The Smoothed Likelihood of Doctrinal Paradoxes

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Abstract

When aggregating logically interconnected judgments from n agents, the result might be inconsistent with the logical connection. This inconsistency is known as the *doctrinal paradox* or *discursive dilemma*, which plays a central role in the field of judgment aggregation. Despite a large body of literature on the worst-case analysis of the doctrinal paradox, little is known about its likelihood under natural statistical models, except for a few i.i.d. distributions [List, 2005].

In this paper, we characterize the likelihood of the doctrinal paradox under a much more general and realistic model called the *smoothed social choice framework* [Xia, 2020b], where agents' ground truth judgments are arbitrarily correlated while the noises are independent. Our main theorem states that under mild conditions, the smoothed likelihood of the doctrinal paradox is either 0, $\exp(-\Theta(n))$, $\Theta(n^{-1/2})$ or $\Theta(1)$. This not only answers open questions by List [2005] for i.i.d. distributions, but also draws clear lines between situations with frequent and with vanishing paradoxes.

1 Introduction

Suppose a defendant is involved in an accident that one person got injured. A jury of n jurors (agents) is making decisions on the following three propositions:

Proposition ω_1: whether the injury is caused by the defendant
Proposition ω_2: whether the injury is serious enough to make the defendant guilty.
Proposition ω_3: whether the defendant is guilty or not.

All jurors are required to be logically consistent in their judgments, i.e. ω_3 is true if and only if both ω_1 and ω_2 are true. Then, the majority rule is applied to aggregate the jurors' judgments on each proposition. However, such aggregation is not always logically consistent as shown in the following simple example, known as the *doctrinal paradox* or *discursive dilemma*.

Example 1 (Doctrinal paradox)**.** *Consider that a jury of three jurors with the judgements in Table 1. We observe that the aggregation result is inconsistent with the common logic $\omega_3 \leftrightarrow \omega_1 \wedge \omega_2$ despite*

Propositions	ω_1	ω_2	ω_3	$\omega_3 \overset{?}{\leftrightarrow} \omega_1 \wedge \omega_2$
Agent 1	Yes	No	No	**True**
Agent 2	No	Yes	No	**True**
Agent 3	Yes	Yes	Yes	**True**
Aggregation	Yes	Yes	No	**False**

Table 1: The judgements of agents in Example 1

the fact that all agents' judgements are consistent.

More generally, the doctrinal paradox refers to the phenomenon where a group of n agents submit logically consistent judgements on p premises (e.g. ω_1 and ω_2 in Example 1) and one conclusion (e.g. ω_3 in Example 1), yet the aggregated judgements are not logically consistent.

The doctrinal paradox Brandt et al. [2016], Grossi and Pigozzi [2014], List [2012] plays a central role in the field of judgement aggregation and its applications. A growing number of literature has shown the negative effects of doctrinal paradox in the application fields of law [Kornhauser and

Sager, 1986, Hanna, 2009, Chilton and Tingley, 2012], economics [Mongin, 2019], computational social choice [Bonnefon, 2010, Brandt et al., 2016], philosophy [Sorensen, 2003, Mongin, 2012] and psychology [Bonnefon, 2011].

Doctrinal paradox is usually unavoidable under some mild assumptions on the aggregation rules [List and Pettit, 2002] However, little is known about the likelihood of doctrinal paradox except for a few i.i.d. distributions [List, 2005]. The i.i.d. assumption has also received a lot of criticisms as it ignores correlations among agents. Therefore, the following research question still remains open:

How likely does the doctrinal paradox happen under realistic models?

The question is already highly challenging under i.i.d. assumptions. To the best of our knowledge, no result is known about the probability of doctrinal paradox for the simple case that corresponds to the *Impartial Culture (IC)* in voting theory, where *all agents' judgments are i.i.d. uniformly distributed*. In this paper, we answer the above question under *smoothed social choice* framework [Xia, 2020b], which is much more realistic and allows agents' votes to be arbitrarily correlated.

In smoothed social choice framework, the "ground truth" of every agent's vote is chosen by an adversary. All agents' "ground truth" distributions are selected from a set of distributions Π over all kinds of votes on the premises. While discussing the probability of doctrinal paradox, we let r be any aggregation rule and f be any logical connection between the conclusion and the premises. We note that the "worst-case" of doctrinal paradox is the cases where its probability is maximized. Under smoothed social choice model, the worst-case assumes that a max-adversary aims to maximize the likelihood of doctrinal paradox, which is called max-smoothed likelihood of doctrinal paradox and denoted by

$$\widetilde{\text{DP}}^{\max}_{\Pi,r,f}(n) \triangleq \sup_{\vec{\pi}\in\Pi^n} \Pr_{P\sim\vec{\pi}}\left(P \text{ is a doctrinal paradox}\right),$$

where $\vec{\pi} = (\pi_1, \cdots, \pi_n) \in \Pi^n$ is the collection of the distributions of n agents chosen by the adversary, and profile P is the collection of n agents' votes generated from $\vec{\pi}$. In contrast, the "best-case" assumes that a min-adversary aims to minimize the probability of doctrinal paradox, which is called min-smoothed likelihood of doctrinal paradox and is denoted by,

$$\widetilde{\text{DP}}^{\min}_{\Pi,r,f}(n) \triangleq \inf_{\vec{\pi}\in\Pi^n} \Pr_{P\sim\vec{\pi}}\left(P \text{ is a doctrinal paradox}\right).$$

Our Contributions. The main merit of this paper is the following dichotomy theorem on the smoothed likelihood of doctrinal paradox.

Theorem 1. (Smoothed Likelihood of Doctrinal Paradox, informal) *Under mild assumptions, for any $n \in \mathbb{Z}_{\geq 0}$, any aggregation rule r and any logical connection f, $\widetilde{\text{DP}}^{\max}_{\Pi,r,f}(n)$ is either 0, $\exp(-\Theta(n))$, $\Theta(n^{-1/2})$ or $\Theta(1)$ and $\widetilde{\text{DP}}^{\min}_{\Pi,r,f}(n)$ is either 0, $\exp(-\Theta(n))$, $\Theta(n^{-1/2})$ or $\Theta(1)$. The condition for each case will be clearly stated in our formal definition.*

A small max-smoothed likelihood of doctrinal paradox implies that doctrinal paradox is rare under all cases, which is good news, see *e.g.*, Example 6. Similarly, a large min-smoothed likelihood of doctrinal paradox implies that doctrinal paradox is common under all cases, which is usually bad news. A direct corollary of Theorem 1 (Corollary 2, when there is only one distribution in Π) answers the open question by List [2005] for i.i.d. distributions. Note that Corollary 2 covers the case that all agents' distributions are uniformly at random, which is not covered in List [2005]. In Section 5, we also numerically verified the results in Theorem 1 under generic settings.

Related Works. List and Pettit [2002] proved the first *impossible theorem* for doctrinal paradox and attracted researchers' attentions on the worst-cases analysis. Pauly and Van Hees [2006], Mongin [2008], Dietrich and List [2008], Awad et al. [2017], Mongin [2019], Baharad et al. [2020] and

Marcoci and Nguyen [2020] proved similar impossible theorems for doctrinal paradox with relaxed requirements or under different settings. Sacrificing some important properties of the aggregation rule is one of the ways to reduce the likelihood of doctrinal paradox [Nehring and Pivato, 2018, Rahwan and Tohmé, 2010, Nehring and Puppe, 2008], but those aggregation rules also received criticisms [Lyon and Pacuit, 2013].

The probability of doctrinal paradox also draws attentions from researchers. List [2005], Bonnefon [2007] and Bonnefon [2010] provides imperial results to the probability of doctrinal paradox. Far less literature studies the theoretical analysis to the probability of doctrinal paradox. List [2005] provided the probability of doctrinal paradox under i.i.d. assumptions. However, their analysis only took limited types of distributions into accounts. Even the simple cases like i.i.d. uniform distributions are missing. Furthermore, the theorems in [List, 2005] does not allow arbitrary logical connection between the premises and the conclusion, which further constrains the applications of their theorem.

The *smoothed social choice* framework [Xia, 2020b] used in this paper was inspired by the *smoothed complexity* theory proposed in Spielman and Teng [2009], and provides a much more realistic setting for social choice than i.i.d. In a position paper, Baumeister et al. [2020] proposed to conduct smoothed analysis in computational social choice without presenting technical results.

2 Preliminaries

Notation. Let $[n] \triangleq \{1, \cdots, n\}$ denote the set of n agents. Let p denote the number of premises. The binary judgement of agent j on the i-th premise is denoted by $\omega_{j,i} \in \{0,1\}$, where $\omega_{j,i} = 1$ means "YES" and $\omega_{j,i} = 0$ means "NO". The conclusion is usually written as ϕ and sometimes written as ω_{p+1} for simplicity. Because all premises are binary, there are $m \triangleq 2^p$ different combinations of *judgements* on the premises. To better present the results, instead of asking the agents to submit their judgements $\vec{\omega}_j = (\omega_{j,1}, \cdots, \omega_{j,p})$, equivalently, we ask each agent to submit an m-dimensional vector $\vec{v}_j$, called her *vote*, whose $\vec{\omega}_j$-th component is 1 and all other components are 0's. Let $\mathcal{V} \triangleq \{\vec{v} \in \{0,1\}^m : ||\vec{v}||_1 = 1\}$ denote the set of all votes whose components are 0's except on one ω_j.

$\vec{v}_j$ can be easily extended to describe *fractional votes*, in particular distributions over judgments. In a fractional profile, each agent can divide the weight of his/her vote $\vec{v}$ to multiple types of judgments, such that for every judgment $\vec{\omega}$, $\vec{v}(\vec{\omega})$ is the weight on $\vec{\omega}$. Moreover, we require that $\vec{v}^\top \vec{1} = 1$ (see Example 2, all vectors in this paper are column vectors by default). Let $\mathcal{V}_{\text{frc}} \triangleq \{\vec{v} \in [0,1]^m : ||\vec{v}||_1 = 1\}$ denote the set of all fractional votes. Clearly, we have $\mathcal{V} \subseteq \mathcal{V}_{\text{frc}}$.

Basic Settings. We let $f : \mathcal{V} \to \{0,1\}$ denotes the *logical connection* between the premises and the conclusion, that is,

$$\forall\, j \in [n],\ \phi_j = f(\vec{v}_j),$$

where $\phi_j \in \{0,1\}$ reprsents agent j's conclusion. We slightly abuse the definition of f and allow it to take agents judgements $\vec{\omega}$ as its input. Mathematically.

$$\forall\, \vec{\omega} \in \{0,1\}^p,\ f(\vec{\omega}) \triangleq f(\vec{v}),\ \text{where } \vec{v}(\vec{\omega}) = 1$$

Let $\boldsymbol{\Omega}_i$ denote the set of all judgements whose i-th premise is 1. Formally, let

$$\boldsymbol{\Omega}_i \triangleq \begin{cases} \{\vec{\omega} \in \{0,1\}^p :\ \vec{\omega}(i) = 1\} & \forall i \in [p] \\ \{\vec{\omega} \in \{0,1\}^p :\ f(\vec{\omega}) = 1\} & \text{if } i = p+1 \end{cases}.$$

For any $n \in \mathbb{Z}_{>0}$, let $P = (\vec{v}_1, \cdots, \vec{v}_n) \in (\mathcal{V}_{\text{frc}})^n$ denote a *(fractional) profile* of n agents. A *(judgement) aggregation rule* is a function $r : (\mathcal{V}_{\text{frc}})^n \to \{0,1\}^{p+1}$, which takes a profile as input and outputs binary values for all premises and the conclusion. For any (fractional) profile

$P \in (\mathcal{V}_{\text{frc}})^n$, we define $\text{Hist}(P) \triangleq \sum_{j\in[n]} \vec{v}_j$ to be the *histogram* of P, which represents the total weight of each combination of judgements in P.

Quota Rules. Quota rule is a natural generalization of the majority rule, where the threshold for win may be different from 0.5. Like the majority rule, quota rule also has the property of *anonymity*, *neutrality*, *independence* and *monotonicity* (see Appendix A.2 for formal definitions). In this paper, we focus on quota rules that independently aggregate agents' judgments on the conclusion and premises.

Formally, given any vector of *acceptance thresholds* (*threshold* in short), denoted by $\vec{q} = (q_1, \cdots, q_{p+1}) \in [0,1]^{p+1}$ and any vector of *breaking criteria* (*breaking* in short), denoted by $\vec{d} = (d_1, \cdots, d_{p+1}) \in \{0,1\}^{p+1}$, we define the quota rule $r_{\vec{q},\vec{d}}(P)$ as follows. For any profile $P \in (\mathcal{V}_{\text{frc}})^n$ and any $i \in [p+1]$, we let $n_i = \sum_{\vec{\omega}\in\boldsymbol{\Omega}_i} \text{Hist}(P)(\vec{\omega})$ denote the total weight of agents whose i-th judgement is 1, then apply the quota rule with threshold q_i and breaking d_i. That is,

$$\forall\, i \in [p+1],\ r_{\vec{q},\vec{d}}(P)(i) \triangleq \begin{cases} 1 & \text{if } n_i > q_i \cdot n \\ d_i & \text{if } n_i = q_i \cdot n \\ 0 & \text{otherwise} \end{cases}, \tag{1}$$

where $r_{\vec{q},\vec{d}}(P)(i)$ is the i-th component of $r_{\vec{q},\vec{d}}(P)$. We say that a profile P is *tied in* ω_i if $n_i = q_i \cdot n$.

The Doctrinal Paradox. We say that a profile P is a doctrinal paradox (under a judgement aggregation rule r) if the $r(P)$ is inconsistent with the agents' logical connection function. Formally,

Definition 1 (Doctrinal paradox). *Given any $n \in \mathbb{Z}_+$, any logical connection function $f : \mathcal{V}_{\text{frc}}^n \to \{0,1\}^{p+1}$, and any quota rule r, a profile $P \in \mathcal{V}_{\text{frc}}^n$ is a doctrinal paradox, if*

$$r(P)(p+1) \neq f\big(r(P)(1), \cdots, r(P)(p)\big).$$

Because doctrinal paradox only depends on the aggregation result, we also say that an *aggregation result* $\vec{\alpha} = (\alpha_1, \cdots, \alpha_{p+1})$ is a doctrinal paradox if $\alpha_{p+1} \neq f(\alpha_1, \cdots, \alpha_p)$. Note that $\vec{\alpha}$ consists of $p+1$ elements, which correspond to the aggregated results of the p premises and one conclusion.

Example 2. *Continuing with the setting of Example 1, the logical connection is $\phi \leftrightarrow \omega_1 \wedge \omega_2$, which is equivalent to the logical connection function*

$$f(\vec{v}) = \begin{cases} 1 & \textit{if } \vec{v} = (0,\, 0,\, 0,\, 1) \\ 0 & \textit{otherwise} \end{cases}.$$

A (fractional) profile P of three votes, a quota rule r (which uses the majority rule with the breaking in favor of 1 *on both premises while in favor of* 0 *on conclusion). Its aggregation are shown in Table 2.*

	ω_1	ω_2	weight	$\vec{v}$	ϕ
Agent 1	1	0	1	$(0,0,1,0)$	0
Agent 2	0	1	1	$(0,1,0,0)$	0
Agent 3	1	0	0.5	$(0,0,0.5,0.5)$	0
	1	1	0.5		1
$\text{Hist}(P)$				$(0,1,1.5,0.5)$	
n_i in (1)	2	1.5			0.5
Breaking d_i	1	1			0
Threshold q_i	0.5	0.5			0.5
$q_i \cdot n$	1.5	1.5			1.5
Aggregation α	1	1			0

Table 2: The profile, rule, and aggregation result for Example 2.

It can be seen from the table that $r(P) = (1,1,0)$, which is inconsistent w.r.t. f. Therefore, P is a doctrinal paradox.

3 Smoothed Likelihood of Doctrinal Paradox

We note that any probability distribution π over $\mathcal{V}$ can be viewed as a fractional profile with a single vote, whose $\vec{\omega}$-th dimension is $\pi(\vec{\omega})$. Therefore, any quota rule r can be applied to the profile that consists of a single vote π, and we let $r(\pi)$ denote the aggregation result for simplicity. To present the main theorem, we first define *effective refinements* of distributions as follows.

Definition 2 (Effective Refinements). *For any distribution π over $\mathcal{V}$, any $n \in \mathbb{Z}_+$, and any quota rule r, we say that a vector $\vec{\alpha} = (\alpha_1, \cdots, \alpha_{p+1}) \in \{0,1\}^{p+1}$ is an* effective refinement *of π, if the following two conditions both hold:*

$$(1)\ \forall\, i \in [p+1] \text{ such that } \pi \text{ is not tied in } \omega_i,\ \alpha_i = r(\pi)(i).$$
$$(2)\ \exists\, P \in \mathcal{V}^n \text{ such that } r(P) = \vec{\alpha}.$$

If condition (1) holds, then we call $\vec{\alpha}$ a refinement *of π. Let*

$$\mathcal{E}_\pi = \left\{\vec{\alpha} \in \{0,1\}^{p+1} : \vec{\alpha} \text{ is an effective refinement of } \pi\right\}$$

denote the set of all effective refinements of π.

In words, condition (1) requires $\vec{\alpha}$ to match the aggregation result on all non-tied propositions. Condition (2) requires the existence of profiles with the aggregation result $\vec{\alpha}$.

Example 3. *Continuing with the setting of Example 2, $\pi = (0.3, 0.2, 0, 0.5)$ is tied in the conclusion and the first premise. There are four refinements of π: $(1,0,0)$, $(1,0,1)$, $(0,0,0)$ and $(0,0,1)$. $(0,0,1)$ and $(1,0,1)$ are not effective, because no profile can make conclusion $\phi = 1$ while keeping both ω_1 to be 0. The other two refinements are effective. For example, $(0,0,0)$ is effective because it is the aggregation of the profile where all agents have 0 judgements on both premises.*

Let CH(Π) denote the convex hull of Π. Next, we define four conditions (κ_1 to κ_4) to present Theorem 1.

$\boldsymbol{\kappa_1}$: $\forall\, P \in \mathcal{V}^n$, P is not a doctrinal paradox. That is, no profile of n votes is a doctrinal paradox.

$\boldsymbol{\kappa_2}$: $\forall\, \pi \in$ CH(Π), $\forall\, \vec{\alpha} \in \mathcal{E}_\pi$, $\vec{\alpha}$ is not a doctrinal paradox. That is, all distributions in the convex hull of Π are "far" from all doctrinal paradoxes.

$\boldsymbol{\kappa_3}$: $\exists\, \pi \in$ CH(Π) such that $\forall \vec{\alpha} \in \mathcal{E}_\pi$, $\vec{\alpha}$ is not a doctrinal paradox. That is, some distribution in the convex hull of Π is "far" from all doctrinal paradoxes.

$\boldsymbol{\kappa_4}$: $\exists\, i \in [m]$ such that " $\omega_{p+1} \leftrightarrow \omega_i$ and $q_{p+1} = q_i$" or " $\omega_{p+1} \leftrightarrow \neg\, \omega_i$ and $q_{p+1} = 1 - q_i$". Intuitively, κ_4 says that the conclusion only relies on one premise and the thresholds are consistent with the relationship between the conclusion and the premise.

Example 4 (Conditions κ_1-κ_4)**.** *Consider the same logical connection and quota rule as in Example 2. Let $\Pi = \{\pi_1 = (0.25, 0.25, 0.25, 0.25), \pi_2 = (0.04, 0.32, 0.32, 0.32)\}$. Let us examine conditions κ_1 to κ_4 as follows:*
$\boldsymbol{\kappa_1}$ is false for any $\boldsymbol{n \geq 2}$ and is true when $\boldsymbol{n = 1}$. *When $n = 2$, $P = (1,0,0,1)$ is the only doctrinal paradox. When $n \geq 3$, $P = (n + 1 - 2\lfloor n/3 \rfloor - \lceil n/3 \rceil, \lfloor n/3 \rfloor, \lfloor n/3 \rfloor, \lceil n/3 \rceil - 1)$ is a doctrinal paradox. We note that κ_2 and κ_3 are false if κ_1 is true. Therefore, we assume $n \geq 2$ for κ_2 and κ_3.*
$\boldsymbol{\kappa_2}$ is False. *CH(Π) $= \{\pi = a \cdot \pi_1 + (1-a) \cdot \pi_2 : a \in [0,1]\}$. When $a \neq 0$, π is a doctrinal paradox and contains no ties. When $a = 0$, $\pi = \pi_1$ is tied in conclusion and both premises. Its effective refinement $\vec{\alpha} = (1,1,0)$ is a doctrinal paradox.*
$\boldsymbol{\kappa_3}$ is False. *π_2 is a doctrinal paradox and contains no ties.*
$\boldsymbol{\kappa_4}$ is False *according to the definitions of f and r.*

We say that a distribution π is *strictly positive*, if there exists $\epsilon > 0$ such that for every $\vec{\omega} \in \{0,1\}^p$, $\pi(\vec{\omega}) \geq \epsilon$. A set of distributions Π is *strictly positive*, if there exists $\epsilon > 0$ such that every $\pi \in \Pi$ is strictly positive (by ϵ). We believe that being strictly positive is a mild requirement for judgement aggregation (c.f. Xia [2020b]'s argument for voting). Π is *closed* if it is a closed set in $\mathbb{R}^m$. We now present the main theorem of this paper.

Theorem 1 (**Smoothed Likelihood of Doctrinal Paradox).** *Given any closed and strictly positive set Π of distributions over $\mathcal{V}$, any logical connection function $f : \mathcal{V} \to \{0,1\}$, and any quota rule r. For any $n \in \mathbb{Z}_{>0}$,*

$$\widetilde{\text{DP}}^{\max}_{\Pi,r,f}(n) = \begin{cases} 0 & \textit{if } \kappa_1 \textit{ is true} \\ \exp(-\Theta(n)) & \textit{otherwise, if } \kappa_2 \textit{ is true} \\ \Theta(n^{-1/2}) & \textit{otherwise, if } \kappa_4 \textit{ is true} \\ \Theta(1) & \textit{otherwise} \end{cases}$$

$$\widetilde{\text{DP}}^{\min}_{\Pi,r,f}(n) = \begin{cases} 0 & \textit{if } \kappa_1 \textit{ is true} \\ \exp(-\Theta(n)) & \textit{otherwise, if } \kappa_3 \textit{ is true} \\ \Theta(n^{-1/2}) & \textit{otherwise, if } \kappa_4 \textit{ is true} \\ \Theta(1) & \textit{otherwise} \end{cases}.$$

We believe that Theorem 1 is quite general, as it works for any logical connection functions and only makes mild assumptions on Π.

Notice that both max and min smoothed likelihood of doctrinal paradox has four cases: the 0 case, the exponential case, the polynomial case, and the constant case. The first three cases are good news, particularly in the max part, which states that the doctrinal paradox vanishes when the number of agents is large. The last case is bad news, in particular for the min part, which states that the doctrinal paradox does not vanish.

Next, let us look at a few applications of Theorem 1. The first is good news, where the max-smoothed likelihood of doctrinal paradox is exponentially small.

Example 5 (Exponentially small case)**.** *Let us consider the same quota rule and the same logical connection as in Example 2 and 4. Let $\Pi = \{\pi_1 = (0.05, 0.05, 0.05, 0.85), \pi_2 = (0.1, 0.1, 0.1, 0.7)\}$. Following a similar reasoning as in Example 4, we have:*

$$(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = \begin{cases} (\text{true}, \text{false}, \text{false}, \text{false}) & \textit{if } n = 1 \\ (\text{false}, \text{true}, \text{true}, \text{false}) & \textit{if } n \geq 2 \end{cases}$$

Therefore, according to Theorem 1, the smoothed likelihood of doctrinal paradox is as shown in Table 3. Also see Figure 2b for our numerical verification.

If applying Theorem 1 to Example 4, we will have the results in Table 3. Also see Figure 2a in section 5 for the numerical verification.

Smoothed likelihood	$n = 1$	$n \geq 2$ (eg. 4)	$n \geq 2$ (eg. 5)
$\widetilde{\text{DP}}^{\max}_{\Pi,r_{\vec{q},\vec{d}},f}(n)$	0	$\Theta(1)$	$\exp(-\Theta(n))$
$\widetilde{\text{DP}}^{\min}_{\Pi,r_{\vec{q},\vec{d}},f}(n)$	0	$\Theta(1)$	$\exp(-\Theta(n))$

Table 3: Smoothed likelihood of doctrinal paradox in Example 4-5.

The next example is also good news because the doctrinal paradox vanishes as $n \to \infty$, though not as fast as in Example 5 w.r.t. the max-adversary.

Example 6 ($\Theta(n^{-1/2})$, $\exp(-\Theta(n))$ and 0 cases)**.** *Let the logical connection be $\phi \leftrightarrow \omega_i$, set of distributions $\Pi = \{\pi_1 = (0.9, 0.1), \pi_2 = (0.3, 0.7)\}$*[1] *and quota rule $r_{\vec{q},\vec{d}}$ such that $(q_i, q_{p+1}) =$*

[1]Because the doctrinal paradox only depends on the votes to premise ω_j (or conclusion ϕ), we use the marginal distribution on ω_i to simplify notations. Here, $\pi = (\text{pr}, 1 - \text{pr})$ mean $\omega_i = 0$ with probability pr while $\omega_i = 1$ with probability $1 - \text{pr}$.

$(0.5, 0.5)$ *and* $(d_i, d_{p+1}) = (1, 0)$. *We have the following results for* κ_1-κ_4*:*

$$(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = \begin{cases} (\text{true}, \text{false}, \text{false}, \text{true}) & \textit{if } n \textit{ is odd} \\ (\text{false}, \text{false}, \text{true}, \text{true}) & \textit{if } n \textit{ is even} \end{cases}$$

Then, we apply Theorem 1 and conclude the smoothed likelihood in Table 4. Also see Figure 2c in section 5 for numerical verification.

Smoothed likelihood	n is odd	n is even
$\widetilde{\text{DP}}^{\max}_{\Pi, r_{\vec{q},\vec{d}}, f}(n)$	0	$\Theta(n^{-1/2})$
$\text{DP}^{\min}_{\Pi, r_{\vec{q},\vec{d}}, f}(n)$	0	$\exp(-\Theta(n))$

Table 4: Smoothed likelihood of doctrinal paradox in Example 6.

The following corollary of Theorem 1 with $\Pi = \{\pi\}$ answers the open questions in [List, 2005] about the probabilities of doctrinal paradox under all i.i.d. distributions.

Corollary 2 (Likelihood of doctrinal paradox under i.i.d. distributions)**.** *Given any strictly positive distribution* π *over* $\mathcal{V}$*, any logical connection function* $f : \mathcal{V} \to \{0, 1\}$*, and any quota rule* r*. For any* $n \in \mathbb{Z}_{>0}$*,*

$$\Pr_{P \sim (\pi, \cdots, \pi)}(P \textit{ is a doctrinal paradox}) = \begin{cases} 0 & \textit{if } \kappa_1 \textit{ is true} \\ \exp(-\Theta(n)) & \textit{otherwise, if } \kappa_5 \textit{ is true} \\ \Theta(n^{-1/2}) & \textit{otherwise, if } \kappa_4 \textit{ is true} \\ \Theta(1) & \textit{otherwise} \end{cases},$$

where κ_5 *is:* $\forall\, \vec{\alpha} \in \mathcal{E}_\pi$*,* $\vec{\alpha}$ *is not a doctrinal paradox.*

In particular, when π is the uniform distribution over $\mathcal{V}$ and the aggregation rule is the majority, the likelihood of doctrinal paradox is either $\Theta(1)$ or 0 depending on the logical connection function f.

4 Our Techniques and the Proof of Theorem 1

In this section, we first introduce the *polyhedra* presentation of doctrinal paradoxes. Then, we will prove Theorem 1 under its *polyhedra* presentation.

Polyhedra Presentation. We define *polyhedron* $\mathcal{H}$ in m-dimensional space by a set of inequalities constrains. That is, $\mathcal{H} = \left\{\vec{x} : \boldsymbol{A}\,\vec{x} \leq \vec{b}\right\}$. Next, we show that the region of doctrinal paradox can be presented by a set of polyhedra. For any $i \in [p+1]$, define vector $\vec{c}_i \in \{q_i - 1, q_i\}^m$ as the characterization vector for proposition ω_i:

$$\vec{c}_i(\vec{\omega}) \triangleq \begin{cases} q_i - 1 & \text{if } \vec{\omega} \in \boldsymbol{\Omega}_i \\ q_i & \text{otherwise} \end{cases}. \tag{2}$$

When calculating the inner product $\vec{c}_i \cdot \text{Hist}(P)$, $\vec{c}_i$ gives the weight of q_i to all votes (sum to $n \cdot q_i$) and gives an extra weight of -1 on all votes that $\omega_i = 1$ (sum up to $-n_i$). One can see that the sign of $\vec{c}_i \cdot \text{Hist}(P)$ is closely related with the quota aggregation results. We define sign function $\text{sign}(\cdot)$ and $\mathbb{1}(\cdot)$ as follows,

$$\text{sign}(x) \triangleq \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{otherwise} \end{cases} \quad \text{and} \quad \mathbb{1}(\kappa) \triangleq \begin{cases} 1 & \text{if } \kappa \text{ is true} \\ 0 & \text{otherwise} \end{cases}.$$

Then, we define

$$\boldsymbol{A}_{\vec{\alpha}} \triangleq \begin{pmatrix} \text{sign}(\alpha_1)\cdot \vec{c}_1^{\mathsf{T}} \\ \vdots \\ \text{sign}(\alpha_{p+1})\cdot \vec{c}_{p+1}^{\mathsf{T}} \end{pmatrix} \quad \text{and} \quad \vec{b}_{\vec{\alpha}} \triangleq \begin{pmatrix} \text{sign}(\alpha_1)\cdot d_1 - \mathbb{1}(\alpha_1 = 1) \\ \vdots \\ \text{sign}(\alpha_{p+1})\cdot d_{p+1} - \mathbb{1}(\alpha_{p+1} = 1) \end{pmatrix}.$$

Let $\mathcal{H}_{\vec{\alpha}} = \{\vec{x} : \ \boldsymbol{A}_{\vec{\alpha}}\,\vec{x} \le \vec{b}_{\vec{\alpha}}\}$ denote the polyhedra and let $\mathcal{H}_{\vec{\alpha},\le 0} = \{\vec{x} : \ \boldsymbol{A}_{\vec{\alpha}}\,\vec{x} \le \vec{0}\}$ denote its *characteristic cone*. Let $\mathcal{C} = \{\mathcal{H}_{\vec{\alpha}} : \ \vec{\alpha} \text{ is a doctrinal paradox}\}$ denote the set of all polyhedra corresponding to doctrinal paradoxes. In Lemma 3, we reveal a connection between $\mathcal{C}$ and the doctrinal paradox.

Lemma 3 (Polyhedra Representation of Doctrinal Paradox)**.** *Given any profile $P \in \mathcal{V}^n$ and any distribution π on $\mathcal{V}$, any quota rule r and any logical connection function f, we have the following two statements for doctrinal paradoxes,*

$$(1).\ \forall \mathcal{H} \in \mathcal{C},\ \text{Hist}(P) \notin \mathcal{H} \ \Leftrightarrow\ P \textit{ is not a doctrinal paradox.}$$
$$(2).\ \forall \mathcal{H} \in \mathcal{C},\ \pi \notin \mathcal{H}_{\le 0} \ \Leftrightarrow\ \textit{all refinements of } \pi \textit{ are not doctrinal paradoxes.}$$

Before presenting the proof, we first use an example to visualizes a polyhedron and its characteristic cone for doctrinal paradoxes.

Example 7 (Polyhedra presentation of doctrinal paradox)**.** *We consider a system with one premise ω_1. The logical connection between conclusion ϕ and premise ω_1 is $\phi \leftrightarrow \omega_1$. The parameters for quota rule are $(q_1, q_C) = (0.25, 0.65)$ and $(d_1, d_C) = (1, 1)$. According to Equation (2), we know that $\vec{c}_1 = (q_1, q_1 - 1)^{\mathsf{T}}$ and $\vec{c}_2 = (q_C, q_C - 1)^{\mathsf{T}}$. Then, the aggregation results of $\vec{\alpha} = (1, 0)$ and $\vec{\alpha} = (0, 1)$ are both doctrinal paradoxes. Thus, $\mathcal{C} = \left\{\mathcal{H}_{(1,0)}, \mathcal{H}_{(0,1)}\right\}$. $\mathcal{H}_{(1,0)}$ is represented by $\boldsymbol{A}_{(1,0)}$ and $\vec{b}_{(1,0)}$ defined as follows.*

$$\boldsymbol{A}_{(1,0)} = \begin{pmatrix} q_1 & q_1 - 1 \\ -q_C & 1 - q_C \end{pmatrix} \quad \textit{and} \quad \vec{b}_{(1,0)} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

It is not hard to verify that for any profile P of n agents, $\boldsymbol{A}_{(1,0)} \cdot \text{Hist}(P) \le \vec{b}_{(1,0)}$ if and only if

$$n_1 \ge q_1 \cdot n \quad \textit{and} \quad n_1 \le q_C \cdot n - 1$$

where n_1 represents the number of votes for $\omega_1 = 1$ in P. Figure 1 illustrates the regions corresponding to $\mathcal{H}_{(1,0)}$ and its characteristic cone.

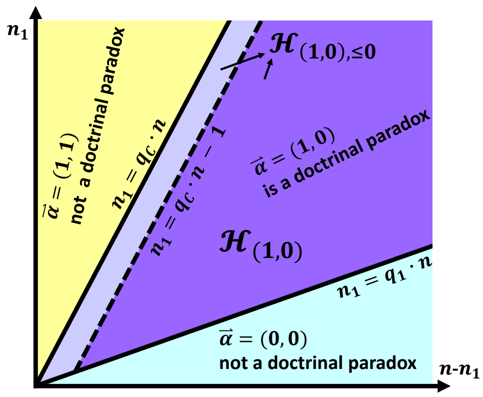


Figure 1: Illustration of the polyhedron $\mathcal{H}_{(1,0)}$ in Example 7. The characteristic cone $\mathcal{H}_{(1,0),\le 0}$ is the combination of $\mathcal{H}_{(1,0)}$ and the region between $n_1 = t_C \cdot n$ and $n_1 = t_C \cdot n - 1$.

Proof for Lemma 3 According to the definition of $\mathcal{C}$, we only need to prove the following statement for polyhedra:

Statement 4. *For any profile $P \in \mathcal{V}^n$ and any distribution π on $\mathcal{V}$, any quota rule $r_{\vec{q},\vec{d}}$ and any logical connection function f,*

$$\begin{aligned} \text{Hist}(P) \in \mathcal{H}_{\vec{\alpha}} &\Leftrightarrow r_{\vec{q},\vec{d}}(P) = \vec{\alpha} \qquad \textit{and} \\ \pi \in \mathcal{H}_{\vec{\alpha},\leq 0} &\Leftrightarrow \vec{\alpha} \textit{ is a refinement of } \pi. \end{aligned}$$

Let $\vec{\omega}_{\vec{\ell}}$ be the corresponding type of vote when the vote on premises is $\vec{\ell}$. Mathematically, $\vec{\omega}_{\vec{\ell}} \in \mathcal{V}$ and $\vec{\omega}_{\vec{\ell}}(\vec{\ell}) = 1$. For all $i \in [p+1]$, the characterization vector $\vec{c}_i$ can be written as, $\vec{c}_i = (q_i - 1) \cdot \left(\sum_{\vec{\ell} \in \boldsymbol{\Omega}_i} \vec{\omega}_{\vec{\ell}}\right) + q_i \left(\sum_{\vec{\ell} \notin \boldsymbol{\Omega}_i} \vec{\omega}_{\vec{\ell}}\right)$. Then,

$$\begin{aligned} \vec{c}_i^{\top} \cdot \text{Hist}(P) &= \vec{c}_i^{\top} \cdot \left(\sum_{\vec{\ell}} \text{Hist}(P)(\vec{\ell}) \cdot \vec{\omega}_{\vec{\ell}}\right) \\ &= (q_i - 1) \cdot \left(\sum_{\vec{\ell} \in \boldsymbol{\Omega}_i} \text{Hist}(P)(\vec{\ell})\right) + q_i \left(\sum_{\vec{\ell} \notin \boldsymbol{\Omega}_i} \text{Hist}(P)(\vec{\ell})\right) \\ &= n \cdot q_i - \underbrace{\sum_{\vec{\ell} \in \boldsymbol{\Omega}_i} \text{Hist}(P)(\vec{\ell})}_{\text{number of votes for } \omega_i = 1} . \end{aligned}$$

According to the definition of quota rule, we know that,

$$\begin{aligned} \vec{c}_i^{\top} \cdot \text{Hist}(P) \leq d_i - 1 &\Leftrightarrow r_{\vec{q},\vec{d}}(P)(i) = 1 \text{ and} \\ -\vec{c}_i^{\top} \cdot \text{Hist}(P) \leq -d_i &\Leftrightarrow r_{\vec{q},\vec{d}}(P)(i) = 0 \end{aligned}$$

Then, the first part of Statement 4 follows by the definition of $\mathcal{H}_{\vec{\alpha}}$. Follow similar procedure as above, we have,

$$\vec{c}_i^{\top} \cdot \pi = q_i - \sum_{\vec{\ell} \in \boldsymbol{\Omega}_i} \pi(\vec{\ell})$$

Thus, for any distribution π over $\mathcal{V}$,

$$\begin{aligned} \vec{c}_i^{\top} \cdot \pi \leq 0 &\Leftrightarrow \exists\, \vec{\alpha} \in \mathcal{E}_\pi \text{ such that } \vec{\alpha}(i) = 1 \text{ and} \\ -\vec{c}_i^{\top} \cdot \pi \leq 0 &\Leftrightarrow \exists\, \vec{\alpha} \in \mathcal{E}_\pi \text{ such that } \vec{\alpha}(i) = 0 \end{aligned}$$

Then, the second part of Statement 4 follows by the definition of $\mathcal{H}_{\vec{\alpha},\leq 0}$. $\square$

Smoothed likelihood of Polyhedra. For any distribution π over $\mathcal{V}$ and any $n \in \mathbb{Z}_0$, the *active dimension* of $\mathcal{H}$ is formally defined as: $\dim_{\mathcal{H},n}(\pi) \triangleq$

$$\begin{cases} \dim(\mathcal{H}_{\leq 0}) & \text{if } \pi \in \mathcal{H}_{\leq 0} \text{ and } \exists\, P \in \mathcal{V}^n \text{ such that } \text{Hist}(P) \in \mathcal{H} \\ -\infty & \text{otherwise} \end{cases} .$$

We say a polyhedra is active (to π at n) if and only if $\dim_{\mathcal{H},n}(\pi) \neq -\infty$. In words, an active polyhedron requires (1) there exists a profile of n votes, whose histogram is in the polyhedron (2) π is in the characteristic cone of the polyhedron. In the next lemma, we reveal a relationship between the active dimensions of polyhedra in $\mathcal{C}$ and the smoothed probability of the doctrinal paradox.

Lemma 5 (Active Dimension $\rightarrow$ Smoothed Probability, a direct application of Theorem 1 in [Xia, 2020a]). *Given any closed and strictly positive set Π of distributions over $\mathcal{V}$, any logical connection function $f : \mathcal{V} \rightarrow \{0,1\}$ and any quota rule r. For any $n \in \mathbb{Z}_{>0}$,*

$$\widetilde{\text{DP}}^{\max}_{\Pi,r,f}(n) = \begin{cases} 0 & \textit{if C1 is true} \\ \exp(-\Theta(n)) & \textit{otherwise, if C2 is true} \\ \max_{\mathcal{H}\in\mathcal{C}} \Theta\left(n^{(dim_{\mathcal{H}}(\pi)-m)/2}\right) & \textit{otherwise} \end{cases}$$
$$\widetilde{\text{DP}}^{\min}_{\Pi,r,f}(n) = \begin{cases} 0 & \textit{if C1 is true} \\ \exp(-\Theta(n)) & \textit{otherwise, if C3 is true} \\ \min_{\mathcal{H}\in\mathcal{C}} \Theta\left(n^{(dim_{\mathcal{H}}(\pi)-m)/2}\right) & \textit{otherwise} \end{cases},$$

where C1 is: $\forall P \in \mathcal{V}^n$, P is not doctrinal paradox.
C2 is: $\forall\, \pi \in CH(\Pi)$ and $\forall\, \mathcal{H} \in \mathcal{C}$, $dim_{\mathcal{H}}(\pi) = -\infty$
C3 is: $\exists\, \pi \in CH(\Pi)$ and $\forall\, \mathcal{H} \in \mathcal{C}$, $dim_{\mathcal{H}}(\pi) = -\infty$.

Note that C1 is the same as κ_1 in Theorem 1. By Lemma 3, we know that κ_2 (or κ_3) is very similar with C2 (or C3), which says that all (or some) distributions in CH(Π) are "far" from all active polyhedra of doctrinal paradoxes.

Proof Sketch for Theorem 1. (See Appendix B.1 for full proof) The only gap left between Lemma 5 and Theorem 1 is that the active dimension $\dim_{\mathcal{H},n}(\pi)$ is still unknown. We sketch the proof in the following two steps:

Step 1: determine dim($\mathcal{H}_{\leq 0}$) (the dimensionality of characteristic cones). We first show that $\vec{c}_1, \cdots, \vec{c}_p$ are linearly independent by contradiction. For any $i \in [p]$, we assume that $\vec{c}_i = \sum_{i' \neq i} a_{i'} \cdot \vec{c}_{i'}$, where $a_{i'}$ are real valued coefficients. Because $\vec{c}_i$ is not a zero vector and different with any $\vec{c}_{i'}$, there must exist a pair of $i'_1, i'_2 \in [p] \setminus \{i\}$ such that $a_{i'_1} \neq 0$ and $a_{i'_2} \neq 0$. Let $\vec{c}_i(\ell_1, \ell_2, \vec{0})$ to be the coordinate of $\vec{c}_i$ corresponds to $\omega_{i'_1} = \ell_1$, $\omega_{i'_2} = \ell_2$ and all other premises are 0. Because $\vec{c}_i$ has the save value in all components that $\omega_i = 0$. We should have $\vec{c}_i(0,0,\vec{0}) = \vec{c}_i(1,0,\vec{0}) = \vec{c}_i(1,1,\vec{0})$, which is contradict with the observation that $\vec{c}_i(0,0,\vec{0}) - \vec{c}_i(1,0,\vec{0}) = a_{i'_1} \neq 0$ and $\vec{c}_i(1,0,\vec{0}) - \vec{c}_i(1,1,\vec{0}) = a_{i'_2} \neq 0$.

Using similar techniques (but the analysis procedure is much more complex), it can be proved that $\vec{c}_{p+1}$ is linearly independent with $\{\vec{c}_1, \cdots, \vec{c}_n\}$ when κ_4 is not true. Then, by the standard conclusion on the dimensionality of polyhedra, we know that $\dim(\mathcal{H}_{\leq 0}) = m$ for all $\mathcal{H} \in \mathcal{C}$ when κ_4 is false. When κ_4 is true, the logical connection can be denoted as either $\phi \leftrightarrow \omega_i$ or $\phi \leftrightarrow \neg\, \omega_i$. For both cases, we have $\text{sign}_{0\rightarrow -1}(\alpha_{p+1}) \cdot \vec{c}_{p+1} = -\text{sign}_{0\rightarrow -1}(\alpha_i) \cdot \vec{c}_i$. Thus, the following equation must hold in $\mathcal{H}_{\vec{\alpha},\,\leq 0}$.

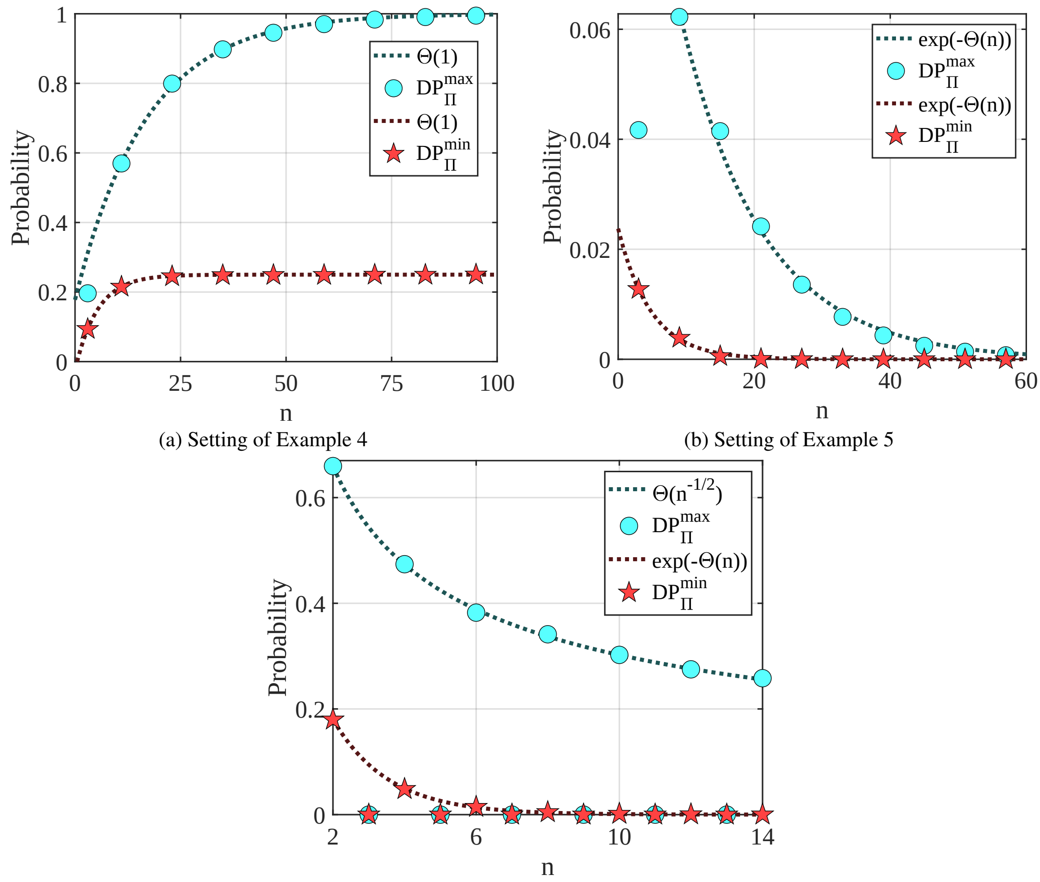
$$\vec{x}^{\mathsf{T}} \cdot \left(\text{sign}_{0\rightarrow -1}(\alpha_{p+1}) \cdot \vec{c}_{p+1}\right) = \vec{x}^{\mathsf{T}} \cdot \left(\text{sign}_{0\rightarrow -1}(\alpha_i) \cdot \vec{c}_i\right) = 0.$$

Thus, when κ_4 is true, $\dim(\mathcal{H}_{\leq 0}) = m - 1$ for all $\mathcal{H} \in \mathcal{C}$.

Step 2: determine whether the polyhedra is active.
From the proof of Lemma 3, we know that $\mathcal{H}_{\vec{\alpha}}$ is active if and only if $\vec{\alpha}$ is an effective refinement of π. Then, we know that none of the polyhedra in $\mathcal{C}$ is active if and only if π has no effective refinements of doctrinal paradoxes. Thus, we know that C2 (and C3) in Lemma 5 is equivalent to κ_2 (and κ_3) in Theorem 1. $\square$

5 Experiments

We conduct numerical experiments to verify the results in Theorem 1. The first three experiments (Figure 2) follows the same setting as Example 4, 5, 6 respectively. In Figure 2, $\text{DP}^{\max}_{\Pi}$ and $\text{DP}^{\min}_{\Pi}$ (the blue circles and red stars) represent the estimated max-smoothed likelihood and



(a) Setting of Example 4

(b) Setting of Example 5

(c) Setting of Example 6

Figure 2: Numerical verification of Theorem 1.

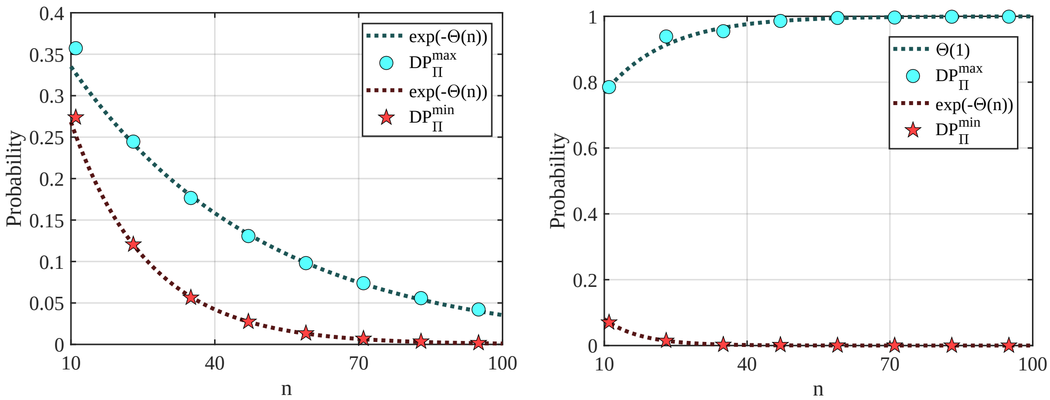
the min-smoothed likelihood of doctrinal paradox. The dot curves illustrate the fittings of the estimated smoothed probabilities. The expressions and fitness of all fitting curves are presented in Appendix C. Recalling the notations used in our definition of $\widetilde{\mathrm{DP}}^{\max}_{\Pi,r,f}(n)$, we say $\vec{\pi} \in \Pi^n$ is one kind of *distribution assignment*. We run one million (10^6) independent trials to estimate the probability of doctrinal paradoxes under each distribution assignment. Then, $\mathrm{DP}^{\max}_{\Pi}$ (or $\mathrm{DP}^{\min}_{\Pi}$) takes the maximum (or the minimum) probability of doctrinal paradoxes among all distribution assignments. It is easy to see that the results are consistent with Theorem 1. For example, Figure 2a shows that both the max-smoothed likelihood and the min-smoothed likelihood of doctrinal paradox are $\Theta(1)$, which matches the result in Table 3.

Then, we present two experiments under more complex settings. Both experiments uses the following logical connection, breaking criteria and set of distributions:
Logical connection: conclusion $\phi = 1$ if $(\omega_1, \omega_2, \omega_3) \in \{(0,0,0),(0,1,0),(1,1,0)\}$ and $\phi = 0$ otherwise.
Breaking criteria: $\vec{d} = (1,0,1,0)$.
Set of distributions: $\Pi = \{\pi_1, \pi_2\}$, where $\pi_1 = (0.25, 0.25, 0.05, 0.05, 0.05, 0.05, 0.05, 0.25)$ and $\pi_2 = (0.32, 0.32, 0.008, 0.008, 0.008, 0.32, 0.008, 0.008)$.
Other settings and the theoretical smoothed likelihoods drawn from Theorem 1 are shown in the Table 5.

In Figure 3, we present the numerical result for two more complex settings, which matches our theoretical results.

Name of setting	Majority	Quota
Threshold $\vec{q}$	$(0.5, 0.5, 0.5, 0.5)$	$(0.2, 0.2, 0.2, 0.2)$
$\widetilde{\text{DP}}^{\max}_{\Pi, r_{\vec{q},\vec{d}}, f}(n)$	$\exp(-\Theta(n))$	$\Theta(1)$
$\widetilde{\text{DP}}^{\min}_{\Pi, r_{\vec{q},\vec{d}}, f}(n)$	$\exp(-\Theta(n))$	$\exp(-\Theta(n))$

Table 5: Settings and results for the experiments with three premises



(a) Majority setting

(b) Quota setting

Figure 3: Numerical verification for the complex setting of three premises.

6 Conclusions and Future Works

This paper provides a highly generic theorem for the likelihood of doctrinal paradox under the natural framework of smoothed social choice. A direct corollary of our theorem solves an open question in the field of judgment aggregations: the likelihood of doctrinal paradox under i.i.d. assumptions. One interesting future direction is to study the likelihood of doctrinal paradox for non-strictly positive distributions. The smoothed probability of doctrinal paradoxes under more general settings remains an open question for the field of judgment aggregations.

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Supplementary Material for COMSOC-21 Submission:
The Smoothed Likelihood of Doctrinal Paradox

A Detailed Settings for Section 2 (Preliminary)

A.1 Vectorized indexing for votes or histogram

We fix the judgement of $\vec{\omega} = \vec{\ell} = (\ell_1, \cdots, \ell_p)$ to correspond to the $(\overline{\ell_1 \cdots \ell_p}_{(2)} + 1)$-th component of $\vec{v}$, where $\overline{\ell_1 \cdots \ell_p}_{(2)}$ is the binary number taking ℓ_i as the i-th digit. For example, if $\omega_{j,1} = 1$ and $\omega_{j,2} = 0$, we will have $\vec{v}_j(\overline{10}_{(2)} + 1) = \vec{v}_j(3) = 1$ while all other components of $\vec{v}_j$ are zero.

A.2 Properties of Quota Rules

The properties of quota rules include :
Anonymity: each pair of agents play interchangeable roles.
Neutrality: each pair of propositions are interchangeable.
Independence: the aggregation of a certain proposition only depends on agents' judgement on this proposition.
Monotonicity: adding vote to the current winner will never change the winner

B Missing Proofs

B.1 Full proof for Theorem 1

Theorem 1 (Smoothed Likelihood for Doctrinal Paradox). *Given any strictly positive set* Π *of distributions over* $\mathcal{V}$*, any logic function* $f : \mathcal{V} \to \{0, 1\}$*, and any quota rule* r*. For any* $n \in \mathbb{Z}_{>0}$*,*

$$\widetilde{\mathrm{DP}}^{\max}_{\Pi,r,f}(n) = \begin{cases} 0 & \textit{if } \kappa_1 \textit{ is true} \\ \exp(-\Theta(n)) & \textit{otherwise, if } \kappa_2 \textit{ is true} \\ \Theta(n^{-1/2}) & \textit{otherwise, if } \kappa_4 \textit{ is true} \\ \Theta(1) & \textit{otherwise} \end{cases}$$

$$\widetilde{\mathrm{DP}}^{\min}_{\Pi,r,f}(n) = \begin{cases} 0 & \textit{if } \kappa_1 \textit{ is true} \\ \exp(-\Theta(n)) & \textit{otherwise, if } \kappa_3 \textit{ is true} \\ \Theta(n^{-1/2}) & \textit{otherwise, if } \kappa_4 \textit{ is true} \\ \Theta(1) & \textit{otherwise} \end{cases}.$$

Proof. The only gap between Lemma 5 (including Lemma 3) and Theorem 1 is that the active dimension $\dim_{\mathcal{H}}(\pi)$ is still unknown. We sketch the proof in the following two steps:

Step 1: determine dim$(\mathcal{H}_{\leq 0})$ (the dimensionality of characteristic cones).

Step 1.1: the linear independence of $\{\vec{c}_1, \cdots, \vec{c}_p\}$. We first show that $\vec{c}_1, \cdots, \vec{c}_p$ are linearly independent by contradiction. For any $i \in [p]$, we assume that $\vec{c}_i = \sum_{i' \neq i} a_{i'} \cdot \vec{c}_{i'}$, where $a_{i'}$ are real valued coefficients. Because $\vec{c}_i$ is not a zero vector and different with any $\vec{c}_{i'}$, there must exist a pair of $i'_1, i'_2 \in [p] \setminus \{i\}$ such that $a_{i'_1} \neq 0$ and $a_{i'_2} \neq 0$. Let $\vec{c}_i(\ell_1, \ell_2, \vec{0})$ to be the coordinate of $\vec{c}_i$ corresponds to $\omega_{i'_1} = \ell_1, \omega_{i'_2} = \ell_2$ and all other premises are 0. Because $\vec{c}_i$ has the same value in all components that $\omega_i = 0$. We should have $\vec{c}_i(0, 0, \vec{0}) = \vec{c}_i(1, 0, \vec{0}) = \vec{c}_i(1, 1, \vec{0})$, which is contradict with the observation that

$$\begin{aligned}
\vec{c}_i(0,0,\vec{0}) &= a_{i'_1}\cdot q_{i'_1} + a_{i'_2}\cdot q_{i'_2} + \sum_{i'\notin\{i'_1.i'_2\}} a_{i'}\cdot q_{i'} \\
\vec{c}_i(1,0,\vec{0}) &= a_{i'_1}(q_{i'_1}-1) + a_{i'_2}\cdot q_{i'_2} + \sum_{i'\notin\{i'_1.i'_2\}} a_{i'}\cdot q_{i'} \\
\vec{c}_i(1,1,\vec{0}) &= a_{i'_1}(q_{i'_1}-1) + a_{i'_2}(q_{i'_2}-1) + \sum_{i'\notin\{i'_1.i'_2\}} a_{i'}\cdot q_{i'} \qquad \text{and thus} \\
\vec{c}_i(0,0,\vec{0}) - \vec{c}_i(1,0,\vec{0}) &= a_{i'_1} \neq 0 \\
\vec{c}_i(1,0,\vec{0}) - \vec{c}_i(1,1,\vec{0}) &= a_{i'_2} \neq 0.
\end{aligned}$$

Step 1.2: the dimension of $\mathcal{H}_{\vec{\alpha}}$ when κ_4 is true. When κ_4 is true, the logical connection can be denoted as either $\phi \leftrightarrow \omega_i$ or $\phi \leftrightarrow \neg\omega_i$. For both cases, we have $\text{sign}_{0\to-1}(\alpha_{p+1})\cdot\vec{c}_{p+1} = -\text{sign}_{0\to-1}(\alpha_i)\cdot\vec{c}_i$. When κ_4 is true, we will have , the following equation must hold in $\mathcal{H}_{\vec{\alpha},\le 0}$.

$$\begin{aligned}
&\vec{x}^{\mathsf{T}}\cdot\big(\text{sign}_{0\to-1}(\alpha_{p+1})\cdot\vec{c}_{p+1}\big) \\
&= \vec{x}^{\mathsf{T}}\cdot\big(\text{sign}_{0\to-1}(\alpha_i)\cdot\vec{c}_i\big) = 0.
\end{aligned}$$

Thus, when κ_4 is not true, we draw the conclusion that

$$\forall\,\mathcal{H}\in\mathcal{C},\ \ \text{dim}(\mathcal{H}_{\le 0}) = m-1 \quad \text{when } \kappa_4 \text{ is true.}$$

Step 1.3: the dimension of $\mathcal{H}_{\vec{\alpha}}$ when κ_4 is not true. We first show the linear independence between $\vec{c}_{p+1}$ and $\{\vec{c}_1,\cdots,\vec{c}_p\}$ when κ_4 is false. We assume that $\vec{c}_{p+1} = \sum_{i'\in[p]} a_{i'}\cdot\vec{c}_{i'}$, where $a_{i'}$ are real valued coefficients. Because $\vec{c}_{p+1}$ is not a zero vector and different with any $\vec{c}_{i'}$ when κ_4 is not true, there must exist a pair of $i'_1, i'_2 \in [p]\setminus\{i\}$ such that $a_{i'_1}\neq 0$ and $a_{i'_2}\neq 0$. Let $\vec{c}_{p+1}(\ell_1,\ell_2,\vec{0})$ to be the coordinate of $\vec{c}_{p+1}$ corresponds to $\omega_{i'_1}=\ell_1$, $\omega_{i'_2}=\ell_2$ and all other premises are 0. Thus, we have the following observations:

$$\begin{aligned}
\vec{c}_{p+1}(0,0,\vec{0}) &= a_{i'_1}\cdot q_{i'_1} + a_{i'_2}\cdot q_{i'_2} + \sum_{i'\notin\{i'_1.i'_2\}} a_{i'}\cdot q_{i'} \\
\vec{c}_{p+1}(1,0,\vec{0}) &= a_{i'_1}(q_{i'_1}-1) + a_{i'_2}\cdot q_{i'_2} + \sum_{i'\notin\{i'_1.i'_2\}} a_{i'}\cdot q_{i'} \\
\vec{c}_{p+1}(1,1,\vec{0}) &= a_{i'_1}(q_{i'_1}-1) + a_{i'_2}(q_{i'_2}-1) + \sum_{i'\notin\{i'_1.i'_2\}} a_{i'}\cdot q_{i'} \qquad \text{and thus} \\
\vec{c}_{p+1}(0,0,\vec{0}) - \vec{c}_{p+1}(1,0,\vec{0}) &= a_{i'_1} \neq 0 \\
\vec{c}_{p+1}(1,0,\vec{0}) - \vec{c}_{p+1}(1,1,\vec{0}) &= a_{i'_2} \neq 0.
\end{aligned} \tag{3}$$

Case 1.3.1: $\vec{c}_{p+1}(0,0,\vec{0}) - \vec{c}_{p+1}(1,1,\vec{0}) = a_{i'_1} + a_{i'_2} \neq 0$. For this case, there must exist at least three different values in $\vec{c}_i$, which contradict with the fact that $\vec{c}_{p+1}\in\{q_{p+1}-1, q_{p+1}\}^m$.
Case 1.3.2: $\vec{c}_{p+1}(0,0,\vec{0}) - \vec{c}_{p+1}(1,1,\vec{0}) = a_{i'_1} + a_{i'_2} = 0$. we note that any nonzero value of $a_{i'_1} + a_{i'_2}$ already results in contradiction according to *Case 1.1.1*. Then, the above relationship of $a_{i'_1} + a_{i'_2} = 0$ must holds for all nonzero $a_{i'}$s, which implies that there cannot exists more than three non-zero values for $a_{i'}$. Now, the only case left is $\vec{c}_i = a(\vec{c}_{i'_1} - \vec{c}_{i'_2})$, where a is a nonzero real number. According to the observations in Inequalities 3, we know that

$$\begin{aligned}
\vec{c}_{p+1}(0,0,\vec{0}) &= a(q_{i'_1} - q_{i'_2}) \\
\vec{c}_{p+1}(1,0,\vec{0}) &= a(q_{i'_1} - q_{i'_2} - 1) \\
\vec{c}_{p+1}(0,1,\vec{0}) &= a(q_{i'_1} - q_{i'_2} + 1),
\end{aligned}$$

which says that $\vec{c}_{p+1}$ has at least three different values and contradict with the fact that $\vec{c}_{p+1} \in \{q_{p+1}-1, q_{p+1}\}^m$.

Thus, when κ_4 is not true, we draw the conclusion that

$$\forall\, \mathcal{H} \in \mathcal{C},\;\; \dim(\mathcal{H}_{\leq 0}) = m \quad \text{when } \kappa_4 \text{ is false.}$$

Step 2: determine whether the polyhedra is active.
Given the first part of Lemma 3, we know that $\mathcal{H}_{\vec{\alpha}} \cap \mathcal{V}^n \neq \emptyset$ is equivalent to the statement of "$\exists\, P \in \mathcal{V}^n$ such that $r_{\vec{q},\vec{d}}(P) = \vec{\alpha}$". Given the first part of Lemma 3, we know that $\pi \in \mathcal{H}_{\vec{\alpha},\leq 0}$ is equivalent to the statement of "$\vec{\alpha}$ is a refinement of π". Combining the above two statements. We know that polyhedra $\mathcal{H}_{\vec{\alpha}}$ is active if and only if $\vec{\alpha}$ is an effective refinement of π. Then, we know that all polyhedra in $\mathcal{C}$ is not active if and only if π has no effective refinements of doctrinal paradoxes. Thus, we know that C2 (and C3) in Lemma 5 is equivalent to κ_2 (and κ_3) in Theorem 1. $\square$

C Additional Numerical Results

In Table 6 - Table 10, we present the expressions of the asymptotic curves under different settings. Root mean square error (RMSE) and coefficient of determination r^2 shows the fitness between the asymptotic curves and the data points (simulated smoothed probability of doctrinal paradox).

Smoothed Probability	Expressions of the asymptotic curves	RMSE	r^2
$\widetilde{\text{DP}}^{\max}_{\Pi,r,f}(n)$	$1 - 0.82227 \cdot \exp(-0.05941 \cdot n)$	9.850×10^{-3}	0.99883398
$\widetilde{\text{DP}}^{\min}_{\Pi,r,f}(n)$	$0.25 - 0.27476 \cdot \exp(-0.18796 \cdot n)$	8.337×10^{-4}	0.99977090

Table 6: The asymptotic curves in Figure 2a for Example 4

Smoothed Probability	Expressions of the asymptotic curves	RMSE	r^2
$\widetilde{\text{DP}}^{\max}_{\Pi,r,f}(n)$	$0.13362 \cdot \exp(-0.08309 \cdot n)$	5.639×10^{-3}	0.93938840
$\widetilde{\text{DP}}^{\min}_{\Pi,r,f}(n)$	$0.02376 \cdot \exp(-0.20700 \cdot n)$	1.106×10^{-4}	0.99934099

Table 7: The asymptotic curves in Figure 2b for Example 5

Smoothed Probability	Expressions of the asymptotic curves	RMSE	r^2
$\widetilde{\text{DP}}^{\max}_{\Pi,r,f}(n)$ when n is even	$0.96124 \cdot (n + 0.12519)^{-1/2}$	3.501×10^{-3}	0.99949065
$\widetilde{\text{DP}}^{\min}_{\Pi,r,f}(n)$ when n is even	$0.65588 \cdot \exp(-0.64539 \cdot n)$	7.508×10^{-4}	0.99989252

Table 8: The asymptotic curves in Figure 2c for Example 6

Smoothed Probability	Expressions of the asymptotic curves	RMSE	r^2
$\mathrm{DP}_{\Pi}^{\max}(n, \mathcal{Q}, f)$	$0.43009 \cdot \exp(-0.02498 \cdot n)$	9.830×10^{-3}	0.99884028
$\mathrm{DP}_{\Pi}^{\min}(n, \mathcal{Q}, f)$	$0.49672 \cdot \exp(-0.06166 \cdot n)$	9.060×10^{-3}	0.99567084

Table 9: The asymptotic curves in Figure 3a for the majority setting of three premises

Smoothed Probability	Expressions of the asymptotic curves	RMSE	r^2
$\mathrm{DP}_{\Pi}^{\max}(n, \mathcal{Q}, f)$	$1 - 0.49443 \cdot \exp(-0.0758 \cdot n)$	5.677×10^{-3}	0.99480574
$\mathrm{DP}_{\Pi}^{\min}(n, \mathcal{Q}, f)$	$0.29644 \cdot \exp(-0.13092 \cdot n)$	4.528×10^{-4}	0.99970318

Table 10: The asymptotic curves in Figure 3b for the quota setting of three premises