Evaluating Committees for Representative Democracies: the Distortion and Beyond

Michał Jaworski, Piotr Skowron

Abstract

We study a model where a group of representatives is elected to make a series of decisions on behalf of voters. The quality of such a representative committee is judged based on the extent to which the decisions it makes are consistent with the voters' preferences. We assume the set of issues on which the committee will make the decisions is unknown—a committee is elected based on the preferences of the voters over the candidates, which only reflect how similar are the preferences of the voters and candidates regarding the issues. In this model we theoretically and experimentally assess qualities of various multiwinner election rules.

1 Introduction

One of the fundamental goals that the (computational) social choice theory sets to itself is to evolve a set of tools that could help societies to improve the processes of making collective decisions. This includes developing frameworks that allow to formally reason about and to meaningfully compare election rules. Typically, such a comparison is done in a context-free manner—the criteria used for comparing rules are generic, not tailored to a specific scenario where the election rule is to be used (cf., the axiomatic approach Arrow *et al.* [2002]). While such an approach has a natural appeal of generality, various applications differ substantially, and a rule which is considered perfect in one situation may be inappropriate in another one. This motivates comparing rules based on context-specific assumptions. In this paper we focus on indirect democracies—we study a framework for comparing rules for selecting committees (e.g., parliaments, representative boards, etc.), assuming that the elected committee will make a series of decisions on behalf of the society.

Our model can be informally described as follows (the formal definitions are provided in Sections 2 and 3). There are three types of objects: *voters*, *candidates*, and *issues*—an issue is an alternative with two possible outcomes: "yes" or "no". Each voter and each candidate has her preferred outcome for each issue—these preferred outcomes induce the preferences of the voters over the candidates—a voter examines how similar is her opinion on the issues to the opinions of the candidates, and ranks them accordingly[1]. Further, the preferred outcomes can be used to measure and to compare the qualities of committees: for a given committee W we can check what kind of decisions would be made by W if elected (we assume that the elected committee uses the majority rule to decide about each issue) and compare these decision with the voters' preferred outcomes. A committee whose decisions coincides with most preferred outcomes is called *optimal*.

Finding an optimal committee would be possible if we knew all the elements of the model. Yet, often the preferred outcomes of the voters (or even the issue space itself) remain unknown during elections, and winning committees are elected based on the preferences of the voters over the candidates. Since the voters' preference rankings only reflect the

[1] The candidates' preferences are usually publicly known—e.g., politicians talk about their preferences over various issues during election campaigns. The voters know their own preferences and so they are able to construct their preferences over the candidates.

similarities between theirs and the candidates' preferred outcomes for the issues, it is not surprising that an election rule might not be able to choose an optimal committee since it has only access to partial information. In this paper we quantify this effect and we ask what is the loss of utility that a committee election rule can cause due to not knowing the issue space.

Our methodology and contribution are the following:

1. In Section 4 we assess the worst-case loss of utility of multiwinner rules—by an analogy to the literature Procaccia and Rosenschein [2006] we call this worst-case loss of utility the *distortion*. We prove that the distortion of any ordinal voting rule $\mathcal{R}$ equals to infinity. Thus, in the most general case, it is inevitable that an error made when selecting a committee based on the preferences of the voters over the candidates, can be arbitrarily bad.

2. Due to the aforementioned negative result, we further focus our theoretical analysis on a specific domain restriction inspired by works from political science on polarized division of ideologies. Under this domain restriction we assume that the societies are centered around two poles, i.e., that there are only two types of preferred outcomes that the voters and the candidates can have. It is quite surprising that already under this seemingly strong assumption the distortion of many known rules, such as SNTV, k-Borda, the Chamberlin-Courant rule, and the Monroe rule can be arbitrary bad once the parameters, such as the number of voters or candidates, are large. On the other hand, in this case the distortion of STV and k-Copeland are constant. In fact, the k-Copeland rule always selects an optimal committee when the societies are polarized.

3. Our negative results show certain limitations of the worst-case analysis when applied to the voting committee model. Consequently, the major (and the main) part of our study aims at understanding the *average* behavior of voting rules. To this end, we performed extensive computer simulations for several natural distribution of individuals' preferred outcomes. In particular, our distributions generalize and extend the polarized model, as described above.

If we judge the rules from the majoritarian perspective (a view that focuses solely on maximizing the total voters' satisfaction), our results show a dichotomy. In the two most extreme cases: (i) when a society is strongly polarized, or (ii) when the opinions with respect to various issues are uniformly and independently distributed in a society, multiwinner rules based on the Condorcet criterion, such as k-Copeland, are the best: this is confirmed both by our experiments and by the theoretical results. On the other hand, when the number of predominant views in a society is larger than two, then proportional rules such as STV select committees whose majoritarian decisions particularly well reflect the opinions of the voters (this is somehow unexpected, given the majoritarian approach to judging the rules). If we use a more proportional metric to assess qualities of voting rules (a metric that puts more emphasis on reducing the societal inequality), then the proportional rules such as STV are always superior.

Related Work

There is a vast literature that considers committees making series of decisions Young [1995]; Feddersen and Pesendorfer [1997, 1998]; Magdon-Ismail and Xia [2018]—this literature dates back to the 18th century, when Condorcet formulated his famous Jury Theorem. However, all these aforementioned works use the assumption that there exists a ground truth, and so, that each decision made by a committee is either objectively correct or wrong. Our model is different—here, the quality of a committee can be judged only with respect to the voters—a good committee is the one that well-represents (subjective) preferences of the society.

Scenarios where a committee makes a sequence of binary decisions were also considered in the context of Colonel Blotto games Laslier and Picard [2002], storable votes Casella [2005, 2011], and voting in multi-attribute domains Brams *et al.* [1998]; Lacy and Niou [2000]; Xia *et al.* [2008, 2010]. None of these works, however, considers an indirect process of decision-making.

The voting committee model that we use in this paper was first introduced by Skowron Skowron [2015]; there, the author argued for the optimality of certain committee election rules, but under strong assumptions on the form of the voters' utility functions. The definition of the concept of the distortion in the voting committee model and its analysis is new to this paper.

Our model is closely related to the one by Koriyama et al. Koriyama *et al.* [2013]. The difference is that Koriyama et al. consider the apportionment problem, i.e., a scenario, where the task is to divide a fixed number of parliamentary seats among political parties according to how the population votes—in our model, on the other hand, the voters vote for individual candidates rather than for political parties.

Recently, the ideas behind proxy voting Miller [1969]; Green-Armytage [2015]; Cohensius *et al.* [2017] and liquid democracy Behrens *et al.* [2014]; Brill and Talmon [2018]; Kahng *et al.* [2018] have attained a considerable attention in the literature. These works consider scenarios where a sequence of decisions on certain issues is to be made through a referendum, but the voters are allowed to transfer their voting rights regarding selected issues to others; thus, for each issue an implicit committee is elected that (for this particular issue) votes on behalf of the whole population of the voters.

Our results allow to formulate conclusions saying how suitable are certain voting rules for electing representative committees. Thus, our research contributes to the vast literature that aims at comparing multi-winner election rules. For an overview of this literature we refer the reader to the paper by Elkind *et al.* [2017], and to the recent book chapter by Faliszewski *et al.* [2017].

The main measure that we use to compare voting rules in this paper is inspired by the popular concept of distortion (see, e.g., Procaccia and Rosenschein [2006]; Caragiannis and Procaccia [2011]; Boutilier *et al.* [2015]; Caragiannis *et al.* [2017]; Anshelevich *et al.* [2018]; Goel *et al.* [2017]; Pierczynski and Skowron [2019]). In these works, however, the unknown primitives are the utilities of the voters over the candidates, and the goal is to compare voting rules that do not have access to the utilities but only to the rankings or approval ballots that are consistent with these utilities. We study a more complex model where the utilities of the voters from the committees are inferred from the decisions made by these committees.

2 Preliminaries

For each $n \in \mathbb{N}$ by $[n]$ we denote the set $\{1, \ldots, n\}$. For a set X we use $S_k(X)$ to denote the set of all k-element subsets of X; by $S(X)$ we denote the set of nonempty subsets of X, i.e., $S(X) = \bigcup_{k \in [|X|]} S_k(X)$. For a logical expression P the term $\mathbb{1}_P$ means 1 if P is true and 0 otherwise.

2.1 Elections, Preferences

Given a set of candidates C we call the elements of $S_k(C)$ size-k *committees* (or simply committees, when k is clear).

A *multiwinner election* (or, in short, an *election*) is a triplet $E = (C, V, k)$, where $C = \{c_1, \dots, c_m\}$ is a set of *candidates*, $V = \{v_1, \dots, v_n\}$ is a set of *voters* and k is an integer representing the intended size of the committee to be elected. We will typically use n and m to represent the numbers of voters and candidates, respectively. We call the elements of $C \cup V$ (the voters and the candidates) *individuals*. For a voter $v_i \in V$ by $\succ_i$ we denote the preference relation of v_i, which is a linear order over C; for example, if v_i prefers a over b, then we write $a \succ_i b$. By $\mathrm{pos}_i(c)$ we denote the position of candidate c in the v_i's preference ranking; for instance, $\mathrm{pos}_i(c) = 1$ if c is v_i's most favorite candidate and $\mathrm{pos}_i(c) = m$ when c is the least preferred candidate for v_i.

A *voting rule* $\mathcal{R}$ is a function that for each election $E = (C, V, k)$ returns a nonempty set of size-k committees, i.e., $\mathcal{R}(E) \in S(S_k(C))$. Throughout the paper we use the parallel-universes tie-breaking mechanism that allows us to obtain all committees that could possibly be built by means of the selected voting rules (see Conitzer *et al.* [2009]).

2.2 Overview of Selected Voting Rules

Below we provide formal definitions of the voting rules that we study in this paper. First, though, let us recall the definitions of the *Plurality* and *Borda* scores. The Plurality score that a voter v_i assigns to a candidate c, denoted by $\mathrm{sc}_P(v_i, c)$, equals 1 if $\mathrm{pos}_i(c) = 1$ and 0, otherwise. The Borda score that c gets from v_i is $\mathrm{sc}_B(v_i, c) = m - \mathrm{pos}_i(c)$. Given an election $E = (C, V, k)$, the Plurality score of a candidate c is the sum of the Plurality scores that c garners from all the voters, $\mathrm{sc}_P(c) = \sum_{v_i \in V} \mathrm{sc}_P(v_i, c)$. Analogously we define the Borda score: $\mathrm{sc}_B(c) = \sum_{v_i \in V} \mathrm{sc}_B(v_i, c)$.

Single nontransferable vote (SNTV). The SNTV rule returns k candidates with the highest Plurality scores, i.e., candidates that are ranked first by the most voters.

k-Borda. It picks k candidates with highest Borda scores.

k-Copeland. The Copeland score of a candidate c is the number of candidates $c' \neq c$ such that c is preferred to c' by a majority of voters. The k-Copeland rule selects k candidates with the highest Copeland scores.

Single transferable vote (STV). Let $q = \left\lfloor \frac{n}{k} \right\rfloor$. STV is an iterative procedure that works as follows. In each iteration we check if there is a candidate with the Plurality score of at least q. If such a candidate exists, call it c, then we:

a) add c to the committee;

b) delete some q voters that rank c first, and

c) delete c from the rankings of the voters, i.e., the candidates that are ranked below c move one position up.

On the other hand, if all the candidates have their Plurality scores lower than q, then we delete the candidate with the lowest Plurality score from each voter's ranking.

The procedure repeats until k candidates are selected.

Chamberlin–Courant (CC). We call a function $\phi : V \to C$ *k-assignment* if $|\phi(V)| = |\{\phi(v) \mid v \in V\}| \leq k$. CC computes the assignment ϕ that maximizes $\sum_{v_i \in V} \mathrm{sc}_B(v_i, \phi(v_i))$ and returns $\phi(V)$ as the winning committee. If $|\phi(V)| < k$, then CC picks $k - |\phi(V)|$ arbitrary candidates to fill the missing slots in the committee.

Intuitively, a k-assignment specifies for each voter v_i, who is v_i's representative in the elected committee; CC finds the committee and the corresponding assignment so that the sum of the voters' happiness from their representatives (measured through Borda scores) is maximized.

Monroe. It works as CC, but puts an additional constraint on assignment functions—each candidate from the set $\phi(V)$ must represent roughly the same number of voters, i.e., $|\phi^{-1}(c)| \in \left\{\left\lfloor \frac{n}{k} \right\rfloor, \left\lceil \frac{n}{k} \right\rceil\right\}$ for each $c \in \phi(V)$. As for CC, Monroe first computes the (balanced) assignment Φ that maximizes $\sum_{v_i \in V} \mathrm{sc}_B(v_i, \phi(v_i))$, and then returns the committee implicitly induced by ϕ.

Greedy–Monroe. This is an iterative procedure. It starts with two empty sets V_0 and W_0. In the i-th iteration it chooses a pair—a candidate $c_i \in C \setminus W_{i-1}$ and subset of voters $V' \subseteq V \setminus V_{i-1}$ with $|V'| \in \left\{\left\lfloor \frac{n}{k} \right\rfloor, \left\lceil \frac{n}{k} \right\rceil\right\}$—that maximizes $\sum_{v_j \in V'} \mathrm{sc}_B(v_j, c_i)$. The rule updates the two sets, $W_i = W_{i-1} \cup \{c_i\}$ and $V_i = V_{i-1} \cup V'$. It stops after k iterations and returns W_k as the winning committee.

3 The Voting Committee Model

An *indirect election* is a quadruple (C, V, I, k), where C and V are sets of candidates and voters, respectively, k is a size of a committee to be elected, and $I = (I_1, \ldots, I_p)$ is a vector of p issues. Each individual $i \in V \cup C$ is represented as a p-dimensional binary vector $(i[j])_{j \in [p]}$, indicating her preferences over the issues—for each $j \in [p]$ we set $i[j] = 1$ if individual i is *for* issue I_j and $i[j] = 0$ if i is *against* issue I_j.

In indirect elections, similarly as in elections, the voters have preferences over the candidates. These preferences are consistent with the preferences of individuals over the issues—a voter v ranks the candidates according to the number of issues for which their preferences coincide. Formally, for each $v_i \in V$ and $c, c' \in C$ we have that $c \succ_i c'$ only if:

$$|\{j \in [p] \mid v_i[j] = c[j]\}| \geq |\{j \in [p] \mid v_i[j] = c'[j]\}|.$$

(if for a given voter v there are multiple candidates with the same numbers of issues on which they agree with v, the voter ranks them in an arbitrary order; intuitively, the voter breaks ties between these candidates according to the preferences over issues of secondary importance, which are not part of the model, according to her personal taste, or simply randomly). Consequently, each instance of indirect elections can be interpreted as a simple election, and so the voting rules from Section 2.2 naturally apply to indirect elections.

To assess the quality of a committee W, we assume that W uses the majority voting to make the decisions regarding the issues. Formally, we define the *decision vector* of W as a binary vector $(W[j])_{j \in [p]}$, where $W[j] = 1$ if $|\{c \in W \mid c[j] = 1\}| > \frac{k}{2}$ and $W[j] = 0$ if $|\{c \in W \mid c[j] = 0\}| > \frac{k}{2}$. In order to avoid tie-breaking issues, hereinafter we assume that the size of the committee k is odd. Naturally, since the voters have preferences over the issues, the decision made by the committee W has an influence on voters' satisfaction. Here, we use perhaps the simplest measure, and define the utility of the voter v_i from a committee W as the number of issues for which the committee's decision coincide with v_i's preferences: $u_i(W) = \sum_{j=1}^{p} \mathbb{1}_{W[j] = v_i[j]}$. The notion of the utility of an individual voter gives us a basis to define the utility of the whole set of voters; we use the following measures:

1. In the utilitarian approach we define the voters' utility from a committee W simply as: $u(W) = \sum_{i=1}^{n} u_i(W)$.
2. In the proportional approach, we use an aggregation based on Nash Welfare: $u(W) = \sum_{i=1}^{n} \log(1 + u_i(W))$. Using the logarithm in the aggregation function is grounded in the broad literature on fair allocation (see, e.g. Conitzer *et al.* [2017]; Caragiannis *et al.* [2016]); intuitively, this metric puts less weight to the total satisfaction of the voters, and more to how the satisfaction is distributed among the voters (promoting reducing the societal inequality).

We will also use the concept of disutility of the voter, $du_i(W)$—it is defined as the number of issues on which the voter disagree with the decisions of the committee. Analogously, for each committee W we define the utilitarian disutility of W as $du(W) = \sum_{i=1}^{n} du_i(W)$. We will not be using proportional aggregations of the disutilities.[2]

4 The Distortion: the Worst-Case Approach

In this section we employ the worst-case approach, formalized through the concept of *distortion*. Informally speaking, the distortion of a rule quantifies the worst-case loss of the utility being the effect of the rule not having access to all the information—here, our rules do not have access to information about issues, but rather choose the winning committees based on the preferences of the voters over the candidates.

This section focuses on the utilitarian distortion; even for this simple aggregation our results are mainly negative. Their primary purpose is to illustrate the limitations of the worst-case analysis when applied to the voting committee model.

Definition 1 (Distortion)**.** *The satisfaction-based distortion of a voting rule $\mathcal{R}$ wrt. a set of indirect elections $\mathcal{E}$ is:*

$$dist^{\mathrm{sat}}_{\{\mathcal{R},\mathcal{E}\}} = \sup_{E \in \mathcal{E}} \max_{W \in \mathcal{R}(E)} \frac{u(OPT(E))}{u(W)},$$

where $OPT(E)$ is a committee with the maximal utility in E, i.e., $OPT(E) \in argmax_{W \in S_k(C)} u(W)$. Analogously, we define the dissatisfaction-based distortion of $\mathcal{R}$ as:

$$dist^{\mathrm{dis}}_{\{\mathcal{R},\mathcal{E}\}} = \sup_{E \in \mathcal{E}} \max_{W \in \mathcal{R}(E)} \frac{du(W)}{du(OPT(E))},$$

In the above definition we take the convention that $0/0 = 1$.

In Definition 1 the distortion is parameterized with a set of instances; this allows us, e.g., to explain how the distortion depends on certain structural properties of voters' preferences. Usually, the considered set of instances will be clear from the context—in such cases we will write $\mathrm{dist}^{\mathrm{sat}}_{\mathcal{R}}$ (resp., $\mathrm{dist}^{\mathrm{dis}}_{\mathcal{R}}$) as an abbreviation for $\mathrm{dist}^{\mathrm{sat}}_{\{\mathcal{R},\mathcal{E}\}}$ (resp., $\mathrm{dist}^{\mathrm{dis}}_{\{\mathcal{R},\mathcal{E}\}}$).

For $k = 1$ the dissatisfaction-based distortion boils down to the well-known concept of the metric distortion Anshelevich *et al.* [2018] in pseudo-metric spaces where the voters and candidates are represented as vertices of hypercubes.

[2]We are not aware of any formula for aggregating disutilities that would share strong fairness properties of the Nash Welfare.

4.1 Universal Hardness of Distortion

We start our analysis by providing a negative result which says that in general, the distortion of any voting rule is arbitrarily bad. On the one hand, this result illustrates the possible inefficiency of the process of making decisions through a representative body, specifically if the committee is elected based on the voters' preferences over the candidates. On the other hand, it justifies introducing additional assumptions to the model—in the subsequent sections we will consider certain restricted (but realistic) structures of voters' preferences.

Theorem 1 (Universal Hardness Theorem)**.** *For each voting rule $\mathcal{R}$ and for each committee size $k \geq 3$:*

1. *the satisfaction-based distortion of $\mathcal{R}$ equals to ∞.*
2. *the dissatisfaction-based distortion of $\mathcal{R}$ equals to ∞.*
3. *there exists an instance E where $du(W)/du(OPT(E)) = \infty$, and $u(OPT(E))/u(W) \geq (k+1)/2 - \varepsilon$, for each $\varepsilon > 0$.*

Proof. We first provide the proof of (3).

Fix a rule $\mathcal{R}$, a committee size k, and $m = k + \frac{k+1}{2}$. Assume that for a single voter, with the preference ranking $c_1 \succ c_2, \ldots, c_m$, the rule picks the committee $W = \{c_{i_1}, \ldots, c_{i_k}\}$.

Fix a constant L and an instance E with $p = L \cdot \frac{k+1}{2}$ issues. We have one voter represented as the vector of p ones, and the following three classes of candidates:

1. C_1: in this class we have $\frac{k-1}{2}$ candidates, each represented as the vector of p ones;
2. C_2: this class contains $\frac{k+1}{2}$ candidates. The i-th candidate is represented as a vector of $(i-1)L$ zeros, followed by L ones, followed by $(\frac{k+1}{2} - i)L$ zeros.
3. C_3: here we have $\frac{k+1}{2}$ candidates, each represented by L ones followed by $(\frac{k+1}{2} - 1)L$ zeros.

Each candidate in $C_2 \cup C_3$ is equally liked by the voter (their preferences coincide for L issues). Now, we slightly perturb the instance, to enforce that the voter will use a particular tie-breaking for the candidates in $C_2 \cup C_3$. For that, for each candidate in $C_2 \cup C_3$, we change at most $k+1$ zeros to ones. We do that in a way that the candidates from C_3 will be put in the voter's preference ranking in the positions from $\{i_1, \ldots, i_k\}$. We give the names to the candidates so that the candidates ranked in the positions from $\{i_1, \ldots, i_k\}$ are $c_{i_1}, \ldots, c_{i_k}$, and the remaining candidates have names from $\{c_1, \ldots, c_m\} \setminus W$.

For this instance, the rule picks a committee containing C_3. This committee will make the decision "one" for at most $L + k + 1$ issues. On the other hand, the committee $C_1 \cup C_2$ would make the decision "one" for all p issues. Thus, we have that:

$$\frac{du(W)}{du(\mathrm{OPT}(E))} = \infty \quad \text{and} \quad \frac{u(\mathrm{OPT}(E))}{u(W)} \geq \frac{p}{L+k+1}.$$

For large L, we have that $u(\mathrm{OPT}(E))/u(W) \geq (k+1)/2 - \varepsilon$.

We now move to the case of (1). Statement (2) follows from (3).

Consider a set of the instances $\mathcal{E}$ such that for each $E \in \mathcal{E}$ we have three issues $I = (I_1, I_2, I_3)$, a single voter v, represented by a vector $v = (1,1,1)$, and five candidates $C = \{c_1, c_2, c_3, c_4, c_5\}$, where $c_1 = (1,0,0)$, $c_2 = (0,1,0)$, $c_3 = (0,0,1)$, $c_4 = (0,0,1)$, $c_5 = (0,0,1)$; the size of the committee is $k = 3$. For each candidate $c \in C$ there is exactly one issue for which c has the same preferred outcome as v. Hence, we can arrange the candidates in the v's preference ranking in any possible way, i.e., we can consider all the permutations of the candidates in the ranking—each such a permutation corresponds to one instance from $\mathcal{E}$. One can observe that in our case the highest possible satisfaction of the voter v equals to *one* and it is the result of the decision $(0, 0, 1)$, which, for example, is made by $W = \{c_3, c_4, c_5\}$.

Let $\mathcal{R}$ be a voting rule. Assume that for an instance with a single voter and five candidates, $\mathcal{R}$ returns a committee that consists of the candidates that are ranked at the i-th, j-th and k-th positions in the v's preference ranking. Consider an instance E such that candidates c_1, c_2, c_3 are ranked at the i-th,j-th and k-th place by v. Hence, one of the committees chosen by $\mathcal{R}$ would be $\widetilde{W} = \{c_1, c_2, c_3\}$—this committee makes the decision $(0, 0, 0)$ and so $u(W) = 0$. Hence, we have $\max_{W \in \mathcal{R}(E)} \frac{u(\mathrm{OPT}(E))}{u(W)} = \infty$; and therefore, $\mathrm{dist}_{\mathcal{R}} = \infty$. $\square$

4.2 Distortion for Polarized Society

We next study a model where the society is centered around two points in the issue space. Formally, we say that an indirect election (C, V, I, k) is centered around two poles if there exists two binary p-dimensional vectors, p_1 and p_2, such that each individual from $V \cup C$ is represented as either p_1 or p_2. When determining the distortion of a rule $\mathcal{R}$ for two poles, we assume that the set of instances $\mathcal{E}$ from Definition 1 consists of all elections that are centered around two poles.

Throughout this section, without loss of generality, we assume that the two poles are the vectors of *zeros* and *ones*, i.e. $p_1 = (0, \dots, 0)$ and $p_2 = (1, \dots, 1)$—hereinafter, we refer to p_1 and p_2 as the *first pole* and the *second pole*, respectively. By V_1 and V_2 we denote the sets of voters that are associated with, respectively, the first pole and second pole. Similarly, by C_1 and C_2 we will refer to the sets of candidates coming from the first and second pole. There are two types of decisions that can be made by a committee: *decision zeros* $((0, \dots, 0))$ and *decision ones* $((1, \dots, 1))$. Without loss of generality, we assume that $|C_1|, |C_2| \geq \frac{k+1}{2}$ as we want to make both decisions possible; otherwise the satisfaction-based distortion equals to 1. Further, we can also assume that $|V_2| > |V_1| > 0$.

Due to space constraints, here we only present our results very briefly. We summarize them in the following theorem:

Theorem 2. *For societies centered around two poles, the satisfaction-based distortion:*

1. *of the k-Copeland rule equals to* 1*.*
2. *of STV is 3, but for large $^n/_k$ and k it approaches 1.*
3. *of SNTV and CC is $\Theta(n)$, of k-Borda is $\Theta(m/k)$, of Monroe is $\Omega(n)$ even if $k = 3$, and of greedy-Monroe is $\Omega(k)$.*

Proof. We include the proof for k-Copeland. The proofs for other rules are given in the appendix. Let us start with k-Copeland.

Let us consider an arbitrary instance where the societies are centered around two poles such that $|V_1| < |V_2|$ and $|C_1|, |C_2| \geq \frac{k+1}{2}$. One can observe that due to the block preferences of voters, i.e. voters from V_1 prefer each candidates from C_1 over each candidate from C_2 (analogous observation occurs in case of voters from V_2), each candidate from C_2 wins a pairwise election against all of the candidates from C_1. Therefore, the worst candidate from C_2 wins at least $|C_1|$ pairwise elections. On the other hand, the best candidate from C_1 can win only against all of the remaining candidates from C_1. Hence, the score of the best candidate from C_1 equals to $|C_1| - 1$. As we consider the instance where $|C_2| \geq \frac{k+1}{2}$, in the winning committee W we include at least $\frac{k+1}{2}$ candidates from C_2 and the decision would be in favor of the voters from V_2 and as a results the satisfaction-based distortion is 1. $\square$

5 Beyond the Worst-Case Analysis

Our theoretical results illuminate particular limitations of the worst-case analysis when applied to the voting committee model. Already Theorem 1 provides a certain barrier to deriving meaningful conclusions that would allow to reason about voting rules and that would apply more broadly. Further, even under a seemingly strong assumption of the society being completely polarized the distortion of some known voting rules is arbitrarily bad in limit. This motivates us to extend our analysis beyond the worst-case and to assess the average qualities of the studied rules for certain natural distributions.

In this section we consider a probabilistic model—we assume that the preferences of the individuals are drawn randomly from a given distribution. For each voting rule $\mathcal{R}$ we run a series of experiments in order to assess the qualities of the outcomes produced by $\mathcal{R}$ for elections drawn from several different distributions. In our experiments we focussed on instances with $n = 500$ voters, $m = 100$ candidates, and where the total number of issues is $p = 100$. We consider the following distributions of individuals' preferences:

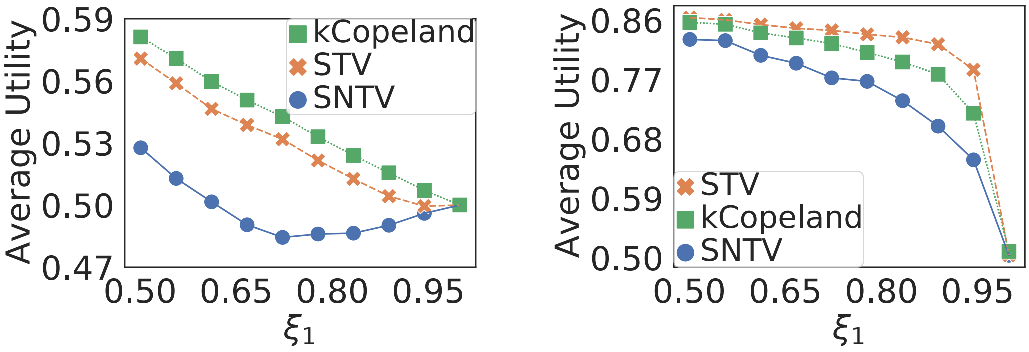
Impartial Culture. For each individual i and issue I_j the i's preferred outcome on I_j is drawn uniformly at random.

(ξ_1, ξ_2)-Polarized Balanced Society ((ξ_1, ξ_2)-PBS). Here we assume that the individuals come from two equal-size groups—each containing 250 voters and 50 candidates. For each group we choose a vector of preferred outcomes, which we call the center of the group. Specifically, the center of the first group is the vector with all 'ones'; for the second group this is the vector of all 'zeros'. For each individual i from group $t \in \{1, 2\}$ we sample a value $\xi(i)$ uniformly at random from $[\xi_t, 1]$. Intuitively, $\xi(i)$ describes how close is i (with respect to her preferences) to the center of his or her group. Formally, for each issue I_j the probability that i's preferred outcome for I_j is consistent with the center of her group equals to $\xi(i)$.

(ξ_1, ξ_2)-Polarized Imbalanced Society ((ξ_1, ξ_2)-PIS). This model is similar to (ξ_1, ξ_2)-PBS; the difference is that the two groups do not have equal sizes. There are 150 voters and 50 candidates in the first group; the second group contains 350 voters and 50 candidates.

(t, ξ)-Poles. In the (t, ξ)-Poles distribution the society is divided into t groups, the sizes of which are derived as follows: We first sample t integers, $x_1, x_2, \ldots, x_t$, i.i.d., uniformly at random from $[0, 1]$. Next, for each $i \in [t]$ we set the size of the i-th group—both the number of candidates and the number of voters—to be proportional to x_i. The total number of voters and candidates must be equal to 500 and 100, respectively; we round up or down the number of individuals within each group when necessary. Second, we sample t central points for the groups—each central point is a p-dimensional binary vector whose coordinates were sampled independently, uniformly at random from $\{0, 1\}$. Finally, for each group $z \in [t]$ and each individual i from the z-th group we derive the preferred outcomes of i as follows. First, we sampled $\xi(i)$ uniformly at random from $[\xi, 1]$. For each issue I_j the individual i will have a preferred outcome consistent with the central point of its group (i.e., equal to the j-th coordinate of the central point) with probability equal to $\xi(i)$.

In our simulations we covered a wide range of possible values of the parameters of the distributions (we give details below). For each distribution with a fixed set of parameters we ran 500 experiments; each experiment was performed as follows. We drew an indirect election and computed the winning committees according to different voting rules. For each such a committee W we computed the vector of decisions made by W, compared these



(a) utilitarian aggregation

(b) proportional aggregation

Figure 1: Comparison of voting rules for elections drawn from the (ξ_1, ξ_2)-PBS for $\xi_1 = \xi_2$ and $k = 31$. The plots depict the average (normalized) utility of committees returned by different voting rules.

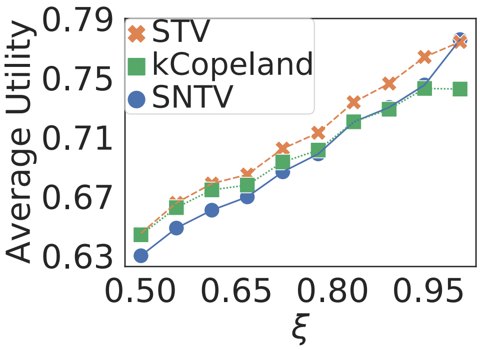
decisions to the preferred outcomes of the voters, and calculated the utility value $u(W)$. We normalized these values, dividing them by $n \cdot p$—for the utilitarian aggregation, or by $n \cdot \log(1 + p)$—for the proportional aggregation (the best possible utility that would be obtained if each voter were perfectly satisfied with every decision made by the committee; an alternative approach would be to divide $u(W)$ by the utility of the optimal committee—cf., Definition 1—but finding such a committee is NP-hard), and we computed the average of these normalized utilities over the 500 runs.

For the sake of clarity, in all the figures in this section we plot only the results for STV, k-Copeland, and SNTV. The plots for greedy Monroe are almost indistinguishable from the plots for STV; similarly, the plots for k-Borda are almost the same as the plots for k-Copeland.

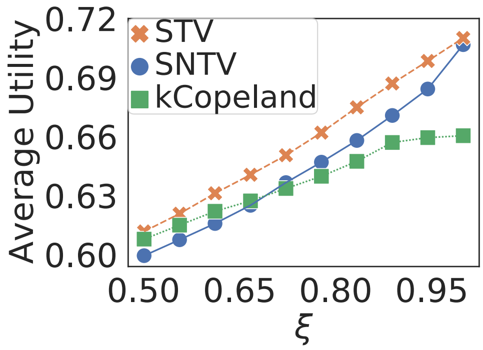
The results of our simulations for (ξ_1, ξ_2)-PBS for $\xi_1 = \xi_2$ and $k = 31$ (recall that we use an odd size of the committee to avoid tie-breaking in the decision-making process) are depicted in Figure 1. The results for (ξ_1, ξ_2)-PBS with fixed $\xi_2 = 0.85$ and ξ_1 ranging from 0.5 to 1, and for (ξ_1, ξ_2)-PIS with $\xi_1 = \xi_2$ lead to the same conclusion, so we do not present the corresponding plots. Further, for all these distributions we also run the experiments for $k = 11$, and obtained results consistent with the ones presented in Figure 1. We also do not include plots for the Impartial Culture model since the results obtained in this model were similar to those obtained for (ξ_1, ξ_2)-PBS when values of ξ_1 and ξ_2 were close to $1/2$.

The results of our experiments performed for the (t, ξ)-Poles distribution are summarized in Figure 2. Due to space constraints, we only present the results for $t \in \{3, 5\}$ and $k = 31$, but we also performed experiments for $t = 10$ and $k = 11$; the results are consistent with those presented.

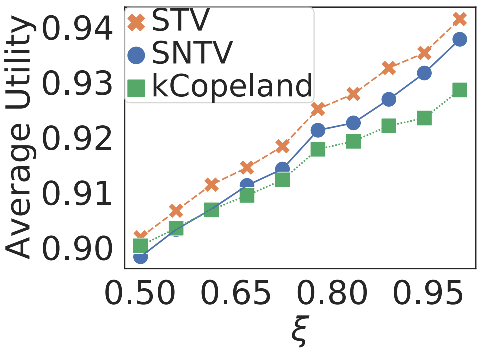
Our interpretations of the results are as follows: For the utilitarian aggregation we observe a dichotomy. In the two most extreme cases: (i) when the society is strongly polarized, or (ii) when the opinions in the society are uniformly distributed with no correlations (the Impartial Culture model), k-Copeland is the best, and performs better than the proportional rules; this conclusion is consistent with our theoretical results (see Theorem 2). SNTV is the worst out of the considered rules. For the proportional aggregation of the utilities the proportional rules, specifically STV, perform better, even when the society is strongly polarized. On the other hand, for societies with more than two predominant opinions, or when the predominant opinions are less extreme, the proportional rules perform better, even



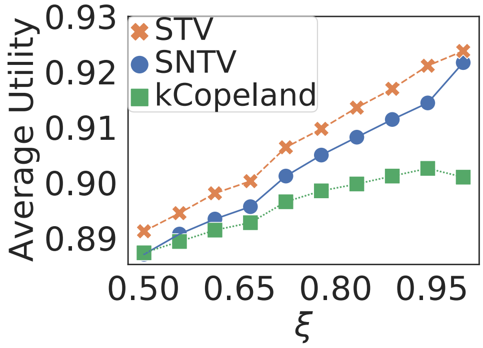
(a) utilitarian aggr., $t = 3$



(b) utilitarian aggr., $t = 5$

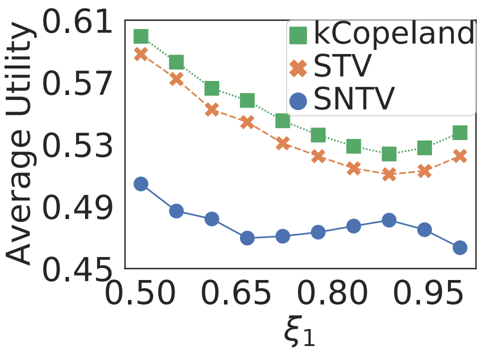


(c) proportional aggr., $t = 3$

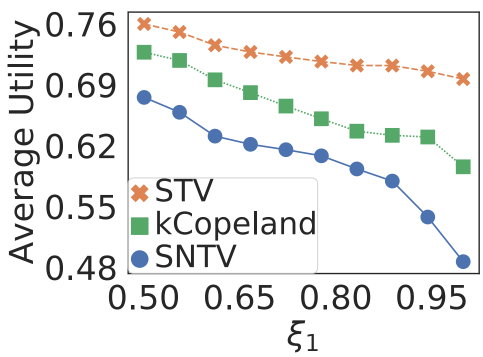


(d) proportional aggr., $t = 5$

Figure 2: Average utilities of voting rules for elections drawn from the (t, ξ)-Poles distribution. The committee size is $k = 31$.



(a) utilitarian aggregation



(b) proportional aggregation

Figure 3: Average qualities of voting rules for elections drawn from (ξ_1, ξ_2)-PBS ($\xi_2 = 0.85$) for the binary distribution of weights.

for the utilitarian aggregation of the voters' satisfactions; this effect is yet magnified when we look at the proportional aggregation of the utilities.

Finally, since in real-life it is often the case that the voters consider some issues more important than the others, we extended our experimental setting in order to capture this intuition. For each voter i and each issue I_j we assume that there is a weight $w_{i,j}$ that measures how important voter i perceives issue I_j. The definition of the utility that a voter i assigns to a committee W changes accordingly (cf., Section 3):

$$u_i(W) = \textstyle\sum_{j=1}^{p} w_{i,j} \cdot \mathbb{1}_{W[j]=v_i[j]}.$$

We consider three different distributions of weights:

Uniform. For each voter and each issue the weight is sampled uniformly at random from $[0, 1]$.

Exponential. We sample the weight a voter assigns to an issue from the exponential distribution with $\lambda = 1$.

Binary. For each voter we randomly pick 10% of issues that she considers important, and for these issues we set the weight to 1. The weights for the remaining issues are 0.

The results of our simulations for (ξ_1, ξ_2)-PBS for the binary distribution of weights for $k = 31$ are depicted in Figure 3—we do not include plots for the remaining distributions, as the results are very similar. Our main conclusion is that adding weights does not change the overall picture—the rules that performed well (resp., badly) in settings without weights still perform well (resp., badly) in the same settings with weights (independently of the distribution from which the weights are drawn).

6 Conclusion

We studied a model, where the voters and the candidates have preferences over a certain set of issues. The preferences of the voters over the candidates are inducted by the individuals' preferred outcomes for the issues. We assumed that the issue space is unknown and the selection of the winning committee is purely based on the preferences of the voters over the candidates. We measured the quality of committees by looking at the majoritarian decisions made by them and by comparing those decisions with the voters' preferred outcomes.

In the most general case, the distortion of any voting rule can be arbitrarily bad. This motivated us to look at special cases inspired by works from political science, where the voters' preferences have certain structure. If the society is extremely polarized or when there is no predominant view in the society, then k-Copeland is the best according to the majoritarian perspective. If we look at the proportional aggregation of voters' utilities, or in the intermediate cases (when there are more predominant views in the society, or when the predominant views are not extreme)—both for the proportional and for the majoritarian aggregation—STV performs much better.

Acknowledgements

The authors were supported by the Foundation for Polish Science within the Homing programme (Project title: Normative Comparison of Multiwinner Election Rules). We are very grateful to IJCAI reviewers for their thorough remarks.

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Michał Jaworski
Department of Mathematics, Informatics and Mechanics
University of Warsaw
Poland
Email: m.jaworski@mimuw.edu.pl

Piotr Skowron
Department of Mathematics, Informatics and Mechanics
University of Warsaw
Poland
Email: p.skowron@mimuw.edu.pl

A Proof of Theorem 3 Omitted From the Main Text

We here give the full proof of the Theorem 3 that is omitted from the main part of the paper due to space constraints.

The proof of the Theorem 2 is based on the following lemmas.

Lemma 1. *For each $n \geq 3$, the satisfaction-based distortion of SNTV and CC rules for the case when the societies are centered around two poles equals to $\frac{1}{n-1}$.*

Proof. Consider an instance with $|V_1| = 1$ and $|V_2| = n-1$, where there are two candidates, $c_1 \in C_1$ and $c_2 \in C_2$, such that the voters from V_1 rank c_1 first, and the voters from V_2 rank c_2 first. The way in which the voters rank the remaining candidates is arbitrary. Both SNTV and CC, among the winning committees, will pick the one than contains at least $\frac{k+1}{2}$ candidates from C_1—for this committee the decision will be zeros and the total utility of the voters will be p (the single voter from V_1 will be fully satisfied). On the other hand, in order to maximize the total utility, the voters should choose the winning committee with at least $\frac{k+1}{2}$ candidates from C_2—this would result in decision ones, which would fully satisfy all the voters from V_2. Hence, it holds that $\mathrm{dist}^{\mathrm{sat}}_{\mathrm{SNTV}} = \mathrm{dist}^{\mathrm{sat}}_{\mathrm{CC}} = \frac{p(n-1)}{p} = n-1$.

In order to show that the distortion cannot be higher than $n-1$ it is sufficient to observe that since $|V_2|, |V_1| > 0$, at least one voter will be always fully satisfied with the decision made by the committee. □

Lemma 2. *The satisfaction-based distortion of k-Copeland*[1] *and NED rule for the case when the societies are centered around two poles equals to 1.*

Proof. Let us consider an arbitrary instance where the societies are centered around two poles such that $|V_1| < |V_2|$ and $|C_1|, |C_2| \geq \frac{k+1}{2}$. One can observe that due to the block preferences of voters, i.e. voters from V_1 prefer each candidates from C_1 over each candidate from C_2 (analogous observation occurs in case of voters from V_2), each candidate from C_2 wins a pairwise election against all of the candidates from C_1. Therefore, the worst candidate from C_2 wins at least $|C_1|$ pairwise elections. On the other hand, the best candidate from C_1 can win only against all of the remaining candidates from C_2. Hence, the score of the best candidate from C_1 equals to $|C_1| - 1$. As we consider the instance where $|C_2| \geq \frac{k+1}{2}$, in the winning committee W we include at least $\frac{k+1}{2}$ candidates from C_2 and the decision would be in favor of the voters from V_2 and as a results the satisfaction-based distortion is 1.

Now we move to the proof of the distortion for the NED rule. We again observe an arbitrary instance with $|V_1| < |V_2|$ and $|C_1|, |C_2| \geq \frac{k+1}{2}$, where the societies are centered around two poles and as a result we observe block preferences. We show that once we have a committee with l_1 candidates from C_1 and l_2 candidates from C_2, then it is always profitable to delete candidate from C_1 and add candidate from C_2 to the winning committee. Indeed, let us delete the candidate from C_1. It results with the loss of at most $|C_1| - l_1$ and as each candidate from C_2 wins the pairwise election with deleted candidate, then we gain l_2. On the other hand, once we add candidate from C_2, then we can loose at most l_2 (the case where all of the candidates from C_2 that are already in the committee win pairwise election with added candidate) and gain at least $|C_1| - l_1 + 1$ (since we have block preferences). Summing up, from the above described operation we have the gain that equals at least to: $-(|C_1| - l_1) + l_2 - l_2 + (|C_1| - l_1 + 1) = 1$. Hence, the selected committee will make the decision in favour of the voters from V_2 and therefore the satisfaction-based distortion is 1. □

Lemma 3. *The satisfaction-based distortion of STV for the case when the societies are centered around two poles and $|V_2| < q\frac{k+1}{2}$ equals to:*

$$\mathrm{dist}^{\mathrm{sat}}_{\mathrm{STV}} = \begin{cases} \frac{q(k-1)+2q-2}{q(k-1)+2} & \text{for } q \le \frac{k+3}{2} \\ \frac{q(k-1)+2\lceil \frac{q(k+1)}{k+3} \rceil}{(k+1)\lceil \frac{q(k+1)}{k+3} \rceil} & \text{for } q > \frac{k+3}{2},\ 1 \equiv q \pmod{\frac{k+3}{2}} \\ \frac{q(k-1)+2\lfloor \frac{q(k+1)}{k+3} \rfloor}{q(k+1)-2\lfloor \frac{q(k+1)}{k+3} \rfloor} & \text{in other cases.} \end{cases}$$

If $|V_2| \ge q\frac{k+1}{2}$, then the satisfaction-based distortion equals to

$$\mathrm{dist}^{\mathrm{sat}}_{\mathrm{STV}} = \begin{cases} \frac{(q+2)(k-1)}{q(k+1)} & \text{for } q \le k-1 \\ 1 & \text{for } q > k-1. \end{cases}$$

Proof. We start by analyzing the case where $|V_2| < q\frac{k+1}{2}$ and $q \le \frac{k+3}{2}$. Let n_1 and n_2 be such that $|V_1| = q\frac{k-1}{2} + n_1$ and $|V_2| = q\frac{k-1}{2} + n_2$. Observe that

$$q = \left\lfloor \frac{|V_1| + |V_2|}{k} \right\rfloor = \left\lfloor \frac{q(k-1) + n_1 + n_2}{k} \right\rfloor$$
$$= q + \left\lfloor \frac{-q + n_1 + n_2}{k} \right\rfloor .$$

Thus, $n_1 + n_2 \in [q, q + k - 1]$. As we want to keep quota q at the selected level and we assumed that $|V_2| < q\frac{k+1}{2}$, then the ratio of the satisfactions is bounded from below by

$$\frac{|V_2|}{|V_1|} = \frac{q\frac{k-1}{2} + n_2}{q\frac{k-1}{2} + n_1} \le \frac{q\frac{k-1}{2} + q - 1}{q\frac{k-1}{2} + 1} = \frac{q(k-1) + 2q - 2}{q(k-1) + 2}.$$

In order to show that the upper bound is achievable, we construct an instance with $|C_1| = |C_2| = \frac{k+1}{2}$, $|V_1| = q\frac{k-1}{2} + 1$ and $|V_2| = q\frac{k-1}{2} + q - 1$. Assume that $\frac{k-1}{2}$ candidates from C_2 have the plurality score q, one candidate has $(q-1)$; and $(q-1)$ candidates from C_1 have the plurality score $(q-1)$ and $(\frac{k+1}{2} - (q-1))$ candidates have the plurality score q. Observe that the number of voters V_1 is as we stated. Indeed, we have $(q-1)(q-1) + (\frac{k+1}{2} - (q-1))q = q\frac{k-1}{2} + 1$. Now, we show that STV selects $\frac{k+1}{2}$ candidates from C_1 to the winning committee W. First, STV chooses $\frac{k-1}{2}$ candidates from C_2 and $(\frac{k+1}{2} - (q-1))$ candidates from C_1 with the plurality score q. At this point, we are left with the candidates that have the plurality scores $(q-1)$. Hence, one candidate has to be deleted—we consider the case such that the candidate from C_2 is deleted. As a result, $\frac{k+1}{2}$ remaining places in W will be filled with the candidates from C_1 and the decision zeros will be made by the winning committee W. Hence, the satisfaction-based distortion is $\frac{q(k-1)+2q-2}{q(k-1)+2}$.

Now, we move to the case where $|V_2| < q\frac{k+1}{2}$, $q > \frac{k+3}{2}$ and $1 \equiv q \pmod{\frac{k+3}{2}}$. Let $|V_2| = q\frac{k-1}{2} + n_2$, where $n_2 = \lceil \frac{q(k+1)}{k+3} \rceil + \ell$, $\ell \in \mathbb{Z}$. We show that the highest value of the ratio of the satisfactions is achieved for $\ell = 0$. At first, let us assume that $\ell \le -1$. As we want to keep a quota q at the selected level, we need $n_2 + |V_1| \in [q\frac{k+1}{2}, q\frac{k+1}{2} + k - 1]$. Indeed, observe that

$$q = \left\lfloor \frac{|V_1| + |V_2|}{k} \right\rfloor = \left\lfloor \frac{q\frac{k-1}{2} + n_2 + |V_1|}{k} \right\rfloor$$
$$= q + \left\lfloor \frac{-q\frac{k+1}{2} + n_1 + |V_2|}{k} \right\rfloor .$$

According to our assumptions, we can write quota as $q = z\frac{k+3}{2} + 1$, where $z \in \mathbb{N}$ and therefore

$$\left\lceil \frac{q(k+1)}{k+3} \right\rceil = \left\lceil z\frac{k+1}{2} + \frac{k+1}{k+3} \right\rceil = z\frac{k+1}{2} + 1$$

Considering the above observations we have

$$\begin{aligned}
\frac{|V_2|}{|V_1|} &\leq \frac{q\frac{k-1}{2} + n_2}{q\frac{k+1}{2} - n_2} = \frac{q\frac{k-1}{2} + \lceil \frac{q(k+1)}{k+3} \rceil + \ell}{q\frac{k+1}{2} - \lceil \frac{q(k+1)}{k+3} \rceil - \ell} \\
&\leq \frac{q\frac{k-1}{2} + \lceil \frac{q(k+1)}{k+3} \rceil - 1}{q\frac{k+1}{2} - \lceil \frac{q(k+1)}{k+3} \rceil + 1} = \frac{z\frac{k+3}{2}\frac{k-1}{2} + \frac{k-1}{2} + z\frac{k+1}{2}}{z\frac{k+3}{2}\frac{k+1}{2} + \frac{k+1}{2} - z\frac{k+1}{2}} \\
&= \frac{z(\frac{k+1}{2})^2 + z\frac{k-1}{2} + \frac{k-1}{2}}{z(\frac{k+1}{2})^2 + \frac{k+1}{2}}.
\end{aligned}$$

Let us now consider $\ell \geq 0$. As we assumed that $|V_1| < |V_2| < q\frac{k+1}{2}$ and in the winning committee W we need to have $\frac{k+1}{2}$ candidates from C_1, in order to make the decision zeros possible, we have to delete $(|C_2| - \frac{k-1}{2})$ candidates from C_2. Hence, at least $\frac{k+1}{2}$ candidates from C_1 need to have the plurality score at least n_2. Therefore, $|V_1| \geq \frac{k+1}{2}n_2$—we have the following inequalities for the ratio of satisfactions

$$\begin{aligned}
\frac{|V_2|}{|V_1|} &\leq \frac{q\frac{k-1}{2} + n_2}{\frac{k+1}{2}n_2} = \frac{q\frac{k-1}{2} + \lceil \frac{q(k+1)}{k+3} \rceil + \ell}{\frac{k+1}{2}\lceil \frac{q(k+1)}{k+3} \rceil + \frac{k+1}{2}\ell} \\
&\leq \frac{q\frac{k-1}{2} + \lceil \frac{q(k+1)}{k+3} \rceil}{\frac{k+1}{2}\lceil \frac{q(k+1)}{k+3} \rceil} = \frac{z\frac{k+3}{2}\frac{k-1}{2} + \frac{k-1}{2} + z\frac{k+1}{2} + 1}{z(\frac{k+1}{2})^2 + \frac{k+1}{2}} \\
&= \frac{z(\frac{k+1}{2})^2 + z\frac{k-1}{2} + \frac{k-1}{2} + 1}{z(\frac{k+1}{2})^2 + \frac{k+1}{2}}.
\end{aligned}$$

Hence, we get the highest bound for $\ell = 0$. Now, we show that above bound is achievable. We consider the following instance. Let $|C_1| = |C_2| = \frac{k+1}{2}$, $|V_1| = \frac{k+1}{2}\lceil \frac{q(k+1)}{k+3} \rceil$ and $|V_2| = q\frac{k-1}{2} + \lceil \frac{q(k+1)}{k+3} \rceil$, where $q = z\frac{k+3}{2} + 1$, $z \in \mathbb{N}$—quota q is indeed satisfied, we have

$$\begin{aligned}
|V_1| + |V_2| &= \frac{k+1}{2}\left\lceil \frac{q(k+1)}{k+3} \right\rceil + q\frac{k-1}{2} + \left\lceil \frac{q(k+1)}{k+3} \right\rceil \\
&= q\frac{k-1}{2} + \frac{k+3}{2}\left(z\frac{k+1}{2} + 1\right) \\
&= q\frac{k-1}{2} + \frac{k+1}{2}\left(z\frac{k+3}{2} + 1\right) + 1 = qk + 1.
\end{aligned}$$

Hence, $q = \lfloor \frac{|V_1|+|V_2|}{k} \rfloor = \lfloor \frac{qk+1}{k} \rfloor$. What is more, let us assume that $\frac{k-1}{2}$ candidates from C_2 have the plurality score q; the remaining one candidate has the plurality score $\lceil \frac{q(k+1)}{k+3} \rceil$ and all of the candidates from C_1 have the plurality score $\lceil \frac{q(k+1)}{k+3} \rceil$. In the procedure of selecting candidates to W, at first, we select $\frac{k-1}{2}$ candidates from C_2, then, as all of the remaining candidates have the same plurality below a quota q, we decide to delete the candidate from C_2. Naturally, next, we fill $\frac{k+1}{2}$ the missing spots in W with the candidates from C_1 and the decision zeros will be made by the winning committee W. Hence, the satisfaction-based distortion in the considered case equals to $\frac{q(k-1)+2\lceil \frac{q(k+1)}{k+3} \rceil}{(k+1)\lceil \frac{q(k+1)}{k+3} \rceil}$.

Next, we consider the case of a quota q such that $1 \neq q \pmod{\frac{k+3}{2}}$ and $|V_2| < q\frac{k+1}{2}$. Let $|V_2| = q\frac{k-1}{2} + n_2$, where $n_2 = \lfloor \frac{q(k+1)}{k+3} \rfloor + \ell$, $\ell \in \mathbb{Z}$. Again, we show that the highest value of the quotient of the satisfactions is for $l = 0$. We start with the assumption that $q = z\frac{k+3}{2}$, where $z \in \mathbb{N}$, which implies that $n_2 = z\frac{k+1}{2} + \ell$. Similarly to the above reasoning, we have to fulfill two inequalities at the same time, namely we need

$$|V_1| \geq q\frac{k+1}{2} - n_2 = z(\frac{k+1}{2})^2 - l$$
$$|V_1| \geq \frac{k+1}{2}n_2 = z(\frac{k+1}{2})^2 + \frac{k+1}{2}l.$$

One can observe that, as both of the above inequalities have to be satisfied, than the highest bound for $|V_1|$ is achieved once $\ell = 0$—it gives us the following inequality for the ratio of the satisfactions

$$\frac{|V_2|}{|V_1|} \leq \frac{q\frac{k-1}{2} + \lfloor \frac{q(k+1)}{k+3} \rfloor}{q\frac{k+1}{2} - \lfloor \frac{q(k+1)}{k+3} \rfloor}.$$

Before we show that above bound is achievable, we consider the case when $q = z\frac{k+3}{2} + s$, where $z \in \mathbb{N}$ and $s \in [\frac{k+1}{2}] \backslash \{0, 1\}$, which implies that $\lfloor q\frac{k+1}{k+3} \rfloor = z\frac{k+1}{2} + s - 1$. We start, slightly different then in the previous case, with the assumption that $\ell \leq 0$. Obviously, the value of the quota q has to be satisfied and therefore we obtain

$$\begin{aligned}
\frac{|V_2|}{|V_1|} &\leq \frac{q\frac{k-1}{2} + n_2}{q\frac{k+1}{2} - n_2} = \frac{q\frac{k-1}{2} + \lfloor \frac{q(k+1)}{k+3} \rfloor + \ell}{q\frac{k+1}{2} - \lfloor \frac{q(k+1)}{k+3} \rfloor - \ell} \\
&\leq \frac{q\frac{k-1}{2} + \lfloor \frac{q(k+1)}{k+3} \rfloor}{q\frac{k+1}{2} - \lfloor \frac{q(k+1)}{k+3} \rfloor} \\
&= \frac{z\frac{k+3}{2}\frac{k-1}{2} + s\frac{k-1}{2} + z\frac{k+1}{2} + s - 1}{z\frac{k+3}{2}\frac{k+1}{2} + s\frac{k+1}{2} - z\frac{k+1}{2} - s + 1} \\
&= \frac{z(\frac{k+1}{2})^2 + z\frac{k-1}{2} + s\frac{k+1}{2} - 1}{z(\frac{k+1}{2})^2 + s\frac{k-1}{2} + 1}.
\end{aligned}$$

For $\ell \geq 1$ we use the inequality $|V_1| \geq \frac{k+1}{2}n_2$—recall that this condition is necessary to make it possible to delete $(|C_2| - \frac{k-1}{2})$ candidates from C_2, which is essential for selecting $\frac{k+1}{2}$ candidates from C_1 to the winning committee W. We have

$$\begin{aligned}
\frac{|V_2|}{|V_1|} &\leq \frac{q\frac{k-1}{2} + n_2}{\frac{k+1}{2}n_2} = \frac{q\frac{k-1}{2} + \lfloor \frac{q(k+1)}{k+3} \rfloor + \ell}{\frac{k+1}{2}\lfloor \frac{q(k+1)}{k+3} \rfloor + \frac{k+1}{2}\ell} \\
&\leq \frac{q\frac{k-1}{2} + \lfloor \frac{q(k+1)}{k+3} \rfloor + 1}{\frac{k+1}{2}\lfloor \frac{q(k+1)}{k+3} \rfloor + \frac{k+1}{2}} \\
&= \frac{z\frac{k+3}{2}\frac{k-1}{2} + s\frac{k-1}{2} + z\frac{k+1}{2} + s}{z(\frac{k+1}{2})^2 + s\frac{k+1}{2}} \\
&= \frac{z(\frac{k+1}{2})^2 + z\frac{k-1}{2} + s\frac{k+1}{2}}{z(\frac{k+1}{2})^2 + s\frac{k+1}{2}} \\
&= \frac{z(\frac{k+1}{2})^2 + z\frac{k-1}{2} + s\frac{k+1}{2} - 1 + 1}{z(\frac{k+1}{2})^2 + s\frac{k-1}{2} + 1 + s - 1}.
\end{aligned}$$

Hence, we get the highest bound for $\ell = 0$. Let us now construct instances such that the lower bound is achievable. Let $|C_1| = |C_2| = \frac{k+1}{2}$, $|V_2| = q\frac{k-1}{2} + n_2$, $|V_1| = q\frac{k+1}{2} - n_2$, where $n_2 = \lfloor \frac{q(k+1)}{k+3} \rfloor$ and $q = z\frac{k+3}{2} + s$, where $z \in \mathbb{N}$ and $s \in [\frac{k+3}{2}] \backslash \{0, 1\}$—observe that we also allow $s = \frac{k+3}{2}$, which coincides with $s = 0$. Assume that $\frac{k-1}{2}$ candidates from C_2 have the plurality score q and one candidate has plurality n_2. What is more, $(\frac{k+3}{2} - s)$ candidates from C_1 have plurality score $n_2 + 1$ and remaining candidates have n_2. Let us check that the number of voters from V_1 is as we stated. Indeed, we have

$$\begin{aligned}
|V_1| &= \Big(\frac{k+3}{2} - s\Big)\Big(n_2 + 1\Big) + \Big(\frac{k+1}{2} - \frac{k+3}{2} + s\Big)n_2 \\
&= \frac{k+3}{2} - s + n_2\frac{k+1}{2} \\
&= \frac{k+3}{2} - s + \Big(z\frac{k+1}{2} + s - 1\Big)\frac{k+3}{2} - n_2 \\
&= \Big(z\frac{k+3}{2} + s\Big)\frac{k+1}{2} - n_2 = q\frac{k+1}{2} - n_2.
\end{aligned}$$

In the procedure of selecting the candidates to the winning committee W, at first, we select $\frac{k-1}{2}$ candidates from C_2, then we are left with one candidate from C_2 and $\frac{k+1}{2}$ candidates from C_1. The remaining candidate from C_2 has the plurality score that is at most the plurality score of each candidate from C_1—we decide to delete candidate from C_2 and obviously $\frac{k+1}{2}$ missing places in the winning committee will be filled with the candidates from C_1. Summing up, the satisfaction-based distortion equals to $\frac{q(k-1)+2\lfloor \frac{q(k+1)}{k+3} \rfloor}{q(k+1)-2\lfloor \frac{q(k+1)}{k+3} \rfloor}$.

Next, we consider the case of $|V_2| \geq q\frac{k+1}{2}$. First, we show that if $|V_2| \geq q\frac{k+1}{2}$ and the decision zeros is possible, then $|V_1| \geq q\frac{k+1}{2}$. For the sake of contradiction assume that $|V_2| \geq q\frac{k+1}{2}$, the decision zeros is possible and $|V_1| < q\frac{k+1}{2}$. As we need at least $\frac{k+1}{2}$ candidates from C_1 in the winning committee W and $|V_1| < q\frac{k+1}{2}$, then it implies that we have to seek for the plurality score of the part of the candidates from C_1 among the voters from V_2 and due to the assumptions concerning the vectors of voters' preferences, in the procedure of selecting candidates, we need to delete at least $(|C_2| - \frac{k-1}{2})$ candidates from C_2. Observe that in each step if there are at least ql voters from V_2 left, then we have at least l candidates from C_2. Indeed, due to pigeonhole principle once we are left with l candidates and at least ql voters, then there exists a candidate with plurality score higher than q and this candidate might be chosen to the winning committee while removing q voters from V_2. Since we start with more than $q\frac{k+1}{2}$ voters from V_2, then we can delete at most $(|C_2| - q\frac{k+1}{2})$ candidates from C_2 and at least $\frac{k+1}{2}$ will be included in W. Hence, the decision zeros is not possible. From now on we assume that $|V_1| = q\frac{k+1}{2} + n_1$, where $n_1 \geq 0$—obviously the decision zeros is now possible. Let us consider the case of $q \leq k - 1$, one may observe that the following equalities hold

$$q = \Big\lfloor \frac{|V_1| + |V_2|}{k} \Big\rfloor = \Big\lfloor \frac{q\frac{k+1}{2} + n_1 + |V_2|}{k} \Big\rfloor.$$

Since we would like to assess the distortion, we have to consider the highest possible value of $|V_2|$. Hence,

$$q = \Big\lfloor \frac{q\frac{k+1}{2} + n_1 + |V_2|}{k} \Big\rfloor = \frac{q\frac{k+1}{2} + n_1 + |V_2|}{k} - \frac{k-1}{k}.$$

Above equalities imply that $|V_2| = q(k - \frac{k+1}{2}) + k - 1 - n_1$. The quotient of satisfactions is as follows

$$\begin{aligned}\frac{|V_2|}{|V_1|} &= \frac{q(k-\frac{k+1}{2})+k-1-n_1}{q\frac{k+1}{2}+n_1} \leq \frac{q(k-\frac{k+1}{2})+k-1}{q\frac{k+1}{2}} \\ &= \frac{(q+2)(k-1)}{q(k+1)},\end{aligned}$$

where higher bound is achievable—the construction of the instance is obvious, we set the vectors of preferences of $q\frac{k+1}{2}$ voters from V_1 in the way that each of the $\frac{k+1}{2}$ candidates from C_1 have the plurality score q and in the procedure we select at first candidates from C_1 to W. Therefore, the satisfaction-based distortion is $\frac{q(k+1)}{(q+2)(k-1)}$. Now, we move into the last case of $q > k-1$. We have $|V_1| + |V_2| = qk + q + n_2 > qk + k - 1$, as we need to keep a quota q at the selected level it holds that $n_2 < 0$ and therefore, $|V_2| < q\frac{k+1}{2}$, which implies that only the decision zeros is possible. Hence, the satisfaction-based distortion of STV in the considered case is 1. □

Lemma 4. *For each $n \geq 3$, the satisfaction-based distortion of SNTV and CC for the case when the societies are centered around two poles equals to $n-1$.*

Proof. Consider an instance with $|V_1| = 1$ and $|V_2| = n-1$, where there are two candidates, $c_1 \in C_1$ and $c_2 \in C_2$, such that the voters from V_1 rank c_1 first, and the voters from V_2 rank c_2 first. The way in which the voters rank the remaining candidates is arbitrary. Both SNTV and CC, among the winning committees, will pick the one than contains at least $\frac{k+1}{2}$ candidates from C_1—for this committee the decision will be zeros and the total utility of the voters will be p (the single voter from V_1 will be fully satisfied). On the other hand, in order to maximize the total utility, the voters should choose the winning committee with at least $\frac{k+1}{2}$ candidates from C_2—this would result in decision ones, which would fully satisfy all the voters from V_2. Hence, it holds that $\mathrm{dist}^{\mathrm{sat}}_{\mathrm{SNTV}} = \mathrm{dist}^{\mathrm{sat}}_{\mathrm{CC}} = \frac{p(n-1)}{p} = n-1$.

In order to show that the distortion cannot be higher than $n-1$ it is sufficient to observe that since $|V_2|, |V_1| > 0$, at least one voter will be always fully satisfied with the decision made by the committee. □

Lemma 5. *For each $n \geq 3$, the satisfaction-based distortion of k-Borda rule for the case when the societies are centered around two poles equals to $\frac{4m-k-3}{k+3}$.*

Proof. First, we observe that in order to maximize the utility we should choose to the winning committee at least $\frac{k+1}{2}$ candidates from the set C_2—this would result in decision ones and would fully satisfy the voters from V_2.

On the other hand, if we apply k-Borda to select the winning committee, it may happen that the decision of the elected committee will be zeros, satisfying the voters from V_1, inducing the satisfaction-based distortion of $|V_2|/|V_1|$. This may happen only if the Borda score of the $\frac{k+1}{2}$-th best candidate from C_1—call this candidate a—is higher than the Borda score of the $\frac{k+1}{2}$-th best candidate from C_2—call this candidate b. One can observe that the highest possible value of the Borda score of candidate a is when it equals to the average Borda score of the $\frac{k+1}{2}$ best candidates from C_1. On the other hand, the lowest possible value of the Borda score of candidate b is when it is equal to the average Borda score of the worst $(|C_2| - \frac{k-1}{2})$ candidates from C_2. Thus, for candidate a being selected to the winning committee the necessary condition is:

$$\begin{aligned}
&\frac{|V_2|\Big[\sum_{i=1}^{\frac{k+1}{2}}(|C_1|-i)\Big]}{\frac{k+1}{2}} \\
&\quad+\frac{|V_1|\Big[\sum_{i=1}^{\frac{k+1}{2}}(|C_1|+|C_2|-i)\Big]}{\frac{k+1}{2}} \\
&\geq \frac{|V_2|\Big[\sum_{i=\frac{k+1}{2}}^{|C_2|}(|C_1|+|C_2|-i)\Big]}{|C_2|-\frac{k-1}{2}} \\
&\quad+\frac{|V_1|\Big[\sum_{i=\frac{k+1}{2}}^{|C_2|}(|C_2|-i)\Big]}{|C_2|-\frac{k-1}{2}}.
\end{aligned} \tag{1}$$

And conversely, if Equation (1), it is always possible to construct an instance where a is selected. We make the following transformation of Equation (1):

$$\begin{aligned}
&\frac{|V_2|\Big[\frac{|C_1|-1+|C_1|-\frac{k+1}{2}}{2}\cdot\frac{k+1}{2}\Big]}{\frac{k+1}{2}} \\
&\quad+\frac{|V_1|\Big[\frac{|C_1|+|C_2|-1+|C_1|+|C_2|-\frac{k+1}{2}}{2}\cdot\frac{k+1}{2}\Big]}{\frac{k+1}{2}} \\
&\geq \frac{|V_2|\Big[\frac{|C_1|+|C_2|-\frac{k+1}{2}+|C_1|}{2}\cdot\big(|C_2|-\frac{k-1}{2}\big)\Big]}{|C_2|-\frac{k-1}{2}} \\
&\quad+\frac{|V_1|\Big[\frac{|C_2|-\frac{k+1}{2}}{2}\cdot\big(|C_2|-\frac{k-1}{2}\big)\Big]}{|C_2|-\frac{k-1}{2}}.
\end{aligned}$$

This implies that:

$$\begin{aligned}
|V_1|\Big[|C_1|+\frac{|C_2|}{2}-\frac{1}{2}\Big] &\geq |V_2|\Big[\frac{|C_2|}{2}+\frac{1}{2}\Big] \qquad \text{,and so:} \\
\frac{2|C_1|+|C_2|-1}{|C_2|+1} &\geq \frac{|V_2|}{|V_1|}.
\end{aligned}$$

Recall, that $|V_2|/|V_1|$ is exactly the value of the satisfaction-based distortion when a is selected. Further, since $|C_2| \geq \frac{k+1}{2}$, we have that the satisfaction-based distortion is:

$$\mathrm{dist}^{\mathrm{sat}}_{k\text{-Borda}} = \frac{2(m-\frac{k+1}{2})+\frac{k+1}{2}-1}{\frac{k+1}{2}+1} = \frac{4m-k-3}{k+3}.$$

□

Lemma 6. *If $k=3$, $\lfloor \frac{n}{3} \rfloor = \frac{n}{3} = q > 3$, then the satisfaction-based distortion of Monroe's rule for the case when societies are centered around two poles is at least $\frac{3q-2}{2}$.*

Proof. In the proof we construct the following example of an instance that gives us lower bound for the distortion. Let $k=3$ denotes the size of the winning committee W and assume that k divides $n = |V_1| + |V_2|$, which implies that quota $q = \frac{n}{3}$—because of the technical reasons we also assume that $q > 3$ and $m \geq 3q+4$, where m denotes the number of candidates. What is more, assume that $C_2 = B = \{b_1, b_2\}$, $|V_1| = 2$ and $C_1 = A \cup A_d$, where $A = \{a_1, a_2\}$ and A_d is a set of *dummy* candidates, i.e. candidates that are used only to spoil the Borda score of the candidates from B. Each voter $v_i \in V_1$ has the following preferences (we use the notation of the set to denote the block of candidates in the ranking, in which order is arbitrary or will be specified later)

$$a_i \succ A_d \succ A \backslash \{a_i\} \succ b_1 \succ b_2$$

and there are two classes of voters from V_2 such that i-th class, denoted by $\widetilde{V}_i$, consists of $(q-1)$ voters with preferences

$$b_1 \succ b_2 \succ a_i \succ A_d \succ A \backslash \{a_i\}.$$

What is more, in addition there are q voters left from V_2 with preferences

$$b_2 \succ b_1 \succ C_1.$$

One can observe that if in W there are all candidates from A and candidate b_2, then Borda score will equal to $2m + 2(q-1)(m-2) + qm = 3qm - 4q + 4$. On the other hand, if set B is included in W, then Borda score is at most $2qm + f(q,m)$, where $f(q,m)$ is a Borda score of the best candidate from the set C_1—obviously it depends on the parameters q and m. Since, we would like the winning committee to make decision zeros, we need the following inequality for $f(q,m)$.

$$\begin{aligned} 2qm + f(q,m) &< 3qm - 4q + 4 \quad \iff \\ f(q,m) &< qm - 4q + 4. \end{aligned}$$

Now, we have to show that such $f(q,m)$ exists. Beside two candidates from B we can either choose candidate from A or from A_d to the winning committee. In the first case, the best possible score of the candidate from A equals to $f(q,m) = m + 3 + (q-2) * (m-2)$—*one* first position; *one* $(m-2)$ position, ahead of two candidates from B; $(q-2)$ third positions among left voters from V_2 behind candidates from B. Let us see, if there exist q and m such that above inequality for $f(q,m)$ holds. We have

$$\begin{aligned} f(q,m) &< qm - 4q + 4 \quad \iff \\ m + 3 + (q-2)(m-2) &< qm - 4q + 4 \quad \iff \\ m + 3 - 2m - 2q &< -4q \quad \iff \\ 3 + 2q &< m. \end{aligned}$$

Above inequality is always true, as we assumed that $m \geq 3q + 4$. Now, we consider the second case, in which a candidate from A_d is chosen. Assume that we have an instance such that $|A_d| = 3q + l$, where $l \in \mathbb{N}$. Hence, number of all candidates equals to $m = 3q + 4 + l$ and each voter v_i has the preferences over candidates from A_d as follows

$$a_{d_i} \succ \ldots \succ a_{d_{3q+l}} \succ a_{d_1} \succ \ldots \succ a_{d_{i-1}}.$$

The highest possible Borda score from the candidate chosen to the winning committee in the considered instance equals to

$$\begin{aligned} &f(q, 3q + 4 + l) \\ &\quad = \frac{3q + 1 + l + 2q + 4 + l}{2}(q-2) \\ &\qquad + (2q + 5 + l) + (2q + 4 + l) \end{aligned}$$

$$= \frac{5q+5+2l}{2}(q-2) + 4q + 9 + 2l,$$

where first part is the aggregated satisfaction among $(q-2)$ voters from V_2—recall that $2q$ voters are represented by the candidates from B and the second part is the satisfaction among three voters from V_1. Now, we have to check, if there exists q such that the inequality for $f(q,m) = f(q, 3q+4+l)$ holds. We have

$$\begin{aligned}
&f(q, 3q+4+l) < q(3q+4+l) - 4q + 4 \quad \Longleftrightarrow \\
&\frac{5q+5+2l}{2}(q-2) + 4q + 9 + 2l < 3q^2 + ql + 4c \quad \Longleftrightarrow \\
&5q^2 - 10q + 5q - 10 + 2ql - 4l + 8q + 18 + 4l \\
&\qquad < 6q^2 + 2ql + 8 \quad \Longleftrightarrow \\
&0 < q^2 - 3q = q(q-3).
\end{aligned}$$

Since we assumed that $q > 3$, above inequality holds for all $q > 3$.

Summing up, if $q > 3$ and $m \geq 3q+4$, then there exists an instance such that the winning committee W that consists of two candidates from A has higher Borda score than each possible committee that consists of candidates from B and the other arbitrary candidate from C_1. Once we consider the quotient of satisfactions in this example we have

$$\frac{|V_2|}{|V_1|} = \frac{n-2}{2} = \frac{3q-2}{2}.$$

Hence, the satisfaction-based distortion for the considered case is bounded from below by $\frac{3q-2}{2}$. ☐

Lemma 7. *The satisfaction-based distortion of greedy-Monroe rule for the case when societies are centered around two poles is at least $k-1$.*

Proof. Let $n = \frac{k(k+1)}{2}$, where n and k denote the number of voters and the size of the winning committee W, respectively. Hence, the quota is $q = \frac{k+1}{2}$. We construct the following instance. Let $C_2 = B = \{b_1, \dots, b_{\frac{k+1}{2}}\}$, $|V_1| = \frac{k+1}{2}$, and $C_1 = A \cup A_d$, where $A = \{a_1, a_2, \dots, a_{\frac{k+1}{2}}\}$ and A_d, $|A_d| = 2q$, is a set of dummy candidates. Each voter $v_i \in V_1$ has the following preference order (a set in a preference ranking denotes the block of candidates ordered in a fixed arbitrary way, or whose ranking will be specified later on):

$$a_i \succ A_d \succ A \backslash \{a_i\} \succ B.$$

The voters from V_2 are divided into k classes. The first $\frac{k-1}{2}$ classes are constructed as follows. The i-th class, $i \in \left[\frac{k-1}{2}\right]$, denoted by $\widetilde{V}_i$, consists of q voters with the following preference rankings:

$$b_i \succ B \backslash \left\{b_i, b_{\frac{k+1}{2}}\right\} \succ b_{\frac{k+1}{2}} \succ A \succ A_d.$$

For the union of the $\frac{k-1}{2}$ above mentioned classes we write $\widetilde{V} := \bigcup_i \widetilde{V}_i$. The remaining $\frac{k+1}{2}$ classes of the voters from V_2 are constructed as follows: the i-th class, $i \in \left[\frac{k+1}{2}\right]$, denoted by $\overline{V}_i$, consists of $(q-1)$ voters with the following preference rankings:

$$b_i \succ B \backslash \left\{b_i, b_{\frac{k+1}{2}}\right\} \succ b_{\frac{k+1}{2}} \succ a_i \succ A \backslash \{a_i\} \succ A_d.$$

Again, we denote the union of these classes as $\overline{V} := \bigcup_i \overline{V}_i$.

Consider the process of building a winning committee by the greedy Monroe rule. Recall that W_i denotes the set of candidates selected by the rule in the first i iterations, and V_i—the set of voters assigned a representative from W_i. In the first $\frac{k-1}{2}$ steps, due to the highest possible Borda score, we select $\frac{k-1}{2}$ candidates from B that are ranked at the first places among all the voters from $\widetilde{V}$—formally, we have $W_{\frac{k-1}{2}} = B \setminus \left\{ b_{\frac{k+1}{2}} \right\}$ and $V_{\frac{k-1}{2}} = \widetilde{V}$. For selecting $(\frac{k+1}{2})$-th candidate to the winning committee we have the following possibilities:

1. Take $b_{\frac{k+1}{2}}$, who will represent q voters from V_2—the Borda score of this candidate is
$$(m - (q-1))q.$$
2. Take a_i, who will represent *one* voter from V_1 and $(q-1)$ voters from $\overline{V}_i$—the Borda score of this candidate is
$$m + (m-q)(q-1).$$
3. Take the best candidate from A_d and she will represent q voters from V_1 (in our case, all the voters from V_1)—in order to compute the Borda score, we have to fix the way in which the voters from V_1 rank the candidates from A_d. We assume that the preferences of each two consecutive voters from V_1 are *shifted by two in the right direction*, i.e. preferences of voter $v_i \in V_1$ over the candidates from A_d are as follows:
$$a_{d_{2q-(2i-2)}} \succ \ldots \succ a_{d_{2q}} \succ a_{d_1} \succ \ldots \succ a_{d_{2q-(2i-3)}}.$$
Hence, the Borda score of the best candidate, without loss of generality $a_{d_{2q}}$, equals to
$$\begin{aligned} &m - 1 + m - 3 + \ldots + m - (2q-1) \\ &= \frac{m - 1 + m - (2q-1)}{2} q = (m-q)q. \end{aligned}$$

Now, we will argue that candidate a_i will be selected to the winning committee. First, observe that the Borda score of a_i is equal to the Borda score of the candidate $b_{\frac{k+1}{2}}$. Indeed:
$$\begin{aligned} m + (m-q)(q-1) &= (m-(q-1))q \quad \iff \\ mq + q - q^2 &= mq - q^2 + q. \end{aligned}$$
Second, observe that the Borda score of a_i is at least as high as the Borda score of $a_{d_{2q}}$. We have:
$$\begin{aligned} m + (m-q)(q-1) &\geq (m-q)q \quad \iff \\ mq + q - q^2 &\geq mq - q^2. \end{aligned}$$
Consequently, in the $(\frac{k+1}{2})$-th step the rule can select a candidate $a_i \in A$. As a result $W_{\frac{k+1}{2}} = W_{\frac{k-1}{2}} \cup \{a_i\}$, and we add to the set of the select voters $V_{\frac{k-1}{2}}$ the voter v_i from V_1 who ranks candidate a_i first, and all of the voters from $\overline{V}_i$. Observe that after the $(\frac{k+1}{2})$-th step all of the remaining candidates from A have the same Borda score as candidate a_i had. Hence, in each next step, we extend the set of the candidates with a candidate a_j from A and she will represent one voter from V_1, that ranks her first, and $(q-1)$ voters from $\overline{V}_j$.

Recall that we have $|V_1| = \frac{k+1}{2} = q$, $(q-1)$ classes with q voters $(\widetilde{V})$, and q classes with $(q-1)$ voters $(\overline{V})$. Hence,
$$\frac{|V_2|}{|V_1|} = \frac{2q(q-1)}{q} = 2(q-1) = k-1.$$
and in general the satisfaction-based distortion in the considered case is at most $k-1$. □

We can now prove Theorem 2, which is formulated as follows:

Theorem 2. *For societies centered around two poles, the satisfaction-based distortion:*

1. *of the k-Copeland rule equals to* 1.
2. *of STV is 3, but for large $^n/_k$ and k it approaches 1.*
3. *of SNTV and CC is $\Theta(n)$, of k-Borda is $\Theta(m/k)$, of Monroe is $\Omega(n)$ even if $k = 3$, and of greedy-Monroe is $\Omega(k)$.*

Proof. We prove the theorem in the above fixed order—let us start with the first point.

(1) Let us consider an arbitrary instance where the societies are centered around two poles such that $|V_1| < |V_2|$ and $|C_1|, |C_2| \geq \frac{k+1}{2}$. One can observe that due to the block preferences of voters, i.e. voters from V_1 prefer each candidates from C_1 over each candidate from C_2 (analogous observation occurs in case of voters from V_2), each candidate from C_2 wins a pairwise election against all of the candidates from C_1. Therefore, the worst candidate from C_2 wins at least $|C_1|$ pairwise elections. On the other hand, the best candidate from C_1 can win only against all of the remaining candidates from C_2. Hence, the score of the best candidate from C_1 equals to $|C_1| - 1$. As we consider the instance where $|C_2| \geq \frac{k+1}{2}$, in the winning committee W we include at least $\frac{k+1}{2}$ candidates from C_2 and the decision would be in favor of the voters from V_2 and as a results the satisfaction-based distortion is 1.

(2) In order to prove the statements in the present point we use Lemma 3. We are going to limit from above each of the quotient presented in Lemma 3 given the restrictions concerning parameters. We have:

a)

$$\frac{q(k-1) + 2q - 2}{q(k-1) + 2} \leq \frac{2q + 2q}{2q} = 2;$$

b) we are going to use the fact that in the second case we can write quota as $q = z\frac{k+3}{2} + 1$, where $z \in \mathbb{N}$

$$\begin{aligned}
&\frac{q(k-1) + 2\lceil\frac{q(k+1)}{k+3}\rceil}{(k+1)\lceil\frac{q(k+1)}{k+3}\rceil} \\
&= \frac{(z\frac{k+3}{2} + 1)(k-1) + 2\lceil z\frac{k+1}{2} + \frac{k+1}{k+3}\rceil}{(k+1)\lceil z\frac{k+1}{2} + \frac{k+1}{k+3}\rceil} \\
&= \frac{(z\frac{k+3}{2} + 1)(k-1) + 2(z\frac{k+1}{2} + 1)}{(k+1)(z\frac{k+1}{2} + 1)} \\
&= \frac{(k+1)(z\frac{k+1}{2} + 1) + z(k-1)}{(k+1)(z\frac{k+1}{2} + 1)} \\
&= \frac{z\frac{k+1}{2} + 1 + z\frac{k-1}{k+1}}{z\frac{k+1}{2} + 1} \leq \frac{z\frac{k+1}{2} + 1 + z}{z\frac{k+1}{2} + 1} \\
&\leq \frac{z\frac{k+1}{2} + z}{z\frac{k+1}{2}} \leq \frac{2z + z}{2z} = \frac{3}{2};
\end{aligned}$$

c)

$$\frac{q(k-1)+2\lfloor\frac{q(k+1)}{k+3}\rfloor}{q(k+1)-2\lfloor\frac{q(k+1)}{k+3}\rfloor} \leq \frac{q(k-1)+2q}{q(k+1)-2q} = \frac{q(k+1)}{q(k-1)} \leq 2;$$

d)

$$\frac{(q+2)(k-1)}{q(k+1)} \leq \frac{3(k-1)}{(k+1)} < 3.$$

Hence, once $q = 1$ and size of the committee $k \to +\infty$, then the satisfaction-based distortion converges to 3. What is more, one can easily observe that for large k and n/k the satisfaction-based distortion approaches 1, as each quotient presented in Lemma 3 converges to 1 if $k \to \infty$ and $n/k \to \infty$ (i.e. $q \to \infty$).

(3) From Lemma 5 we know that the satisfaction-based distortion of k-Borda equals to $\frac{4m-k-3}{k+3}$. As the number of candidates m has to be at least k (the size of the committee), then for $m \geq 3$ and $k \geq 3$ we have

$$\frac{4m-k-3}{k+3} \geq \frac{4m-2m}{k+k} = \frac{m}{k}.$$

On the other hand,

$$\frac{4m-k-3}{k+3} \leq 4\frac{m}{k}.$$

Hence, the satisfaction-based distortion for k-Borda is $\Theta(m/k)$. The rest of the results in the present point follow directly from Lemma 4, Lemma 6 and Lemma 7.

□