Four Faces of Altruistic Hedonic Games

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Abstract

We consider coalition formation games where agents have to form partitions based on their preferences. In the subclass of hedonic (coalition formation) games, we assume that the agents' preferences over different partitions only depend on the coalitions they are part of. While these games have been broadly studied in the literature, most models assume that agents are always selfish. By contrast, we consider *altruistic* hedonic and coalition formation games which model how agents behave altruistically to their friends based on an underlying network of friends. Doing so, we present three further variations of the original model due to Nguyen et al. [17] by varying to which of their friends players behave altruistically and how these friends' opinions are evaluated. We study these four models with respect to desirable axiomatic properties. Overall, we show that the extended altruistic models that include all friends into the altruistic behavior of an agent fulfill more of these properties than the hedonic models. In particular, the extended models fulfill unanimity while most of the altruistic hedonic models fail to satisfy this property. Furthermore, the extended models provide stronger properties in terms of monotonicity.

1 Introduction

In *coalition formation games*, agents form a partition into coalitions (subsets of agents) based on their preferences. In the subclass of *hedonic games* [12, 7, 5], players' preferences are required to only depend on the coalitions they are part of.

Many approaches of how to represent those hedonic preferences efficiently have been studied in the literature (see, e.g., Aziz and Savani [4]). One of these representations is the *friend-and-enemy encoding* by Dimitrov et al. [11] which allows to compactly represent hedonic games. In their model, each player divides the set of other players into friends and enemies which then leads to a so-called *network of friends* that represents the mutual friendship relations among the players. Based on this encoding, they introduce *friend-oriented hedonic games*, where players preferences over coalitions are determined by the number of friends in the coalitions and, only in the case of a tie, also by the number of enemies. Based on friend-oriented hedonic games, Nguyen et al. [17] introduced *altruistic hedonic games* where agents assign values to coalitions based on the friend-oriented encoding but do not decide their preferences solely based on their own valuations. Instead, they also consider the valuations of their friends that are in the same coalitions. Depending on the order in which players look at their own or their friends' valuations, Nguyen et al. distinguish three degrees of altruism: selfish first, equal treatment, and altruistic treatment.

We study three variants of altruistic hedonic games. First, we vary the way in which players aggregate their friends' valuations by taking the minimum instead of the average. This change can be seen as taking an egalitarian instead of an utilitarian approach. Second, we vary the scope of friends that are included into an agent's altruistic behavior. While, in altruistic hedonic games, an agent only considers the friends that are in the same coalition as she is, we extend the altruistic behavior of the agents to all their friends, independent of the current coalition structure. By doing so, we release the restriction to hedonic games and consider more general coalition formation games. Third, we combine these two approaches and consider minimum-based altruism over the set of all friends. We compare the four approaches, with respect to some desirable axiomatic properties.

Related work. Hedonic games were introduced by Drèze and Greenberg [12] and formally modeled by Banerjee et al. [5] and Bogomolnaia and Jackson [7]. They have been broadly studied in the literature, addressing the issue of compactly representing preferences, conducting axiomatic analyses, dealing with different notions of stability, and investigating the computational complexity of the associated problems (see, e.g., Aziz and Savani [4]).

This work considers altruistic hedonic games due to Nguyen et al. [17] and studies three variants of these games with respect to desirable axiomatic properties. With respect to various stability notions, minimum-based altruistic hedonic games and altruistic coalition formation games have already been studied by Wiechers and Rothe [21] and Kerkmann and Rothe [15], respectively. Other related work is due to Schlueter and Goldsmith [20]. They introduced *super altruistic hedonic games* where players in the same coalition have a different impact on a player's preference based on their distances in the underlying network of friends. This model is related to social distance games by Brânzei and Larson [9].

In *noncooperative* game theory, altruism has been considered, e.g., by Ashlagi et al. [3] who introduced *social context games* where a strategic game is embedded in a social context that is modeled by a graph of neighborhood. Examples include *ranking games* [8] and *coalitional congestion games* [13, 16]. In particular, in the social context that Ashlagi *et al.* call "surplus collaboration," players seek to maximize the average payoff of themselves and their friends. Other work studying altruism in noncooperative games is due to Hoefer and Skopalik [14], Chen et al. [10], Apt and Schäfer [2], and Rahn and Schäfer [18]. A survey of altruism in (cooperative and noncooperative) game theory is due to Rothe [19].

2 Preliminaries

Let $N = \{1, \dots, n\}$ be a set of players. For any $i \in N$, $\mathcal{N}^i = \{C \subseteq N \mid i \in C\}$ denotes the set of coalitions containing i. A *coalition structure* is a partition $\Gamma = \{C_1, \dots, C_k\}$ of the players into k coalitions $C_1, \dots, C_k \subseteq N$ (i.e., $\bigcup_{r=1}^{k} C_r = N$ and $C_r \cap C_s = \emptyset$ for all distinct $r, s \in \{1, \dots, k\}$). The unique coalition in Γ containing player $i \in N$ is denoted by $\Gamma(i)$. The set of all coalition structures for a set of agents N is denoted by $\mathcal{C}_N$.

A *coalition formation game* is a pair $(N, \succeq)$, where $N = \{1, \dots, n\}$ is a set of agents, $\succeq = (\succeq_1, \dots, \succeq_n)$ is a profile of preferences, and every preference $\succeq_i \subseteq \mathcal{C}_N \times \mathcal{C}_N$ is a complete weak order over all coalition structures. For two coalition structures $\Gamma, \Delta \in \mathcal{C}_N$, we say that agent i *weakly prefers* Γ *to* Δ if $\Gamma \succeq_i \Delta$, that i *prefers* Γ *to* Δ ($\Gamma \succ_i \Delta$) if $\Gamma \succeq_i \Delta$ but not $\Delta \succeq_i \Gamma$, and that i is *indifferent between* Γ *and* Δ ($\Gamma \sim_i \Delta$) if $\Gamma \succeq_i \Delta$ and $\Delta \succeq_i \Gamma$.

A *hedonic game* is a coalition formation game $(N, \succeq)$ where the preference $\succeq_i$ of any $i \in N$ only depends on the coalitions that she is part of. This means that i is indifferent between any two coalition structures $\Gamma, \Delta \in \mathcal{C}_N$ as long as her coalition is the same, i.e., $\Gamma(i) = \Delta(i) \implies \Gamma \sim_i \Delta$. i's preference can then be represented by a complete weak order over the set $\mathcal{N}^i$ of coalitions containing i. For coalitions $A, B \in \mathcal{N}^i$, we say that player i *weakly prefers* A *to* B if $A \succeq_i B$ (and analogously for (strict) preference and indifference).

Friend-Oriented Preferences. As a compact representation of hedonic games, Dimitrov et al. [11] proposed friend-oriented hedonic games where the players' preferences are represented by a *network of friends.* Each player $i \in N$ has a set of friends $F_i \subseteq N \setminus \{i\}$ and a set of enemies $E_i = N \setminus (F_i \cup \{i\})$. This encoding can be represented by a graph $G = (N, H)$, where two players are connected by an edge if and only if they are friends of each other.[1] It then holds that $F_i = \{j \mid \{i, j\} \in H\}$. In the *friend-oriented* preference extension [11], more friends are preferred to fewer friends, and in case of an equal number of friends, fewer enemies are preferred. These preferences can be represented additively, by assigning a value

[1] Note that we only consider mutual friendship relations.

of $n = |N|$ to each friend and a value of -1 to each enemy: For any player $i \in N$ and any coalition $A \in \mathcal{N}^i$, define the *value of a coalition* by

$$v_i(A) = n|A \cap F_i| - |A \cap E_i|. \tag{1}$$

For $A, B \in \mathcal{N}^i$, we then have

$$A \succeq_i^F B \iff v_i(A) \geq v_i(B). \tag{2}$$

Slightly abusing notation, we will write $v_i(\Gamma)$ instead of $v_i(\Gamma(i))$ for i's valuation of $\Gamma(i)$.

Altruistic Hedonic Games. To introduce altruism to a player's (hedonic) preference, Nguyen et al. [17] consider the friends of this agent that are in the same coalition as she is. As they consider all friends to be equally important, they consider the average of their valuations. To denote the average friend-oriented valuation of i's friends in coalition $A \in \mathcal{N}^i$, they use

$$\mathrm{avg}_i^F(A) = \sum_{a \in A \cap F_i} \frac{v_a(A)}{|A \cap F_i|}. \tag{3}$$

We further define the average friend-oriented valuation of i and her friends in A by

$$\mathrm{avg}_i^{F+}(A) = \sum_{a \in (A \cap F_i) \cup \{i\}} \frac{v_a(A)}{|(A \cap F_i) \cup \{i\}|}. \tag{4}$$

Nguyen et al. [17] distinguish three *degrees of altruism* that differ in the order in which an agent considers her own and her friends' valuations. We define these three degrees of altruism using a player i's *utility* for a coalition as a measure of comparison. These utilities combine i's valuation v_i and her friends' average valuation $\mathrm{avg}_i^F(A)$. A constant $M \geq n^4$ is used as a weight to ensure that an agents preference is first determined by her own valuation in the selfish-first model and first determined by her friends' valuations in the altruistic-treatment model. For equal-treatment preferences, no weight is needed since the agent treats herself and her friends the same.

Selfish First (SF): Using the utility function $u_i^{SF}(A) = M \cdot v_i(A) + \mathrm{avg}_i^F(A)$, we define the *selfish-first preferences* of agent i by

$$A \succeq_i^{SF} B \iff u_i^{SF}(A) \geq u_i^{SF}(B) \tag{5}$$

Hence, under SF the agent first looks at her own valuation and, only in the case of a tie, considers the average valuation of her friends that are in the same coalition.

Equal Treatment (EQ): Using the utility function $u_i^{EQ}(A) = \mathrm{avg}_i^{F+}(A)$, we define the *equal-treatment preferences* of agent i by

$$A \succeq_i^{EQ} B \iff u_i^{EQ}(A) \geq u_i^{EQ}(B). \tag{6}$$

Hence, under EQ the agent treats her own valuation and the valuation of each of her friends that are in the same coalition equally.

Altruistic Treatment (AL): Using the utility function $u_i^{AL}(A) = v_i(A) + M \cdot \mathrm{avg}_i^F(A)$, we define the *altruistic-treatment preferences* of agent i by

$$A \succeq_i^{AL} B \iff u_i^{AL}(A) \geq u_i^{AL}(B). \tag{7}$$

Under AL the agent first considers the average valuation of her friends and, only in the case of a tie, decides according to her own valuation.

Indeed, the definitions of the selfish-first preferences and altruistic-treatment preferences indeed capture the intuitive ideas behind them: For $M \geq n^5$, $i \in N$, and $A, B \in \mathcal{N}^i$, $v_i(A) > v_i(B)$ implies $A \succ_i^{SF} B$ and $\mathrm{avg}_i^F(A) > \mathrm{avg}_i^F(B)$ implies $A \succ_i^{AL} B$ [17, Thm. 1 & 2].

We will sometimes abuse notation and just write u_i for player i's utility (or $\succeq_i$ for i's preference) when the degree of altruism is clear from the context or when we talk about all three degrees. We will now give an example of an altruistic hedonic game.

Example 1. *Consider an altruistic hedonic game with $n = 4$ agents whose network of friends is given by a path: $1 — 2 — 3 — 4$. We consider the coalitions $A = \{1, 2, 3\}$, $B = \{1, 2, 4\}$, and $C = \{1, 2\}$. It then holds that $v_1(A) = n - 1 = 3$, $v_2(A) = 2n = 8$, $v_3(A) = n - 1 = 3$, $v_1(B) = v_2(B) = n - 1 = 3$, $v_4(B) = -2$, and $v_1(C) = v_2(C) = n = 4$.*

Under friend-oriented preferences agent 1 is indifferent between A and B ($A \sim_i^F B$) because $v_1(A) = v_1(B)$. Under selfish-first altruistic preferences, however, agent 1 resolves this tie by looking a her friend 2's valuations. Since $\mathrm{avg}_1^F(A) = v_2(A) = 8 > 3 = v_2(B) = \mathrm{avg}_1^F(B)$, 1 prefers A to B ($A \succ_1^{SF} B$) under selfish-first altruistic hedonic preferences.

Comparing A and C, 1 prefers C to A ($C \succ_1^{SF} A$) under selfish-first preferences because $v_1(C) = 4 > 3 = v_1(A)$ but prefers A to C ($A \succ_1^{AL} C$) under altruistic treatment because of her friend 2's valuation: $\mathrm{avg}_1^F(A) = v_2(A) = 8 > 4 = v_2(C) = \mathrm{avg}_1^F(C)$.

3 Three Variants of Altruistic Games

When agents behave altruistically in altruistic hedonic games [17], they consider the average valuation of their friends that are in the same coalition. This can be seen as an utilitarian approach since valuations are added up and then divided by the size of the coalition to prevent a "tyranny of the many" (larger coalitions dominating just because of their size). The egalitarian analogue can be achieved by considering the minimum of the friends' valuations instead. Furthermore, altruistic hedonic preferences incorporate the hedonic restriction by only considering the friends of an agent that are in the same coalition as she is. Dropping the restriction to hedonic games, we consider preferences of extended altruism where all friends of an agent are considered, also if they are part of other coalitions.

For any agent $i \in N$ and any coalition $A \in \mathcal{N}^i$, we define the minimum of the valuations of i's friends that are in the same coalition as i (and of the valuation of i) by

$$\mathrm{min}_i^F(A) = \min_{a \in A \cap F_i} v_a(A) \quad \text{and} \quad \mathrm{min}_i^{F+}(A) = \min_{a \in (A \cap F_i) \cup \{i\}} v_a(A). \tag{8}$$

For readability, we let $\mathrm{min}_i^F(\Gamma) = \mathrm{min}_i^F(\Gamma(i))$ and $\mathrm{min}_i^{F+}(\Gamma) = \mathrm{min}_i^{F+}(\Gamma(i))$.

For any coalition structure $\Gamma \in \mathcal{C}_N$, we denote the sum of the valuations of i's friends for *their* coalitions in Γ (plus the valuation of i) by

$$\mathrm{sum}_i^{extF}(\Gamma) = \sum_{a \in F_i} v_a(\Gamma) \quad \text{and} \quad \mathrm{sum}_i^{extF+}(\Gamma) = \sum_{a \in F_i \cup \{i\}} v_a(\Gamma). \tag{9}$$

Analogously, we define the minimum of the valuations by

$$\mathrm{min}_i^{extF}(\Gamma) = \min_{a \in F_i} v_a(\Gamma) \quad \text{and} \quad \mathrm{min}_i^{extF+}(\Gamma) = \min_{a \in F_i \cup \{i\}} v_a(\Gamma). \tag{10}$$

In these definitions, we define the minimum of the empty set as zero.

Still distinguishing the same three degrees of altruism as Nguyen et al. [17], we define the different altruistic preferences by the following utilities. For each player $i \in N$, any coalition structures $\Gamma, \Delta \in \mathcal{C}_N$, a constant $M \geq n^3$, and $\alpha \in \{\min, \mathrm{sum}^{ext}, \min^{ext}\}$, we define

- the α *selfish-first preferences* by $u_i^{\alpha SF}(\Gamma) = M \cdot v_i(\Gamma) + \alpha_i^F(\Gamma)$ and
$$\Gamma \succeq_i^{\alpha SF} \Delta \iff u_i^{\alpha SF}(\Gamma) \geq u_i^{\alpha SF}(\Delta). \tag{11}$$
- the α *equal-treatment preferences* by $u_i^{\alpha EQ}(\Gamma) = \alpha_i^{F+}(\Gamma)$ and
$$\Gamma \succeq_i^{\alpha EQ} \Delta \iff u_i^{\alpha EQ}(\Gamma) \geq u_i^{\alpha EQ}(\Delta). \tag{12}$$
- the α *altruistic-treatment preferences* by $u_i^{\alpha AL}(\Gamma) = v_i(\Gamma) + M \cdot \alpha_i^F(\Gamma)$ and
$$\Gamma \succeq_i^{\alpha AL} \Delta \iff u_i^{\alpha AL}(\Gamma) \geq u_i^{\alpha AL}(\Delta). \tag{13}$$

The factor M, which is used for the selfish-first model and for altruistic treatment, again ensures that an agent's utility is first determined by the agent's own valuation in the selfish-first model and by the friends' valuations in the altruistic model. Similarly as before, we can show that for $M \geq n^3$, $v_i(\Gamma) > v_i(\Delta)$ implies $\Gamma \succ_i^{\alpha SF} \Delta$, and $\alpha_i^F(\Gamma) > \alpha_i^F(\Delta)$ implies $\Gamma \succ_i^{\alpha AL} \Delta$. We omit the (straightforward) proofs here.

A pair $(N, \succeq^\alpha)$ where $\succeq^\alpha$ is a profile of preferences defined by one of the degrees of altruism in (11)–(13) and with

- $\alpha = \min$ is said to be a *min-based altruistic hedonic game* (MBAHG, for short) with *min-based altruistic (hedonic) preferences* $\succeq^{\min}$.
- $\alpha = \text{sum}^{ext}$ is said to be an *altruistic coalition formation game* (ACFG, for short) with *extended altruistic preferences* $\succeq^{\text{sum}^{ext}}$ (or just $\succeq^{ext}$).
- $\alpha = \min^{ext}$ is said to be a *min-based altruistic coalition formation game* (MBACFG, for short) with *extended min-based altruistic preferences* $\succeq^{\min^{ext}}$.

Note that min-based altruistic preferences fulfill the hedonic requirement, since the utility $u_i^{\min}(\Gamma)$ of an agent i for a coalition structure Γ only depends on $\Gamma(i)$. Hence, we can also use these preferences to compare coalitions: $\Gamma(i) \succeq_i^{\min} \Delta(i) \iff \Gamma \succeq_i^{\min} \Delta$.

Example 2. *Again, consider a game with four agents whose network of friends is given by a path: 1 — 2 — 3 — 4. Further consider the coalition structures* $\Gamma = \{\{1\},\{2,3\},\{4\}\}$ *and* $\Delta = \{\{1\},\{2,4\},\{3\}\}$.

Under any hedonic preferences, like (min-based) altruistic hedonic preferences, it is clear that agent 1 *is indifferent between* Γ *and* Δ *because any hedonic preference only depends on* $\Gamma(1) = \{1\} = \Delta(1)$. *Under extended (min-based) altruistic preferences, however, agent* 1 *acts altruistically to all her friends. We have* $\text{sum}_1^{extF}(\Gamma) = v_2(\Gamma) = 4$ *and* $\text{sum}_1^{extF}(\Delta) = v_2(\Delta) = -1$. *Hence,* 1 *prefers* Γ *to* Δ *(*$\Gamma \succ_1^{extAL} \Delta$*) under extended altruistic treatment. Actually,* 1 *prefers* Γ *to* Δ *under all degrees of extended (min-based) altruistic preferences.*

4 Properties of Altruistic Hedonic Games

We now list some desirable properties of preference extensions that are inspired by various related topics such as voting theory and resource allocation. Let $G = (N, H)$ be a given network of friends. We say that player i's hedonic preference $\succeq_i$ on $\mathcal{N}^i$ is *reflexive* if $A \succeq_i A$ for each coalition $A \in \mathcal{N}^i$; $\succeq_i$ is *transitive* if for any three coalitions $A, B, C \in \mathcal{N}^i$, $A \succeq_i B$ and $B \succeq_i C$ implies $A \succeq_i C$; $\succeq_i$ is *polynomial-time computable* if for two given coalitions $A, B \in \mathcal{N}^i$, it can be decided in polynomial time whether or not $A \succeq_i B$; and $\succeq_i$ is *anonymous* if renaming the players in $N \setminus \{i\}$ does not change $\succeq_i$. Clearly, the first three properties are necessary to have efficiently computable and rational preferences, and anonymity means that only the structure of the friendship network is important.

Weak Friend-Orientedness: If coalition A is acceptable for i (i.e., $A \succeq_i \{i\}$), then $A \cup \{f\}$ is also acceptable for i, where $f \in F_i \setminus A$.

Favoring Friends: If $x \in F_i$ and $y \in E_i$ then $\{x, i\} \succ_i \{y, i\}$.

Indifference between Friends: If $x, y \in F_i$ then $\{x, i\} \sim_i \{y, i\}$.

Indifference between Enemies: If $x, y \in E_i$ then $\{x, i\} \sim_i \{y, i\}$.

Note that these four properties hold for friend-oriented preferences, see the work of Alcantud and Arlegi [1].[2] The next property is inspired by the property "citizens' sovereignty" from voting theory which says that only the voters shall decide on who has won an election, so for a voting rule to satisfy this property it is required that every candidate can be made a winner for suitably chosen voter preferences (see, e.g., [6]). Similarly, we require that only the players shall decide on which coalitions turn out to be their most preferred ones, under a suitably chosen network of friends.

Sovereignty of Players: For a fixed player i and each $C \in \mathcal{N}^i$, there exists a network of friends such that C ends up as i's most preferred coalition.

We now introduce monotonicity which says that an agent's preference of one coalition over another should not become worse if an enemy of that agent is turned into a friend.

Monotonicity: Let $j \neq i$ be a player with $j \in E_i$ and let $A, B \in \mathcal{N}^i$. Let further $\succeq_i'$ be the preference relation resulting from $\succeq_i$ when j turns from being i's enemy to being i's friend (all else remaining equal). We call $\succeq_i$

- *type-I-monotonic* if it holds that (1) if $A \succ_i B$, $j \in A \cap B$, and $v_j(A) \geq v_j(B)$, then $A \succ_i' B$, and (2) if $A \sim_i B$, $j \in A \cap B$, and $v_j(A) \geq v_j(B)$, then $A \succeq_i' B$;
- *type-II-monotonic* if it holds that (1) if $A \succ_i B$ and $j \in A \setminus B$, then $A \succ_i' B$, and (2) if $A \sim_i B$ and $j \in A \setminus B$, then $A \succeq_i' B$.

Another important property is symmetry. It says that, if two agents j and k have the same friends, then adding either j or k to a coalition should have the same impact on the utilities of the players in that coalition.

Symmetry: Let j and k be two distinct players with $j \neq i \neq k$. We say that $\succeq_i$ is *symmetric* if it holds that if swapping the positions of j and k in G is an automorphism then $\left(\forall\, C \in \mathcal{N}^i \setminus (\mathcal{N}^j \cup \mathcal{N}^k)\right)\ [C \cup \{j\} \sim_i C \cup \{k\}]$.

The next property is local friend dependence. It says that an agent's preference over some coalitions can change if the sets of friends' friends change. These friends also have to be members of the coalition that is under consideration. Thus local friend dependence is a crucial property that tries to capture the essence of the proposed approach to altruism in hedonic games.

Local Friend Dependence: The preference order $\succeq_i$ can depend on the sets of friends $F_1, \ldots, F_n$ of some agents. Let $A, B \in \mathcal{N}^i$. We say that comparison (A, B) is

- *friend-dependent in* $\succeq_i$ if $A \succeq_i B$ is true (or false) and can be made false (or true) by changing the set of friends of some players in $N \setminus \{i\}$ (while not changing any relation to i);

[2] Alcantud and Arlegi [1] define so-called weighted GNB rankings (where objects are classified into three categories: good, neutral, and bad), which are a generalization of friend-oriented preferences in hedonic games.

Table 1: Properties satisfied (✓) or not satisfied (✗) by (min-based) altruistic hedonic games, and (min-based) altruistic coalition formation games as defined in Sections 2 and 3. A dash (—) indicates that the property is not defined for the model.

Model	AHG			MBAHG			ACFG			MBACFG		
Degree of Altruism	*SF*	*EQ*	*AL*	*SF*	*EQ*	*AL*	*SF*	*EQ*	*AL*	*SF*	*EQ*	*AL*
Reflexivity	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
Transitivity	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
Pol.-time comp.	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
Anonymity	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
Weak Friend-Orient.	✓	✓	✓	✓	✓	✓	—	—	—	—	—	—
Favoring Friends	✓	✓	✓	✓	✓	✓	—	—	—	—	—	—
Indiff. bw. Friends	✓	✓	✓	✓	✓	✓	—	—	—	—	—	—
Indiff. bw. Enemies	✓	✓	✓	✓	✓	✓	—	—	—	—	—	—
Sovereignty of Players	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
Type-I-Monotonicity	✓	✗	✗	✗	✗	✗	✓	✓	✓	✗	✗	✗
Type-II-Monotonicity	✓	✗	✗	✓	✗	✗	✓	✓	✓	✓	✓	✓
Symmetry	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
Friend Dependence	✓[1]	✓[1]	✓[1]	✓[1]	✓[2]	✓[1]	✓[1]	✓[1]	✓[1]	✓[1]	✓[1]	✓[1]
Local Friend Dep.	✓[1]	✓[1]	✓[1]	✓[1]	✓[2]	✓[1]	✗	✗	✗	✗	✗	✗
Unanimity	✓	✗	✗	✓	✗	✗	✓	✓	✓	✓	✓	✓
Local Unanimity	✓	✓	✓	✓	✓	✓	—	—	—	—	—	—

[1] If there are at least four agents and the considered agent's set of friends is nonempty.
[2] If the considered agent has at least two friends.

- *locally friend-dependent in* $\succeq_i$ if $A \succeq_i B$ is true (or false), can be made false (or true) by changing the set of friends of some players in $(A \cup B) \cap F_i$ (while not changing any relation to i), and changing the set of friends of any of the other players in $N \setminus (\{i\} \cup (F_i \cap (A \cup B)))$ (while not changing any relation to any player in $\{i\} \cup (F_i \cap (A \cup B)))$ does not affect the status of the comparison.

We say $\succeq_i$ is *friend-dependent* if there are $A, B \in \mathcal{N}^i$ such that (A, B) is friend-dependent in $\succeq_i$.

We say $\succeq_i$ is *locally friend-dependent* if $\succeq_i$ is friend-dependent and every (A, B) that is friend-dependent in $\succeq_i$ is locally friend-dependent in $\succeq_i$.

Finally, we turn to unanimity. We first define *local unanimity* which is suited for the restricted scope of hedonic games: If two coalitions A and B contain the same friends of a player i, and if i and all these friends value A higher than B, then we want i to prefer A over B. We further define *unanimity* which is a broader notion.

Local Unanimity: Let $A, B \in \mathcal{N}^i$ and $A \cap F_i = B \cap F_i$. We say that $\succeq_i$ is *locally unanimous* if $v_a(A) > v_a(B)$ for each $a \in (F_i \cup \{i\}) \cap A$ implies that $A \succ_i B$.

Let $\Gamma, \Delta \in \mathcal{C}_N$. We say that $\succeq_i$ is *unanimous* if $v_a(\Gamma(a)) > v_a(\Delta(a))$ for each $a \in F_i \cup \{i\}$ implies $\Gamma(i) \succ_i \Delta(i)$.

We now study which of these desirable properties are satisfied by the (min-based) altruistic hedonic games. In Section 5, we will further consider which of the properties hold for (min-based) altruistic coalition formation games. Table 1 summarizes our results.

We start with some basic properties that hold for altruistic hedonic preferences as well as for min-based altruistic hedonic preferences. We omit the straightforward proof.

Proposition 3. *Under all three degrees of altruism, any (min-based) altruistic hedonic preference satisfies reflexivity, transitivity, polynomial-time computability, and anonymity.*

Nguyen et al. [17] already showed that weak friend-orientedness, favoring friends, indifference between friends, indifference between enemies, sovereignty of players, symmetry, and local unanimity are satisfied for all three degrees of hedonic altruism (5)–(7). All these properties also hold for the three degrees of min-based altruistic hedonic preferences. We omit the proof, which works quite similar to the corresponding proof by Nguyen et al. [17].

Nguyen et al. [17] further state that all hedonic altruistic preferences are locally friend-dependent. We substantiate their statement with the following theorem that says that local friend dependence holds under all three degrees of hedonic altruism except for some edge cases where there are not enough agents or no friends to consider. In particular, if i has no friends, her preference coincides with her friend-oriented preference (as defined in (2)).

Theorem 4. *The preference $\succeq_i^\star$ of agent $i \in N$ is (locally) friend-dependent if and only if i has at least one friend and*

1. *in case $\star = SF$, there are at least four players in N;*
2. *in case $\star = EQ$, there are at least four players in N or there are exactly three players while i has exactly one friend;*
3. *in case $\star = AL$, there are at least three players in N.*

Proof. First, any preference $\succeq_i$ obtained under any of the three degrees of altruism (5)–(7) is locally friend-dependent if and only if it is friend-dependent: Agent i's utilities for two coalitions A and B can only be changed by changing the set of i's friends or the sets of friends of i's friends in A and B, respectively.

Second, we show that there exists a pair (A, B) of coalitions that is friend-dependent under $\succeq_i$ if and only if i has at least one friend and N is sufficiently large.

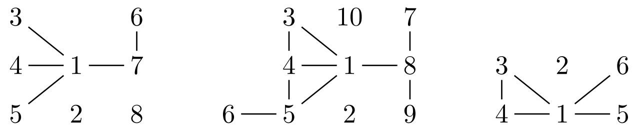
Only if: If i has no friends or if $n \leq 2$, there are no friends whose sets of friends could be changed. So, there is obviously no pair of coalitions that is friend-dependent under any degree of altruism. For $n = 3$, it is easy check whether there exist friend-dependent pairs of coalitions. Omitting the details, the results are as stated in the theorem.

If: We show that there is a pair (A, B) that is friend-dependent (under any degree of altruism) if $n \geq 4$ and $|F_i| > 0$. **Case 1:** There are at least two agents $e_1, e_2 \in N \setminus \{i\}$ that are i's enemies (and one $f \in F_i$ due to $|F_i| > 0$). It holds that $v_i(\{i, f, e_1\}) = v_i(\{i, f, e_2\})$. Hence, i's utility depends on $\text{avg}_i^F(\{i, f, e_1\})$ and $\text{avg}_i^F(\{i, f, e_2\})$. If $\text{avg}_i^F(\{i, f, e_1\}) = \text{avg}_i^F(\{i, f, e_2\})$, we change F_f such that $\text{avg}_i^F(\{i, f, e_1\}) \neq \text{avg}_i^F(\{i, f, e_2\})$, and vice versa. (This is possible by adding e_1 to F_f or deleting e_1 from F_f.) This changes i's preference over $\{i, f, e_1\}$ and $\{i, f, e_2\}$ under all three degrees of altruism. Hence, $(\{i, f, e_1\}, \{i, f, e_2\})$ is friend-dependent. **Case 2:** i is friends with all but one agent $e_1 \in N \setminus \{i\}$ and thus has at least two friends $f_1, f_2 \in F_i$ (due to $n \geq 4$). Then $(\{i, f_1, e\}, \{i, f_2, e\})$ is friend-dependent (by adding e to F_{f_1} or deleting e from F_{f_1}). **Case 3:** If i is friends with all (at least $n - 1 \geq 3$) agents $f_1, \dots f_{n-1} \in N \setminus \{i\}$, then $(\{i, f_1, f_2\}, \{i, f_1, f_3\})$ is friend-dependent (by adding f_2 to F_{f_1} or deleting f_2 from F_{f_1}). ❑

Similarly as for average-based AHGs, we can show that local friend dependence holds under all three degrees of min-based hedonic altruism except for some edge cases. Here, $\succeq_i^{minEQ}$ additionally coincides with i's friend-oriented preference if i has only one friend.

Theorem 5. *The preference $\succeq_i^\star$ of agent $i \in N$ is (locally) friend-dependent if and only if*

1. *in case $\star = minSF$, i has at least one friend and there are at least four players in N;*
2. *in case $\star = minEQ$, i has at least two friends (thus there are at least three players);*
3. *in case $\star = minAL$, i has at least one friend and there are at least three players in N.*



(a) Network of $\mathcal{G}_1$ (b) Network of $\mathcal{G}_2$ (c) Network of $\mathcal{G}_3$

Figure 1: Networks of friends in the proofs of Theorems 7 and 8

Theorem 5 can be shown similarly to Theorem 4; we defer its proof to the appendix.

We now turn to monotonicity. Interestingly, both types of monotonicity hold for selfish-first preferences but for none of the other two degrees of altruism.

Theorem 6 (Nguyen et al. [17])**.** *SF preferences* (5) *are type-I-monotonic and type-II-monotonic. EQ preferences* (6) *and AL preferences* (7) *are not type-II-monotonic.*

Theorem 7. *EQ preferences* (6) *and AL preferences* (7) *are not type-I-monotonic.*

Proof. Let $\mathcal{G}_1$ be a game with the network of friends shown in Figure 1a. We consider players $i = 1$, $j = 2 \notin F_1$ and coalitions $A = \{1,2,3,4,5\}$ and $B = \{1,2,6,7,8\}$. Then $v_1(A) = 3n-1$, $v_1(B) = n-3$, $v_f(A) = n-3$ for all $f \in A \cap F_1$, and $v_f(B) = 2n-2$ for all $f \in B \cap F_1$. Calculating 1's equal-treatment utilities, we get $u_1^{EQ}(A) = \frac{6n-10}{4}$ and $u_1^{EQ}(B) = \frac{3n-5}{2}$. Hence, $A \sim_1^{EQ} B$. We further have $2 \in A \cap B$, and $v_2(A) \geq v_2(B)$. Let $\mathcal{G}_1'$ be the game that results from $\mathcal{G}_1$ when making 2 a friend of 1's. In $\mathcal{G}_1'$ we have $B \succ_1^{EQ} A$, violating type-I-monotonicity for equal-treatment preferences.

Analogously, for the game $\mathcal{G}_2$ (illustrated in Figure 1b), players 1 and $2 \notin F_1$, and the coalitions $A = \{1,2,7,8,9,10\}$ and $B = \{1,2,3,4,5,6\}$, AL preferences are not type-I-monotonic because $A \succ_1^{AL} B$, $2 \in A \cap B$, and $v_2(A) \geq v_2(B)$ in $\mathcal{G}_2$ but $B \succ_1^{AL} A$ in $\mathcal{G}_2'$. ❑

Theorems 6 and 7 are conform with the intuition behind the definitions of equal treatment and altruistic treatment: If a player i gets an additional friend who is unsatisfied in i's coalition then this should diminish player i's utility under equal-treatment and altruistic-treatment preferences. For min-based altruistic hedonic preferences, we get similar results.

Theorem 8. *Min-based SF preferences are type-II-monotonic, but not type-I-monotonic. Min-based EQ and AL preferences are neither type-I- nor type-II-monotonic.*

Proof. We omit the proof that min-based SF preferences are type-II-monotonic since it is very similar to the proof of Theorem 6 saying that SF preferences are type-II-monotonic.

To see that none of the three degrees of min-based altruistic preferences is type-I-monotonic, consider the game $\mathcal{G}_3$ with the network of friends in Figure 1c, players $i = 1$ and $j = 2$, and coalitions $A = \{1,2,3,4\}$ and $B = \{1,2,5,6\}$. Then $v_1(A) = v_1(B) = 11$, $v_2(A) = v_2(B) = -3$, $v_3(A) = v_4(A) = 11$, and $v_5(B) = v_6(B) = 4$. Hence, $\min_i^F(A) = 11$ and $\min_1^F(B) = 4$. It follows that $A \succ_1^{minSF} B$, $A \succ_1^{minEQ} B$, and $A \succ_1^{minAL} B$.

Consider the game $\mathcal{G}_3'$ that results from $\mathcal{G}_3$ by making 2 a friend of 1's. For this game, $v_1(A) = v_1(B) = 18$, $v_2(A) = v_2(B) = 4$, $v_3(A) = v_4(A) = 11$, and $v_5(B) = v_6(B) = 4$. Then $\min_1^F(A) = 4 = \min_1^F(B)$, so $A \sim_1^{minSF} B$, $A \sim_1^{minEQ} B$, and $A \sim_1^{minAL} B$, which contradicts type-I-monotonicity for the three degrees of min-based altruistic hedonic games.

To see that $\succeq_1^{minEQ}$ and $\succeq_1^{minAL}$ violate type-II-monotonicity, consider the same game $\mathcal{G}_3$ from Figure 1c, players $i = 1$ and $j = 2$, and coalitions $A = \{1,2,3,4\}$ and $B = \{1,5,6\}$. Then $A \succ_i^{minEQ} B$ and $A \succ_i^{minAL} B$. However, considering $\mathcal{G}_3'$, we get $B \succ_i^{minEQ} A$ and $B \succ_i^{minAL} A$, violating type-II-monotonicity for min-based EQ and AL preferences. ❑

Intuitively, the results of Theorem 8 do make sense because in min-based altruistic hedonic games a player's preference is determined by the friend that is worst off. Only in the case of min-based SF preferences, agent i still first looks at her own valuation. Hence, an additional friend will always increase her utility. Under the two other degrees of min-based altruism, however, an additional friend might decrease her utility because this friend might have a lower valuation for the coalition than i and all other friends of i's in this coalition.

Finally, it is easy to see that unanimity holds for (min-based) SF preferences. Unanimity will be discussed in more detail in the next section on altruistic coalition formation games and we will see that (min-based) EQ and AL preferences are not unanimous (see Example 13).

5 Properties of Altruistic Coalition Formation Games

We now study the properties from Section 4 for extended (min-based) altruistic preferences ($\succeq^{ext}$ and $\succeq^{\min^{ext}}$). We redefine some of the properties for hedonic preferences from Section 4 for the more general setting of coalition formation games. Reflexivity, transitivity, polynomial-time computability, and anonymity are defined as before with the only difference that we now compare coalition structures instead of coalitions. It is easy to see that our extended models still satisfy these four properties under all three degrees of altruism.

Sovereignty of Players: For a fixed player $i \in N$ and each $\Gamma \in \mathcal{C}_N$, there exists a network of friends such that Γ ends up as i's most preferred coalition structure.

Sovereignty of players still holds for all degrees of altruism of our extended models. This can be shown with an analogous construction as in the proof of Nguyen et al. [17, Theorem 5]: For a given player $i \in N$ and a coalition structure $\Gamma \in \mathcal{C}_N$, we construct a network of friends where all players in $\Gamma(i)$ are friends of each other while there are no other friendship relations. Then Γ is i's (nonunique) most preferred coalition structure.

Monotonicity: Let $j \neq i$ be a player with $j \in E_i$ and let $\Gamma, \Delta \in \mathcal{C}_N$. For $\alpha \in \{\text{sum}^{ext}, \min^{ext}\}$, let further $\succeq_i^{\alpha\prime}$ be the preference relation resulting from $\succeq_i^{\alpha}$ when j turns from being i's enemy to being i's friend (all else remaining equal). We call $\succeq_i^{\alpha}$

- *type-I-monotonic* if it holds that (1) if $\Gamma \succ_i^{\alpha} \Delta$, $j \in \Gamma(i) \cap \Delta(i)$, and $v_j(\Gamma) \geq v_j(\Delta)$, then $\Gamma \succ_i^{\alpha\prime} \Delta$, and (2) if $\Gamma \sim_i^{\alpha} \Delta$, $j \in \Gamma(i) \cap \Delta(i)$, and $v_j(\Gamma) \geq v_j(\Delta)$, then $\Gamma \succeq_i^{\alpha\prime} \Delta$;
- *type-II-monotonic* if it holds that (1) if $\Gamma \succ_i^{\alpha} \Delta$, $j \in \Gamma(i) \setminus \Delta(i)$, and $v_j(\Gamma) \geq v_j(\Delta)$, then $\Gamma \succ_i^{\alpha\prime} \Delta$, and (2) if $\Gamma \sim_i^{\alpha} \Delta$, $j \in \Gamma(i) \setminus \Delta(i)$, and $v_j(\Gamma) \geq v_j(\Delta)$, then $\Gamma \succeq_i^{\alpha\prime} \Delta$.

Type-I-monotonicity and type-II-monotonicity as defined here hold for all three degrees of extended altruism.[3] All degrees of extended min-based altruism satisfy type-II-monotonicity. The proofs of Theorems 9 and 10 can be found in the appendix.

Theorem 9. *Under all three degrees of altruism, extended altruistic preferences $\succeq^{ext}$ are type-I-monotonic and type-II-monotonic.*

Theorem 10. *Under all three degrees of altruism, extended min-based altruistic preferences $\succeq^{\min^{ext}}$ are type-II-monotonic but not type-I-monotonic.*

[3] It is quite remarkable that all three degrees of our extended model of altruism satisfy both type-I- and type-II-monotonicity, unlike in the hedonic models of altruism which fail to satisfy either of them for equal and altruistic treatment (and min-based altruistic selfish-first preferences even violate type-I-monotonicity).

Next, symmetry requires that if two agents j and k have the same friends, then for any coalition structure, where j and k are together with the same set of friends, moving j or k to another coalition should have the same impact on an agent i in that coalition.

Symmetry: Let j and k be two distinct players with $j \neq i \neq k$. For $\alpha \in \{\text{sum}^{ext}, \min^{ext}\}$, we say that $\succeq_i^\alpha$ is *symmetric* if it holds that if swapping the positions of j and k in G is an automorphism then $\big(\forall\, \Gamma \in \mathcal{C}_N, \Gamma(j) \setminus \{j,k\} = \Gamma(k) \setminus \{j,k\}\big)\, [\Gamma_{j \to \Gamma(i)} \sim_i^\alpha \Gamma_{k \to \Gamma(i)}]$.

Theorem 11. *Under all three degrees of altruism, extended (min-based) altruistic preferences $\succeq_i^\alpha$, $\alpha \in \{\text{sum}^{ext}, \min^{ext}\}$, are symmetric.*

The proof of Theorem 11 is omitted. We now turn to friend dependence.

Local Friend Dependence: Let $\alpha \in \{\text{sum}^{ext}, \min^{ext}\}$ and $\Gamma, \Delta \in \mathcal{C}_N$. We say that comparison (Γ, Δ) is

- *friend-dependent in* $\succeq_i^\alpha$ if $\Gamma \succeq_i^\alpha \Delta$ is true (or false) and can be made false (or true) by changing the set of friends of some players in $N \setminus \{i\}$ (while not changing any relation to i);
- *locally friend-dependent in* $\succeq_i^\alpha$ if $\Gamma \succeq_i^\alpha \Delta$ is true (or false), can be made false (or true) by changing the set of friends of some players in $F_i \cap (\Gamma(i) \cup \Delta(i))$ (while not changing any relation to i), and changing the set of friends of any of the other players in $N \setminus (\{i\} \cup (F_i \cap (\Gamma(i) \cup \Delta(i))))$ (while not changing any relation to any player in $\{i\} \cup (F_i \cap (\Gamma(i) \cup \Delta(i))))$ does not affect the status of the comparison.

We say $\succeq_i^\alpha$ is *friend-dependent* if there are $\Gamma, \Delta \in \mathcal{C}_N$ such that (Γ, Δ) is friend-dependent in $\succeq_i^\alpha$.

We say $\succeq_i^\alpha$ is *locally friend-dependent* if $\succeq_i^\alpha$ is friend-dependent and every (Γ, Δ) that is friend-dependent in $\succeq_i^\alpha$ is locally friend-dependent in $\succeq_i^\alpha$.

As under extended (min-based) altruistic preferences the agents behave altruistically to all their friends, even if these friends are not in the same coalition, it intuitively makes sense that the extended models fulfill friend dependence but not local friend dependence.

Theorem 12. *Under all three degrees of altruism, extended (min-based) altruistic preferences are not locally friend-dependent. But they are friend-dependent if and only if there are at least three agents and the considered agent has at least one friend.*

Proof. It is easy to see that there is no friend-dependent comparison if there are at most two agents or if the considered agent i has no friends. For the case of $n \geq 3$ agents where i has at least one friend, there exists a comparison (Γ, Δ) that is friend-dependent but not locally friend-dependent. Consider three players $i, f, x \in N$ with $f \in F_i$ and two coalition structures $\Gamma = \{\{i\}, \{f\}, \{x\}, \dots\}$ (singleton coalitions only) and $\Delta = \{\{i\}, \{f, x\}, \dots\}$ (singleton coalitions except for $\{f, x\}$). Then, by adding x to F_f or deleting x from F_f, i's preference over Γ and Δ changes under all three degrees of extended (min-based) altruism. ❑

Unanimity: Let $\Gamma, \Delta \in \mathcal{C}_N$. We say that $\succeq_i^{ext}$ is *unanimous* if $v_a(\Gamma) > v_a(\Delta)$ for each $a \in F_i \cup \{i\}$ implies $\Gamma \succ_i^{ext} \Delta$.

It is easy to see that, for all three degrees of altruism, the extended (min-based) altruistic preferences fulfill unanimity. This crucially distinguishes our models for altruism in coalition formation games from the models for altruism in hedonic games. As the following example shows, unanimity is not satisfied by hedonic (min-based) EQ and AL preferences.

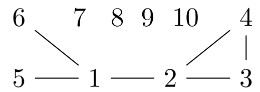


Figure 2: Network of friends for the game in Example 13

Table 2: Relevant values for the game in Example 13 with the network of friends in Figure 2

v_i	1	2	5	6
Γ_1	10	10	0	0
Γ_2	16	20	5	5

	sum_1^{extF}	sum_1^{extF+}	$\min_1^{extF}$	$\min_1^{extF+}$
Γ_1	10	20	0	0
Γ_2	30	46	5	5
	avg_1^{F}	avg_1^{F+}	$\min_1^{F}$	$\min_1^{F+}$
Γ_1	$\frac{10}{1} = 10$	$\frac{10+10}{2} = 10$	10	10
Γ_2	$\frac{5+5}{2} = 5$	$\frac{16+5+5}{3} = 8,\overline{6}$	5	5

Example 13. *Consider the game given by the network of friends in Figure 2 and the coalition structures* $\Gamma_1 = \{\{1,2\},\{3\},\{4\},\dots,\{10\}\}$ *and* $\Gamma_2 = \{\{1,5,\dots,10\},\{2,3,4\}\}$. *Table 2 shows all relevant values that are needed to compute the utilities of agent* 1 *under extended (min-based) altruistic preferences and (min-based) altruistic hedonic preferences.*

One can observe that agent 1 *and all her friends assign a higher value to* Γ_2 *than to* Γ_1. *Hence, it seems reasonable if* 1 *would prefer* Γ_2 *to* Γ_1 *under all degrees of altruism. This is actually the case for our extended (min-based) altruistic preferences which are unanimous.*

However, the hedonic models are blind to the fact that agent 1 *and all her friends are better off in* Γ_2 *than in* Γ_1. *Under the (min-based) altruistic hedonic preferences, only the friends in the current coalition are considered which leads to* 1 *preferring* $\Gamma_1(1) = \{1,2\}$ *to* $\Gamma_2(1) = \{1,5,\dots,10\}$ *under (min-based) hedonic equal-treatment and altruistic-treatment preferences. This shows that all these preferences are not unanimous.*

6 Discussion

We have studied the four models of altruism in (hedonic) coalition formation games and have seen that all four models satisfy some basic properties, namely reflexivity, transitivity, polynomial-time computability, anonymity, sovereignty of players, and symmetry.

Out of the altruistic hedonic models, the average-based model is the only one that fulfills all the properties considered here at least under SF preferences (but not under EQ or AL).

The extended sum-based altruistic model is the strongest concerning its properties. It is the only model that fulfills both types of monotonicity for all degrees of altruism. In contrast to that, both min-based models naturally fail to satisfy type-I-monotonicity: An additional friend might lead to a decrease in utility when taking the minimum over the friends. Both hedonic models further fail to satisfy type-II-monotonicity under EQ and AL.

The extended altruistic models also fulfill a stronger notion of unanimity than the hedonic models do. We consider unanimity to be very important since it is a very natural property: If all considered agents have a unanimous opinion about two coalition structures, it seems very counterintuitive to decide contrary to this opinion. Unfortunately, as we have seen in Example 13, such a situation can occur under altruistic hedonic preferences. Arguably, whenever we are not restricted to hedonic games, it would be reasonable to rather consider the extended altruistic model and seek to increase *all* friends' valuations.

Acknowledgments

This work was supported in part by DFG grants RO-1202/14-2 and RO-1202/21-1. The first author is a member of the PhD-programme "Online Participation," supported by the North Rhine-Westphalian funding scheme "Forschungskollegs."

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Appendix

Omitted Proofs

Below we give some proofs that were previously omitted.

Proposition 3. *Under all three degrees of altruism, any (min-based) altruistic hedonic preference satisfies reflexivity, transitivity, polynomial-time computability, and anonymity.*

Proof. Reflexivity follows immediately from the definition.

Transitivity follows from the fact that the relation $\geq$ is transitive for rational numbers.

Furthermore, each valuation (1) of an agent for a coalition can obviously be computed in polynomial time. Hence, each summand in (3) and (4) (or each value over which the minimum is taken in (8)) can be computed in polynomial time. The number of summands (or values over which the minimum is taken) is bounded by n, which implies that both sums (or minimums) can be computed in polynomial time, which in turn allows to determine the utilities for any two coalitions in polynomial time.

Finally, by renaming the players, the numbers of friends and enemies of the players do not change. Therefore, the calculations of the utilities do not change either, leading to no change in the relation between two coalitions. ❑

Theorem 5. *The preference $\succeq_i^\star$ of agent $i \in N$ is (locally) friend-dependent if and only if*

1. *in case $\star = minSF$, i has at least one friend and there are at least four players in N;*
2. *in case $\star = minEQ$, i has at least two friends (thus there are at least three players);*
3. *in case $\star = minAL$, i has at least one friend and there are at least three players in N.*

Proof. First, similarly as for average-based AHGs (see the proof of Theorem 4), it is not hard to see that every min-based hedonic altruistic preference that is friend-dependent is also locally friend-dependent.

Second, we again show that $\succeq_i \in \{\succeq_i^{minSF}, \succeq_i^{minEQ}, \succeq_i^{minAL}\}$ is friend-dependent, i.e., there exists a pair of coalitions that is friend-dependent under $\succeq_i$, if and only if i has enough friends and N is sufficiently large.

Again, the results for $n \leq 3$ players are not stated in detail as they can easily be verified.

Only if: It is obvious that $|F_i| = 0$ implies that all min-based altruistic preferences are not friend-dependent. Moreover, we consider $\succeq_i^{minEQ}$ for the case that $|F_i| = 1$. Then $v_i(C)$ is the minimum valuation in $u_i^{minEQ}(C)$ for any coalition C because i has at most one friend in C while this friend might have more friends in C. Hence, $\succeq_i^{minEQ}$ coincides with $\succeq_i^F$ and is not friend-dependent.

If: The argumentation for showing that $\succeq_i^{minSF}$ and $\succeq_i^{minAL}$ are friend-dependent if $n \geq 4$ and $|F_i| > 0$ is very similar to the argumentation in the proof of Theorem 4 and is therefore omitted.

We now show that $\succeq_i^{minEQ}$ is friend-dependent if $|F_i| \geq 2$. Assuming $|F_i| \geq 2$, there are $f_1, f_2 \in F_i$. If f_1 and f_2 are friends of each other, then $\{i, f_1, f_2\} \succ_i^{minEQ} \{i, f_1\}$. Otherwise, $\{i, f_1\} \succ_i^{minEQ} \{i, f_1, f_2\}$. Hence, we can change $\succeq_i^{minEQ}$ by changing the friendship relation between f_1 and f_2, which means that $\succeq_i^{minEQ}$ is friend-dependent. ❑

Theorem 9. *Under all three degrees of altruism, extended altruistic preferences $\succeq^{ext}$ are type-I-monotonic and type-II-monotonic.*

Proof. Let $G = (N, H)$ be a network of friends and let $i \in N$, $\Gamma, \Delta \in \mathcal{C}_N$, and $j \in E_i$. We denote with $G' = (N, H \cup \{(i, j)\})$ the network of friends resulting from G when j turns from being i's enemy to being i's friend (all else being equal). Then, for any player $a \in N$ and coalition $\Gamma \in \mathcal{N}^a$, we denote a's value for Γ in G' with $v'_a(\Gamma)$, her preference in G' with $\succeq_a^{ext\prime}$ and her new friend and enemy sets with F'_a and E'_a. Hence, we have $F'_i = F_i \cup \{j\}$, $E'_i = E_i \setminus \{j\}$, $F'_j = F_j \cup \{i\}$, and $E'_j = E_j \setminus \{i\}$. Further, v'_i, v'_j, and $\succeq_i^{ext\prime}$ might differ from v_i, v_j, and $\succeq_i^{ext}$, while the friends, enemies, and values of all other players stay the same, i.e., $F'_a = F_a$, $E'_a = E_a$, and $v'_a = v_a$ for all $a \in N \setminus \{i, j\}$.

Type-I-Monotonicity: Let $j \in \Gamma(i) \cap \Delta(i)$ and $v_j(\Gamma) \geq v_j(\Delta)$. It then holds that $v'_i(\Gamma) = n|\Gamma(i) \cap F'_i| - |\Gamma(i) \cap E'_i| = n|\Gamma(i) \cap F_i| + n - |\Gamma(i) \cap E_i| + 1 = v_i(\Gamma) + n + 1$. Equivalently, $v'_i(\Delta) = v_i(\Delta) + n + 1$, $v'_j(\Gamma) = v_j(\Gamma) + n + 1$, and $v'_j(\Delta) = v_j(\Delta) + n + 1$. Furthermore,

$$\begin{aligned}\mathrm{sum}_i^{extF\prime}(\Gamma) &= \sum_{a \in F'_i} v'_a(\Gamma) = \sum_{a \in F_i \cup \{j\}} v'_a(\Gamma) = \sum_{a \in F_i} v_a(\Gamma) + v'_j(\Gamma) \\ &= \mathrm{sum}_i^{extF}(\Gamma) + v_j(\Gamma) + n + 1 \text{ and} \qquad (14)\\ \mathrm{sum}_i^{extF\prime}(\Delta) &= \mathrm{sum}_i^{extF}(\Delta) + v_j(\Delta) + n + 1. \qquad (15)\end{aligned}$$

(1) Selfish First: If $\Gamma \succ_i^{extSF} \Delta$ then either (i) $v_i(\Gamma) = v_i(\Delta)$ and $\mathrm{sum}_i^{extF}(\Gamma) > \mathrm{sum}_i^{extF}(\Delta)$, or (ii) $v_i(\Gamma) > v_i(\Delta)$. In case (i), $v_i(\Gamma) = v_i(\Delta)$ implies $v'_i(\Gamma) = v'_i(\Delta)$. Applying $\mathrm{sum}_i^{extF}(\Gamma) > \mathrm{sum}_i^{extF}(\Delta)$ and $v_j(\Gamma) \geq v_j(\Delta)$ to (14) and (15), we get $\mathrm{sum}_i^{extF\prime}(\Gamma) > \mathrm{sum}_i^{extF\prime}(\Delta)$. This together with $v'_i(\Gamma) = v'_i(\Delta)$ implies $\Gamma \succ_i^{extSF\prime} \Delta$. In case (ii), $v_i(\Gamma) > v_i(\Delta)$ implies $v'_i(\Gamma) > v'_i(\Delta)$. Hence, $\Gamma \succ_i^{extSF\prime} \Delta$.

If $\Gamma \sim_i^{extSF} \Delta$ then $v_i(\Gamma) = v_i(\Delta)$ and $\mathrm{sum}_i^{extF}(\Gamma) = \mathrm{sum}_i^{extF}(\Delta)$. $v_i(\Gamma) = v_i(\Delta)$ implies $v'_i(\Gamma) = v'_i(\Delta)$. Applying $\mathrm{sum}_i^{extF}(\Gamma) = \mathrm{sum}_i^{extF}(\Delta)$ and $v_j(\Gamma) \geq v_j(\Delta)$ to (14) and (15), we get $\mathrm{sum}_i^{extF\prime}(\Gamma) \geq \mathrm{sum}_i^{extF\prime}(\Delta)$. This together with $v'_i(\Gamma) = v'_i(\Delta)$ implies $\Gamma \succeq_i^{extSF\prime} \Delta$.

(2) Equal Treatment: If $\Gamma \succ_i^{extEQ} \Delta$ then $\mathrm{sum}_i^{extF}(\Gamma) + v_i(\Gamma) > \mathrm{sum}_i^{extF}(\Delta) + v_i(\Delta)$. Using (14), (15), $v'_i(\Gamma) = v_i(\Gamma) + n + 1$, $v'_i(\Delta) = v_i(\Delta) + n + 1$, and $v_j(\Gamma) \geq v_j(\Delta)$, this implies $\mathrm{sum}_i^{extF\prime}(\Gamma) + v'_i(\Gamma) > \mathrm{sum}_i^{extF\prime}(\Delta) + v'_i(\Delta)$. Hence, $\Gamma \succ_i^{extEQ\prime} \Delta$.

If $\Gamma \sim_i^{extSF} \Delta$, using the same equations, $\Gamma \succeq_i^{extEQ\prime} \Delta$ is implied.

(3) Altruistic Treatment: If $\Gamma \succ_i^{extAL} \Delta$ then either (i) $\mathrm{sum}_i^{extF}(\Gamma) = \mathrm{sum}_i^{extF}(\Delta)$ and $v_i(\Gamma) > v_i(\Delta)$, or (ii) $\mathrm{sum}_i^{extF}(\Gamma) > \mathrm{sum}_i^{extF}(\Delta)$. In case (i), $\mathrm{sum}_i^{extF}(\Gamma) = \mathrm{sum}_i^{extF}(\Delta)$ together with (14), (15), and $v_j(\Gamma) \geq v_j(\Delta)$, implies $\mathrm{sum}_i^{extF\prime}(\Gamma) \geq \mathrm{sum}_i^{extF\prime}(\Delta)$. Further, $v_i(\Gamma) > v_i(\Delta)$ together with $v'_i(\Gamma) = v_i(\Gamma) + n + 1$ and $v'_i(\Delta) = v_i(\Delta) + n + 1$ implies $v'_i(\Gamma) > v'_i(\Delta)$. Altogether, this implies $\Gamma \succ_i^{extAL\prime} \Delta$. In case (ii), $\mathrm{sum}_i^{extF\prime}(\Gamma) > \mathrm{sum}_i^{extF\prime}(\Delta)$ is implied and $\Gamma \succ_i^{extAL\prime} \Delta$ follows.

If $\Gamma \sim_i^{extAL} \Delta$ then $\mathrm{sum}_i^{extF}(\Gamma) = \mathrm{sum}_i^{extF}(\Delta)$ and $v_i(\Gamma) = v_i(\Delta)$. Using the same equations as before, $\Gamma \succeq_i^{extAL\prime} \Delta$ is implied.

Type-II-Monotonicity: Let $j \in \Gamma(i) \setminus \Delta(i)$ and $v_j(\Gamma) \geq v_j(\Delta)$. It follows that $v'_i(\Gamma) = v_i(\Gamma) + n + 1$, $v'_i(\Delta) = v_i(\Delta)$, $v'_j(\Gamma) = v_j(\Gamma) + n + 1$, and $v'_j(\Delta) = v_j(\Delta)$. Furthermore,

$$\begin{aligned}\mathrm{sum}_i^{extF\prime}(\Gamma) &= \mathrm{sum}_i^{extF}(\Gamma) + v_j(\Gamma) + n + 1 \text{ and} \qquad (16)\\ \mathrm{sum}_i^{extF\prime}(\Delta) &= \mathrm{sum}_i^{extF}(\Delta) + v_j(\Delta). \qquad (17)\end{aligned}$$

(1) Selfish First: If $\Gamma \succeq_i^{extSF} \Delta$ then $v_i(\Gamma) \geq v_i(\Delta)$. Hence, $v'_i(\Gamma) = v_i(\Gamma) + n + 1 \geq v_i(\Delta) + n + 1 > v_i(\Delta) = v'_i(\Delta)$, implying $\Gamma \succ_i^{extSF\prime} \Delta$.

(2) Equal Treatment: If $\Gamma \succeq_i^{extEQ} \Delta$ then $\mathrm{sum}_i^{extF}(\Gamma) + v_i(\Gamma) \geq \mathrm{sum}_i^{extF}(\Delta) + v_i(\Delta)$. Together with (16), (17), and $v_j(\Gamma) \geq v_j(\Delta)$ this implies $\mathrm{sum}_i^{extF\prime}(\Gamma) + v'_j(\Gamma) > \mathrm{sum}_i^{extF\prime}(\Delta) + v'_j(\Delta)$. Hence, $\Gamma \succ_i^{extEQ\prime} \Delta$.

(3) Altruistic Treatment: If $\Gamma \succeq_i^{extAL} \Delta$ then $\mathrm{sum}_i^{extF}(\Gamma) \geq \mathrm{sum}_i^{extF}(\Delta)$. Together with (16), (17), and $v_j(\Gamma) \geq v_j(\Delta)$ this implies $\mathrm{sum}_i^{extF\prime}(\Gamma) > \mathrm{sum}_i^{extF\prime}(\Delta)$, so $\Gamma \succ_i^{extAL\prime} \Delta$.

This completes the proof. ❑

Theorem 10. *Under all three degrees of altruism, extended min-based altruistic preferences $\succeq^{\min^{ext}}$ are type-II-monotonic but not type-I-monotonic.*

Proof. We use the same notation as in the proof of Theorem 9 and begin with showing that extended min-based preferences are type-II-monotonic under all three degrees of altruism.

Type-II-Monotonicity: Let $j \in \Gamma(i) \setminus \Delta(i)$ and $v_j(\Gamma) \geq v_j(\Delta)$. It then holds that $v_i'(\Gamma) = v_i(\Gamma)+n+1$, $v_i'(\Delta) = v_i(\Delta)$, $v_j'(\Gamma) = v_j(\Gamma)+n+1$, and $v_j'(\Delta) = v_j(\Delta)$. Furthermore,

$$\begin{aligned}
\min_i^{extF\prime}(\Gamma) &= \min_{a \in F_i \cup \{j\}} v_a'(\Gamma) \\
&= \min\bigl(\min_i^{extF}(\Gamma), v_j'(\Gamma)\bigr) \\
&= \min\bigl(\min_i^{extF}(\Gamma), v_j(\Gamma) + n + 1\bigr), \\
\min_i^{extF\prime}(\Delta) &= \min_{a \in F_i \cup \{j\}} v_a'(\Delta) \\
&= \min\bigl(\min_i^{extF}(\Delta), v_j'(\Delta)\bigr) \\
&= \min\bigl(\min_i^{extF}(\Delta), v_j(\Delta)\bigr), \\
\min_i^{extF+\prime}(\Gamma) &= \min\bigl(\min_i^{extF\prime}(\Gamma), v_i'(\Gamma)\bigr) \\
&= \min\bigl(\min_i^{extF}(\Gamma), v_j(\Gamma) + n + 1, v_i(\Gamma) + n + 1\bigr), \text{ and} \\
\min_i^{extF+\prime}(\Delta) &= \min\bigl(\min_i^{extF\prime}(\Delta), v_i'(\Delta)\bigr) \\
&= \min\bigl(\min_i^{extF}(\Delta), v_j(\Delta), v_i(\Delta)\bigr).
\end{aligned}$$

(1) SF: Assume that $\Gamma \succeq_i^{minextSF} \Delta$. Then $v_i(\Gamma) \geq v_i(\Delta)$. Hence, $v_i'(\Gamma) = v_i(\Gamma)+n+1 \geq v_i(\Delta) + n + 1 > v_i(\Delta) = v_i'(\Delta)$, which implies $\Gamma \succ_i^{minextSF\prime} \Delta$.

(2) EQ: First, assume that $\Gamma \succ_i^{minextEQ} \Delta$. Then

$$\min\bigl(\min_i^{extF}(\Gamma), v_i(\Gamma)\bigr) > \min\bigl(\min_i^{extF}(\Delta), v_i(\Delta)\bigr).$$

It follows that

$$\begin{aligned}
\min_i^{extF+\prime}(\Gamma) &= \min\bigl(\min_i^{extF}(\Gamma), v_j(\Gamma) + n + 1, v_i(\Gamma) + n + 1\bigr) \\
&\geq \min\bigl(\min_i^{extF}(\Gamma), v_j(\Gamma) + n + 1, v_i(\Gamma)\bigr) \\
&> \min\bigl(\min_i^{extF}(\Delta), v_j(\Gamma), v_i(\Delta)\bigr) \\
&\geq \min\bigl(\min_i^{extF}(\Delta), v_j(\Delta), v_i(\Delta)\bigr) \\
&= \min_i^{extF+\prime}(\Delta).
\end{aligned} \tag{18}$$

Hence, $\Gamma \succ_i^{minextEQ\prime} \Delta$.

Second, assume that $\Gamma \sim_i^{minextEQ} \Delta$. Then

$$\min\bigl(\min_i^{extF}(\Gamma), v_i(\Gamma)\bigr) = \min\bigl(\min_i^{extF}(\Delta), v_i(\Delta)\bigr).$$

Similarly as in (18), it follows that

$$\min_i^{extF+\prime}(\Gamma) \geq \min_i^{extF+\prime}(\Delta).$$

Hence, $\Gamma \succeq_i^{minextEQ\prime} \Delta$.

(3) AL: First, assume that $\Gamma \succ_i^{minextAL} \Delta$. Then (i) $\min_i^{extF}(\Gamma) > \min_i^{extF}(\Delta)$ or (ii) $\min_i^{extF}(\Gamma) = \min_i^{extF}(\Delta)$ and $v_i(\Gamma) > v_i(\Delta)$.

In case of (i), we get

$$\begin{aligned} \min_i^{extF\prime}(\Gamma) &= \min\bigl(\min_i^{extF}(\Gamma), v_j(\Gamma) + n + 1\bigr) \\ &\geq \min\bigl(\min_i^{extF}(\Gamma), v_j(\Delta) + n + 1\bigr) \\ &> \min\bigl(\min_i^{extF}(\Delta), v_j(\Delta)\bigr) \\ &= \min_i^{extF\prime}(\Delta). \end{aligned} \tag{19}$$

Hence, $\Gamma \succ_i^{minextAL\prime} \Delta$.

In case of (ii), similarly as in (19), we get

$$\min_i^{extF\prime}(\Gamma) \geq \min_i^{extF\prime}(\Delta).$$

Furthermore, $v_i(\Gamma) > v_i(\Delta)$ implies $v_i'(\Gamma) > v_i'(\Delta)$. Hence, $\Gamma \succ_i^{minextAL\prime} \Delta$.

Second, we assume that $\Gamma \sim_i^{minextAL} \Delta$. Then $\min_i^{extF}(\Gamma) = \min_i^{extF}(\Delta)$ and $v_i(\Gamma) = v_i(\Delta)$. Similarly as in (19), we get $\min_i^{extF\prime}(\Gamma) \geq \min_i^{extF\prime}(\Delta)$. Furthermore, $v_i(\Gamma) = v_i(\Delta)$ implies $v_i'(\Gamma) > v_i'(\Delta)$. Hence, $\Gamma \succ_i^{minextAL\prime} \Delta$.

This completes the proof for type-II-monotonicity.

Type-I-Monotonicity: To see that $\succeq^{minextSF}$ is not type-I-monotone, consider the game $\mathcal{G}_4$ with the network of friends in Figure 3a.

(a) Network of $\mathcal{G}_4$ (b) Network of $\mathcal{G}_4'$ (c) Network of $\mathcal{G}_5$ (d) Network of $\mathcal{G}_5'$

Figure 3: Networks of friends in the proof of Theorem 10

Furthermore, consider the coalition structures $\Gamma = \{\{1,2\},\{3,4,5\},\{6\}\}$ and $\Delta = \{\{1,2\},\{3,4,5,6\}\}$ and players $i = 1$ and $j = 2$ with $2 \in \Gamma(1) \cap \Delta(1)$, and $v_2(\Gamma) = -1 = v_2(\Delta)$. It holds that $v_1(\Gamma) = v_1(\Delta) = -1$, $\min_1^{extF}(\Gamma) = 2n$, and $\min_1^{extF}(\Delta) = 2n - 1$. Hence, $\Gamma \succ_1^{minextSF} \Delta$.

Now, making 2 a friend of 1's leads to game $\mathcal{G}_4'$ with the network of friends in Figure 3b. For this game, we have $v_1(\Gamma) = v_1(\Delta) = n$ and $\min_1^{extF}(\Gamma) = \min_1^{extF}(\Delta) = n$. This implies $\Gamma \sim_1^{minextSF} \Delta$ which contradicts type-I-monotonicity.

To see that $\succeq^{minextEQ}$ and $\succeq^{minextAL}$ are not type-I-monotone, consider the game $\mathcal{G}_5$ with the network of friends in Figure 3c. Consider the coalition structures $\Gamma = \{\{1,2,3,4\},\{5\}\}$ and $\Delta = \{\{1,2,3,4,5\}\}$ and players $i = 1$ and $j = 2$ with $2 \in \Gamma(1) \cap \Delta(1)$, and $v_2(\Gamma) = -3 > -4 = v_2(\Delta)$. It holds that $\min_1^{extF+}(\Gamma) = \min_1^{extF}(\Gamma) = 2n - 1$, and $\min_1^{extF+}(\Delta) = \min_1^{extF}(\Delta) = 2n - 2$. Hence, $\Gamma \succ_1^{minextEQ} \Delta$ and $\Gamma \succ_1^{minextAL} \Delta$.

Now, making 2 a friend of 1's leads to game $\mathcal{G}_5'$ with the network of friends in Figure 3d. For this game, we have $\min_1^{extF+}(\Gamma) = \min_1^{extF}(\Gamma) = n$ and $\min_1^{extF+}(\Delta) = \min_1^{extF}(\Delta) = n$. This implies $\Gamma \sim_1^{minextEQ} \Delta$ and $\Gamma \sim_1^{minextAL} \Delta$ which contradict type-I-monotonicity.

This completes the proof for type-I-monotonicity. ❑

Theorem 11. *Under all three degrees of altruism, extended (min-based) altruistic preferences $\succeq_i^{\alpha}$, $\alpha \in \{\mathrm{sum}^{ext}, \min^{ext}\}$, are symmetric.*

Proof. Let $i, j, k \in N$ be three distinct players and $\Gamma \in \mathcal{C}_N, \Gamma(j) \setminus \{j,k\} = \Gamma(k) \setminus \{j,k\}$. Suppose that swapping the positions of j and k in G is an automorphism. Then j is a friend of a player $a \in N \setminus \{j,k\}$ if and only if k is. Therefore, all players $a \in \Gamma(j) \setminus \{j,k\}$ value $\Gamma(j) \setminus \{j\}$ and $\Gamma(j) \setminus \{k\}$ the same and all players $a \in \Gamma(i) \setminus \{j,k\}$ value $\Gamma(i) \cup \{j\}$ and $\Gamma(i) \cup \{k\}$ the same. Further, j values $\Gamma(i) \cup \{j\}$ the same as k values $\Gamma(i) \cup \{k\}$ and j values $\Gamma(j) \setminus \{k\}$ the same as k values $\Gamma(j) \setminus \{j\}$ because j and k have the same friends in $\Gamma(i)$ and $\Gamma(j)$, respectively. Hence, $v_a(\Gamma_{j \to \Gamma(i)}) = v_a(\Gamma_{k \to \Gamma(i)})$ for all $a \in F_i \cup \{i\}$. This implies $\Gamma_{j \to \Gamma(i)} \sim_i^{\alpha SF} \Gamma_{k \to \Gamma(i)}$, $\Gamma_{j \to \Gamma(i)} \sim_i^{\alpha EQ} \Gamma_{k \to \Gamma(i)}$, and $\Gamma_{j \to \Gamma(i)} \sim_i^{\alpha AL} \Gamma_{k \to \Gamma(i)}$. ❑

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