On the nucleolus of irreducible minimum cost spanning tree games

Leanne Streekstra

Abstract

Computing the nucleolus of a minimum cost spanning tree game has been known to be NP-hard. In this paper, I therefore consider the computational complexity for a subset of these games; minimum cost spanning tree games with an irreducible cost matrix. I show that these games are a special case of connected balanced games and thus the existing algorithm which efficiently computes the nucleolus for these games, can be used here as well. Moreover, I show that the computational complexity of this algorithm can be reduced from $\mathcal{O}(n^4)$ to $\mathcal{O}(n^3)$ when applied to minimum cost spanning tree games with irreducible cost matrix, by reducing the size of the input set.

1 Introduction

The nucleolus, introduced by Schmeidler [14], is one of the most well-known solution concepts for allocating the cost of a joint project, as it suggests a single allocation satisfying some natural properties of equality and fairness. A downside of the nucleolus is however, that in many cases it cannot be computed efficiently.

It has been shown that for any n-player game, the value of the nucleolus is determined by at most $2n-2$ coalitions ([4], [13]). However, it is unfortunately no less hard to identify these coalitions than it is to compute the nucleolus itself. Some games though, due to specific characteristics, allow for the identification of a specific subset of coalitions sufficient for computing the nucleolus, which then allows us to compute the nucleolus in polynomial time in the number of players. Some games which have been found to allow such restriction are peer group games ([5]), standard tree games ([12]), assignment games ([16]), matching games ([11]) and bankruptcy games ([1]).

Minimum cost spanning tree problems model situations where agents, located at different points, need to be connected to a source. The problem of finding a minimum cost spanning tree (mcst) is a familiar combinatorial optimization problem, which can be solved in quadratic time. It has been shown that computing the nucleolus of a mcst games is however NP-hard ([7]).

In this paper we study a specific subclass of mcst games; mcst games with an irreducible cost matrix. An irreducible cost matrix is such that lowering the cost of any edge would lower the cost of the mcst. Bergantiños and Vidal-Puga [2] have shown that these games are convex, unlike mcst games in general.

We study the computational complexity of the nucleolus for this subset of mcst games. We find that mcst games with an irreducible cost matrix are a special case of connected balanced games. An algorithm to compute the nucleolus of connected balanced games in $\mathcal{O}(n^4)$ has been introduced by Solymosi et al. [17]. We are able to show that in our setting, a smaller set of coalitions is sufficient for computing the nucleolus than for connected balanced games in general. This allows us to reduce the complexity of the algorithm to $\mathcal{O}(n^3)$.

The rest of this paper is organised as follows. In section 2 we introduce mcst games along with other relevant technical definitions. In section 3 we provide a new notation for mcst games with an irreducible cost matrix, which recognizes the large number of mcsts these games allow for. Finally, in section 4 we show the relation with connected balanced games and the fact that only 2n-2 coalitions need to be considered to compute the nucleolus of mcst games with an irreducible cost matrix.

2 The model

Let $N = \{1, ..., n\}$ be a finite set of *agents*. Let $G = (V, E)$ denote a complete graph over the nodes V of which E is the set of edges consisting of all unordered pairs $\{i, j\}$, $i, j \in N$. In minimum cost spanning tree problems the node set V is $N_0 = N \cup \{0\}$ where 0 is a special node called the *source*. A non-negative, symmetric *cost matrix* $C = (c_{ij})_{i,j \in N_0}$ represents the cost associated with the edge between any pair of nodes, where $c_{ij} = c_{ji}$ is the cost associated with the undirected edge $\{i, j\}$. Furthermore $c_{ii} = 0$ for all $i \in N_0$. Given a *subgraph* $G' = (N_0, E')$ with $E' \subseteq E$, a *path* from i to j in G' is a sequence of distinct edges $(\{i_s, i_{s+1}\})_{s=1}^{k}$ s.t. $\{i_s, i_{s+1}\} \in E'$ for all $s \in \{1, ..., k\}$, $i_1 = i$ and $i_{k+1} = j$. We say that i and j are *connected* in G' if there is a path from i to j in G'.
A *minimum cost spanning tree* (mcst) $\Gamma_N = (N_0, E_N)$ is a graph without cycles that connects all n agents with the source 0 and is such that $\Sigma_{\{i,j\} \in E_N} c_{ij}$ is minimal. Similarly, $\Gamma_S = (S_0, E_S)$ is a mcst for S where $S_0 = S \cup \{0\}$ and $E_S \subset S_0 \times S_0$ is such that it has the smallest total cost over all edge sets in $S_0 \times S_0$ connecting all agents in S to the source. We abuse notation and often use $e \in \Gamma_N$ instead of $e \in E_N$.
We denote the unique path in Γ_N between i and j with $p_{ij}^{\Gamma_N}$ and omit the superscript whenever there is no risk of confusion. Slightly abusing notation, we use $\{i_s, i_{s+1}\} \in p_{ij}$ to denote that $\{i_s, i_{s+1}\}$ is an element of p_{ij}. We say that l is on p_{ij}, when there is an $s \in \{2, ..., k\}$ s.t. $\{i_s, i_{s+1}\} \in p_{ij}$ and $l = i_s$.
Let Γ_N be a mcst for (N, C). The *irreducible cost matrix* C^* is defined as follows: for all $\{i, j\} \in E$, $c^*_{ij} = max_{\{k,l\} \in p_{ij}^{\Gamma_N}} \{c_{kl}\}$. C^* has the property that lowering c^*_{ij} any further for any $\{i, j\} \in E$ would lower the total cost of the mcst. We call (N, C^*) the *irreducible form* of the mcst problem (N, C). Although there may be more than one mcst for (N, C), it has been shown that C^* does not depend on which mcst is chosen([2]).
Given a mcst problem (N, C), the associated cooperative mcst game is (N, c), where the characteristic function $c : 2^n \to \mathbb{R}$ is now defined by:
$c(S, C) = \Sigma_{\{i,j\} \in E_S} c_{ij}$ for $S \subseteq N$.
Note that the agents in a coalition S, when building a mcst on their own, are not allowed to use the nodes of agents outside of S.
In the current setting it will be more convenient to work with the dual of a mcst game: $v(S, C) = c(N, C) - c(N \backslash S, C)$. The dual $v(S, C)$ can be seen as a measure of the contribution of $S \subseteq N$ to the total cost. Note that $v(N, C) = c(N, C)$.
An *allocation* $x \in \mathbb{R}^n$ assigns a cost share to each agent s.t. $\Sigma_{i \in N}\, x_i = v(N, C)$. We use $x(S) = \Sigma_{i \in S}\, x_i$ to denote the total amount allocated to coalition S. The *satisfaction* $f(S, x) = x(S) - v(S)$ of a coalition $\emptyset \neq S \subset N$ can be seen as the level of content of coalition S with its total payoff. An allocation x is said to be a *core allocation* if $f(S, x) \geq 0$ for all $S \subseteq N$. The set $\mathcal{C}(N, C)$ denotes the set of all core allocations. It has been shown that the core of a mcst game is never empty ([8, 9]).
A collection $\emptyset \notin \mathcal{B} \subseteq 2^n$ is said to be *balanced* on N, if there exist positive numbers $\gamma_s, S \in \mathcal{B}$ such that

$$\Sigma_{S \in \mathcal{B}}\, \gamma_S\, v(S) = e^N$$

where e^N denotes the indicator vector, i.e. $e^S_i = 1$ if $i \in S$ and 0 otherwise. A cooperative game (N, v) is called balanced if for every balanced collection $\mathcal{B}$ and every set of balancing weights $\gamma_S, S \in \mathcal{B}$

$$v(N) \geq \Sigma_{S \in \mathcal{B}}\, \gamma_S\, v(S).$$

It was shown by Bondareva [3] and Shapley [15] that the core of a cooperative game is non-empty if and only if it is balanced.
Let $\theta(x)$ be the vector of the satisfactions of all $\emptyset \neq S \subset N$ arranged in non-decreasing order. The nucleolus of a game (N, v) is the allocation that lexicographically maximizes the vector $\theta(x)$:

$$Nu(N, v) = \{x \in \mathbb{R}^n |\, \theta(x) \geq^\ell \theta(y), \forall y \text{ s.t. } y(N) = x(N)\}$$

where $\geq^\ell$ denotes the lexicographic order in $\mathbb{R}^{2^n}$. Schmeidler [14] showed that the nucleolus consists of a single point for every cooperative game. It follows from the definition that the nucleolus is a core allocation whenever the core is non-empty.
We use $\mathcal{N}$ to denote the nucleolus of the irreducible form of a mcst problem i.e. $\mathcal{N}(N, v) = Nu(N, v^*)$, where $v^*(S, C) = v(S, C^*)$ for all $S \subseteq N$.
A coalition S is called *inessential* in game (N, v) if there is a proper partition of $S = \{S_1, ...S_t\}$, $t \geq 2$ such that $v(S) \leq \Sigma_{j=1}^t v(S_j)$. For such S, $f(S, x) \geq \Sigma_{j=1}^t f(S_j, x)$ for all $x \in \mathbb{R}^n$. A coalition that is not inessential is called *essential*. We denote the set of essential coalitions with $\mathcal{E}$. Note that singleton coalitions are always essential. It was shown by Huberman [10] that for games with a non-empty core, the nucleolus is independent of inessential coalitions.

3 Mcst games as cluster notation

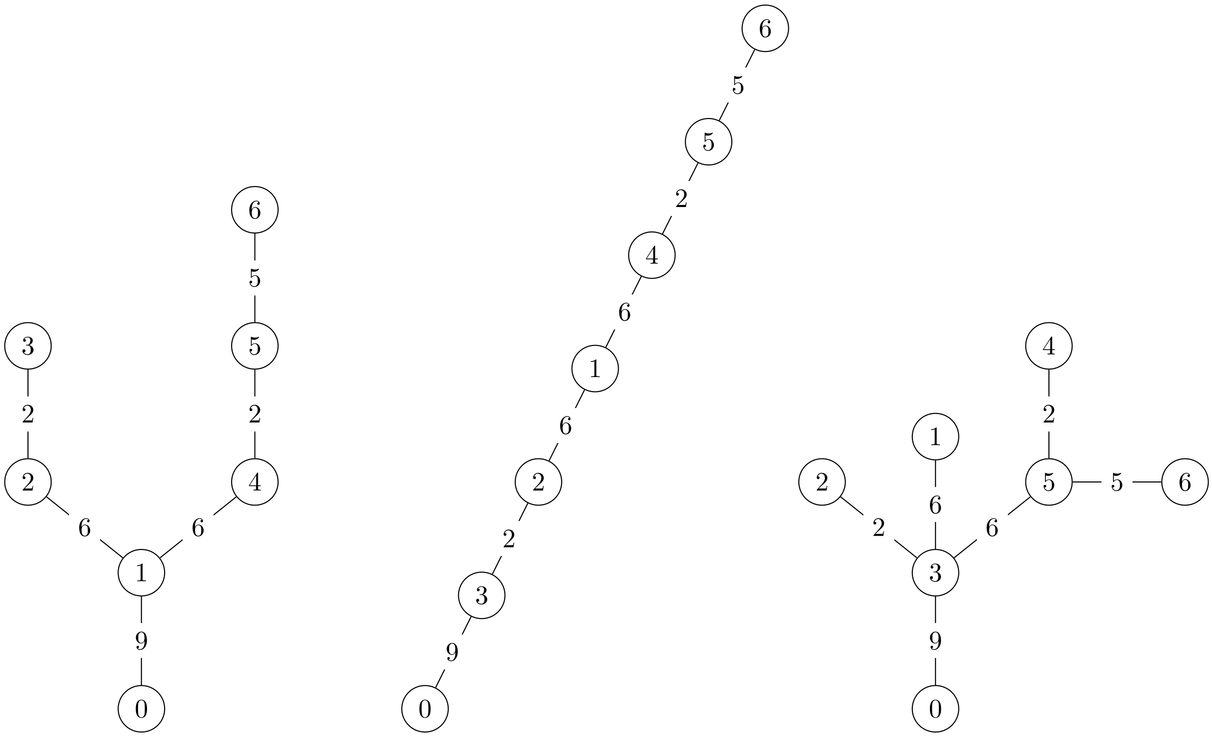

Figure 1: Example of 3 mcst's with the same irreducible cost matrix

The irreducible matrix allows for many different mcst's related with the same cost matrix. Consider the mcst's in Figure 1, where the circled numbers represent agents and the numbers on the edges represent the cost of the respective edge. It can be checked that, although the structure of these trees is quite different, they are all related to the irreducible cost matrix in Figure 2.

For this reason we propose an alternative representation in the form of clusters that

	0	1	2	3	4	5	6
0	0	9	9	9	9	9	9
1		0	6	6	6	6	6
2			0	2	6	6	6
3				0	6	6	6
4					0	2	5
5						0	5

Figure 2: Irreducible cost matrix for Figure 1

abstracts away from a specific mcst. Before doing so, it is useful to note that in any mcst game with irreducible cost matrix and $n \geq 2$ there is at least one pair of symmetric agents. We say that two agents $i, j \in N$ are *symmetric* if for all $k \in N\backslash\{i,j\}$, $c_{ik} = c_{jk}$. Let $c_S^- := min_{j \in N\backslash S,\, i \in S}\, c_{ij}^*$ denote the cost of the cheapest edge connecting coalition S to any agent not in S.

Lemma 1. *Two agents i and j are symmetric under C^* iff $c_j^- = c_{ij}^* = c_i^-$.*

Proof. Let $i, j \in N$ be symmetric and let $q \in N_0\backslash\{j\}$ be such that $c_{jq}^* = min_{k \in N\backslash\{i,j\}}\, c_{jk}^*$. As $c_{jq}^* = c_{iq}^*$ by assumption, we now get that:

$$max_{e \in p_{jq}}\{c_e\} = c_{jq}^* = c_{iq}^* = max_{e \in p_{iq}}\{c_e\}$$

From which it follows that

$$c_{ij}^* = max_{e \in p_{ij}}\{c_e\} \leq c_{jq}^*$$

As we chose q such that $c_{jq}^* = min_{k \in N\backslash\{i,j\}}\, c_{jk}^*$, it follows that $c_{ij}^* = c_j^-$. By the same reasoning we get that $c_{ij}^* = c_i^-$.

Next, assume that $c_j^- = c_{ij}^* = c_i^-$. Bergantiños and Vidal-Puga [2] have shown that for any $\{i,j\}$ there exists a mcst Γ_N' for (N, C^*) s.t. $\{i,j\} \in \Gamma_N'$. As C^* does not depend on the exact tree chosen, $c_{kl}^* = max_{e \in p_{kl}'}\{c_e^*\}$ for all $k, l \in N_0$, where p_{kl}' denotes the unique path between k and l on Γ_N'.

Now take any $q \in N_0\backslash\{i,j\}$. W.l.g. assume i is on p_{jq}'. Combining that $c_{jq}^* = max_{e \in p_{jq}'}\{c_e\}$ and that $c_{ij}^* \leq c_{iq}^*$ by assumption, it follows that $c_{iq}^* = c_{jq}^*$. As we chose $q \in N_0\backslash\{i,j\}$ arbitrarily, it follows that i and j are symmetric. $\square$

From Lemma 1 it easily follows that if $c_{ij} = min_{e \in \Gamma_N}\{c_e\}$, i and j are symmetric agents. We are now ready to introduce the cluster representation for mcst problems with irreducible cost matrix.

We can recursively construct a cluster representation from a mcst Γ_N as follows, where we denote the clusters in with $I_{\mathbf{e}}$ for $\mathbf{e} \in \mathbf{E} := \{1, ..., l\}, l \leq n$.

Step 1 Select the edge with the lowest cost, i.e. $e = \{i,j\} = argmin_{e \in \Gamma_N}\{c_e\}$. Define $I_{\mathbf{e}} := \{k \in N_0 | c_{ik}^* = c_e\} \cup \{i\}$ for $\mathbf{e} := 1$ and set $c_{\mathbf{e}} := c_{ij}$. By choice of e, we know that all i, $j \in I_{\mathbf{e}}$ are symmetric. $I_{\mathbf{e}}$ is our first cluster. In figure 3 this corresponds to the circle containing agents 2 and 3 and the orange number on the circle corresponds to $c_{\mathbf{e}}$. Note that more than one $e \in E$ may be related to the same $\mathbf{e} \in \mathbf{E}$. These are the edges that can be replaced by each other without changing the total cost of the mcst.

Step m+1 Pick the cheapest edge $e = \{i,j\}$ that is not yet related to any $\mathbf{e}' < m+1$. Define $I_{\mathbf{e}} := \{k \in N_0 | c_{ik}^* = c_e\} \cup \{i\}$ for $\mathbf{e} = m+1$ and set $c_{\mathbf{e}} := c_{ij}$. Note that for all $i, j \in I_{\mathbf{e}}$

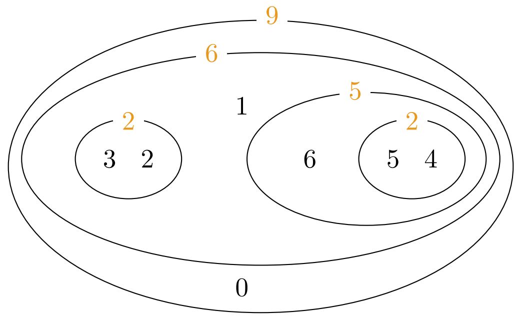

Figure 3: Cluster representation for the irreducible cost matrix in Figure 2

and $k \in N \backslash I_{\mathbf{e}}$, $c^*_{ik} = c^*_{jk}$, as by choice of $\mathbf{e}$, $c^*_{ij} < c^*_{ik}$. We say that the agents in $I_{\mathbf{e}}$ are symmetric with respect to $N \backslash I_{\mathbf{e}}$. From this observation it follows that if $I_{\mathbf{e}} \cap I_{\mathbf{e}'} \neq \emptyset$ for some $\mathbf{e}' < m+1$, then $I_{\mathbf{e}'} \subset I_{\mathbf{e}}$.
We repeat this until all $e \in \Gamma_N$ are associated with an $\mathbf{e} \in \mathbf{E}$.
Figure 3 shows the cluster representation we get from any of the trees in Figure 1. The black numbers denote the nodes, while the orange numbers on the circles denote the associated cost. For any two nodes, the cost of connecting them directly to each other is the cost of the smallest circle encapsulating them both. For example, $c_{46} = 5$ and $c_{26} = 6$.
Let $g : E \to \mathbf{E}$ denote a function mapping each edge e to its corresponding cluster edge $\mathbf{e}$.
Let $\mathscr{I}_{\mathbf{e}} := I_{\mathbf{e}} \backslash (\bigcup_{\mathbf{e}' < \mathbf{e}} I_{\mathbf{e}'})$ denote the set of $i \in N$ for which $c_{\mathbf{e}} = c^-_i$.
The following observation follows easily from lemma 1 our recursive construction of the cluster representation.

Corollary 1. *For any $\mathbf{e} \in \mathbf{E}$, all $i, j \in \mathscr{I}_{\mathbf{e}}$ are symmetric.*

Lemma 2. *For every irreducible cost matrix C^*, there exists a mcst Γ_N s.t. for all $p_{i0} = (e^1, ..., e^t)$ with $t \geq 2$, $c_{e^s} \leq c_{e^r}$ for every $1 \leq s < r \leq t$.*

Proof. Let Γ_N be a mcst related to C^* in which the edge costs are not monotonically decreasing as we move away from the source. We will show that we can recursively build a mcst from Γ_N that does satisfy this condition.
Let i be such that for $ij, jk \in \Gamma_N$, k is on p_{i0} and $c^*_{ij} > c^*_{jk}$. By definition of C^*, $c^*_{ik} = c^*_{ij}$ and so by replacing ij with ik in Γ_N, we do not change the total cost of the tree, but we do have one less pair of edges e and e' s.t. e' is closer to the source than e and $c^*_e > c^*_{e'}$.
We repeat this process until we've established a tree Γ'_N in which for all $p_{i0} = (e^1, ..., e^t)$ with $t \geq 2$, $s < r$ implies $c_{e^s} < c_{e^r}$. ☐

Granot and Huberman [8] have shown that the nucleolus of mcst games satisfies the following property:

Separability
For all mcst problems (N, C) and $S \subset N$ s.t. $v(N, C) = v(S, C) + v(N \backslash S, C)$ it holds that:

$$\mathcal{N}_i(N, v) = \begin{cases} \mathcal{N}_i(S, v) & \text{if } i \in S \\ \mathcal{N}_i(N \backslash S, v) & \text{otherwise} \end{cases}$$

Let T_i denote the set of agents in the subtree of Γ_N rooted at i. Clearly, if there are two distinct agents connected directly to the source in Γ_N, i.e. $\{i, 0\}, \{j, 0\} \in \Gamma_N$ for $i \neq j$, then

T_i and T_j are such that $v(T_i \cup T_j, C) = v(T_i, C) + v(T_j, C)$. We can therefore assume that there is a unique edge connecting to the source in Γ_N and we denote it with e^0. Moreover, due to lemma 2 we can assume this edge to be strictly more expensive than all other edges in Γ_N and hence $\mathbf{e} = l$ is the cluster edge corresponding to edge e^0 and c_l denotes the related cost.

4 The nucleolus of the irreducible form of a mcst game

We start by showing that in a mcst game with irreducible cost matrix, coalitions that do not consist of a single, entire cluster are inessential.

Lemma 3. *A coalition S, $|S| \geq 2$, such that $S \neq I_\mathbf{e}$ for any $\mathbf{e} \in \mathbf{E}$, is inessential.*

Proof. Bergantiños and Vidal-Puga [2] have shown for mcst with irreducible cost matrix that $c^*(N \backslash \{i\}) = c^*(N) - c_i^-$. From this it follows that $c^*(N \backslash (S \cup \{i\})) = c^*(N \backslash S) - c^-_{i,\Gamma_{N \backslash S}}$ where $c^-_{i,\Gamma_{N \backslash S}} = min_{j \in N \backslash (S \cup \{i\})} c^*_{ij}$. Moreover, by definition of v^*, we get that $v^*(\{i\}) = c_i^-$, for all $i \in N$.

First note that for any $\mathbf{e}$ such that $I_\mathbf{e} \neq \mathscr{I}_\mathbf{e}$, we have that $|\{e \in \Gamma_N | g(e) = \mathbf{e}\} = |\mathscr{I}_\mathbf{e}|$ and for $\mathbf{e}$ such that $I_\mathbf{e} = \mathscr{I}_\mathbf{e}$, $|\{e \in \Gamma_N | g(e) = \mathbf{e}\} = |\mathscr{I}_\mathbf{e}| - 1$

Let S be such that $S \neq I_\mathbf{e}$ for any $\mathbf{e} \in \mathbf{E}$. First, suppose there is no $\mathscr{I}_\mathbf{e} = I_\mathbf{e}$, $\mathscr{I}_\mathbf{e} \subseteq S$. By our previous observation, it follows that $v^*(S) = \Sigma_{i \in S}\, c_i^- = \Sigma_{i \in S}\, v^*(\{i\})$. Hence S is inessential.

Now suppose there is a $\mathscr{I}_\mathbf{e} = I_\mathbf{e}$, $\mathscr{I}_\mathbf{e} \subset S$, . We distinguish two cases:

1. For all $i \in S$ and $j \in N$, if $c_i^- = c^*_{ij}$ then $j \in S$.

In other words, S consists of the union of several clusters. Let $S_1, ..., S_k$ be the partition of S into its disjoint clusters; i.e. for all $1 \leq j \leq k$, $S_j = I_\mathbf{e}$ for some $\mathbf{e} \in \mathbf{E}$. Let Γ_N be a mcst for (N, c) where the edge cost monotonically decreases as we move away from the source. Each S_j is then a separate subtree T_k in Γ_N and therefore $c^*(N \backslash S) = c^*(N) - v^*(S_1) - .... - v^*(S_k)$. It then follows that, $v^*(S) = v^*(S_1) + ... + v^*(S_k)$ and S is thus inessential.

2. There is an $i \in S$ and some $j \in N \backslash S$ s.t. $c_{ij} = c_i^-$.

Let $S_{-i} := S \backslash \{i\}$. We know that $c^*(N \backslash S) = c^*(N \backslash S_{-i}) - c^-_{i,\Gamma_{N \backslash S_{-i}}}$. As there is a $j \in N \backslash S$ s.t. $c^*_{ij} = c_i^-$, it follows that $c^-_{i,\Gamma_{N \backslash S_{-i}}} = c_i^-$. Thus $c^*(N \backslash S) = c^*(N \backslash S_{-i}) - c_i^-$. We get that $c^*(N) - c^*(N \backslash S) = (c^*(N) - c^*(N \backslash S_{-i})) + c_i^-$, which means that $v^*(S) = v^*(S_{-i}) + v^*(\{i\})$ and thus S is inessential. □

If $S = I_\mathbf{e}$ for some $\mathbf{e} \in \mathbf{E}$, then for all $i \in S$, $c^-_{i,\Gamma_{N \backslash (S \backslash \{i\})}} > c_\mathbf{e}$ and thus S is essential. Hence $\mathcal{E}$ consists of all $S = I_\mathbf{e}$ together with all singleton coalitions. A game is called *connected* if we can arrange agents in a linear order and the essential coalitions are only those that are connected with respect to this order. It turns out that mcst games with irreducible cost matrix are connected games.

Lemma 4. *A mcst game with irreducible cost matrix is a connected game.*

Proof. To see that there always exists a mcst that is a chain, consider the following procedure.

Step 1 Pick an $\mathscr{I}_\mathbf{e} = I_\mathbf{e}$. Connect all $i \in \mathscr{I}_\mathbf{e}$ in a chain. As $c_{ij} = c_{ik}$ for all $i, j, k \in \mathscr{I}_\mathbf{e}$, $i \neq j \neq k$ the order does not matter.

Step i+1 Pick an $\mathscr{I}_\mathbf{e}$ such that all $I_{\mathbf{e}'} \subset I_\mathbf{e}$ were already chosen in previous steps. Connect any $I_{\mathbf{e}'}, I_{\mathbf{e}''} \subset I_\mathbf{e}$, $I_{\mathbf{e}'} \cap I_{\mathbf{e}''} = \emptyset$ with each other in a chain and connect all $i \in \mathscr{I}_\mathbf{e}$ to the end of this chain. As $c_{ij} = c_{ik} = c_{kl}$ for all $i, j \in \mathscr{I}_\mathbf{e}, k \in I_{\mathbf{e}'}, l \in I_{\mathbf{e}''}$ all newly added edges have the same cost and the order does not matter.

We repeat this until all $\mathscr{I}_{\mathbf{e}}$ are covered.
Clearly, by construction, all $S \in \mathcal{E}$ are connected in the mcst we just built. $\square$

In fact, all $S \in \mathcal{E}$ are connected in *any* mcst that is a chain.
Derks and Kuipers [6] have presented an algorithm for computing the nucleolus of a connected balanced game and showed that it runs within $\mathcal{O}(n^5)$ time. Solymosi et al. [17] improved on this result and presented an algorithm that computes the nucleolus of a connected balanced game in $\mathcal{O}(n^4)$. By lemma 4 and the fact that mcst games are balanced, we can thus compute $\mathcal{N}$ in $\mathcal{O}(n^4)$ with the help of the algorithm of Solymosi et al. [17]. However, as mcst games with an irreducible cost matrix are a special case of connected balanced games, we can in fact compute the nucleolus even faster.
In any mcst game with irreducible cost matrix there are at most n distinct clusters. However, as we can assume there to be a unique edge connecting to the source, we know that $\mathscr{I}_{\mathbf{e}} = 0$ for some $\mathbf{e} \in \mathbf{E}$. Moreover there is an $\mathbf{e}' \neq \mathbf{e}$ such that $I_{\mathbf{e}'} = N$. Although N is not inessential according to definition, as $f(N, x) = 0$ always holds, we can ignore it nonetheless. We thus have at most $n-2$ essential cluster coalitions and n singleton coalitions, so $|\mathcal{E}| = 2(n-1)$.
As the amount of essential coalitions in mcst games with irreducible cost matrix is $2(n-1)$ instead of the $n(n+1)/2$ required in the case of general balanced connected games, the algorithm presented by Solymosi et al. [17] can run in $\mathcal{O}(n^3)$ in our case.

Corollary 2. *The nucleolus of a mcst game with irreducible cost matrix can be computed in $\mathcal{O}(n^3)$.*

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Leanne Streekstra
Fernuniversität Hagen
Hagen, Germany
Email: `Leanne.Streekstra@outlook.com`