Threshold Task Games: Theory, Platform and Experiments

Kobi Gal, Ta Duy Nguyen, Quang Nhat Tran and Yair Zick[1]

Abstract

Threshold task games (TTGs) are a class of cooperative games in which participants form coalitions to complete tasks associated with different rewards and thresholds for success. We provide efficient algorithms for computing approximately optimal coalition structures in TTGs. We also present non-trivial bounds on the cost of stability for this class. We put our theoretical results to practice; we design a web-based framework which allows human players to interact in a collaborative task-based model. Our analysis of human play in two different countries shows that players succeed in general to form optimal coalition structures, and converge to approximately stable payoff divisions.

1 Introduction

A group of players needs to collaborate in order to complete a set of tasks. Their objective is twofold: first, they must form *coalitions* — disjoint groups of players, each working on a separate task; second, if players are self-interested, they must decide on a reasonable way of dividing revenue from their tasks.

Example 1. Alice, Bob and Claire sign up for an online freelance website (e.g. fiverr.com). The website currently has two tasks: a script task t_1 that requires a total of 3 hours to complete, and pays \$5, and a programming task t_2 requiring 7 hours of work, paying \$15. It is possible to complete a task more than once (so t_1 can be assigned to all three players). Alice and Bob can contribute 3 hours each, whereas Claire can contribute 4 hours. Assuming that the task hour load is easily divisible between the players, the (non-unique) best partition of players into work groups would be to assign Alice and Claire to t_2, and have Bob work alone on t_1. The next step would be to decide how one should divide task revenue One could reasonably argue that Claire should receive a higher share of the revenue than Alice, as she contributed more hours to the task; moreover, in order to complete t_2, Claire has to be assigned to it. However, each person could have worked on t_1 alone and receive \$5 for their efforts. Clearly, finding an appropriate payoff division is not a straightforward task.

Cooperative game theory studies situations in which players form coalitions and share revenue. One can model scenarios like that described in Example 1 using a *resource-task based model*. Player i controls a resource (its weight w_i), and a groups of players can complete a task (and obtain a reward) if their combined weight exceeds a certain *threshold*. The value of a group of players (also called a *coalition*) is the value of the best task it can complete given its resources. In the literature, these are known as *threshold task games* (TTGs) [12, 22]. Despite their intuitive appeal, there has been little work analyzing solution concepts for TTGs, nor how humans behave when playing them. Our work directly addresses this gap.

1.1 Our Contribution

We study the *cost of stability* in TTGs: this measures the relative overhead required in order to stabilize the underlying TTG; that is, the amount of additional subsidy required to

[1]Work done while Yair Zick was at the National University of Singapore.

guarantee that there exists some payoff division that assigns each coalition a payoff exceeding its value. In Section 3.1, we present a simple algorithm for computing *coalition structures* (partitions of the player set) that guarantee at least $\frac{1}{2}$ of the optimal social welfare for TTGs (note that computing an optimal coalition structure for TTGs is NP-complete [17]). Next, in Section 3.2 we provide a tight bound on the cost of stability for TTGs.

Finally, we describe an online platform called Business Cats which provides a negotiation environment for playing TTGs simply and intuitively (Section 4).[2] We recruit participants in two countries to play TTG sessions and analyze the results. Our analysis (Section 5) shows that human players form nearly-optimal coalition structures, and arrive at core-stable payoff divisions. We identify key criteria contributing to successful play; for example, players tend to favor *power preserving* proposals: players will often refuse proposals that do not assign them a payoff that is commensurate with their value in the game, as measured by their relative weight value. These results provide evidence for the use of empirical negotiation frameworks to support theoretical results and induce good play from people.

1.2 Related Work

Our work examines both theoretical properties and actual gameplay in TTGs. On the theoretical side, Nguyen and Zick [24] bound the cost of stability in TTGs, and provide efficient algorithms for computing the optimal coalition structure. Their analysis requires several limiting assumptions on the structure of player weights and tasks. We do away with these assumptions in Sections 3.1 and 3.2 via careful analysis and novel techniques.

Threshold task games were introduced by Chalkiadakis et al. [12]; their work departs from the classic cooperative game model, allowing players to split resources amongst several tasks. The only work we are aware of that studies a TTG model in the classic cooperative game setting is by Balcan et al. [7], who establish the PAC learnability of TTGs. The optimal coalition structure generation problem is also well-studied (see [25] for a recent overview); other related works include [4, 5]. Other task-based models include [1, 4, 28].

Weighted voting games (WVGs), a subclass of TTGs, are extremely well-studied. This is most likely because they carry the best of both worlds: they have a succinct representation (requiring only n weights and a threshold to describe), but their analysis is complex. For example, computing solution concepts for WVGs is intractable [17, 18, 19]. The complexity of solution concepts for general cooperative games is well studied in the literature, dating back to [16]; more recent works include [8, 14, 20, 21] (see [13] and [11] for an overview).

Bachrach et al. [3] introduce and study the cost of stability, and bound it for WVGs; however, they study a model where coalition structures do not form, thus their results do not directly apply to our setting. Other works that study the cost of stability either make assumptions on the class of cooperative games [26], or on the underlying player interaction structure [10, 23].

Works on human coalition formation are relatively sparse. Bitan et al. [9] study coalition formation in human-computer teams using voting. The line of empirical work most closely resembling ours studies coalition formation in WVG [6, 22]. These WVG platforms are significantly different than Business Cats in that they constrain the game to a single coalition and do not allow players to choose different tasks, or form multiple coalitions.

2 Preliminaries

In what follows, sets of players are given in uppercase letters $(S, T, \dots)$, vectors are denoted by $\vec{x}, \vec{y}, \dots$, and for $k \in \mathbb{Z}^+$ $[k]$ denotes the set $\{1, \dots k\}$. Given a vector $\vec{x} \in \mathbb{R}^n$ and a set

[2] The platform is available at http://business-cats.comp.nus.edu.sg/

$S \subseteq \{1, \dots, n\}$ we let $x(S) = \sum_{i \in S} x_i$. A *cooperative game* $G = \langle N, v \rangle$ consists of a set of players $N = \{1, \dots, n\}$ and a function $v : 2^N \to \mathbb{R}$, called the *characteristic function*. Given a set of players $S \subseteq N$ (also known as a *coalition*), $v(S)$ is the value of S; we make the standard assumption that $v(\emptyset) = 0$, and that v is monotone: for every two coalitions $S \subseteq T \subseteq N$, we have $v(S) \leq v(T)$ (see [13] for a discussion of these assumptions).

2.1 Cooperative Solution Concepts

Given a cooperative game $G = \langle N, v \rangle$, a partition of players into coalitions is called a *coalition structure*. We say that a coalition structure CS^* is *optimal* if it maximizes social welfare; that is, $CS^* \in \operatorname{argmax}_{CS \in \Pi(N)} \sum_{S \in CS} v(S)$, where $\Pi(N)$ denotes the set of partitions of N. For brevity, we overload notation and write $v(CS) = \sum_{S \in CS} v(S)$. We let $OPT(G)$ be the value of an optimal coalition structure over G. We refer to the problem of finding an optimal coalition structure (also known as the *coalition structure generation problem*) as OptCS. We say that a coalition structure CS^* is *β-optimal* for G if $v(CS^*) \geq \beta OPT(G)$. As a solution concept, OptCS assumes that a (benevolent) central planner tries to find the best way of partitioning players into groups, ignoring any strategic considerations. If one assumes that players form coalitions in order to receive part of the obtained reward, it is reasonable to assume selfish player behavior. Given a coalition structure CS, an *imputation* for CS is a vector $\vec{x} \in \mathbb{R}^n_+$ satisfying $\sum_{i \in S} x_i = v(S)$ for all $S \in CS$. In other words, players in $S \in CS$ generate the revenue $v(S)$, and may divide it amongst themselves in any way they see fit; however, they may not transfer any of the utility generated by S to non-members of S. The tuple $\langle CS, \vec{x} \rangle$ is called an *outcome* of G. Let us denote by $\mathcal{I}(G)$ the set of all outcomes for G. The core is a subset of outcomes in $\mathcal{I}(G)$ defined by,

$$Core(G) = \{\langle CS, \vec{x} \rangle \in \mathcal{I}(G) : x(S) \geq v(S), \forall S \subseteq N\}.$$

The constraints describing the core are also referred to as *stability conditions*, and outcomes in the core are referred to as *stable*. Core stability is a natural requirement: if $\langle CS, \vec{x} \rangle$ is not stable, then there exists some coalition T whose members can generate more revenue than what they are assigned. In other words, T's members could ensure a strictly better outcome for themselves if they choose to work together, rather than under the coalitions they were assigned to under CS. It is well-known that if $\langle CS, \vec{x} \rangle \in Core(G)$, then CS must be optimal [13]; in that respect, the core resolves both the OptCS objective and coalitional strategic considerations. Unfortunately, the core may be empty.

Example 2. Consider a 3 player game where $v(S) = 0$ if $|S| \leq 1$, and is 1 otherwise. An optimal coalition structure has a value of 1. Consider any valid imputation (x_1, x_2, x_3), such that $\sum_{i=1}^{3} x_i = 1$. It must be the case that x_1, x_2 or x_3 are strictly greater than 0; with no loss of generality, assume that $x_1 > 0$. In that case, $x_2 + x_3 < 1 = v(\{2, 3\})$, and therefore (x_1, x_2, x_3) cannot satisfy the stability conditions. This implies the core is empty. It is easy to verify that any imputation that pays players strictly less than $\frac{3}{2}$ in total cannot stabilize the game, whereas the payoff $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is stable. Therefore, paying each player $\frac{1}{2}$ is a minimal stable payoff division.
We note that this game can be described in the language of weights and thresholds: each player has a weight of 1, and there is a single task. If the total weight of players in a coalition S is ≥ 2, then they can complete the task and obtain a reward of 1; otherwise they coalition has a value of 0.

As Example 2 implies, stabilizing a game may require an additional external subsidy. The minimal total payoff needed can be found by solving the following linear program

$$\min \sum_{i \in N} x_i \qquad \text{s.t. } x(S) \geq v(S) \qquad \forall S \subseteq N. \tag{1}$$

Let V^* be the optimal value of (1); the *relative cost of stability* (CoS) of a game G is the ratio between V^* and $OPT(G)$[3]: $CoS(G) = \frac{V^*}{OPT(G)}$. We note that if the optimal solution to (1) has a value $V^* > OPT(G)$ — i.e. the core is empty — then the corresponding payoff division $\vec{x}^*$ (the output of (1)) satisfies the stability constraints, i.e. $\sum_{i \in S} x_i^* \geq v(S)$ for all $S \subseteq N$. However, $\vec{x}^*$ must violate the coalitional efficiency constraints: if CS^* is an optimal coalition structure for G, it cannot be the case that for any $S \in CS^*$, $\sum_{i \in S} x_i^* = v(S)$; if that were the case, then the core would not be empty. In other words, stable payoffs that pay a total of V^* do not describe viable payoff divisions.

2.2 Threshold Task Games

Threshold Task Games (TTGs) are a subclass of cooperative games. In TTGs, each player $i \in N$ has a *weight* $w_i \in \mathbb{Z}_+$. In addition, we are given a set of *tasks* $\mathcal{T} = \{t_1, \dots, t_m\}$, where each task $t_j \in \mathcal{T}$ has a *threshold* $q_j \in \mathbb{Z}_+$ and a *value* $v_j \in \mathbb{Z}_+$. The characteristic function v is $v(S) = \max_{j \in [m]}\{v_j : w(S) \geq q_j\}$. In other words, the value of a coalition S is the value of the best task it can complete, given the resources it controls. It is useful to think of $\mathcal{T}$ as a set of task *types* rather than actual tasks: if a coalition S completes the task t_j, other coalitions are free to complete it as well. We assume that if two tasks t_j and t_k have $q_j \leq q_k$ then $v_j \leq v_k$; otherwise no coalition will ever actually complete t_k. Finally, we assume that all tasks in $\mathcal{T}$ can be completed; that is, for every $t_j \in \mathcal{T}$, $q_j \leq w(N)$.
Weighted voting games (WVGs) are perhaps the best known example of TTGs; they are simply TTGs with one task with a value of 1. We forgo that assumption, and allow the value of winning coalitions in WVGs to be any positive number.

3 Approximate Solution Concepts in TTGs

We now present the main theoretical results in this paper: bounds on the cost of stability, and efficient approximation algorithms for the OptCS problem in TTGs. Throughout this section, we assume that $w_1 \geq \cdots \geq w_n$ and that $q_1 \geq \cdots \geq q_m$.

3.1 Approximately Optimal Coalition Structures

Nguyen and Zick [24] present a $\frac{1}{2}$ approximation algorithm for WVGs, and a $< \frac{1}{2}$ approximation for OptCS in general TTGs; in what follows, we present an efficient algorithm for computing a $\frac{1}{2}$ approximation to the OptCS problem in TTGs. The proof of Theorem 3.3 employs a simple dynamic programming algorithm for computing a $\frac{1}{2}$ optimal coalition structure; however, the proof of its correctness (in particular, Lemma 3.1) is non-trivial. We are interested in a special type of coalition structures: we say that a coalition structure CS is *contiguous* if there exist values $i_1 < \cdots < i_\ell = n$ such that CS equals

$$\{\{1, \dots, i_1\}, \{i_1 + 1, \dots, i_2\}, \dots, \{i_{\ell-1} + 1, \dots, i_\ell = n\}\}.$$

We begin by proving that there exists a contiguous coalition structure that has value at least $\frac{1}{2}$ of the optimal value (Lemma 3.1). The proof of Theorem 3.3 is completed by a dynamic programming algorithm that finds the best contiguous coalition structure.

Lemma 3.1. Given a TTG $G = \langle N, v \rangle$, there exists a contiguous coalition structure CS^* such that $v(CS^*) \geq \frac{1}{2} OPT(G)$.

[3]In the original work defining CoS [3], the cost of stability is defined as $V^* - OPT(G)$. Subsequent works (e.g. [10, 23]) utilize the relative definition we use here.

Proof. Let $CS^* = \{C_1, \dots, C_k, L\}$ be an optimal coalition structure where L is the only coalition of value 0. We assume that CS^* is an optimal coalition structure that assigns minimal weight to singleton coalitions, and that no players can be transferred to L without decreasing the value of CS^*. We assume w.l.o.g. that $C_1, \dots, C_k$ work on tasks $t_1^*, \dots, t_k^*$ such that $q_1^* \geq \dots \geq q_k^*$; in particular, this implies that the task values are weakly decreasing as well: $v_1^* \geq \dots \geq v_k^*$. We now construct a $\frac{1}{2}$-optimal contiguous coalition structure, where players complete tasks in $t_1^*, \dots, t_k^*$. The idea is to show that under the contiguous coalition structure, all of the odd-numbered tasks are completed. If this is the case, $\frac{1}{2}$ optimality is guaranteed. Suppose that k is even: since $v_1^* \geq \dots \geq v_k^*$, we have

$$\sum_{j=1}^{\frac{1}{2}k} 2v_{2j-1}^* \geq \sum_{j=1}^{k} v_j^* = OPT(G) \iff \sum_{j=1}^{\frac{1}{2}k} v_{2j-1}^* \geq \frac{1}{2} OPT(G)$$

When k is odd the above bound holds for the indices $1, \dots, k-1$, with the addition of the (odd numbered) k-th task.

We begin with a simple observation: since $C_1, \dots, C_k$ complete tasks $t_1^*, \dots, t_k^*$, it must be the case that $w(N \setminus L) \geq \sum_{j=1}^{k} q_j^*$. In other words, the combined weight of the coalitions $C_1, \dots, C_k$ must be, at the very least, as great as the combined thresholds of the tasks that they complete. Furthermore, it is no loss of generality to assume that if $\{i\}, \{i'\}$ are singleton coalitions in CS^* working on $t_{j(i)}^*, t_{j(i')}^*$, respectively, then $q_{j(i)}^* \geq q_{j(i')}^*$ implies that $w_i \geq w_{i'}$ — in other words, heavier singletons are assigned to higher-threshold tasks (otherwise, we can simply switch singleton players' assigned tasks and both tasks can still be completed). We utilize the following proposition (See Appendix A for the proof).

Proposition 3.2. *Suppose that CS^* is an optimal coalition structure assigning minimal weight to singleton coalitions, where heavier singleton players work on heavier tasks; let t_r^* be the task completed by a singleton s_r. Let*

$$\bar{w}_i = \begin{cases} w_i & \text{if } i \notin \{s_1, \dots, s_q\} \\ q_r^* & \text{if } i = s_r \end{cases}$$

Then for every $i, i' \in N$, $w_i \geq w_{i'} \iff \bar{w}_i \geq \bar{w}_{i'}$.

We assume w.l.o.g. that weights are already reduced to $\bar{w}_i$ as per Proposition 3.2; this reduction still respects the weight order, as argued in Proposition 3.2. Under this weight configuration, all tasks in CS^* can still be completed by the same players; furthermore, for all $j \in [k]$ and all $i \in C_j$, $w_i \leq q_j^*$; in particular, if we let $A(j) = \{i \in N : w_i \leq q_j^*\}$, then $C_j \subseteq A(j)$ for all $C_j \in CS^*$.

Consider Algorithm 1; its input is the thresholds $q_1^* \geq \dots \geq q_k^*$ of the tasks completed in an optimal coalition structure, and player weights $w_1 \geq \dots \geq w_n$; its output is a $\frac{1}{2}$ optimal contiguous coalition structure. Algorithm 1 first splits players' weight amongst tasks, forming a *fractional coalition structure* (lines 7–22). Player i allocates a weight of $c(j, i)$ to task t_j^*; $c(j, i)$ equals either the remaining weight that player i can allocate — this occurs if $w_i > q_{j+1}^*$, or if the task t_j^* requires more resources than what player i has — or just enough to complete t_j^*. Due to this condition, we are guaranteed that player i never assigns positive weight to tasks such that $w_i > q_j^*$, and in particular, players may not split their weight between more than two tasks. Since we assume that $C_j \subseteq A(j)$, the following invariant must hold:

$$\sum_{i \in N} r_i \geq \sum_{j \in [k]} \max\{q_j^* - y_j, 0\}$$

Note that $R_j = \max\{q_j^* - y_j, 0\}$ is the weight required to complete task t_j^*; simply put, the invariant states that players always maintain enough weight to complete all tasks. The coalition structure formed during this process is guaranteed to complete all optimal tasks, and is contiguous; however, it is not a valid partition of the players, as some players might commit fractions of their weight to multiple tasks.

Algorithm 1 Construct a $\frac{1}{2}$ optimal Contiguous CS

Input: thresholds $q_1^* \geq \cdots \geq q_k^*$; weights $w_1 \geq \cdots \geq w_n$
1: **for** $i \in N$ **do**
2: $\quad r_i \leftarrow w_i$ $\qquad\triangleright$ *r_i maintains the remaining weight player i can allocate to tasks*
3: **end for**
4: **for** $j \in [k]$ **do**
5: $\quad y_j \leftarrow 0$ $\qquad\triangleright$ *y_j maintains weight allocated to task t_j^**
6: **end for**
7: $j \leftarrow 1$; $i \leftarrow 1$
8: **while** $i < n$ and $j < k$ **do**
9: $\quad$ **if** $r_i < q_j^* - y_j$ or $w_i > q_{j+1}^*$ **then**
10: $\quad\quad c(j,i) \leftarrow r_i$ $\qquad\triangleright$ *allocate all of i's remaining resources to task j*
11: $\quad$ **else**
12: $\quad\quad c(j,i) \leftarrow q_j^* - y_j$
13: $\quad$ **end if**
14: $\quad r_i \leftarrow r_i - c(j,i)$
15: $\quad y_j \leftarrow y_j + c(j,i)$
16: $\quad$ **if** $r_i = 0$ **then**
17: $\quad\quad i \leftarrow i + 1$ $\qquad\triangleright$ *player i has nothing more to allocate; proceed.*
18: $\quad$ **end if**
19: $\quad$ **if** $y_j \geq q_j^*$ **then**
20: $\quad\quad j \leftarrow j + 1$ $\qquad\triangleright$ *task t_j^* complete; proceed to the next task*
21: $\quad$ **end if**
22: **end while**
23: **return** DeFrac($(c(j,i))_{i \in N, j=1,\ldots,k}$)
24: **function** DeFrac($(c(j,i))_{i \in N, j=1,\ldots,k}$))
25: $\quad$ **for** $\ell = 1, \ldots, k$ **do**
26: $\quad\quad$ **if** ℓ is odd **then**
27: $\quad\quad\quad S_\ell \leftarrow \{i \in N : c(\ell, i) > 0;\}$
28: $\quad\quad$ **else**
29: $\quad\quad\quad S_\ell \leftarrow \{i \in N : c(\ell, i) = w_i\}$
30: $\quad\quad$ **end if**
31: $\quad$ **end for**
32: $\quad L \leftarrow N \setminus \bigcup_{\ell=1}^k S_\ell$
33: $\quad$ **return** $CS = \{S_\ell : S_\ell \neq \emptyset\} \cup \{L\}$
34: **end function**

In other words, when we terminate in line 22, we complete all tasks $t_1^*, \ldots, t_k^*$ using contiguous blocks of players; however, some tasks may have fractional player weights assigned to them. However, note that our weight allocation procedure never assigns fractions of a player's weight to more than two contiguous blocks. In the DeFrac procedure, we ensure that every odd fractional coalition is assigned all players that assigned *any* weight to it. Since no player splits her weight to more than two coalitions, we are guaranteed that the procedure in lines 25–32 results in a valid coalition structure, where all odd-numbered tasks are completed. Since the value of the odd-numbered tasks is at least $\frac{1}{2}$ the value of the optimal coalition structure (as argued above), we are done. $\square$

Lemma 3.1 efficiently constructs a $\frac{1}{2}$-optimal contiguous coalition structure using Algorithm 1; however, in order to do so, it requires a *given* non-contiguous optimal coalition structure, which is NP-hard to compute, even when there is a single task [17]. To overcome this issue, we show that it is possible to compute an optimal *contiguous* coalition structure in polynomial time.

Theorem 3.3. *There exists an efficient algorithm computing a $\frac{1}{2}$ optimal coalition structure for any TTG G.*

Proof. We show that an optimal contiguous coalition structure CS^* can be found in polynomial time; the value of CS^* is at least the value of the contiguous coalition structure output by Algorithm 1, and thus by Lemma 3.1 has a value $\geq \frac{1}{2}OPT(G)$. Let $X(i,j)$ be the revenue extracted by an optimal contiguous coalition structure using only players in $\{i,\dots,n\}$ and tasks in $\{t_j,\dots,t_m\}$. Let ℓ be the first index such that $\{i,\dots,\ell-1\}$ can complete t_j, and generate a value of v_j; in this case the optimal revenue is $v_j + X(\ell,j)$. If t_j is not completed by a coalition of the form $\{i,\dots,\ell-1\}$, then the contiguity constraints require that no other coalition complete t_j; in this case, the optimal value is $X(i,j+1)$. Therefore, $X(i,j) = \max\{v_j + X(\ell,j), X(i,j+1)\}$ where $\ell = \min\{s : w_i + \cdots + w_{s-1} \geq q_j\}$. It is easy to keep track of the actual coalitions formed from the $X(i,j)$ values; since $X(1,1)$ is the value of the optimal contiguous coalition structure, we are done. $\square$

3.2 The Cost of Stability in TTGs

In what follows, we analyze the cost of stability for TTGs. Nguyen and Zick [24] show that $CoS(G) \leq 2$ for TTGs where players are not allowed to form singleton coalitions. We now show how one can forgo this assumption. Given a task $t_j = \langle q_j, v_j\rangle$, and a coalition $S \subseteq N$, let $OPT(S,t_j)$ be the optimal revenue of the WVG restricted to S, where players may only complete the task t_j. We begin with the following lemma (See Appendix B for the proof).

Lemma 3.4. Suppose we are given k disjoint subsets of players $T_1,\dots,T_k$ and k tasks $t_1,\dots,t_k$ such that for every $i \in T_s$, $w_i \leq q_s$ and $q_1 < \cdots < q_k$. Assume that for every $s \in [2..k]$ and every player $i \in T_s$, $w_i \geq q_{s-1}$. If $i \in T_s$, let $x_i = \alpha_s w_i$ with $\alpha_s \leq \frac{v_s}{q_s}$. Then

$$\sum_s OPT(T_s,t_s) \geq \frac{1}{2}\sum_s x(T_s) - \frac{v_k}{2}.$$

We are now ready to prove our main result for this section.

Theorem 3.5. *For any TTG G, $CoS(G) \leq 2$.*

Proof. We can assume with no loss of generality that $\max_i w_i \leq \max_j q_j$. Given a payoff division $\vec{x} \in \mathbb{R}^n$, we call a task $t = (q,v)$ critical if there exists $S \subset N$ such that such that $w(S) \geq q$ and $x(S) = v$. A coalition S whose value is derived by completing a critical task will also be referred to as critical.

Let α_1 be the minimum value such that the payoff $x_1(i) = \alpha_1 w_i$ is stable. There must exist a critical task for $\vec{x}_1$. Let $t_1 = (q_1, v_1)$ be the critical task with the highest threshold, and let S_1 be a critical task for t_1 under $\vec{x}_1$. Since $x_1(S_1) = \alpha_1 w(S_1)$, we have $\alpha_1 = \frac{x_1(S_1)}{w(S_1)} = \frac{v_1}{w(S_1)} \leq \frac{v_1}{q_1}$. Let $T_1 = \{i \in N : w_i \leq q_1\}$. Because we assume that t_1 is the highest-threshold critical task, every critical set must be a subset of T_1.

Given j pairs $(T_1,\alpha_1),\dots,(T_j,\alpha_j)$, if $\cup_{s=1}^{j} T_s \neq N$, we form (T_{j+1},α_{j+1}) as follows: Let α_{j+1} be the minimum number such that the following payoff division is stable:

$$x_{j+1}(i) = \begin{cases} \alpha_s w_i \text{ if } i \in T_s \text{ for some } s \leq j, \\ \alpha_{j+1} w_i \text{ otherwise.} \end{cases} \tag{2}$$

Let $t_{j+1} = (q_{j+1}, v_{j+1})$ be the critical task with the highest threshold under $\vec{x}_{j+1}$. Let $T_{j+1} = \{i \in N \setminus \cup_{s=1}^{j} T_s : w_i \leq q_{j+1}\}$.
We note that $q_1 < \cdots < q_j < q_{j+1}$ and $\alpha_1 > \cdots > \alpha_j > \alpha_{j+1}$. Let S_{j+1} be a critical set for t_{j+1}; then $x_{j+1}(S_{j+1}) = v_{j+1} \geq \alpha_{j+1} w(S) \geq \alpha_{j+1} q_{j+1}$. Hence we have $\alpha_{j+1} \leq \frac{v_{j+1}}{q_{j+1}}$.
Suppose that in the end we have k pairs $(T_1, \alpha_1), \ldots, (T_k, \alpha_k)$. Consider the payoff $x(i) = \alpha_s w_i$ if $i \in T_s$; this payoff division is stable by our construction: we always maintain the stability constraints in the reduction process, as per (2). Under $\vec{x}$, the critical task with highest threshold is t_k corresponding to T_k. Let S_k be a critical set corresponding to t_k. In particular we have $x(S_k) = v_k$. Let $T'_s = T_s \setminus S_k$. We can verify that $T'_1, \ldots, T'_k$ satisfy the conditions in Lemma 3.4. Therefore we have

$$\sum_s OPT(T'_s, t_s) \geq \frac{1}{2} \sum_s x(T_s) - \frac{v_k}{2}.$$

Hence

$$\begin{aligned} OPT(G) &\geq v(S_k) + \sum_s OPT(T'_s, t_s) \geq v_k + \frac{1}{2} \sum_s x(T_s) - \frac{v_k}{2} \\ &= \frac{1}{2}(x(S_k) + v_k) + \frac{1}{2} \sum_s x(T_s) - \frac{v_k}{2} = \frac{1}{2} x(N). \end{aligned}$$

and $x(N) \leq 2 OPT(G)$. □

Nguyen and Zick [24] additionally show that for any $\varepsilon > 0$ there exists some WVG G for which $CoS(G) \geq 2 - \varepsilon$, thus the bound in Theorem 3.5 is tight.
To conclude, when one allows players to form coalition structures (rather than sticking to forming the grand coalition as in [3]), the cost of stability in threshold task games decreases dramatically. In addition, there exist efficient algorithms for computing $\frac{1}{2}$-optimal coalition structures. These results indicate that while exact solutions for TTGs may not exist (in the case of the core), or are computationally intractable to compute (in the case of the optimal coalition structure), arriving at good approximate solutions is possible. However, theoretical guarantees say very little about how humans actually play TTGs. We explore this issue in Section 4.

4 The Business Cats Platform

To investigate human play in TTG, we design and implement a web-based negotiation game called Business Cats[4]. Our objective was to create a game environment that 1. closely emulates TTGs 2. is intuitive and fun to play and 3. allows players to strategically reason about their bargaining power. As seen in Figure 1, players play the roles of cats; each cat owns a ladder (corresponding to the weight of the player) that can be used to climb ledges of different heights (corresponding to task thresholds). There are fish of different values on top of the ledges (corresponding to task rewards). Players negotiate to form coalitions, combining their ladders to reach the fish and share the rewards.
Our design closely emulates the TTG formulation as described in Section 2.2; moreover, it allows for intuitive and simple gameplay: this is evidenced from extensive feedback and design sessions with our experimental subjects, both in lab settings and on Amazon Mechanical Turk. Our interface models an iterative bargaining process, allowing players to easily reason about their bargaining power. Indeed, prior work on collaborative interactions among

[4] Available at http://business-cats.comp.nus.edu.sg/. In order to play you will need at least three participants. Alternatively, one can open the website on different browsers/devices to simulate multiple players.

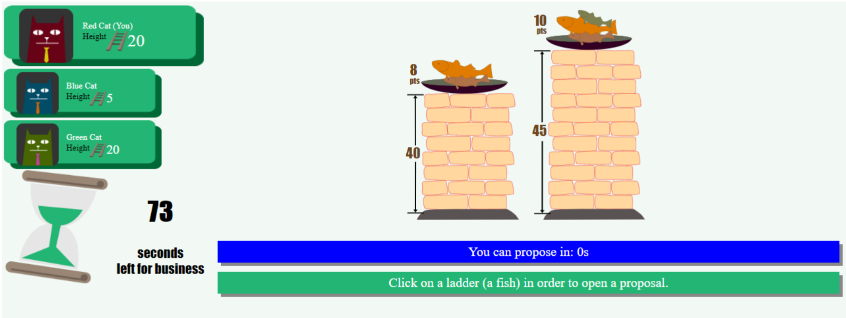

Figure 1: Snapshot of the Business Cats GUI (3 players and 2 tasks)

human players uses repeated interaction in order to facilitate strategic gameplay [6, 22]. This is not unreasonable: as Aumann and Maschler [2] describe, stable payoff divisions can be arrived at via a natural process of coalitions offering proposals and counter-proposals. In particular, such a sequence of proposals and counter-proposals converges to an outcome in the *bargaining set* [15]; this cooperative solution concept contains the core; thus, if the core is not empty, payoff divisions in the core can be arrived at via a simple bargaining process. The bargaining process described by Aumann and Maschler [2] is very similar to the one we implement in the Business Cats platform, with the exception that *individual players* (rather than coalitions) propose and counter-propose. To conclude, the Business Cats platform satisfies all of our stated desiderata.

The game consists of proposal and response phases. Unlike the turn-based process used by Mash et al., we allow players to initiate a proposal at any time they wish, *as long as no other proposal is active*; in other words, there is only one 'live' proposal at any given point in time. A proposal specifies a prospective task to work on, the coalition members that will work on it, and the proposed share of rewards. In order for a proposal to be submitted, the invited members must be able to complete the proposed task; the total payoff must equal the task value, and the proposer must receive a positive payoff. Upon submitting a proposal, all players see the proposal (and cannot submit additional proposals); players who were offered a positive share of the payoffs must *respond* to the proposal with an accept/reject[5]. Responders cannot see each other's response until all players respond; if some responder does not accept/reject within 30 seconds, the proposal is automatically rejected. If all responders accept the proposal, the coalition is formed, involved players receive the share they accepted, and are removed from the game. If the remaining players can still complete some task, the remaining players can still submit proposals; otherwise the game terminates, and uninvolved players receive a payoff of 0. If *some* responder rejects the proposal, a cool-down period (30 seconds for proposer, 15 seconds for the rest) is initiated, after which new proposals can be submitted. The cool-down period ensures that players have time to consider the game state, and prevents players from spamming proposals. The proposal history (including responses), and the current pending proposal are visible to every player. The game automatically terminates after 5 minutes; however, the vast majority of instances ended well before the 5-minute mark, taking less than 1 minute on average.

[5]The proposing player is naturally assumed to accept their own proposal.

5 Experiments

We generate TTGs instances, varying the number of players (3–5); weights (multiples of 5, no greater than 25); number of tasks (1–4); task thresholds (multiples of 5, no greater than 100), task values (from the number of players to 10) and whether singleton coalitions were available. We constrain that greater task thresholds imply greater task values. During the experiments, games instances were randomly drawn from this pre-generated pool of games.

5.1 Design

We recruit 104 participants (undergraduate students from the National University of Singapore and Ben-Gurion University of the Negev, Israel). IRB approval was obtained from the institutions running the study. Participants were given a detailed tutorial, and were required to pass a comprehension quiz in order to play. Participants play a random series of games with different configurations, and are randomly matched to other players. Each participant receives a show-up fee equivalent to \$6.5 US as well as a bonus (between 0 and \$6.5) dependent on their total revenue in all the games they play. In total, we collected data from 857 game configurations.

We wish to study how people play the game in terms of how they form and respond to proposals. Specifically, we form the following hypotheses: first, that people would generally form "good" coalition structures in terms of total revenue, and coalitional stability, as defined in Sections 3.1 and 3.2; second, that people respond to offers in a way that respects the power relationship in the game, as defined by their relative weight.

5.2 Analysis

We analyze the games played in terms of the number of optimal and stable coalitions formed, and how people accepted and made offers with respect to the power relationships in the game. We present aggregate results across the two countries as there was no significant difference in player behavior.

Successful Coalitions and Optimality: Tables 1a and 1b show the games collected with respect to number of players and tasks, and the percentage of games for which players reach the optimal coalition structure. These tables show that as the number of players and tasks grow, player performance decreases (as measured by overall welfare).

# players	3	4	5	Total
# games	392	277	188	857
Opt. (%)	54.6	49.5	48.9	51.7

(a) # of players.

# tasks	1	2	3	4	Total
# games	88	273	283	213	857
Opt. (%)	92.0	56.4	43.5	39.9	51.7

(b) # of tasks.

Table 1: # of games collected and percentage optimal coalition structures by game type

Figure 2 shows a histogram of the welfare achieved by players during gameplay, normalized by $OPT(G)$. Here, the x axis is the ratio between the actual social welfare achieved in the game and the optimal social welfare. Figure 2 shows that in more than 75% of instances, players were able to extract at least 75% of the optimal revenue. In 51.7% of the game instances, players were able to reach the optimal coalition value. On average, players were able to extract 87% of the optimal value.

Not shown in Figure 2 is the fact that in 86% of the cases, participants formed a non-trivial coalition structure — i.e. one containing more than one coalition. When the grand coalition

N was formed, it was the optimal choice 86% of the time. Together these results support the first part of hypothesis 1, showing that players were able to create approximately optimal coalition structures for different number of players and tasks.

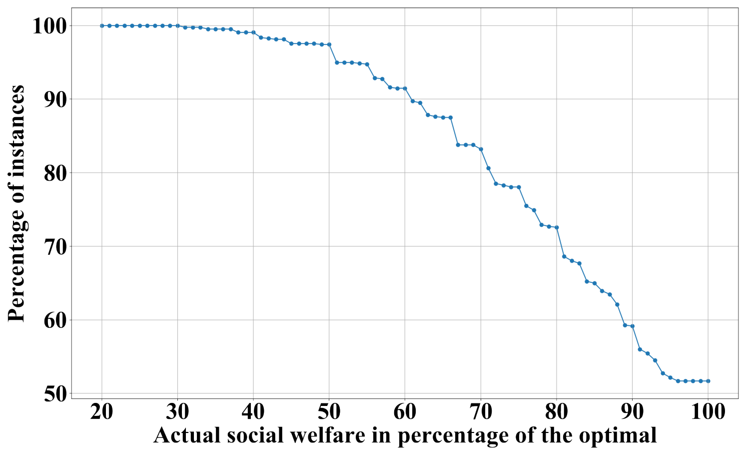

Figure 2: Welfare achieved during gameplay, as a percentage of the optimal welfare.

Stability Analysis: In this section we constrain the analysis to the games in which there was a non empty core (537 games). To measure the distance from a payoff division $\vec{x}$ to a stable payoff, we solve the following optimization problem (using a linear program):

$$\min \sum_{i \in N} |\alpha_i| \quad \text{s.t. } \forall S \subseteq N : \sum_{i \in S} x_i + \alpha_i \geq v(S), \sum_{i \in N} x_i + \alpha_i = OPT(G) \tag{3}$$

where $OPT(G)$ is the value of an optimal coalition structure. Given a solution to (3) $\vec{\alpha}^*$, we let $\frac{\sum_{i \in N} |\alpha_i^*|}{OPT(G)}$ be the distance of $\vec{x}$ from a stable payoff division.

In general, 32.5% of the coalitions achieved stable outcomes, and more than 50% were at a distance of $<$ 20% from a stable payoff. This result supports the second half of the first hypothesis, showing that our bargaining process was able, on average, to arrive at approximately stable outcomes.

Power Analysis: To study the second hypothesis, we first consider the acceptance ratio for players of different weights in the game. Figure 3 shows the CDF of acceptance rates for the players with the highest and lowest weight in a game. For example, players with the highest weight in a game instance accepted a 40% share of the profits in approximately 20% of instances, whereas the lowest-weight players did so for approximately 70% of instances. In both cases, acceptance rates rise with share percentage, with a "jump" to more than 50% acceptance rate for shares above 30% for lowest-weight players, and approximately 50% for highest-weight players. This confirms that players with higher weights do, in general, understand their relative power in the game.

We also analyze how players respond to offers in the game. We apply the definition of Mash et al. [22] and say an offer (x_i, x_j) made to responders i and j with weights (w_i, w_j), is *power preserving* if $x_i \leq x_j$ whenever $w_i \leq w_j$. Overall, 213 out of 1824 (11.7%) were non power preserving, which is considerably lower than the 40% ratio reported for WVGs [22]. In addition, 63.8% of non power preserving offers were rejected, whereas the rejection rate for power preserving offers was just 32.6%. Put together, these results confirm the second

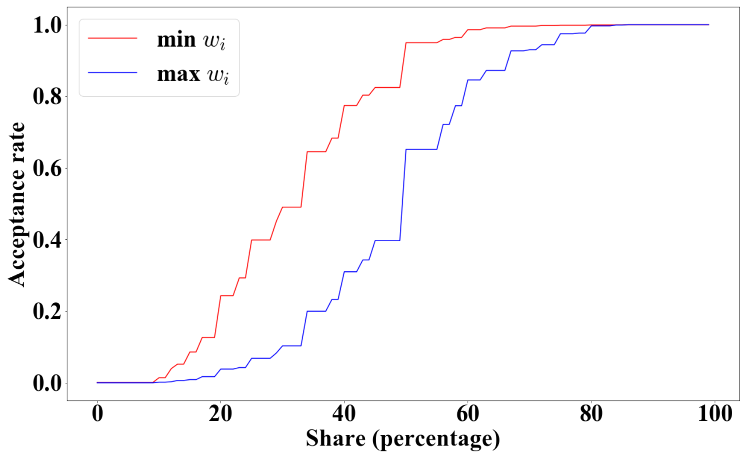

Figure 3: Cumulative acceptance rate of the most and the least powerful players in a game

hypothesis: people generally make offers that align with responders' weight, and responders were more likely to accept such offers. We mention that similar results hold when other measures of player power (such as the Shapley or Banzhaf values) are used.

Active Participation: While player negotiation tactics are diverse, we do note that by actively proposing, players can significantly improve their prospects. That is, when players initiate proposals, they are more likely to receive better payoffs. Within the 857 games played, there were 1655 instances of (passive) players ending the game without making a proposal, and 1569 instances of (active) players making at least one offer. Active participation directly affects one's prospective payoff; one simple measure of this property is one's likelihood to benefit from cooperation. If player i does form coalitions with others, the highest payoff that they can secure is $v(\{i\})$. If player i secures a payoff strictly greater than $v(\{i\})$ then they strictly benefit from cooperation. Passive players, who make no proposals, strictly benefit from cooperation in only 61% of instances; on the other hand, active players (those who made at least one proposal) strictly benefit from cooperation in 75% of the instances.
The results of our empirical study indicate that human players reach efficient, approximately stable outcomes in TTGs, and that *actual* bargaining processes, mirroring theoretical bargaining processes [2], yield approximately stable outcomes.

6 Conclusions and Future Work

In this work, we conduct a thorough theoretical and empirical analysis of threshold task games. We provide an efficient algorithm computing a $\frac{1}{2}$-optimal coalition structure, and provide tight bounds on the cost of stability. The Business Cats platform is the first publicly available interactive simulation of a TTG environment; the code and data are currently anonymized (to maintain a proper review process); however, we intend to release them to the research community upon publication. Our analysis indicates that people tend to form coalitions that are nearly stable and optimal, and that players who actively make proposals tend to obtain higher rewards. It is our hope that more platforms like ours will be designed and used to test cooperative game-theoretical models and hypotheses in the wild. This is an important direction to pursue if one is interested in seeing the translation of theoretical notions from cooperative game theory into practical applications.

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A Proof of Proposition 3.2

Proposition 3.2. *Suppose that CS^* is an optimal coalition structure assigning minimal weight to singleton coalitions, where heavier singleton players work on heavier tasks; let t_r^* be the task completed by a singleton s_r. Let*

$$\bar{w}_i = \begin{cases} w_i & \text{if } i \notin \{s_1, \dots, s_q\} \\ q_r^* & \text{if } i = s_r \end{cases}$$

Then for every $i, i' \in N$, $w_i \geq w_{i'} \iff \bar{w}_i \geq \bar{w}_{i'}$.

Proof. Since heavier singletons work on heavier tasks, reducing singletons' weight to be their task thresholds preserves their relative weight. Let $Q(s_r) = \{i \notin \{s_1, \dots, s_q\} : w_i \leq w_{s_r}\}$. Suppose that $q_r^* < w_i$ for some $i \in Q(s_r)$; since i can complete t_r^*, and s_r has a greater weight than i, we can assign i to t_r^*, and assign s_r to the coalition that i was in (since $w_{s_r} \geq w_i$, that coalition has a weakly higher value). This results in an optimal coalition structure that puts a strictly lower weight on singletons, a contradiction. □

B Proof of Lemma 3.4

Lemma 3.4. Suppose we are given k disjoint subsets of players $T_1, \dots, T_k$ and k tasks $t_1, \dots, t_k$ such that for every $i \in T_s$, $w_i \leq q_s$ and $q_1 < \dots < q_k$. Assume that for every $s \in [2..k]$ and every player $i \in T_s$, $w_i \geq q_{s-1}$. If $i \in T_s$, let $x_i = \alpha_s w_i$ with $\alpha_s \leq \frac{v_s}{q_s}$. Then

$$\sum_s OPT(T_s, t_s) \geq \frac{1}{2} \sum_s x(T_s) - \frac{v_k}{2}.$$

Proof. For $s \in [2..k]$, let the players in T_s with weight $w_i \leq q_s$ play a WVG with task t_s. Each winning coalition weighs less than $2q_s$ and there is at most one losing coalition. If there is no losing coalition, we have $w(T_s) \leq 2q_s \frac{OPT(T_s,t_s)}{v_s}$ or, $OPT(T_s, t_s) \geq \frac{v_s}{2q_s} w(T_s) \geq \frac{x(T_s)}{2}$. If there is one losing coalition, any player i in this losing coalition has $w_i \geq q_{s-1}$ and thus $v(\{i\}) \geq v_{s-1}$. Therefore $w(T_s) \leq 2q_s \frac{OPT(T_s,t_s)-v_{s-1}}{v_s} + q_s$ or, $OPT(T_s, t_s) \geq \frac{v_s}{2q_s} w(T_s) - \frac{v_s}{2} + v_{s-1} \geq \frac{x(T_s)}{2} - \frac{v_s}{2} + \frac{v_{s-1}}{2}$.
Since $v_s > v_{s-1}$, we have in any case for $s \in [2, \dots, k]$, $OPT(T_s, t_s) \geq \frac{x(T_s)}{2} - \frac{v_s}{2} + \frac{v_{s-1}}{2}$. Similarly, if $s = 1$, we have $OPT(T_1, t_1) \geq \frac{x(T_1)}{2} - \frac{v_1}{2}$. Summing up the inequalities, we have

$$\sum_s OPT(T_s, t_s) \geq \frac{1}{2} \sum_s x(T_s) - \frac{v_k}{2}.$$

□

C An Example Run of the Business Cats Gameplay

Example 3. Figure 4 shows an example run of the Business Cats gameplay of the instance shown in Figure 1. This instance has three players: the blue cat has a weight of 5, the red and green cats have a weight of 20. There are two tasks: t_1 has a threshold $q_1 = 40$ and value $v_1 = 8$, t_2 has a threshold $q_2 = 45$ and value $v_2 = 10$. The green cat made the first proposal, offering to have all players complete t_1 with the payoff division $(2, 2, 4)$; however, the red cat rejected this offer. The green cat subsequently proposed all players completing t_2 with the payoff division $(2, 2, 6)$; the red cat rejected this offer as well. Finally, the red cat proposed all players completing t_2, with the payoff division $(2, 4, 4)$; all players accepted this proposal, and the game ended.

		Player:				
		Weight:	5	20	20	
Proposer	**Task**					**Result**
Initial Cool-off period – 15 sec.						
1st	(40, 8)	Proposal	2	2	4	✗
		Response	✓	✗	✓	
Cool-off period – 15/30 sec.						
2nd	(45, 10)	Proposal	2	2	6	✗
		Response	✓	✗	✓	
Cool-off period – 15/30 sec.						
3rd	(45, 10)	Proposal	2	4	4	✓
		Response	✓	✓	✓	
No more tasks can be completed – game ends.						

Figure 4: An example run of the Business Cats gameplay.

D Case Studies

As we show in Section 5.2, Business Cats players were able to achieve overall good performance. However, as one might expect, human play exhibits a high level of complexity. We observe instances where players (a) propose suboptimal coalitions (as was the case in the first proposal in Example 3); (b) accept unreasonable proposals, especially later during game progression; or (c) reject reasonable proposals. We provide two examples from our collected data that illustrate some good/bad behaviors.

Example 4. The game consists of four players with weights $w_1 = w_2 = w_3 = 15, w_4 = 20$, and two tasks $t_1 = \langle 35, 4 \rangle; t_2 = \langle 45, 7 \rangle$. Player 4 has a strictly higher weight, and greater influence on the game (as measured, by the Shapley value [27]) than the other players. However, player 4 consistently refused to form coalitions where their share was equal to that of others. Eventually, players 1,2 and 3 agreed to complete t_2, excluding player 4, and receiving a reward of 2, 3, 2, respectively.

In Example 4, player 4 had a potentially decisive role, but received zero payoff. Indeed, player weights are not the sole determinant of payoffs. In section 5.2 we show that payoffs are affected by other factors such as player propensity to actively initiate proposals.

Example 5. The game consists of three players ($w_1 = 25, w_2 = 10, w_3 = 15$), and two tasks ($t_1 = \langle 25, 3\rangle; t_2 = \langle 45, 8\rangle$). In this example, Player 2 offered to form a coalition with players 1 and 3 to complete t_2, offering both players a payoff of 2. Player 1 rejected this offer. Player 2 subsequently offered to form the same coalition offering player 1 a value of 4 and player 3 a value of 2. This coalition was successful.

In Example 5, player 1 could form a singleton coalition and complete t_1 with a revenue of 3. However, player 1 accepted an offer to form a coalition with players 2 and 3, once they were offered a better payoff. Indeed, the resulting outcome is core stable (and, in particular, optimal).

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Kobi Gal
Ben-Gurion University of the Negev, Israel & University of Edinburgh, U.K.
Edinburgh, U.K.
Email: kgal@inf.ed.ac.uk

Ta Duy Nguyen
National University of Singapore
Singapore
Email: taduy@comp.nus.edu.sg

Quang Tran Nhat
National University of Singapore
Singapore
Email: e0072418@u.nus.edu

Yair Zick
University of Massachusetts, Amherst
Amherst, MA, USA
Email: yzick@umass.edu