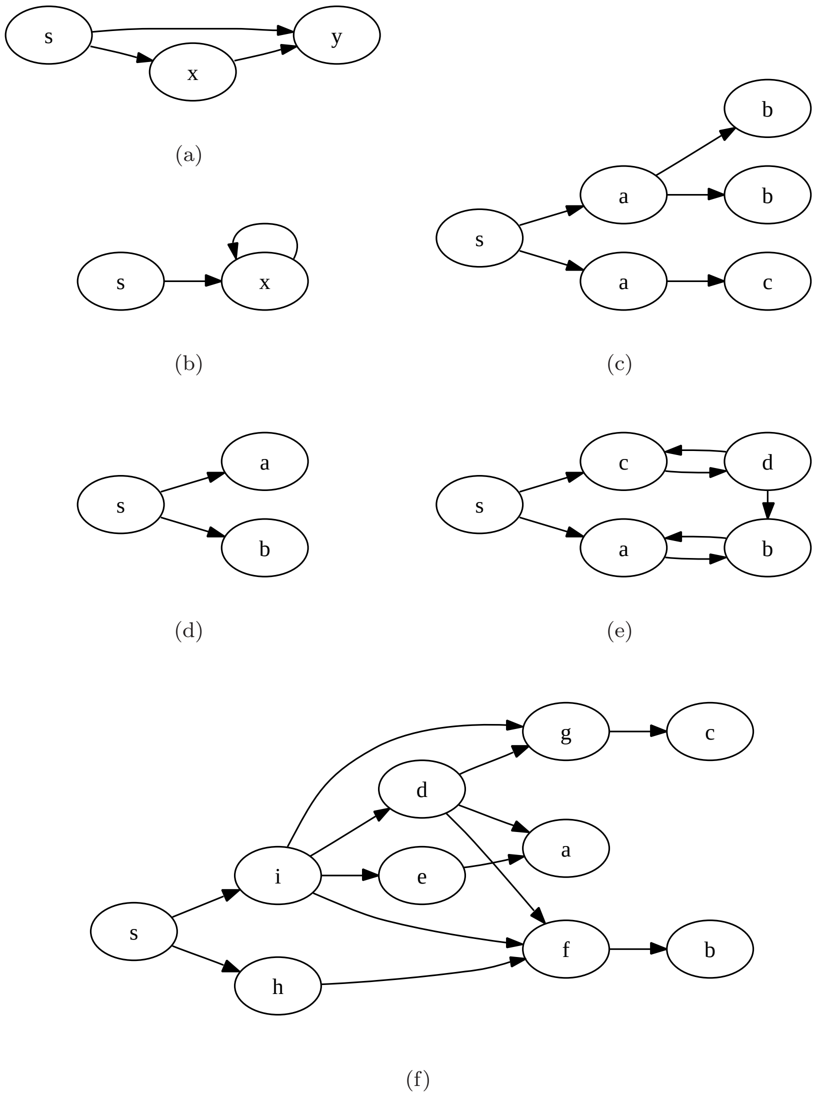


Figure 5.1: Graphs proving PRS do not satisfy axioms.

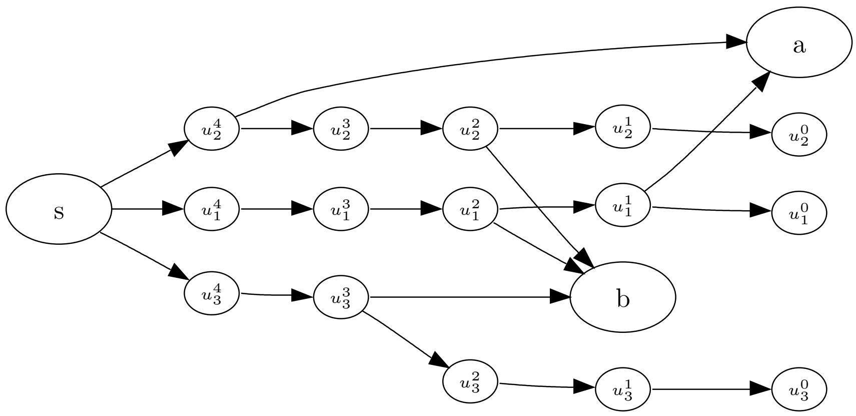


Figure 5.2: Example graph from proof of Lemma 5.7.

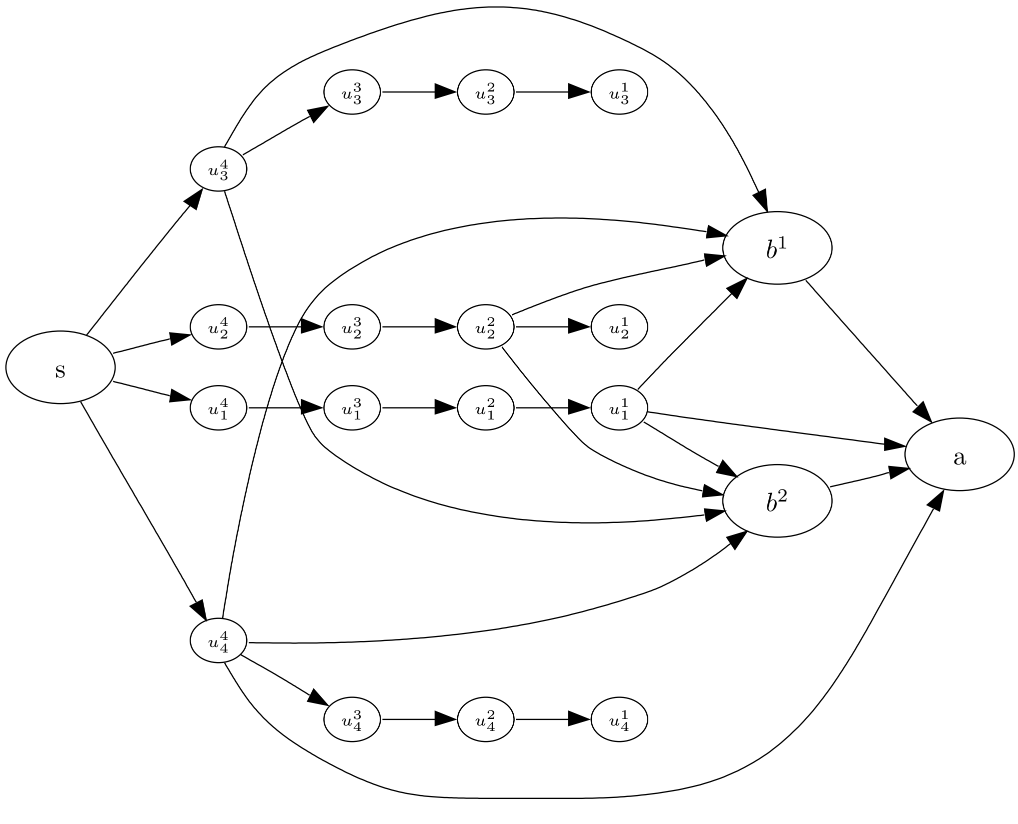


Figure 5.3: Example graph from the proof of Theorem 5.6 case 1.

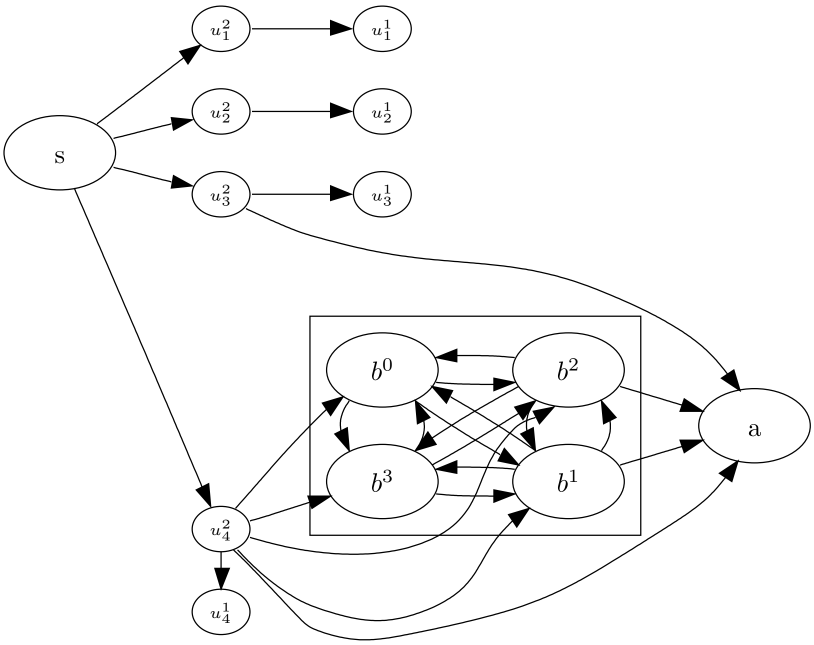


Figure 5.4: Example graph from the proof of Theorem 5.6 case 2.

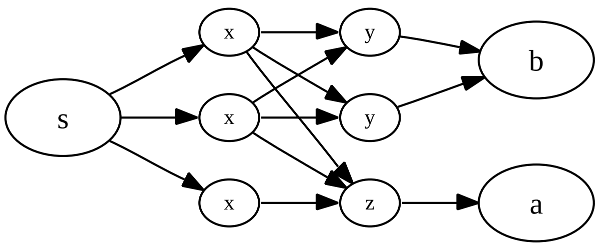


Figure 5.5: Graph proving F_P does not satisfy axioms.

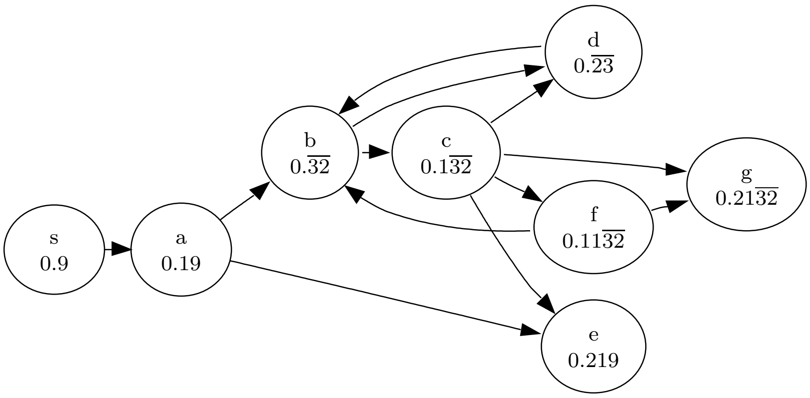
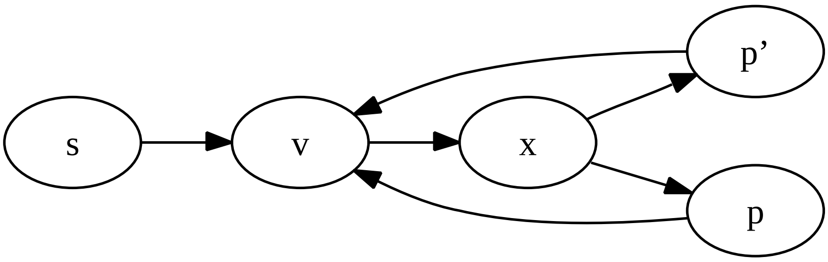


Figure 5.6: Values assigned by the personalized recursive in-degree algorithm



$$\begin{aligned} \mathbf{a} &= (p, x, v, p', x) \\ \mathbf{b} &= (p, x, v, p) \\ \mathbf{c} &= (p', x, v, p') \end{aligned}$$

Figure 5.7: Example paths from the proof of Proposition 5.11.

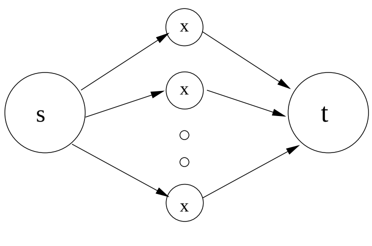


Figure 5.8: Graph proving that Recursive In-degree does not satisfy axioms

Chapter 6

Conclusions

Reasoning about preferences and preference aggregation is a fundamental task in reasoning about multi-agent systems (see e.g. Boutilier et al. (2004); Conitzer and Sandholm (2002); LaMura and Shoham (1998)). A typical instance of preference aggregation is the setting of ranking systems. Ranking systems are fundamental ingredients of some of the most famous tools/techniques in the Internet (e.g. Google's page rank and eBay's reputation systems, among many others).

Moreover, the task of building successful and effective on-line trading environments has become a central challenge to the AI community (Boutilier et al., 1997; Monderer et al., 2000; Sandholm, 2003). Ranking systems are believed to be fundamental for the establishment of such environments. Although reputation has always been a major issue in economics (see e.g. Kreps and Wilson (1982); Milgrom and Roberts (1982)), reputation systems have become so central recently due to the fact that some of the most influential and powerful Internet sites and companies have put reputation systems in the core of their business.

Our aim in this thesis was to treat ranking systems from an axiomatic perspective. The classical theory of social choice lay the foundations to a large part of the rigorous work on multi-agent systems. Indeed, the most classical results in the theory of mechanism design, such as the Gibbard-Satterthwaite Theorem (Gibbard, 1973; Satterthwaite, 1975))are applications of the theory of social choice. Moreover, previous work in AI has employed the theory of social choice for obtaining foundations for reasoning tasks (Doyle and Wellman, 1989) and multi-agent coordination (Kfir-Dahav and Tennenholtz, 1996). It is however interesting to note that ranking systems suggest a novel and new type of theory of social choice. We see this point as especially attractive, and as a main reason for concentrating on the study of the axiomatic foundations of ranking systems.

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- מונוטוניות — צומת המוצבע על ידי תת-קבוצה של המצביעים של צומת אחר לא יהיה חזק ממנו; ו-
- תכונת האיחוד — פיצול גרף לא קשיר לרכיבים זרים לא משנה את הדרוג הפנימי בכל רכיב.

בהמשך, אנו דנים בתכונות חזקות יותר של מערכות דרוג:

- חוסר כפיה (חיזוק של אי-טריביאליות) — לכל דרוג ללא תיקו של הצמתים קיים פרופיל הצבעות שמייצר אותו; ו-
- איזומורפיזם (חיזוק של הוגנות מינימלית) — אין משמעות לשמות הצמתים אלא רק להצבעות ביניהם.

אנו מראים שלא קיימות מערכות דרוג המקיימות חוסר כפיה, אלא אם אנו מגבילים את עצמנו לבדיוק שלושה סוכנים. עבור מקרה מיוחד זה אנו מציגים אקסיומטיזציה של מערכת דרוג כזו.

המסקנה המתקבלת מהסיווג לעיל היא שלא נוכל למצוא מערכת דרוג סבירה אשר תואמת-תמריצים בצורה מלאה. אי לכך, אנו מרחיבים את הדיון לכימות רמת תאימות-התמריצים של מערכות דרוג. בהקשר זה אנו רואים מספר תוצאות חיוביות של מערכות דרוג שהן תואמות-תמריצים כמעט לחלוטין אשר מקיימות גם חוסר כפיה. אנחנו מוכיחים גם תוצאה שלילית משמעותית לגבי מערכות דרוג המקיימות מונוטוניות חזקה (קבוצה זו כוללת את PageRank ומגוון מערכות דרוג נוספות). התוצאה מראה שבכל מערכת דרוג כזו תיתכן מניפולציה משמעותית של הסוכנים.

לבסוף, אנו מציגים מודל שונה של מערכות דרוג בו המערכת מספקת דרוג מותאם-אישית לכל משתתף במערכת. אנו מתאימים את אקסיומות הטרנזיטיביות, אי-התלות המדורגת באפשרויות לא-רלוונטיות, ותאימות-התמריצים מהמודל הכללי ומוכיחים תוצאה חיובית מפתיעה — משפט ייצוג עבור המערכות המקיימות את כל האקסיומות לעיל. תוצאה זו מפתיעה במיוחד לאור העובדה שבמודל הכללי לא מצאנו אף מערכת דרוג סבירה שהנה תואמת-תמריצים. בהמשך אנו מראים כי כל האקסיומות הן נחוצות לצורך ההוכחה, והחלשה של אחת מהאקסיומות מולידה מערכות דרוג חדשות ומעניינות ברובן.

אקסיומטיזציה כזו נקרת משפט ייצוג והיא תופסת את התמצית המדייקת והנחות העומדות מאחורי כלל הדרוג.

- הגישה הנורמטיבית, בה בוחרים קבוצת דרישות אותן אנו מצפים שכלל דרוג צריך למלא, ומנסים לראות אם קיים כלל דרוג המקיים את הדרישות.

במחקר זה אנו מיישמים את שתי הגישות לעיל בהקשר של מערכות דרוג. ראשית, יישמנו את הגישה התיאורית לאלגוריתם PageRank העומד בבסיס טכנולוגיית החיפוש של גוגל. בהקשר זה אנו מציגים משפט ייצוג עבור גרסה מפושטת של אלגוריתם PageRank באמצעות חמש אקסיומות:

- איזומורפיזם — שינוי שמות המשתתפים אינו משנה את תוצאת הדרוג;
- קשת עצמית — הוספת הצבעה עצמית מחזקת את המצביע אך פרט לזה לא משנה את הדרוג;
- הצבעה על ידי ועדה — הצבעה דרך "ועדה" של צמתים חדשים במקום הצבעה ישירה לא משנה את הדרוג;
- קריסה — איחוד שני צמתים המצביעים על אותה קבוצת צמתים לא משנה את הדרוג אלא של הצמתים שאוחו; ו-
- מתווך — הסרת צומת מתווך בין קבוצות שוות גודל של צמתים כאשר המצביעים שווים בדרוגם לא משנה את הדרוג.

אנו מוכחים כי PageRank מקיים את כל האקסיומות הנ"ל וכי זו המערכת היחידה המקיימת את כל האקסיומות. ההוכחה היא קונסטרוקטיבית ומציגה אלגוריתם (לא יעיל) לחישוב PageRank.

בהמשך, אנו חוקרים את הגישה הנורמטיבית. בגישה זו אנו מציגים שתי תכונות חשובות של מערכות דרוג: תכונות טרנזיטיביות, ואי-תלות מדורגת באפשרויות לא-רלוונטיות. תכונות הטרנזיטיביות תופסות את ההשפעות הטרנזיטיביות שתוארו לעיל, ואילו אי-התלות המדורגת באפשרויות לא-רלוונטיות תופסת את העובדה שדרוג של סוכן לא צריך להיקבע אלא לפי הדרוג של אלה שהצביעו עבורו. אנו מוכיחים תוצאה מפתיעה לפיה שתי הדרישות לעיל אינו ניתנות לסיפוק יחדיו. כמו כן, אנו מראים שבהחלשה של תכונת הטרנזיטיביות, ניתן לספק את שתי התכונות במערכת דרוג מעניינת. אנו מציגים מערכת דרוג זו ומציעים אלגוריתם יעיל לחשבה.

עוד בגישה הנורמטיבית, אנו תוקפים את הסוגיה של תמריצים. ספציפית, אנו מתייחסים למקרה בו סוכן אנוכי מנסה לתמרן את המערכת לדרגו בדרוג גבוה יותר מהמגיע לו על ידי שינוי הצבעותיו. בהקשר זה אנו מציגים סיווג מלא של קיום מערכות דרוג תואמות-תמריצים תחת חמש אקסיומות בסיסיות, כל אחת עם גרסה חלשה וחזקה. הגרסאות החלשות של האקסיומות הן:

- אי-טריביאליות — המערכת לא תמיד מדרגת את הצמתים באותו אופן;
- הוגנות מינימלית — כאשר אין הצבעות, כל הצמתים מדורגים בצורה שווה;
- תגובה חיובית — הוספת קשת אינה מחלישה את הצומת שמוצבע על ידי הקשת החדשה;

תקציר

דרוג של סוכנים לפי קלט של סוכנים אחרים הוא רכיב חשוב במערכות מרובות סוכנים. כמו כן, דרוג זה הפך מרכיב מרכזי במגוון אתרי אינטרנט. הדוגמאות המפורסמות ביותר הן ככל הנראה אלגוריתם PageRank של גוגל ומערכת המוניטין של אתר eBay.

התאוריה הקלסית של בחירה חברתית, כפי שהוצגה על-ידי ארו בשנת 1963 מתארת מצב בו קבוצה של סוכנים (או מצביעים) נקראת לדרג קבוצה של אפשרויות. בהינתן הקלט של הסוכנים, כלומר דרוגים אישיים של הסוכנים על האפשרויות, מעוניינים לייצר דרוג חברתי של האפשרויות. תאוריית הבחירה החברתית חוקרת את התכונות הרצויות מכללי דרוג חברתיים כאלה. בפרט, משפט אי האפשרות המפורסם של ארו מוכיח כי לא קיים כלל דרוג המקיים כמה דרישות מינימליות.

מערכות דרוג מציגות מודל חדש של בחירה חברתית. החידוש במערכות דרוג הוא שבעולם זה הסוכנים והאפשרויות הם היינו הך. אי לכך, בסביבה זו יש לקחת בחשבון השפעות טרנזיטיביות של הצבעה. כלומר, אם סוכן a מצביע עבור סוכן b, הצבעה זו יכולה להשפיע על האמינות ועל המשקל המיוחס להצבעה מטעם b על c. יש להתייחס להשפעות עקיפות כגון אלה בבואנו לבחון כללי דרוג בסביבה זו.

לשם פשטות הדיון, וכדי להימנע מתוצאות אי אפשרות בסגנון ארו, נתמקד במודל בו ההצבעה מתבצעת בשתי דרגות בלבד. כלומר, כל סוכן יכול לבחור האם להצביע או לא להצביע עבור סוכן אחר, אך לא מעבר לכך. פרוש טבעי למודל זה הוא דרוג של אתרים באינטרנט. במקרה זה, קבוצת הסוכנים מייצגת את קבוצת האתרים באינטרנט והקישורים בין דף p לקבוצת דפים Q מיוצגים על ידי הצבעות מסוכן p לקבוצת הסוכנים Q. הבעיה של מציאת דרוג חברתי במקרה זה מתלכדת עם בעיית דרוג הדפים. גישות שונות לפתרון בעיה זה ממומשות על ידי מנועי חיפוש שונים כדוגמת גוגל.

פורמלית, מערכת דרוג מוגדרת באופן הבא:

הגדרה (מערכת דרוג)**:** *תהי $\mathbb{G}_V$ קבוצת כל הגרפים המכוונים ללא קשתות מקבילות, אך עם אפשרות לקשתות עצמיות.* מערכת דרוג F *היא פונקציונל הממפה עבור כל קבוצת קדקודים סופית V כל גרף $G \in \mathbb{G}_V$ ליחס סדר חלש מלא $\preceq_G^F$ על קבוצת הקדקודים V.*

דוגמה פשוטה למערכת דרוג היא הצבעת אישור (Approval Voting) בה הסוכנים מדורגים לפי מספר ההצבעות שהם קיבלו, מבלי להתייחס לזהות המצביעים.

ניתן לחלק את התאוריה של בחירה חברתית לשתי גישות אקסיומטיות משלימות:

- הגישה התיאורית, בה בהינתן כלל דרוג מסוים r מנסה למצוא קבוצת אקסיומות נאותות ושלמות עבור r. כלומר, למצוא קבוצה של דרישות ש- r מקיימת, ובנוסף כל כלל דרוג המקיים את הדרישות הנ"ל חייב להתלכד עם r. תוצאה המראה

המחקר נעשה בהנחיית פרופ' משה טננהולץ בפקולטה להנדסת תעשיה וניהול.

מחקר זה נתמך באופן חלקי על ידי הקרן הלאומית למדע.

אני מודה לחברת IBM על מלגת הדוקטורט הנדיבה.

אני מודה לטכניון על התמיכה הכספית הנדיבה בהשתלמותי.

הגישה האקסיומטית למערכות דרוג

חיבור על מחקר

לשם מילוי חלקי של הדרישות לקבלת התואר
דוקטור לפילוסופיה

אלון אלטמן

הוגש לסנט הטכניון – מכון טכנולוגי לישראל

תמוז תשס"ז   חיפה   יוני 2007

הגישה האקסיומטית
למערכות דרוג

אלון אלטמן