Approval-Based Apportionment*

Markus Brill, Paul Gölz, Dominik Peters,
Ulrike Schmidt-Kraepelin, and Kai Wilker

Abstract

In the apportionment problem, a fixed number of seats must be distributed among parties in proportion to the number of voters supporting each party. We study a generalization of this setting, in which voters cast approval ballots over parties, such that each voter can support multiple parties. This approval-based apportionment setting generalizes traditional apportionment and is a natural restriction of approval-based multiwinner elections, where approval ballots range over individual candidates. Using techniques from both apportionment and multiwinner elections, we identify rules that generalize the D'Hondt apportionment method and that satisfy strong axioms which are generalizations of properties commonly studied in the apportionment literature. In fact, the rules we discuss provide representation guarantees that are currently out of reach in the general setting of multiwinner elections: First, we show that core-stable committees are guaranteed to exist and can be found in polynomial time. Second, we demonstrate that extended justified representation is compatible with committee monotonicity (also known as house monotonicity).

1 Introduction

The fundamental fairness principle of *proportional representation* is relevant in a variety of applications ranging from recommender systems (Lu and Boutilier, 2011) to digital democracy (Brill, 2018). It features most explicitly in the context of political elections, which is the language we adopt for this paper. In this context, proportional representation prescribes that the number of representatives championing a particular opinion in a legislature be proportional to the number of voters who favor that opinion.

In most democratic institutions, proportional representation is implemented via *party-list elections*: Candidates are members of political parties and voters are asked to indicate their favorite party; each party is then allocated a number of seats that is (approximately) proportional to the number of votes it received. The problem of transforming a voting outcome into a distribution of seats is known as *apportionment*. Analyzing the advantages and disadvantages of different apportionment methods has a long and illustrious political history and has given rise to a deep and elegant mathematical theory (Balinski and Young, 1982; Pukelsheim, 2014).

Unfortunately, forcing voters to choose a single party prevents them from communicating any preferences beyond their most preferred alternative. For example, if a voter feels equally well represented by several political parties, there is no way to express this preference within the voting system. In the context of single-winner elections, *approval voting* has been put forward as a solution to this problem as it strikes an attractive compromise between simplicity and expressivity (Brams and Fishburn, 2007; Laslier and Sanver, 2010). Under approval voting, each voter is asked to specify a set of candidates she "approves of," i.e., voters can arbitrarily partition the set of candidates into approved candidates and disapproved ones. Proponents of approval voting argue that its introduction could increase voter turnout, "help elect the strongest candidate," and "add legitimacy to the outcome" of an election (Brams and Fishburn, 2007, pp. 4–8).

Due to the practical and theoretical appeal of approval voting in single-winner elections, a number of scholars have suggested to also use approval voting for multiwinner elections, in which a fixed number of candidates needs to be elected (Kilgour and Marshall, 2012). In contrast to the single-winner setting, where the straightforward voting rule "choose the candidate approved by the highest

---

*An earlier version of this paper appeared in the proceedings of AAAI-2020 (Brill et al., 2020).

number of voters" enjoys a strong axiomatic foundation (Fishburn, 1978), several ways of aggregating approval ballots have been proposed in the multiwinner setting (Kilgour and Marshall, 2012; Lackner and Skowron, 2020).

Most studies of approval-based multiwinner elections assume that voters directly express their preference over individual candidates; we refer to this setting as *candidate-approval* elections. This assumption runs counter to widespread democratic practice, in which candidates belong to political parties and voters indicate preferences over these parties (which induce implicit preferences over candidates). In this paper, we therefore study *party-approval* elections, in which voters express approval votes over parties and a given number of seats must be distributed among the parties. We refer to the process of allocating these seats as *approval-based apportionment*.

We believe that party-approval elections are a promising framework for legislative elections in the real world. Allowing voters to express approval votes over parties enables the aggregation mechanism to coordinate like-minded voters. For example, two blocks of voters might currently vote for parties that they mutually disapprove of. Using approval ballots could reveal that the blocks jointly approve a party of more general appeal; allocating more seats to this party leads to mutual gain. This cooperation is particularly necessary for small minority opinions that are not centrally coordinated. In such cases, finding a commonly approved party can make the difference between being represented or votes being wasted because the individual parties receive insufficient support.

One aspect that makes it easier to transition from party-list elections to party-approval elections (rather than to candidate-approval elections) is that party-approval elections can be implemented as *closed-list* systems. That is, parties can retain the power to choose the ordering in which their candidates are allocated seats, which parties have in many current democratic systems around the world. By contrast, candidate-approval elections necessarily confer this power to the voters (leading to an *open-list* system), which might give parties an incentive to oppose a change of the voting system. Of course, party-approval elections are compatible with an open-list approach, since we can run a secondary mechanism alongside the party-approval election to determine the order of party candidates.

1.1 Related Work

To the best of our knowledge, this paper is the first to formally develop and systematically study approval-based apportionment. That is not to say that the idea of expressing and aggregating approval votes over parties has not been considered before. Indeed, several scholars have explored possible generalizations of existing aggregation procedures.

For instance, Brams et al. (2019) study multiwinner approval rules that are inspired by classical apportionment methods. Besides the setting of candidate approval, they explicitly consider the case where voters cast party-approval votes. They conclude that these rules could "encourage coalitions across party or factional lines, thereby diminishing gridlock and promoting consensus."

Such desire for compromise is only one motivation for considering party-approval elections, as exemplified by recent work by Speroni di Fenizio and Gewurz (2019). To allow for more efficient governing, they aim to concentrate the power of a legislature in the hands of few big parties, while nonetheless preserving the principle of proportional representation. To this end, they let voters cast party-approval votes and transform these votes into a party-list election by assigning each voter to one of her approved parties. One method for doing this (referred to as *majoritarian portioning* later in this paper) assigns voters to parties in such a way that the strongest party has as many votes as possible.

Several other papers consider extensions of approval-based voting rules to accommodate party-approval elections: In their paper introducing the *satisfaction approval voting* rule, Brams and Kilgour (2014) also discuss a variant based on party-approval votes. Mora and Oliver (2015) study a version of Phragmén's sequential rule (Phragmén, 1894) in which so-called "collective candidatures" can be allocated more then one seat, and analyze the rule's limit behavior as the number of seats grows

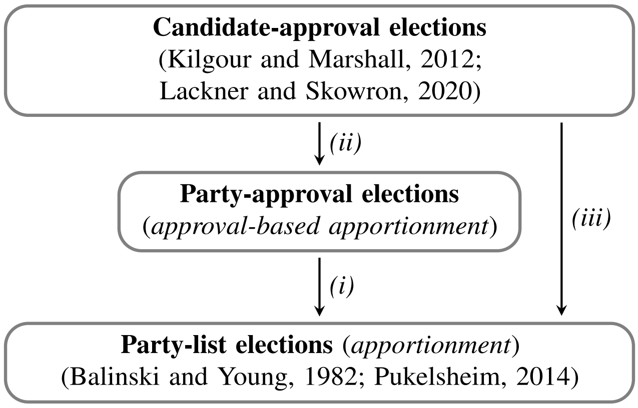

Figure 1: Relations between the different settings of multiwinner elections. An arrow from X to Y signifies that X is a generalization of Y. The relationship corresponding to arrow *(iii)* has been explored by Brill et al. (2018). We establish and explore the relationship *(i)* in Section 3 and the relationship *(ii)* in Section 4.

large. This party version of Phragmén's rule is further analyzed by Janson and Öberg (2019), who also consider a party version of Thiele's sequential rule (Thiele, 1895). Each of these papers studies a specific rule, whereas our aim is to explore the party-approval setting in its own right.

1.2 Relation to Other Settings

Party-approval elections can be positioned between two well-studied voting settings (see Figure 1).

First, approval-based apportionment *generalizes standard apportionment*, which corresponds to party-approval elections in which all approval sets are singletons. This relation (depicted as arrow *(i)* in Figure 1) provides a generic two-step approach to define aggregation rules for approval-based apportionment problems: transform a party-approval instance to an apportionment instance, and then apply an apportionment method. In Section 3, we employ this approach to construct approval-based apportionment methods satisfying desirable properties.

Second, our setting can be viewed as a *special case of approval-based multiwinner voting*, in which voters cast *candidate-approval* votes. A party-approval election can be embedded in this setting by replacing each party by multiple candidates belonging to this party, and by interpreting a voter's approval of a party as approval of all of its candidates. This embedding establishes party-approval elections as a subdomain of candidate-approval elections (see arrow *(ii)* in Figure 1). In Section 4, we explore the axiomatic and computational ramifications of this domain restriction.

1.3 Contributions

In this paper, we formally introduce the setting of approval-based apportionment and explore different possibilities of constructing axiomatically desirable aggregation methods for this setting. Besides its conceptual appeal, this setting is also interesting from a technical perspective.

Exploiting the relations described in Section 1.2, we resolve problems that remain open in the more general setting of candidate-approval elections. First, we prove that committee monotonicity is compatible with extended justified representation (a representation axiom proposed by Aziz et al., 2017) by providing a rule that satisfies both properties. Second, we show that the core of an approval-based apportionment problem is always nonempty, and that a popular multiwinner rule known as Proportional Approval Voting (PAV) always returns a core-stable committee. In addition, we present a polynomial-time algorithm that constructs a core-stable committee.

While some familiar multiwinner rules (in particular, PAV) provide stronger representation guarantees when applied in the party-approval setting, this is not the case for most other known rules. For a wide range of multiwinner voting rules, we give examples that show that their axiomatic guarantees do not improve in the party-approval setting. From a computational complexity perspective, we show that some rules known to be NP-hard in the candidate-approval setting remain NP-hard to evaluate in the party-approval setting. While calculating voting rules does not appear to become easier in party-approval, we find that it becomes computationally easier to reason about proportionality axioms. In particular, we show that it becomes tractable to check whether a given committee is proportional, in the sense of providing extended justified representation or the weaker axiom of proportional justified representation. The analogs of these problems for candidate-approval elections are NP-hard.

2 The Model

A *party-approval election* is a tuple (N, P, A, k) consisting of a set of voters $N = \{1, \ldots, n\}$, a finite set of parties P, a ballot profile $A = (A_1, \ldots, A_n)$ where each ballot $A_i \subseteq P$ is the set of parties approved by voter i, and the committee size $k \in \mathbb{N}$. We assume that $A_i \neq \emptyset$ for all $i \in N$. When considering computational problems, we assume that k is encoded in unary (see Footnote 5).

A *committee* in this setting is a multiset $W : P \to \mathbb{N}$ over parties, which determines the number of seats $W(p)$ assigned to each party $p \in P$. The size of a committee W is given by $|W| = \sum_{p \in P} W(p)$, and we denote multiset addition and subtraction by $+$ and $-$, respectively. A *party-approval rule* is a function that takes a party-approval election (N, P, A, k) as input and returns a committee W of valid size $|W| = k$.[1]

In our axiomatic study of party-approval rules, we focus on axioms capturing proportional representation: proportional justified representation (Sánchez-Fernández et al., 2017b), extended justified representation, and core stability (Aziz et al., 2017). These axioms are derived from their analogs in multiwinner elections (see Section 4.2) and can be defined in terms of quota requirements.

For a party-approval election (N, P, A, k) and a subset $S \subseteq N$ of voters, define the *quota of S* as $q(S) = \lfloor k \cdot |S|/n \rfloor$. Intuitively, $q(S)$ corresponds to the number of seats that the group S "deserves" to be represented by (rounded down).

Definition 1. *A committee $W : P \to \mathbb{N}$ provides* extended justified representation (EJR) *for a party-approval election (N, P, A, k) if there is no subset $S \subseteq N$ of voters such that $\bigcap_{i \in S} A_i \neq \emptyset$ and $\sum_{p \in A_i} W(p) < q(S)$ for all $i \in S$. W provides* proportional justified representation (PJR)*, if there is no subset $S \subseteq N$ of voters such that $\bigcap_{i \in S} A_i \neq \emptyset$ and $\sum_{p \in \bigcup_{i \in S} A_i} W(p) < q(S)$.*

In words, EJR requires that for every voter group S with a commonly approved party, at least one voter of the group should be represented by $q(S)$ many candidates. The PJR requirement for such groups is strictly weaker: the committee should contain at least $q(S)$ candidates from parties approved by some voter in S. Therefore, a committee providing EJR also provides PJR. A party-approval rule is said to *satisfy EJR* (respectively, *PJR*) if it only produces committees providing EJR (respectively, PJR).

We can obtain a stronger representation axiom by removing the requirement of a commonly approved party.

Definition 2. *A committee $W : P \to \mathbb{N}$ is* core stable *for a party-approval election (N, P, A, k) if there is no nonempty subset $S \subseteq N$ and committee $T : P \to \mathbb{N}$ of size $|T| \leqslant q(S)$ such that $\sum_{p \in A_i} T(p) > \sum_{p \in A_i} W(p)$ for all $i \in S$. The* core *of a party-approval election is defined as the set of all core-stable committees.*

[1] This definition implies that rules are *resolute*, that is, only a single committee is returned. In the case of a tie between multiple committees, a tiebreaking mechanism is necessary. Our results hold independently of the choice of a specific tiebreaking mechanism.

Core stability requires adequate representation even for voter groups that cannot agree on a common party, by ruling out the possibility that the group can deviate to a smaller committee that represents all voters in the group strictly better. It follows from the definitions that core stability is a stronger requirement than EJR: If a committee violates EJR, there is a group S that would prefer any committee of size $q(S)$ that assigns all seats to the commonly approved party.

A final, non-representational axiom that we will discuss is *committee monotonicity* (e.g., Barberà and Coelho, 2008; Elkind et al., 2017). A party-approval rule f satisfies this axiom if, for all party-approval elections (N, P, A, k), it holds that $f(N, P, A, k) \subseteq f(N, P, A, k + 1)$. An analogous property is known as *house monotonicity* in the apportionment literature. Committee monotonic rules avoid the so-called *Alabama paradox*, in which a party loses a seat when the committee size increases. Besides avoiding this paradox, committee monotonic rules have the advantage that they can be used to construct proportional rankings (Skowron et al., 2017).

3 Constructing Party-Approval Rules via Portioning and Apportionment

Party-approval elections are a generalization of party-list elections, which can be thought of as party-approval elections in which all approval sets are singletons. Since there is a rich body of research on apportionment methods, it is natural to examine whether we can employ these methods for our setting as well. To use them, we will need to translate party-approval elections into the party-list domain on which apportionment methods operate. This translation thus needs to transform a collection of approval votes over parties into vote shares for each party. Motivated by time sharing, Bogomolnaia et al. (2005) have developed a theory of such transformation rules, further studied by Duddy (2015) and Aziz et al. (2019). We will refer to this framework as *portioning*.

The approach explored in this section, then, divides the construction of a party-approval rule into two independent steps: (1) portioning, which maps a party-approval election to a vector of parties' shares; followed by (2) apportionment, which transforms the shares into a seat distribution.

Both the portioning and the apportionment literature have discussed representation axioms similar in spirit to EJR, PJR, and core stability. For both settings, several rules have been found to satisfy these properties. One might hope that by composing two rules that are each representative, we obtain a party-approval rule that is also representative (and satisfies, say, EJR). If we succeed in finding such a combination, it is likely that the resulting voting rule will automatically satisfy committee monotonicity since most apportionment methods satisfy this property. In the general candidate-approval setting (considered in Section 4), the existence of a rule satisfying both EJR and committee monotonicity is an open problem.

3.1 Preliminaries

We start by introducing relevant notions from the literature of portioning (Bogomolnaia et al., 2005; Aziz et al., 2019) and apportionment (Balinski and Young, 1982; Pukelsheim, 2014), with notation suitably adjusted to our setting.

Portioning

A *portioning problem* is a triple (N, P, A), just as in party-approval voting but without a committee size. A *portioning* is a function $r : P \to [0, 1]$ with $\sum_{p \in P} r(p) = 1$. We interpret $r(p)$ as the vote share of party p. A *portioning method* maps each portioning problem (N, P, A) to a portioning.

Our minimum requirement on portioning methods will be that they uphold proportionality if all approval sets are singletons, i.e., if we are already in the party-list domain. Formally, we say that a portioning method is *faithful* if for all (N, P, A) with $|A_i| = 1$ for all $i \in N$, the resulting portioning

r satisfies $r(p) = |\{i \in N \mid A_i = \{p\}\}|/n$ for all $p \in P$. Among the portioning methods considered by Aziz et al. (2019), only three are faithful. They are defined as follows.

Conditional utilitarian portioning selects, for each voter i, p_i as a party in A_i approved by the highest number of voters. Then, $r(p) = |\{i \in N \mid p_i = p\}|/n$ for all $p \in P$.

Random priority computes $n!$ portionings, one for each permutation σ of N, and returns their average. The portioning for $\sigma = (i_1, \dots, i_n)$ maximizes $\sum_{p \in A_{i_1}} r(p)$, breaking ties by maximizing $\sum_{p \in A_{i_2}} r(p)$, and so forth.

Nash portioning selects the portioning r maximizing the Nash welfare $\prod_{i \in N} \big(\sum_{p \in A_i} r(p)\big)$.

On first sight, Nash portioning seems particularly promising because it satisfies portioning versions of core stability and EJR (Aziz et al., 2019; Guerdjikova and Nehring, 2014). Concretely, it satisfies a property called *average fair share* introduced by Aziz et al. (2019), which requires that there is no subset $S \subseteq N$ of voters such that $\bigcap_{i \in S} A_i \neq \emptyset$ and $\frac{1}{|S|} \sum_{i \in S} \sum_{p \in A_i} r(p) < |S|/|N|$. However, despite these promising properties, we will see that Nash portioning does not work for our purposes. Instead, we will need to make use of a more recent portioning approach, which was proposed by Speroni di Fenizio and Gewurz (2019) in the context of party-approval voting.

Majoritarian portioning proceeds in rounds $j = 1, 2, \dots$. Initially, all parties and voters are *active*. In iteration j, we select the active party p_j that is approved by the highest number of active voters. Let N_j be the set of active voters who approve p_j. Then, set $r(p_j)$ to $|N_j|/n$, and mark p_j and all voters in N_j as inactive. If active voters remain, the next iteration is started; else, r is returned.

Under majoritarian portioning, the approval preferences of voters who have been assigned to a party are ignored in further iterations. Note that conditional utilitarian portioning can similarly be seen as a sequential method, in which the preferences of inactive voters are not ignored.

Apportionment

An *apportionment problem* is a tuple (P, r, k), which consists of a finite set of parties P, a portioning $r : P \to [0, 1]$ specifying the vote shares of parties, and a committee size $k \in \mathbb{N}$. Committees are defined as for party-approval elections, and an *apportionment method* maps apportionment problems to committees W of size k.

An apportionment method satisfies *lower quota* if each party p is always allocated at least $\lfloor k \cdot r(p) \rfloor$ seats in the committee. Furthermore, an apportionment method f is *committee monotonic* if $f(P, r, k) \subseteq f(P, r, k + 1)$ for every apportionment problem (P, r, k).

Among the standard apportionment methods, only two satisfy both lower quota and committee monotonicity: the *D'Hondt method* (aka *Jefferson method*) and the *quota method*.[2] The D'Hondt method assigns the k seats iteratively, each time giving the next seat to the party p with the largest quotient $r(p)/(s(p) + 1)$, where $s(p)$ denotes the number of seats already assigned to p. The *quota method* (Balinski and Young, 1975) is identical to the D'Hondt method, except that, in the jth iteration, only parties p satisfying $s(p)/j < r(p)$ are eligible for the allocation of the next seat.

Composition

If we take any portioning method and any apportionment method, we can compose them to obtain a party-approval rule. Formally, the composition of portioning method R and apportionment method M maps each party-approval election (N, P, A, k) to a committee $M(P, R(N, P, A), k)$. Note that if

[2] All other *divisor methods* fail lower quota, and the *Hamilton method* is not committee monotonic (Balinski and Young, 1982).

the apportionment method is committee monotonic then so is the composed rule, since the portioning is independent of k.

3.2 Composed Rules That Fail EJR

Perhaps surprisingly, many pairs of portioning and apportionment methods fail EJR. This is certainly true if the individual parts are not representative themselves. For example, if an apportionment method M properly fails lower quota (in the sense that there is a rational-valued input r on which lower quota is violated), then one can construct an example profile on which any composed rule using M fails EJR: Construct a party-approval election with singleton approval sets in which the voter counts are proportional to the shares in the counter-example r. Then any faithful portioning method, applied to this election, must return r. Since M fails lower quota on r, the resulting committee will violate EJR. By a similar argument, an apportionment method that violates committee monotonicity on some rational portioning will, when composed with a faithful portioning method, give rise to a party-approval rule that fails committee monotonicity.

As mentioned above, among the named and studied apportionment methods, only two satisfy both lower quota and committee monotonicity: D'Hondt and the quota method. However, it turns out that the composition of either option with the conditional-utilitarian, random-priority, or Nash portioning methods fails EJR, as the following examples show.

Example 1. *Let $n = k = 6$, $P = \{p_0, p_1, p_2, p_3\}$, and consider the ballot profile $A = (\{p_0\}, \{p_0\}, \{p_0, p_1, p_2\}, \{p_0, p_1, p_2\}, \{p_1, p_3\}, \{p_2, p_3\})$.*

Then, the conditional utilitarian solution sets $r(p_0) = 4/6$, $r(p_1) = r(p_2) = 1/6$, and $r(p_3) = 0$. Any apportionment method satisfying lower quota allocates four seats to p_0, one each to p_1 and p_2, and none to p_3. The resulting committee does not provide EJR since the last two voters, who jointly approve p_3, have a quota of $q(\{5, 6\}) = 2$ that is not met.

Example 2. *Let $n = k = 6$, $P = \{p_0, p_1, p_2, p_3\}$, and consider the ballot profile $A = (\{p_0\}, \{p_0\}, \{p_0, p_1, p_2\}, \{p_0, p_1, p_3\}, \{p_1\}, \{p_2, p_3\})$.*

Random priority chooses the portioning $r(p_0) = 23/45$, $r(p_1) = 23/90$, and $r(p_2) = r(p_3) = 7/60$. Both D'Hondt and the quota method allocate four seats to p_0, two seats to p_1, and none to the other two parties. This clearly violates the claim to representation of the sixth voter (with $q(\{6\}) = 1$).

Nash portioning produces a fairly similar portioning, with $r(p_0) \approx 0.5302$, $r(p_1) \approx 0.2651$, and $r(p_2) = r(p_3) \approx 0.1023$. D'Hondt and the quota method produce the same committee as above, leading to the same EJR violation.

At first glance, it might be surprising that Nash portioning combined with a lower-quota apportionment method violates EJR (and even the weaker axiom PJR). Indeed, Nash portioning satisfies core stability in the portioning setting, which is a strong notion of proportionality, and the lower-quota property limits the rounding losses when moving from the portioning to a committee. As expected, in the election of Example 2, the portioning produced by Nash gives sufficient representation to the sixth voter since $r(p_2) + r(p_3) \approx 0.2047 > 1/6$. However, since both $r(p_2)$ and $r(p_3)$ are below $1/6$ on their own, lower quota does not apply to either of the two parties, and the sixth voter loses all representation in the apportionment step.

3.3 Composed Rules That Satisfy EJR

As we have seen, several initially promising portioning methods fail to compose to a rule that satisfies EJR. One reason is that these portioning methods are happy to assign small shares to several parties. The apportionment method may round several of those small shares down to zero seats. This can lead to a failure of EJR when not enough parties obtain a seat. It is difficult for an apportionment method to avoid this behavior since the portioning step obscures the relationships between different parties that are apparent from the approval ballots of the voters.

Majoritarian portioning is designed to maximize the seat allocations to the largest parties. Thus, it tends to avoid the problem we have identified. While it fails the strong representation axioms that Nash portioning satisfies, this turns out not to be crucial: Composing majoritarian portioning with any apportionment method satisfying lower quota yields an EJR rule. If we use an apportionment method that is also committee monotonic, such as D'Hondt or the quota method, we obtain a party-approval rule that satisfies both EJR and committee monotonicity.[3]

Theorem 1. *Let M be a committee monotonic apportionment method satisfying lower quota. Then, the party-approval rule composing majoritarian portioning and M satisfies EJR and committee monotonicity.*

Proof. Consider a party-approval election (N, P, A, k) and let r be the outcome of majoritarian portioning applied to (N, P, A). Let $N_1, N_2, \dots$ and $p_1, p_2, \dots$ be the voter groups and parties in the construction of majoritarian portioning, so that $r(p_j) = |N_j|/n$ for all j.

Consider the committee $W = M(P, r, k)$ and suppose that EJR is violated, i.e., that there exists a group $S \subseteq N$ with $\bigcap_{i \in S} A_i \neq \emptyset$ and $\sum_{p \in A_i} W(p) < q(S)$ for all $i \in S$.

Let j be minimal such that $S \cap N_j \neq \emptyset$. We now show that $|S| \leqslant |N_j|$. By the definition of j, no voter in S approves of any of the parties $p_1, p_2, \dots p_{j-1}$; thus, all those voters remain active in round j. Consider a party $p^* \in \bigcap_{i \in S} A_i$. In the jth iteration of majoritarian portioning, this party had an approval score (among active agents) of at least $|S|$. Therefore, the party p_j that is chosen in the jth iteration has an approval score that is at least $|S|$ (of course, $p^* = p_j$ is possible). The approval score of party p_j equals $|N_j|$. Therefore, $|N_j| \geqslant |S|$.

Since $|N_j| \geqslant |S|$, we have $q(N_j) \geqslant q(S)$. Since M satisfies lower quota, it assigns at least $\lfloor k \cdot r(p_j) \rfloor = \lfloor k\,(|N_j|/n) \rfloor = q(N_j)$ seats to party p_j. Now consider a voter $i \in S \cap N_j$. Since this voter approves party p_j, we have $\sum_{p \in A_i} W(p) \geqslant W(p_j) \geqslant q(N_j) \geqslant q(S)$, a contradiction.

This shows that EJR is indeed satisfied; committee monotonicity follows from the committee monotonicity of M. ☐

While the party-approval rules identified by Theorem 1 satisfy EJR and committee monotonicity, they do not quite reach our gold standard of representation, i.e., core stability.[4]

Example 3. *Let $n = k = 16$, $P = \{p_0, \dots, p_4\}$, and consider a ballot profile with the following approval sets: 4 times $\{p_0, p_1\}$, 3 times $\{p_1, p_2\}$, once $\{p_2\}$, 4 times $\{p_0, p_3\}$, 3 times $\{p_3, p_4\}$, and once $\{p_4\}$. Note the symmetry between p_1 and p_3, and between p_2 and p_4. Majoritarian portioning allocates $1/2$ to p_0 and $1/4$ each to p_2 and p_4. Any lower-quota apportionment method must translate this into 8 seats for p_0 and 4 seats each for p_2 and p_4. This committee is not in the core: Let S be the coalition of all 14 voters who approve multiple parties, and let T allocate 4 seats to p_0 and 5 seats each to p_1 and p_3. This gives strictly higher representation to all members of the coalition.*

The example makes it obvious why majoritarian portioning cannot satisfy the core: All voters approving of p_0 get deactivated after the first round, which makes p_2 seem universally preferable to p_1. However, p_1 is a useful vehicle for cooperation between the group approving $\{p_0, p_1\}$ and the group approving $\{p_1, p_2\}$. Since majoritarian portioning is blind to this opportunity, it cannot guarantee core stability.

The example also illustrates the power of core stability: The deviating coalition does not agree on any single party they support, but would nonetheless benefit from the deviation. There is room for collaboration, and core stability is sensitive to this demand for better representation.

[3] As long as the apportionment method is computable in polynomial time (which is the case for D'Hondt and the quota method), the same holds for the resulting party-approval rule.

[4] We will show later (see Section 4.3) that every party-approval election has a nonempty core. However, it remains an open problem whether core stability and committee monotonicity are compatible in the party-approval setting.

4 Constructing Party-Approval Rules via Multiwinner Voting Rules

In the previous section, we applied tools from apportionment, a more restrictive setting, to our party-approval setting. Now, we go in the other direction, and apply tools from a more general setting: As mentioned in Section 1.2, party-approval elections can be viewed as a special case of candidate-approval elections, i.e., multiwinner elections in which approvals are expressed over individual candidates rather than parties. After introducing relevant candidate-approval notions, we show how party-approval elections can be translated into candidate-approval elections. This embedding allows us to apply established candidate-approval rules to our setting. Exploiting this fact, we will prove the existence of core-stable committees for party-approval elections.

4.1 Preliminaries

A *candidate-approval election* is a tuple (N, C, A, k). Just as for party-approval elections, $N = \{1, \ldots, n\}$ is a set of voters, C is a finite set, A is an n-tuple of nonempty subsets of C, and $k \in \mathbb{N}$ is the committee size. The conceptual difference is that C is a set of individual candidates rather than parties. This difference manifests itself in the definition of a committee because a single candidate cannot receive multiple seats. That is, a *candidate committee* W is now simply a subset of C with cardinality k. (Therefore, it is usually assumed that $|C| \geqslant k$.) A *candidate-approval rule* is a function that maps each candidate-approval election to a candidate committee.

A diverse set of such voting rules has been proposed since the late 19th century (Kilgour and Marshall, 2012; Janson, 2016; Lackner and Skowron, 2020), out of which we will only introduce the one which we use for our main positive result. Let H_j denote the jth harmonic number, i.e., $H_j = \sum_{t=1}^{j} 1/t$. Given (N, C, A, k), the candidate-approval rule *proportional approval voting (PAV)*, introduced by Thiele (1895), chooses a candidate committee W maximizing the *PAV score* $\mathrm{PAV}(W) = \sum_{i \in N} H_{|W \cap A_i|}$.

We now describe EJR, PJR, and core stability in the candidate-approval setting, from which our versions of these axioms are derived. Recall that $q(S) = \lfloor k\,|S|/n \rfloor$. A candidate committee W provides *EJR* if there is no subset $S \subseteq N$ and no integer $\ell > 0$ such that $q(S) \geqslant \ell$, $|\bigcap_{i \in S} A_i| \geqslant \ell$, and $|A_i \cap W| < \ell$ for all $i \in S$. A candidate committee W provides *PJR* if there is no subset $S \subseteq N$ and no integer $\ell > 0$ such that $q(S) \geqslant \ell$, $|\bigcap_{i \in S} A_i| \geqslant \ell$, and $|\bigcup_{i \in S} A_i \cap W| < \ell$. (The requirement $|\bigcap_{i \in S} A_i| \geqslant \ell$ is often referred to as *cohesiveness*.) A candidate-approval rule satisfies EJR (respectively, PJR) if it always produces EJR (respectively, PJR) committees.

The definition of core stability is even closer to the version in party-approval elections: A candidate committee W is *core stable* if there is no nonempty group $S \subseteq N$ and no set $T \subseteq C$ of size $|T| \leqslant q(S)$ such that $|A_i \cap T| > |A_i \cap W|$ for all $i \in S$. The *core* consists of all core-stable candidate committees.

4.2 Embedding Party-Approval Elections

We have informally argued in Section 1.2 that party-approval elections constitute a subdomain of candidate-approval elections. We formalize this notion by providing an embedding of party-approval elections into the candidate-approval domain. Our approach is similar to that of Brill et al. (2018), who have formalized how apportionment problems can be phrased as candidate-approval elections.

For a given party-approval election (N, P, A, k), we define a corresponding candidate-approval election (N, C, A', k) with the same set of voters N and the same committee size k. The set of candidates contains k many "clone" candidates $p^{(1)}, \ldots, p^{(k)}$ for each party $p \in P$, so $C = \bigcup_{p \in P} \{p^{(1)}, \ldots, p^{(k)}\}$. Voter i approves a candidate $p^{(j)}$ in the candidate-approval election if and only if she approves the corresponding party p in the party-approval election. Thus, $A'_i = \bigcup_{p \in A_i} \{p^{(1)}, \ldots, p^{(k)}\}$. This embedding establishes party-approval elections as a subdomain of candidate-approval elections. As a consequence, we can apply rules and axioms from the more general candidate-approval setting also in the party-approval setting.

In particular, the generic way to apply a candidate-approval rule for a party-approval election consists in (1) translating the party-approval election into a candidate-approval election, (2) applying the candidate-approval rule, and (3) counting the number of chosen clones per party to construct a committee over parties. For computational purposes, note that since we insisted that k is encoded in unary, the size of the candidate-approval election is bigger than the size of its corresponding party-approval election by at most a polynomial factor.[5]

Having established party-approval elections as a subdomain of candidate-approval elections, our variants of EJR and core stability (Definitions 1 and 2) are immediately induced by their candidate-approval counterparts. In particular, any candidate-approval rule satisfying an axiom in the candidate-approval setting will satisfy the corresponding axiom in the party-approval setting as well. Note that, by restricting our view to party approval, the cohesiveness requirement of EJR is reduced to requiring a single commonly approved party.

Since party-*list* elections constitute a subdomain of party-approval elections (and thus of candidate-approval elections), candidate-approval rules can also be applied to *apportionment* problems (see Figure 1). Several candidate-approval rules including PAV coincide with the D'Hondt method when applied to party-list elections (Brill et al., 2018).

4.3 PAV Guarantees Core Stability

Core stability is particularly attractive because blocking coalitions are not required to unanimously approve any single party. They only need to be able to coordinate for mutual gain. Our earlier Example 3 illustrates how a coalition might deviate in spite of not agreeing on any approved party.

Unfortunately, it is still unknown whether core-stable candidate committees exist for all candidate-approval elections.[6] All standard candidate-approval rules either already fail weaker representation axioms such as EJR, or are known to fail core stability. In particular, Aziz et al. (2017) have shown that PAV satisfies EJR, but may produce non-core-stable candidate committees even for candidate-approval elections for which core-stable candidate committees are known to exist. Peters and Skowron (2020) show that a large class of candidate-approval rules (so-called welfarist rules) must all fail core stability.

In contrast, we show that core stability can always be achieved in the party-approval setting. In particular, the committee selected by PAV is always core stable for party-approval elections. Our proof uses a similar technique to the existing proof that PAV satisfies EJR for candidate-approval elections (Aziz et al., 2017, Theorem 10).

Theorem 2. *For every party-approval election, PAV chooses a core-stable committee.*

Proof. Consider a party-approval election (N, P, A, k) and let $W : P \to \mathbb{N}$ be the committee selected by PAV. Assume for contradiction that W is not core stable. Then, there is a nonempty coalition S and a committee $T : P \to \mathbb{N}$ such that $|T| \leqslant k\,|S|/n$ and $\sum_{p \in A_i} T(p) > \sum_{p \in A_i} W(p)$ for every voter $i \in S$.

Let $u_i(W)$ denote the number of seats in W that are allocated to parties approved by voter i, i.e., $u_i(W) = \sum_{p \in A_i} W(p)$. Furthermore, for a party p with $W(p) > 0$, we let $\Delta(p, W)$ denote the *marginal contribution* to the PAV score of allocating a seat to p, i.e., $\Delta(p, W) = \mathrm{PAV}(W) -$

[5] For candidate-approval elections, it does not make sense to have more seats than candidates, whereas for party-approval elections it is natural to have more seats than parties. If k was encoded in binary, then the derived candidate-approval instance would have exponential size. Thus, even simple sequential candidate-approval algorithms would then have exponential running time. This would complicate running-time comparisons between the candidate-approval and party-approval setting and would blur the intuitive distinction between simple and complex algorithms. Encoding k in unary sidesteps this technical complication.

[6] However, it is known that *approximately* core-stable committees exist, for several different ways of approximating the core notion (Fain et al., 2018; Cheng et al., 2019; Jiang et al., 2020; Peters and Skowron, 2020).

$\mathrm{PAV}(W - \{p\})$. Observe that $\Delta(p, W) = \sum_{i \in N_p} 1/u_i(W)$, where $N_p = \{i \in N \mid p \in A_i\}$. The sum of all marginal contributions satisfies

$$\begin{aligned}\sum_{p \in P} W(p)\, \Delta(p, W) &= \sum_{p \in P} \sum_{i \in N_p} \frac{W(p)}{u_i(W)} \\ &= \sum_{i \in N} \sum_{p \in A_i} \frac{W(p)}{u_i(W)} = |\{i \in N \mid u_i(W) > 0\}| \leqslant n.\end{aligned}$$

Note that terms $\Delta(p, W)$ for $W(p) = 0$ and quotients $1/u_i(W)$ for $u_i(W) = 0$ are undefined in the calculation above, but that they only appear with factor 0.

It follows that the average marginal contribution of all k seats in W is at most n/k, and consequently, that there has to be a party p_1 with a seat in W such that $\Delta(p_1, W) \leqslant n/k$. Using a similar argument, we show that there is also a party p_2 with $T(p_2) > 0$ which would increase the PAV score by at least n/k if it received an additional seat in W:

$$\begin{aligned}\sum_{p \in P} T(p)\, \Delta(p, W + \{p\}) &= \sum_{i \in N} \sum_{p \in A_i} \frac{T(p)}{u_i(W + \{p\})} \\ \geqslant \sum_{i \in S} \sum_{p \in A_i} \frac{T(p)}{u_i(W + \{p\})} &= \sum_{i \in S} \sum_{p \in A_i} \frac{T(p)}{u_i(W) + 1} \\ \geqslant \sum_{i \in S} \sum_{p \in A_i} \frac{T(p)}{u_i(T)} &= |\{i \in S \mid u_i(T) > 0\}| = |S|.\end{aligned}$$

The second inequality holds because every voter in S *strictly* increases their utility when deviating from W to T; the last equality holds because every voter in S must get some representation in T to deviate. As desired, it follows that there has to be a party p_2 in the support of T with $\Delta(p_2, W + \{p_2\}) \geqslant |S|/|T| \geqslant n/k$.

If any of these inequalities would be strict, that is, if $\Delta(p_1, W) < n/k$ or $\Delta(p_2, W + \{p_2\}) > n/k$, then the committee $W - \{p_1\} + \{p_2\}$ would have a PAV score of

$$\begin{aligned}&\mathrm{PAV}(W) - \Delta(p_1, W) + \Delta(p_2, W - \{p_1\} + \{p_2\}) \\ &\geqslant \mathrm{PAV}(W) - \Delta(p_1, W) + \Delta(p_2, W + \{p_2\}) \\ &> \mathrm{PAV}(W),\end{aligned} \tag{1}$$

which would contradict the choice of W.

Else, suppose that $\Delta(p, W) = n/k$ for all parties p in the support of W and $\Delta(p, W + \{p\}) = n/k$ for all parties p in the support of T. If there is a party p_1 in W that is approved by some voter $i \in S$, we can choose an arbitrary party p_2 from the support of T that i approves as well. Then, for voter i, the marginal contribution of p_2 in $W - \{p_1\} + \{p_2\}$ is $\frac{1}{u_i(W - \{p_1\} + \{p_2\})} = \frac{1}{u_i(W)}$, but the marginal contribution of p_2 in $W + \{p_2\}$ for i is only $\frac{1}{u_i(W + \{p_2\})} = \frac{1}{u_i(W)+1}$. This implies $\Delta(p_2, W - \{p_1\} + \{p_2\}) > \Delta(p_2, W + \{p_2\})$, which makes inequality (1) strict and again contradicts the optimality of W.

Thus, one has to assume that no voter in S approves any party in the support of W. Pick an arbitrary p in the support of T, and recall that $\Delta(p', W + \{p'\}) = n/k$ for all p' in the support of T. Thus, all inequalities in the derivation of $\sum_{p \in P} T(p)\, \Delta(p, W + \{p\}) \geqslant |S|$ above must be equalities, which implies that this increase in PAV score must solely come from voters in S. Thus, there are at least n/k voters in S who are not represented at all in W, but commonly approve p. This would be a violation of EJR, contradicting the fact that PAV satisfies this axiom. $\square$

We have thus obtained the main result of this paper.

Corollary 3. *The core of a party-approval election is nonempty.*

Theorem 2 motivates the question of whether other candidate-approval rules satisfy stronger representation axioms when restricted to the party-approval subdomain. We have studied this question for various rules besides PAV, and the answer was always negative; see Appendix C for details.[7]

An immediate follow-up question to Corollary 3 is whether core-stable committees can be computed efficiently. PAV committees are known to be NP-hard to compute in the candidate-approval setting, and we confirm in Appendix B that hardness still holds in the party-approval subdomain.

Equally confronted with the computational complexity of PAV, Aziz et al. (2018) proposed a local-search variant of PAV, which runs in polynomial time and guarantees EJR in the candidate-approval setting. Using the same approach, we can find a core-stable committee in the party-approval setting. We defer the proof to Appendix A.

Theorem 4. *Given a party-approval election, a core-stable committee can be computed in polynomial time.*

While the party-approval setting does not reduce the complexity of computing PAV, it allows us to check efficiently check whether a given committee provides EJR or PJR; both problems are coNP-hard in the candidate-approval setting (Aziz et al., 2015a, 2018). For EJR, this follows from coherence becoming simpler for party-approval elections. Our algorithm for checking PJR employs submodular minimization. For details, we refer to Appendix B.

5 Discussion

In this paper, we have initiated the axiomatic analysis of approval-based apportionment. On a technical level, it would be interesting to see whether the party-approval domain allows us to satisfy other combinations of axioms that are not known to be attainable in candidate-approval elections. For instance, the compatibility between strong representation axioms and certain notions of support monotonicity is an open problem (Sánchez-Fernández and Fisteus, 2019).

We have presented our setting guided by the application of apportioning parliamentary seats to political parties. While we believe that this is an attractive application worthy of practical experimentation, our formal setting has other interesting applications. An example would be participatory budgeting settings in which the provision of items of equal cost is decided, where the items come in different types. For instance, a university department could decide how to allocate Ph.D. scholarships across different research projects, in a way that respects the preferences of funding organizations.

As another example, the literature on multiwinner elections suggests many applications to recommendation problems (Skowron et al., 2016). For instance, one might want to display a limited number of news articles, movies, or advertisements in a way that fairly represents the preferences of the audience. These preferences might be expressed not over individual pieces of content, but over content producers (such as newspapers, studios, or advertising companies), in which case our setting provides rules that decide how many items should be contributed by each source. Expressing preferences on the level of content producers is natural in repeated settings, where the relevant pieces of content change too frequently to elicit voter preferences on each occasion. Besides, content producers might reserve the right to choose which of their content should be displayed.

In the general candidate-approval setting, the search continues for rules that satisfy EJR and committee monotonicity, or core stability. But for the applications mentioned above, these guarantees are already achievable.

[7] We present relevant counterexamples for the candidate-approval rules seq-Phragmén, leximax-Phragmén, Eneström-Phragmén, Rule X, and the Maximin Support Method. In addition, for the candidate-approval rules SeqPAV, RevSeqPAV, var-Phragmén, Approval Voting (AV), SatisfactionAV, MinimaxAV, MonroeAV, GreedyMonroeAV, GreedyAV, HareAV, and Chamberlin–CourantAV, existing counterexamples can easily be adjusted to the party-approval setting.

Acknowledgements

We would like to thank AAAI and COMSOC reviewers for their insightful comments. This work was partially supported by the Deutsche Forschungsgemeinschaft under grant BR 4744/2-1. We thank Steven Brams and Piotr Skowron for suggesting the setting of party approval to us, and we thank Rupert Freeman, Levi Geiser, Anne-Marie George, Ayumi Igarashi, Svante Janson, Jérôme Lang, and Ariel Procaccia for helpful comments and discussions.

References

H. Aziz, M. Brill, V. Conitzer, E. Elkind, R. Freeman, and T. Walsh. Justified representation in approval-based committee voting. In *Proceedings of the 29th AAAI Conference on Artificial Intelligence (AAAI)*, pages 784–790. AAAI Press, 2015a.

H. Aziz, S. Gaspers, J. Gudmundsson, S. Mackenzie, N. Mattei, and T. Walsh. Computational aspects of multi-winner approval voting. In *Proceedings of the 14th International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, pages 107–115. IFAAMAS, 2015b.

H. Aziz, M. Brill, V. Conitzer, E. Elkind, R. Freeman, and T. Walsh. Justified representation in approval-based committee voting. *Social Choice and Welfare*, 48(2):461–485, 2017.

H. Aziz, E. Elkind, S. Huang, M. Lackner, L. Sánchez-Fernández, and P. Skowron. On the complexity of extended and proportional justified representation. In *Proceedings of the 32nd AAAI Conference on Artificial Intelligence (AAAI)*, pages 902–909. AAAI Press, 2018.

H. Aziz, A. Bogomolnaia, and H. Moulin. Fair mixing: The case of dichotomous preferences. In *Proceedings of the 2019 ACM Conference on Economics and Computation (EC)*, pages 753–781, 2019.

M. Balinski and H. P. Young. *Fair Representation: Meeting the Ideal of One Man, One Vote*. Yale University Press, 1982.

M. L. Balinski and H. P. Young. The quota method of apportionment. *The American Mathematical Monthly*, 82(7):701–730, 1975.

S. Barberà and D. Coelho. How to choose a non-controversial list with k names. *Social Choice and Welfare*, 31(1):79–96, 2008.

A. Bogomolnaia, H. Moulin, and R. Stong. Collective choice under dichotomous preferences. *Journal of Economic Theory*, 122(2):165–184, 2005.

S. J. Brams and P. C. Fishburn. *Approval Voting*. Springer-Verlag, 2nd edition, 2007.

S. J. Brams and D. M. Kilgour. Satisfaction approval voting. In *Voting Power and Procedures*, Studies in Choice and Welfare, pages 323–346. Springer, 2014.

S. J. Brams, D. M. Kilgour, and M. R. Sanver. A minimax procedure for electing committees. *Public Choice*, 132:401–420, 2007.

S. J. Brams, D. M. Kilgour, and R. F. Potthoff. Multiwinner approval voting: an apportionment approach. *Public Choice*, 178(1–2):67–93, 2019.

M. Brill. Interactive democracy. In *Proceedings of the 17th International Conference on Autonomous Agents and Multiagent Systems (AAMAS) Blue Sky Ideas track*, pages 1183–1187. IFAAMAS, 2018.

M. Brill, R. Freeman, S. Janson, and M. Lackner. Phragmén's voting methods and justified representation. In *Proceedings of the 31st AAAI Conference on Artificial Intelligence (AAAI)*, pages 406–413. AAAI Press, 2017.

M. Brill, J.-F. Laslier, and P. Skowron. Multiwinner approval rules as apportionment methods. *Journal of Theoretical Politics*, 30(3):358–382, 2018.

M. Brill, P. Gölz, D. Peters, U. Schmidt-Kraepelin, and K. Wilker. Approval-based apportionment. In *Proceedings of the 34th AAAI Conference on Artificial Intelligence (AAAI)*, pages 1854–1861. AAAI Press, 2020.

R. Camps, X. Mora, and L. Saumell. The method of Eneström and Phragmén for parliamentary elections by means of approval voting. Technical report, arXiv:1907.10590 [econ.TH], 2019.

Y. Cheng, Z. Jiang, K. Munagala, and K. Wang. Group Fairness in Committee Selection. In *Proceedings of the 2019 ACM Conference on Economics and Computation (EC)*, pages 263–279, 2019.

C. Duddy. Fair sharing under dichotomous preferences. *Mathematical Social Sciences*, 73:1–5, 2015.

E. Elkind, P. Faliszewski, P. Skowron, and A. Slinko. Properties of multiwinner voting rules. *Social Choice and Welfare*, 48:599–632, 2017.

B. Fain, K. Munagala, and N. Shah. Fair allocation of indivisible public goods. In *Proceedings of the 2018 ACM Conference on Economics and Computation (EC)*, pages 575–592, 2018.

P. C. Fishburn. Axioms for approval voting: Direct proof. *Journal of Economic Theory*, 19(1): 180–185, 1978.

M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman, 1979.

A. Guerdjikova and K. Nehring. Weighing experts, weighing sources: The diversity value. 2014. Mimeo.

S. Janson. Phragmén's and Thiele's election methods. Technical report, arXiv:1611.08826 [math.HO], 2016.

S. Janson and A. Öberg. A piecewise contractive dynamical system and Phragmén's election method. *Bulletin de la Société Mathématique de France*, 147(3):395–441, 2019.

Z. Jiang, K. Munagala, and K. Wang. Approximately stable committee selection. In *Proceedings of the 52nd Annual ACM SIGACT Symposium on Theory of Computing (STOC)*, pages 463–472, 2020.

D. M. Kilgour and E. Marshall. Approval balloting for fixed-size committees. In *Electoral Systems*, Studies in Choice and Welfare, pages 305–326. Springer, 2012.

B. Korte and J. Vygen. *Combinatorial Optimization*. Springer, 6th edition, 2018.

M. Lackner and P. Skowron. Approval-based committee voting: Axioms, algorithms, and applications. Technical report, arXiv:2007.01795 [cs.GT], 2020.

J.-F. Laslier and M. R. Sanver, editors. *Handbook on Approval Voting*. Studies in Choice and Welfare. Springer-Verlag, 2010.

T. Lu and C. Boutilier. Budgeted social choice: From consensus to personalized decision making. In *Proceedings of the 22nd International Joint Conference on Artificial Intelligence (IJCAI)*, pages 280–286. AAAI Press, 2011.

X. Mora and M. Oliver. Eleccions mitjançant el vot d'aprovació. El mètode de Phragmén i algunes variants. *Butlletí de la Societat Catalana de Matemàtiques*, 30(1):57–101, 2015.

D. Peters and P. Skowron. Proportionality and the limits of welfarism. In *Proceedings of the 21st ACM Conference on Economics and Computation (ACM-EC)*, pages 793–794, 2020.

E. Phragmén. Sur une méthode nouvelle pour réaliser, dans les élections, la représentation proportionnelle des partis. *Öfversigt af Kongliga Vetenskaps-Akademiens Förhandlingar*, 51(3):133–137, 1894.

E. Phragmén. *Proportionella val. En valteknisk studie*. Svenska spörsmål 25. Lars Hökersbergs förlag, Stockholm, 1895.

E. Phragmén. Sur la théorie des élections multiples. *Öfversigt af Kongliga Vetenskaps-Akademiens Förhandlingar*, 53:181–191, 1896.

E. Phragmén. Till frågan om en proportionell valmetod. *Statsvetenskaplig Tidskrift*, 2(2):297–305, 1899.

F. Pukelsheim. *Proportional Representation: Apportionment Methods and Their Applications*. Springer, 2014.

L. Sánchez-Fernández and J. A. Fisteus. Monotonicity axioms in approval-based multi-winner voting rules. In *Proceedings of the 18th International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, pages 485–493, 2019.

L. Sánchez-Fernández, E. Elkind, and M. Lackner. Committees providing EJR can be computed efficiently. Technical report, arXiv:1704.00356v3 [cs.GT], 2017a.

L. Sánchez-Fernández, E. Elkind, M. Lackner, N. Fernández, J. A. Fisteus, P. Basanta Val, and P. Skowron. Proportional justified representation. In *Proceedings of the 31st AAAI Conference on Artificial Intelligence (AAAI)*, pages 670–676. AAAI Press, 2017b.

L. Sánchez-Fernández, N. Fernández, J. A. Fisteus, and M. Brill. The maximin support method: An extension of the D'Hondt method to approval-based multiwinner elections. In *Proceedings of the 35th AAAI Conference on Artificial Intelligence (AAAI)*. AAAI Press, 2021. Forthcoming.

P. Skowron, M. Lackner, M. Brill, D. Peters, and E. Elkind. Proportional rankings. In *Proceedings of the 26th International Joint Conference on Artificial Intelligence (IJCAI)*, pages 409–415. IJCAI, 2017.

P. K. Skowron, P. Faliszewski, and J. Lang. Finding a collective set of items: From proportional multirepresentation to group recommendation. *Artificial Intelligence*, 241:191–216, 2016.

P. Speroni di Fenizio and D. A. Gewurz. The space of all proportional voting systems and the most majoritarian among them. *Social Choice and Welfare*, 52(4):663–683, 2019.

T. N. Thiele. Om flerfoldsvalg. *Oversigt over det Kongelige Danske Videnskabernes Selskabs Forhandlinger*, pages 415–441, 1895.

Markus Brill
Technische Universität Berlin
Berlin, Germany
Email: brill@tu-berlin.de

Paul Gölz
Carnegie Mellon University
Pittsburgh, PA, USA
Email: pgoelz@cs.cmu.edu

Dominik Peters
Harvard University
Cambridge, MA, USA
Email: dpeters@seas.harvard.edu

Ulrike Schmidt-Kraepelin
Technische Universität Berlin
Berlin, Germany
Email: u.schmidt-kraepelin@tu-berlin.de

Kai Wilker
Technische Universität Berlin
Berlin, Germany
Email: wilker@campus.tu-berlin.de

A Proof of Theorem 4

In the following we prove that in the party-approval subdomain, core-stable committees can be computed in polynomial time. We make use of a local search procedure, introduced for the candidate-approval setting by Aziz et al. (2018), which approximates a local maximum of the PAV score function. Aziz et al. (2018) show that their algorithm runs in polynomial time and returns committees providing EJR. For party-approval elections, we show that, by a minor adjustment of the algorithm, committees computed by LS-PAV satisfy core stability. In Algorithm 1 we slightly adjust the original definition by parameterizing the procedure by the approximation threshold ϵ. Note that, once again, the algorithm is defined in terms of candidate-approval elections; in Section 4.2 we show how to apply candidate-approval rules to party-approval elections.

Algorithm 1 LS-PAV (Candidate-Approval Rule)

> **function** LS-PAV(N, C, A, k, ϵ)
> $W \leftarrow k$ arbitrary candidates from C
>   **while** $\exists\, c \in W, c' \in C \setminus W$ such that $PAV(W \setminus \{c\} \cup \{c'\}) \geqslant PAV(W) + \epsilon$ **do**
> $W \leftarrow W \setminus \{c\} \cup \{c'\}$
>   **end while**
>   **return** W
> **end function**

Theorem 4. *Given a party-approval election, a core-stable committee can be computed in polynomial time.*

Proof. We show that LS-PAV with threshold $\epsilon = \frac{1}{(1+2(k-1))(k-1)k}$ always selects a committee from the core when $k > 1$ and that the procedure runs in polynomial time for this specific choice of ϵ. This suffices to prove the Theorem as computing a core-stable committee for $k = 1$ is trivial. The proof is an extension of the proof of Theorem 2.

We define the utility of voter i with respect to committee W as $u_i(W) = \sum_{p \in A_i} W(p)$. For some party-approval election (N, P, A, k) let W be a committee selected by LS-PAV with ϵ and assume that W is not core stable. Hence, there exists $S \subseteq N, T : P \to \mathbb{N}, \ell \in [k]$ with $|S| \geqslant \ell\, n/k, |T| = \ell$ and

$$u_i(T) > u_i(W)\ \forall\ i \in S. \tag{2}$$

We start by reproducing some observations which were done within the beginning of the proof of Theorem 2. For more detailed arguments, we refer to this proof.

We define the marginal contribution of a party p to the PAV score of W

$$\Delta(p, W) = \text{PAV}(W) - \text{PAV}(W - \{p\}) = \sum_{i \in N\,:\, p \in A_i} \frac{1}{u_i(W)}$$

and obtain an upper bound for the sum of the marginal contribution of all seats in W w.r.t. W, i.e.,

$$\sum_{p \in P} W(p) \cdot \Delta(p, W) \leqslant n \tag{3}$$

and a lower bound for the sum of marginal contribution of all seats in T to $W + \{p\}$, where p is the party corresponding to the seat, i.e.,

$$|S| \leqslant \sum_{p \in P} T(P) \cdot \Delta(p, W + \{p\}). \tag{4}$$

Hence, there exists a party p_1 in the support of W for which

$$\Delta(p_1, W) \leqslant n/k \tag{5}$$

holds and there exists a party p_2 in the support of T for which

$$n/k \leqslant \Delta(p_2, W + \{p_2\}) \tag{6}$$

holds. We distinguish three cases:

Case 1: It holds that

1. there exists p_1 in the support of $W : \Delta(p_1, W) \leqslant n/k - \epsilon$ **or**
2. there exists p_2 in the support of $T : n/k + \epsilon \leqslant \Delta(p_2, W + \{p_2\})$.

smallskip

First assume that the first condition is satisfied. Then, let p_1 be a such a party from the support of W and p_2 such that (6) holds. We define the multiset $W' = W - \{p_1\} + \{p_2\}$ and observe that

$$\begin{aligned}\mathrm{PAV}(W') &= \mathrm{PAV}(W) - \Delta(p_1, W) + \Delta(p_2, W - \{p_1\} + \{p_2\}) \\ &\geqslant \mathrm{PAV}(W) - \Delta(p_1, W) + \Delta(p_2, W + \{p_2\}) \geqslant \mathrm{PAV}(W) + \epsilon,\end{aligned}$$

a contradiction to the assumption that LS-PAV with parameter ϵ returned W. The last inequality follows from case condition 1 and (6).

If the second condition holds, an analogous argument yields a contradiction.

Case 2: It holds that

1. for all p_1 in the support of $W : \Delta(p_1, W) > n/k - \epsilon$ **and**
2. for all p_2 in the support of $T : \Delta(p_2, W + \{p_2\}) < n/k + \epsilon$ **and**
3. there exists $i \in S : u_i(W) > 0$.

Applying (3) and case condition 1, we obtain an upper bound for $\Delta(p_1, W)$ for any p_1 in the support of W:

$$\begin{aligned}\Delta(p_1, W) &\leqslant n - \sum_{p \in P \setminus p_1} W(p)\Delta(p, W) - (W(p_1) - 1)\,\Delta(p_1, W) \\ &< n - (k-1)\left(\frac{n}{k} - \epsilon\right) = \frac{n}{k} + (k-1)\epsilon.\end{aligned} \tag{7}$$

Analogously, applying (4) and case condition 2, we obtain an lower bound for $\Delta(p_2, W + \{p_2\})$ for any p_2 in the support of T. That is,

$$\begin{aligned}\Delta(p_2, W + \{p_2\}) &\geqslant |S| - \sum_{p \in P \setminus p_2} T(p)\,\Delta(p, W + \{p\}) - (T(p_2) - 1)\,\Delta(p_2, W + \{p_2\}) \\ &> |S| - (\ell - 1)\left(\frac{n}{k} + \epsilon\right) \geqslant \frac{n}{k} - (k-1)\epsilon.\end{aligned} \tag{8}$$

Subsequently, choose some $i \in S$ with $u_i(W) > 0$ (existence guaranteed by case condition 3) and a party p_1 from the support of W which is also included in the approval set of voter i, A_i. Then, choose a party p_2 in the support of T which is also approved by voter i but $W(p_2) <$

$T(p_2)$ (existence guaranteed by the fact that voter i prefers committee T to committee W). Note that in particular, the restrictions made by case conditions 1 and 2 already imply that p_1 and p_2 are different parties.[8]

For this choice of p_1 and p_2 we aim quantify the gap between the contribution of p_2 with respect to $W - \{p_1\} + \{p_2\}$ and the contribution of p_2 with respect to $W + \{p_2\}$. More precisely, we will show that

$$\Delta(p_2, W - \{p_1\} + \{p_2\}) \geqslant \Delta(p_2, W + \{p_2\}) + \frac{1}{k(k-1)}. \tag{9}$$

To this end recall that voter i supports party p_1 and hence

$$u_i(W - \{p_1\}) = u_i(W) - 1. \tag{10}$$

Moreover, for all remaining voters $j \in N \setminus \{i\}$ it holds that

$$u_j(W - \{p_1\}) \leqslant u_j(W). \tag{11}$$

Lastly, from i being in the deviator set S, we know that

$$u_i(W) \leqslant k - 1. \tag{12}$$

Let $N_p = \{i \in N : p \in A_i\}$ denote the set of supporters of party p and $N_p^{-i} = N_p \setminus \{i\}$. Putting it all together, we get

$$\begin{aligned}
\Delta(p_2, W - \{p_1\} + \{p_2\}) &= \sum_{j \in N_{p_2}} \frac{1}{u_j(W - \{p_1\}) + 1} \\
&= \sum_{j \in N_{p_2}^{-i}} \frac{1}{u_j(W - \{p_1\}) + 1} + \frac{1}{u_i(W - \{p_1\}) + 1} \\
&\geqslant \sum_{j \in N_{p_2}^{-i}} \frac{1}{u_j(W) + 1} + \frac{1}{u_i(W)} \\
&= \sum_{j \in N_{p_2}^{-i}} \frac{1}{u_j(W) + 1} + \frac{1}{u_i(W) + 1} + \frac{1}{(u_i(W) + 1)u_i(W)} \\
&\geqslant \Delta(p_2, W + \{p_2\}) + \frac{1}{k(k-1)}.
\end{aligned}$$

The first inequality holds due to (10) and (11) and the second due to (12).

Finally, making use of (7),(8), and (9), we can show

$$\begin{aligned}
\mathrm{PAV}(W') &= \mathrm{PAV}(W) - \Delta(p_1, W) + \Delta(p_2, W - \{p_1\} + \{p_2\}) \\
&\geqslant \mathrm{PAV}(W) - \Delta(p_1, W) + \Delta(p_2, W + \{p_2\}) + \frac{1}{k(k-1)} \\
&> \mathrm{PAV}(W) - \frac{n}{k} - (k-1)\epsilon + \frac{n}{k} - (k-1)\epsilon + \frac{1}{k(k-1)} \\
&= \mathrm{PAV}(W) - 2(k-1)\epsilon + \frac{1}{k(k-1)} \\
&= \mathrm{PAV}(W) + \epsilon,
\end{aligned}$$

[8] Assume for contradiction that p_1 and p_2 are the same parties. Then, in particular it holds that $\Delta(p_1, W) = \Delta(p_2, W)$ and $\Delta(p_1, W + \{p_1\}) = \Delta(p_2, W + \{p_2\})$. Consider the difference $\Delta(p_1, W) - \Delta(p_1, W + \{p_1\})$. Note that we do not do any further assumptions in order to derive (9). Preempting (9), we know that $\Delta(p_1, W) - \Delta(p_1, W + \{p_1\}) \geqslant \frac{1}{k(k-1)}$ holds, but on the other hand we get from (7) and (8) that $\Delta(p_1, W) - \Delta(p_1, W + \{p_1\}) \leqslant 2(k-1)\epsilon = \frac{1}{1/2 + k(k-1)} < \frac{1}{k(k-1)}$ holds, a contradiction.

a contradiction to the termination of LS-PAV. The first inequality is due to (9) and the second due to (7) and (8).

Case 3: Finally, suppose that we are neither in Case 1 nor in Case 2. It follows that $\sum_{i \in S} u_i(W) = 0$ but $\sum_{i \in S} u_i(T) \geqslant |S|$. Hence, there exists some p_2 in the support of T with at least $|S|/|T| \geqslant n/k$ supporters in S. This is a contradiction to the fact that LS-PAV satisfies EJR which was shown by Aziz et al. (2018).[9]

Lastly, we show that LS-PAV for $\epsilon = \frac{1}{(1+2(k-1))(k-1)k}$ runs in polynomial time in $|P|$, n, and k. We follow the proof by Aziz et al. (2018) showing that LS-PAV runs in polynomial time for $\epsilon' = n/k^2$. Per iteration of the while loop, the algorithm computes at most mk PAV scores, which can be done in polynomial time. In order to bound the number of while loops, observe that the PAV score of a committee is upper bounded by $nH_k \in \mathcal{O}(n \ln k)$ and the algorithm improves the PAV score of the best committee found so far in every iteration by at least ϵ. Hence, there are $\mathcal{O}(nk^3 \ln k)$ iterations of the while loop, which suffices to prove the claim. $\square$

B Computational Aspects

PAV is NP-hard to compute in the candidate-approval setting (Aziz et al., 2015b). Since party-approval is a restricted domain, it is in principle possible that PAV is easy for that domain, but, as we show next, hardness still holds for party-approval elections.

We show NP-hardness by reduction from the NP-complete problem INDEPENDENT SET (Garey and Johnson, 1979).

INDEPENDENT SET
Input: Undirected graph $G = (V, E)$, $t \in \mathbb{N}$.
Question: Is there a vertex subset $V' \subseteq V$ of size $|V'| = t$ such that no two vertices in V' are connected by an edge in G?

This problem is NP-hard even when restricted to cubic graphs (where every vertex has degree 3) (Garey and Johnson, 1979). Our reduction is a simplified version of the reduction proposed by Aziz et al. (2015b, Theorem 1).

Theorem 5. *For a given threshold $s \in \mathbb{R}$, deciding whether there exists a committee with PAV score at least s is NP-hard in the party-approval subdomain.*

Proof. For a given cubic graph $G = (V, E)$ and independent set size $t \in [|V|]$, we construct a party-approval election (N, P, A, k) in the following. For each vertex $v \in V$, there is a party $p_v \in P$. For every edge $e = \{u, v\} \in E$, there is one voter in N who approves exactly p_u and p_v. Lastly, we set $k = t$.

This construction is clearly polynomial in the size of G. We show that G has an independent set of size t iff there is a committee W for the election (N, P, A, k) with $\mathrm{PAV}(W) \geqslant s = 3t$.

"$\Rightarrow$": Assume that G has an independent set $V' \subseteq V$ of size $|V'| = t$. Consider the committee W where for every vertex $v \in V'$, the party p_v receives exactly one seat (thus, the committee has size $k = t$). Each party p_v is approved by three voters, namely all those voters corresponding to edges that are incident to v. Because V' is an independent set, no voter approves more than one party in the committee, and thus only has a single seat on the committee belonging to an approved party. Consequently, the total PAV score of W is exactly $3t$.

"$\Leftarrow$": Assume that W is a committee with $\mathrm{PAV}(W) \geqslant 3t$ for the constructed election. The PAV score of a given committee can be computed by starting with the empty committee and then iteratively

[9] Note that Aziz et al. (2018) show that LS-PAV satisfies EJR when $\epsilon' = \frac{n}{k^2}$. Since $\epsilon \leqslant \epsilon'$ for all $k \geqslant 2$, their result carries over to LS-PAV with ϵ.

adding up the marginal PAV score of each seat within the committee. Every party p_v is approved by exactly three voters and therefore, giving one seat to p_v in the committee can increase the PAV score by at most three. As there are only t seats available, every seat assignment has to increase the PAV score by exactly three. In order to achieve an increase of three when adding a seat to p_v, all the voters who approve p_v must have been previously completely unrepresented. Thus, all parties present in W receive only one seat and do not have any common approving voters. By construction, this implies that the set of vertices $\{v \in V : W(p_v) > 0\}$ corresponding to W is an independent set of size t. □

In the candidate-approval setting, checking whether a given committee satisfies PJR or EJR is coNP-complete (Aziz et al., 2017, 2018). In other words, given a committee, it is hard to find a cohesive coalition of voters that is underrepresented. Interestingly, this task is tractable in party-approval elections. Intuitively, checking becomes easier in party-approval elections because groups of voters are already cohesive when they have only one approved party in common.

Theorem 6. *Given a party-approval election (N, P, A, k) and a committee $W : P \to \mathbb{N}$, it can be checked in polynomial time whether W satisfies EJR.*

Proof. We describe a procedure to check whether a given committee W violates EJR. For each party $p \in P$ and each $\ell \in [k]$, define

$$S_{p,\ell} = \{i \in N \mid p \in A_i \text{ and } \textstyle\sum_{p \in A_i} W(p) \leqslant \ell\}$$

and check whether $\ell < q(S_{p,\ell})$ holds. If so, the set $S_{p,\ell}$ induces an EJR violation. This is because $\bigcap_{i \in S_{p,\ell}} A_i \neq \emptyset$ and $\sum_{p \in A_i} W(p) \leqslant \ell < q(S_p, \ell)$ holds for all $i \in N$.

Now, assume that the condition is not satisfied for any party $p \in P$ and any $\ell \in [k]$. We claim that this proves the nonexistence of an EJR violation. Assume for contradiction that there exists a group $S \subseteq N$ inducing an EJR violation. Let $p \in \bigcap_{i \in S} A_i$ and $\ell = \max_{i \in S}\{\sum_{p \in A_i} W(p)\}\}$. By definition, $S \subseteq S_{p,\ell}$ and hence $q(S_{p,\ell}) \geqslant q(S) > \ell$, a contradiction. A straightforward implementation of this algorithm has polynomial running time $\mathcal{O}(|P|\, k\, n)$. □

Checking PJR is more involved. We use techniques from submodular optimization. Recall that, given a finite set U, a function $f : 2^U \to \mathbb{R}$ is *submodular* if for all subsets $X, Y \subseteq U$ with $X \subseteq Y$ and for every $x \in U \setminus Y$, it holds that

$$f(X \cup \{x\}) - f(X) \geqslant f(Y \cup \{x\}) - f(Y).$$

A submodular function $f : 2^U \to \mathbb{Z}$ can be minimized in time polynomial in $|U| + \log \max\{|f(S)| : S \subseteq U\}$ (Korte and Vygen, 2018, Theorem 14.19). Applying this result, one can check whether a party-approval committee provides PJR in polynomial time.

Theorem 7. *Given a party-approval election (N, P, A, k) and a committee $W : P \to \mathbb{N}$, it can be checked in polynomial time whether W satisfies PJR.*

Proof. We fix a committee $W : P \to \mathbb{N}$ and define the function $h : 2^N \to \mathbb{N}$ by

$$h(S) = \sum_{p \in \bigcup_{i \in S} A_i} W(p),$$

i.e., for a voter group $S \subseteq N$, $h(S)$ is the total number of seats that W allocates to parties approved by some voter in S. Moreover, for each party $p \in P$, we let $N_p = \{i \in N \mid p \in A_i\}$ denote the set of supporters of p. Observe that the committee W satisfies PJR if and only if there is no party $p \in P$ and group of voters $S \subseteq N_p$ with $h(S) < q(S)$.

We show how to check in polynomial time for a fixed party $p \in P$, whether there exists such a group of voters $S \subseteq N_p$. Then, this procedure can be repeated for every party in P.

We define the function $f : 2^{N_p} \to \mathbb{R}$ by

$$f(S) = h(S) - |S|\,\frac{k}{n}$$

and show that f is submodular. To this end let $X, Y \subseteq N_p$ with $X \subseteq Y$ and $x \in N_p \setminus Y$. Then,

$$\begin{aligned} f(X \cup \{x\}) - f(X) &= \sum_{p \in A_x} W(p) - \sum_{p \in A_x \cap (\bigcup_{i \in X} A_i)} W(p) - \frac{k}{n} \\ &\geqslant \sum_{p \in A_x} W(p) - \sum_{p \in A_x \cap (\bigcup_{i \in Y} A_i)} W(p) - \frac{k}{n} \\ &= f(Y \cup \{x\}) - f(Y), \end{aligned}$$

which suffices to prove the submodularity of f.

By multiplying f by n, we obtain an integer-valued submodular function with $\max\{n \cdot |f(S)| : S \subseteq N_p\} \leqslant kn$; thus, we can minimize f in time $\mathcal{O}(n + \log(kn))$.

We show in the following that any $S \subseteq N_p$ is the witness of a PJR violation if and only if $f(S) \leqslant -1$.

For the direction from left to right, assume that $S \subseteq N_p$ shows a violation of PJR, i.e., $h(S) < q(S)$. Since both values are integers, we know in particular that $h(S) \leqslant q(S) - 1 = \lfloor |S| \frac{k}{n} \rfloor - 1 \leqslant |S| \frac{k}{n} - 1$ holds. This implies $f(S) \leqslant -1$.

For the direction from right to left, fix some $S \subseteq N_p$ with $f(S) \leqslant -1$. It follows that $h(S) \leqslant |S| \frac{k}{n} - 1 < q(S)$, a violation of PJR for the group S.

The above observation implies a natural procedure to check for a PJR violating group within the supporters of some party p: Minimize the function f and check whether its minimum is larger than -1. If not, we have found a violation. If the minimum of f is larger than -1 for all $p \in P$, then W satisfies PJR. The described algorithm runs in time $\mathcal{O}\big(|P|(n + \log(kn))\big)$. □

C Results on Further Multiwinner Voting Rules

In this section we consider other approval-based multiwinner voting rules from the literature and study their axiomatic properties in the party-approval subdomain. Note that we use the language of the candidate-approval setting and in particular, W is a set (not a multiset) of candidates. In order to apply the described rules in the party-approval setting, we can transform any party-approval election to a candidate-approval election by introducing k clones of each party (see Section 4.2).

We focus on five rules that satisfy PJR in the candidate-approval setting, and briefly comment on rules not satisfying PJR in Section C.4. For all five rules, we show that they do *not* satisfy stronger proportionality axioms in the party-approval subdomain; see Table 1 for a summary of our observations. Furthermore, we show that leximax-Phragmén remains computationally intractable when restricting the domain to party-approval elections.

C.1 Phragmén's Rules

The first three rules we consider are (at least partially) due to Swedish mathematician Lars Edvard Phragmén.[10] The first two rules, leximax-Phragmén and seq-Phragmén, are based on the concept of *load distributions*: It is assumed that adding a candidate to the committee incurs one unit of "load," which needs to be distributed among the approvers of this candidate. The rules aim to select

[10]Phragmén's original papers are written in French or Swedish (Phragmén, 1894, 1895, 1896, 1899); an English account of this work was composed by Janson (2016).

Rule	PJR	EJR	Core Stability
PAV	✓	✓	✓
seq-Phragmén	✓	-	-
leximax-Phragmén	✓	-	-
Eneström-Phragmén	✓	-	-
Rule X	✓	✓	-
Maximin support method	✓	-	-

Table 1: The table contains a summary of the axiomatic properties of candidate-approval rules within the subdomain of party-approval elections.

committees for which the associated load can be distributed as evenly as possible among the voters, where the balancedness of a load distribution is measured by the maximal total load of a voter.

Formally, a real-valued vector $(x_{i,c})_{i \in N, c \in C}$ is a *load distribution* for a candidate-approval election (N, C, A, k) if the following properties hold (Brill et al., 2017):

$$0 \leqslant x_{i,c} \leqslant 1 \qquad \text{for } i \in N, c \in C, \tag{13}$$

$$x_{i,c} = 0 \qquad \text{if } c \notin A_i, \tag{14}$$

$$\sum_{i \in N} \sum_{c \in C} x_{i,c} = k, \tag{15}$$

$$\sum_{i \in N} x_{i,c} \in \{0, 1\} \qquad \text{for } c \in C. \tag{16}$$

In this definition, $x_{i,c}$ represents the load of candidate c that is assigned to voter i. The total load of voter i is given by $\sum_{c \in C} x_{i,c}$. Properties (15) and (16) ensure that each load distribution corresponds to a committee of size k: candidate c is in the committee if and only if $\sum_{i \in N} x_{i,c} = 1$.

The rule **leximax-Phragmén** globally minimizes the balancedness of load distributions and returns committees corresponding to load distributions $(x_{i,c})$ such that $\max_{i \in N} \sum_{c \in A_i} x_{i,c}$ is minimal. (Ties are broken in a leximax fashion; for details, we refer to Brill et al. (2017)). In candidate-approval elections, leximax-Phragmén satisfies EJR and is NP-hard to compute (Brill et al., 2017). We first show that the computational intractability still holds for party-approval elections.

Theorem 8. *Computing a winning committee for leximax-Phragmén is NP-hard in the party-approval subdomain.*

Proof. The notion of load distributions can be adapted in a straightforward manner to party-approval elections, by replacing constraint (13) with $0 \leqslant x_{i,p} \leqslant k$ and constraint (16) with $\sum_{i \in N} x_{i,p} \in [k]$ for all $p \in P$. We prove that the following problem is NP-hard:

PARTY APPROVAL LEXIMAX-PHRAGMÉN
Input: party-approval election (N, P, A, k), distribution bound $s \in \mathbb{R}$
Question: Is there a load distribution $(x_{i,p})$ such that $\max_{i \in N} \sum_{p \in A_i} x_{i,p} \leqslant s$?

Similarly to the proof of Theorem 5, we use a polynomial reduction from INDEPENDENT SET on cubic graphs. The reduction is a variant of the one by Brill et al. (2017), which shows that leximax-Phragmén is NP-hard in the candidate-approval setting.

Given a cubic graph $G = (V, E)$ and independent set size $t \in \mathbb{N}$, we define the following party-approval election: For every vertex $v \in V$, there is a party $p_v \in P$. Additionally, for every edge $e = \{u, v\} \in E$, there is a voter in N who approves exactly p_u and p_v. The committee shall be as large as the independent set, that is, $k = t$. To prove that this reduction is sound, we show that G has an independent set of size t if and only if there is a load distribution $(x_{i,p})$ with $\max_{i \in N} \sum_{p \in A_i} x_{i,p} \leqslant \frac{1}{3}$.

"$\Rightarrow$": Assume G has an independent set $V' \subseteq V$ of size $|V'| = t$. Because G is cubic, every party in the created election is approved by exactly 3 voters. We define a valid load distribution, in which every party corresponding to a vertex in V' creates a load of $\frac{1}{3}$ on every approving voter. (This also implies that in the induced committee, the parties corresponding to V' receive exactly one seat.) Because V' is an independent set, no voter receives load from multiple parties, and hence the maximal total load of every voter is $\frac{1}{3}$.

"$\Leftarrow$": Assume there is a load distribution $(x_{i,p})$ such that $\max_{i \in N} \sum_{p \in A_i} x_{i,p} \leqslant \frac{1}{3}$. Since every party is approved by exactly 3 voters, it follows that $x_{i,p} = \frac{1}{3}$ for a voter i who approves a party p that receives a seat in the induced committee. Consequently, no party receives more than one seat in the induced committee and no voter approves more than one party in the committee. Thus, the committee induces the independent set $\{v \in V : x_{i,p_v} > 0 \text{ for some } i \in N\}$ of size t. ☐

In order to prove that leximax-Phragmén does not satisfy EJR in the party-approval setting, we use straightforward adaptation of an example by Aziz et al. (2017) (which also used by Sánchez-Fernández et al. (2017b) and Brill et al. (2017)).

Proposition 9. *leximax-Phragmén does not satisfy EJR for party-approval elections.*

Proof. Let $n = 8$, $k = 4$, and $P = \{A, B, C, D, X\}$. The ballot profile is given by

$$\begin{array}{llll} 1 \times \{A, X\}, & 1 \times \{B, X\}, & 1 \times \{C, X\}, & 1 \times \{D, X\}, \\ 1 \times \{A\}, & 1 \times \{B\}, & 1 \times \{C\}, & 1 \times \{D\}. \end{array}$$

In this election, leximax-Phragmén gives one seat each to the parties A, B, C, D and thus achieves a perfectly balanced load distribution. Consider the group consisting of the four voters approving party X. This group has a quota of 2, but no voter in this group is represented twice in the leximax-Phragmén committee. ☐

The instance from the proof of Proposition 9 also shows that the incompatibility of EJR and *proportional representation (PR)*, a proportionality axiom proposed by Sánchez-Fernández et al. (2017b), remains intact in the party-approval subdomain.

The rule **seq-Phragmén** constructs committee sequentially, starting with the empty committee and iteratively adding a candidate that increases the maximum voter load the least. For a formal definition, we again refer to Brill et al. (2017). Seq-Phragmén does not satisfy EJR in candidate-elections, and the same is true for the party-approval subdomain.

Proposition 10. *seq-Phragmén fails EJR for in party-approval elections.*

Proof. Fix a natural number $k \geqslant 282$. We construct a party-approval election with parties A, B, C, D, E, X. The ballot profile of the $n = 2k$ many voters is as follows:

$$\begin{array}{llll} 1 \times \{A, X\}, & 1 \times \{B, X\}, & 1 \times \{C, X\}, & 1 \times \{D, X\}, \\ 7 \times \{A, B, C, D\}, & & (2k - 11) \times \{E\}. & \end{array}$$

We first ignore the voters approving E and focus on the 11 remaining voters. Initially, adding a seat to A, B, C, or D would increase the maximal voter load to $\frac{1}{8}$, while giving X one seat would increase it to $\frac{1}{4}$. Without loss of generality, assume A receives this seat. Then, giving the next seat to B, C, or D would increase the maximal load to $(\frac{7}{8} + 1) \cdot \frac{1}{8} = \frac{15}{64}$; giving it to A would increase it to $\frac{1+1}{8} = \frac{16}{64}$, and giving the seat to X would increase it to $\frac{\frac{1}{8}+1}{4} = \frac{18}{64}$. Thus, we can assume B receives the seat. Analogously, the next two seats are allocated to C and D, respectively—the exact computations can

Party	Iteration 1	Iteration 2	Iteration 3	Iteration 4	Iteration 5
A	**0.125**	0.25	0.34570	0.42944	**0.50272**
B	0.125	**0.23438**	0.35938	0.44312	0.51639
C	0.125	0.23438	**0.33008**	0.45508	0.52835
D	0.125	0.23438	0.33008	**0.41382**	0.53882
X	0.25	0.28125	0.33984	0.42236	0.52582

Table 2: The seq-Phragmén computation for the profile in Proposition 10, when party E is ignored. The table shows, for each iteration, the maximal voter load that would result from assigning the next seat to a given party, rounded to five significant digits. The bold entries denote which party receives a seat (with lexicographic tie-breaking).

Party	Round 1	Round 2	Round 3	Round 4	Round 5	Round 6	Round 7
A	240.0	179.86	179.86	103.26	103.26	103.26	**103.26**
X_1	**242.0**	120.71	120.71	120.71	120.71	120.71	60.14
X_2	190.0	190.0	**190.0**	68.71	45.96	45.96	45.96
X_3	183.0	121.86	121.86	121.86	121.86	**121.86**	0.57
X_4	240.0	240.0	179.61	**134.92**	13.64	13.64	13.64
X_5	241.0	**241.0**	119.71	119.71	66.13	7.86	7.86
X_6	186.0	186.0	125.11	125.11	**125.11**	3.82	3.82

Table 3: The Eneström-Phragmén computation for the restricted profile in Proposition 11. The table shows the scores of the parties in the first seven iterations. The bold entries denote the party with the highest score.

be found in Table 2. The fifth seat would then be allocated again to A, increasing the maximal voter load to $\frac{16473}{32768} \approx 0.50272$.

Taking the voters for E into account would not affect the computation above, because all voters who approve E do not approve any other party. Thus, in every seq-Phragmén iteration, either E receives a seat or one of A, B, C, D receives a seat, until A, B, C, D all have one seat. Adding a seat to E increases the load of a voter approving E by $\frac{1}{2k-11}$. Thus, if $k-4$ seats are allocated to E, each E-voter would have a load of $\frac{k-4}{2k-11}$. Observe that $\lim_{k\to\infty} \frac{k-4}{2k-11} = \frac{1}{2}$ and indeed, $\frac{k-4}{2k-11} < 0.50272$ for all $k \geqslant 282$. Therefore, seq-Phragmén returns a committee where A, B, C, D each receive one seat and E receives the remaining $k-4$ seats.

This is a contradiction to EJR: Since $n/k = 2$, EJR demands one of the four voters approving X to be represented at least twice in the committee. This is not the case for the committee selected by seq-Phragmén. □

Additionally, Phragmén developed a voting rule which adapts the well-known single transferable vote (STV) system to approval ballots. Following Camps et al. (2019), we refer to this method as **Eneström-Phragmén**. Like seq-Phragmén, this method selects candidates iteratively. Initially, every voter has *weight* of 1 and a candidate's score is the sum of all approving voters' weights. In every round, the candidate with the highest score s is added to the committee. If a voter i with weight f_i approves the candidate who is added to the committee, then their weight will be updated to $f_i \cdot (s - n/k)/s$ if $s > n/k$, and to 0 otherwise. This process is repeated until all k seats are assigned.

The Eneström-Phragmén rule does not satisfy EJR in candidate-approval elections (Sánchez-Fernández et al., 2017a; Camps et al., 2019), and the same holds for party-approval elections.

Proposition 11. *Eneström-Phragmén fails EJR for in party-approval elections.*

Proof. For $k \geqslant 18$, consider an election with $n = 120k$ voters and $k+1$ parties $A, X_1, \dots, X_k$. The ballot profile is as follows:

$$\begin{aligned}
&120 \times \{A, X_1\}, && 120 \times \{A, X_2\}, && 122 \times \{X_1, X_3\},\\
&70 \times \{X_2, X_4\}, && 120 \times \{X_4, X_5\}, && 121 \times \{X_5, X_6\},\\
&61 \times \{X_3\}, && 50 \times \{X_4\}, && 65 \times \{X_6\},\\
&109 \times \{X_j\} \text{ for } j \in \{7, \dots, 15\}, && && \\
&110 \times \{X_j\} \text{ for } j \in \{16, 17, 18\}. && &&
\end{aligned}$$

If $k > 18$, we also add 120 voters approving $\{X_j\}$ for every $j \in \{19, \dots, k\}$.

First, consider the parties $A, X_1, \dots, X_6$ only. In Table 3, the Eneström-Phragmén calculation for an election restricted to these parties is described. Note that in the first 6 iterations, the parties $X_1, \dots, X_6$ receive one seat each and all have, when selected as winners, a score that exceeds 120. Afterwards, every party has a score strictly smaller than 109.

Furthermore, observe that the parties $X_7, \dots, X_k$ are all approved by voters who only approve this one particular party. As a result, their scores are not affected when other parties receive a seat. The parties $X_7, \dots, X_{15}$ have a score of 109, X_{16}, X_{17}, X_{18} a score of 110, and $X_{19}, \dots, X_k$ (if they exist) a score of 120. When any of these parties receive a seat, their score is decreased to 0, as they are all approved by at most n/k voters.

Together, this shows that Eneström-Phragmén firstly allots one seat each to $X_1, \dots, X_6$. Then, the score of the parties $A, X_1, \dots, X_6$ is always smaller than 109, and therefore, $X_7, \dots, X_k$ all receive a seat, which fills the committee. Thus, in the committee selected by Eneström-Phragmén, $X_1, \dots, X_k$ each receive one seat. However, the $240 = 2n/k$ voters who approve A form a cohesive group, where at least one voter should be represented by at least two seats according to EJR. This is not the case in the committee generated by Eneström-Phragmén . □

C.2 Rule X

Rule X has been proposed by Peters and Skowron (2020). Rule X is similar to seq-Phragmén, but satisfies stronger proportionality guarantees. In particular, Rule X satisfies EJR, but not core stability (Peters and Skowron, 2020). The same holds for the party-approval setting.

Proposition 12. *Rule X fails core stability in party-approval elections.*

Proof. Consider again the party-approval election from Example 3. In this election, Rule X gives 8 seats to party p_0 and 4 seats each to parties p_2 and p_4. As explained in Example 3, this committee is not in the core. □

C.3 Maximin Support Method

The *maximin support method (MMS)* has been proposed by Sánchez-Fernández et al. (2021). The method has strong similarities to seq-Phragmén and selects candidates sequentially. Sánchez-Fernández et al. (2021) show that MMS satisfies PJR, but not EJR. We adapt their EJR counterexample to the party-approval setting.

Proposition 13. *The maximin support method fails EJR in party-approval elections.*

Proof. Let $n = 8$, $k = 4$, and $P = \{A, B, C, X\}$. The ballot profile is given by

$$\begin{aligned}
&5 \times \{A, X\}, && 4 \times \{B, X\}, && 3 \times \{C, X\},\\
&2 \times \{A\}, && 1 \times \{B\}, && 1 \times \{C\}.
\end{aligned}$$

In this election, MMS gives one seat each to the parties A, B, C, X. Consider the group consisting of the 12 voters approving party X. This group has a quota of 3, but no voter in this group is represented three times in the leximax-Phragmén committee. □

C.4 Other Rules

We also considered several other rules from the literature that are known to violate PJR in the candidate setting: sequential PAV and reverse sequential PAV (Thiele, 1895; Janson, 2016), satisfaction approval voting (Brams and Kilgour, 2014), minimax approval voting (Brams et al., 2007), var-Phragmén (Brill et al., 2017), GreedyMonroeAV (Sánchez-Fernández et al., 2017b), Approval Voting, MonroeAV, GreedyAV, HareAV, and Chamberlin–CourantAV (see (Aziz et al., 2017) for definitions of the latter five rules). For each of these rules, we verified that they do not satisfy PJR in the party-approval subdomain either. Since existing counterexamples can be easily adjusted to the party-approval setting, we omit the details.