Multistage Committee Elections

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Abstract

Electing a single committee of a small size is a classical and well-understood voting situation. Being interested in a sequence of committees, we introduce and study two time-dependent multistage models based on simple scoring-based voting. Therein, we are given a sequence of voting profiles (stages) over the same set of agents and candidates, and our task is to find a small committee for each stage of high score. In the *conservative model* we additionally require that any two consecutive committees have a small symmetric difference. Analogously, in the *revolutionary model* we require large symmetric differences. We prove both models to be NP-hard even for a constant number of agents, and, based on this, initiate a parameterized complexity analysis for the most natural parameters and combinations thereof. Among other results, we prove both models to be in XP yet W[1]-hard regarding the number of stages, and that being revolutionary seems to be "easier" than being conservative: If the (upper- resp. lower-) bound on the size of symmetric differences is constant, the conservative model remains NP-hard while the revolutionary model becomes polynomial-time solvable.

Keywords: Computational social choice · Plurality · SNTV · symmetric difference · parameterized complexity.

1 Introduction

In well-studied classical committee election scenarios, given a set of candidates, we aim at selecting a small committee that is, in a certain sense, most suitable for a given collection of preferences over the candidates [7, 14, 26]. However, typically these scenarios concentrate solely on electing a committee in a single election to the neglect of a time dimension. This neglect results in serious limitations of the model. For instance, it is not possible to ensure a relationship (e.g., a small number of changes) between any two consecutive committees. We tackle this issue by introducing a *multistage* [21] variant of the problem. In this variant, a *sequence* of voting profiles *is given*, and we seek a *sequence* of small committees, each collecting a reasonable number of approvals, such that the *difference between consecutive committees* is upper-bounded.

For instance, assume a research community to seek organizers of a series of events (say those scheduled for next year/term). The organizers must be fixed in advance to allow enough preparation time. Since events may differ in location and type, not every candidate fits equally well to every event. Thus, each member (voter) of the community is asked to name one suitable organizer for each event and, based on this, the goal is to determine a sequence of organizing committees (one for each event). Naturally, there are three constraints: (i) each committee is bounded in size, (ii) each committee has enough support from the voters, and (iii) at least a certain number of candidates in consecutive committees overlap to avoid a lack of knowledge transfer jeopardizing effectiveness. We arrive at our main problem (where SNTV comes from the multiwinner rule "single non-transferable vote" to which our model boils down in case of just one stage) which is formally defined next.[1]

[1] Although most of our result do transfer to the setting of general approval profiles as we will discuss in the conclusion of the paper, we restrict our voting profiles to be simple "Plurality profiles", where each agent approves exactly one candidate, for three reasons. (I) Aiming for positive algorithmic results, it is most natural to start with the simplest relevant scenario. In the conclusion we discuss that many results actually transfer to more complex profiles. (II) Plurality profiles are widely accepted in practice. They are used as

Model and Notation. We denote by $\mathbb{N}$ and $\mathbb{N}_0$ the natural numbers excluding and including zero, respectively. For a function $f\colon A \to B$, let $f^{-1}(B') = \{a \in A \mid f(a) \in B'\}$ for every $B' \subseteq B$. We use basic notation from graph theory [11] and parameterized algorithmics [10]. We are ready to define the main problem of this work as follows.

Multistage SNTV (MSNTV)

Input: A set of agents $A = \{a_1, \ldots, a_n\}$, a set of candidates $C = \{c_1, \ldots, c_m\}$, a sequence $U = (u_1, \ldots, u_\tau)$ of τ voting profiles with $u_t\colon A \to C \cup \{\emptyset\}$, $t \in \{1, \ldots, \tau\}$, and three integers $k \in \mathbb{N}$, $\ell \in \mathbb{N}_0$, and $x \in \mathbb{N}$.

Question: Is there a sequence $(C_1, \ldots, C_\tau)$ of committees $C_t \subseteq C$ such that for all $t \in \{1, \ldots, \tau\}$ it holds true that $|C_t| \le k$ and $\text{score}_t(C_t) := |u_t^{-1}(C_t)| \ge x$, and

$$|C_t \triangle C_{t+1}| \le \ell \tag{1}$$

holds true for all $t \in \{1, \ldots, \tau - 1\}$?

In fact, MSNTV models various practical scenarios, for example in context of buffet selection and exhibition composition.

Buffet Selection. Suppose we are asked to organize the venue's breakfast buffet of a multiday event (like a workshop seminar). We offer different disjoint food bundles (candidates) for breakfast and ask the participants of the event to share their preferences of which bundle is their favorite for which day. Due to space constraints, we can offer only at most some number of bundles in the buffet (committee) simultaneously. Moreover, to stay at low cost and to avoid food waste, we want that the bundles changing from one day to the next to be of small number. Clearly, under these constraints, given the collected preferences (voting profiles) we want at all days to have a high number of participants whose voted bundle made it into the buffet.

Exhibition Composition. As another example, when planning a multiday exhibition of sculptures (candidates) in a lobby of a hotel where we are enabled neither to show at once all the sculptures that we want to exhibit, nor to exchange arbitrarily many sculptures between consecutive exhibitions days (due to, e.g., limited capacity of transporting sculptures between some depot and the hotel). To nevertheless offer an enjoyable experience for numerous visitors, we ask the visitors to vote for each day they plan to visit for their favorite sculpture to be exhibited (to keep the poll simple and robust).

Our main goal is to study MSNTV from a computational complexity and parameterized algorithmics perspective to detect tractable cases.

Note that if we drop condition (1) (or, equivalently, setting $\ell = n$), then on the one hand, we have no control over changes between consecutive committees, yet on the other hand, we obtain a linear-time solvable problem. Thus, control comes with a computational cost, bound in the value of ℓ. As a side goal, we aim at a better understanding of condition (1). To this end, we additionally study a problem variant of MSNTV, referred to as Revolutionary Multistage SNTV (RMSNTV), obtained by replacing (1) by $|C_t \triangle C_{t+1}| \ge \ell$. In words, in RMSNTV we request a change of size at least ℓ between consecutive committees. By this, while we complemented the meaning of ℓ, it still expresses a control over the changes, and hence comes possibly again with a computational cost. We intend to understand whether the "*conservative*" (MSNTV) and the revolutionary (RMSNTV) variant differ, and if so, then how and why.

basis for complex voting procedures such as S(N)TV, as well as the two-vote or two-stage voting systems used for the German or French parliament. (III) For the agents to select just their favorite candidate for each time step seems to be only a weak (cognitive) barrier for applicability of the model. Obviously, our definition can be easily extended to more expressive scoring-based voting profiles.

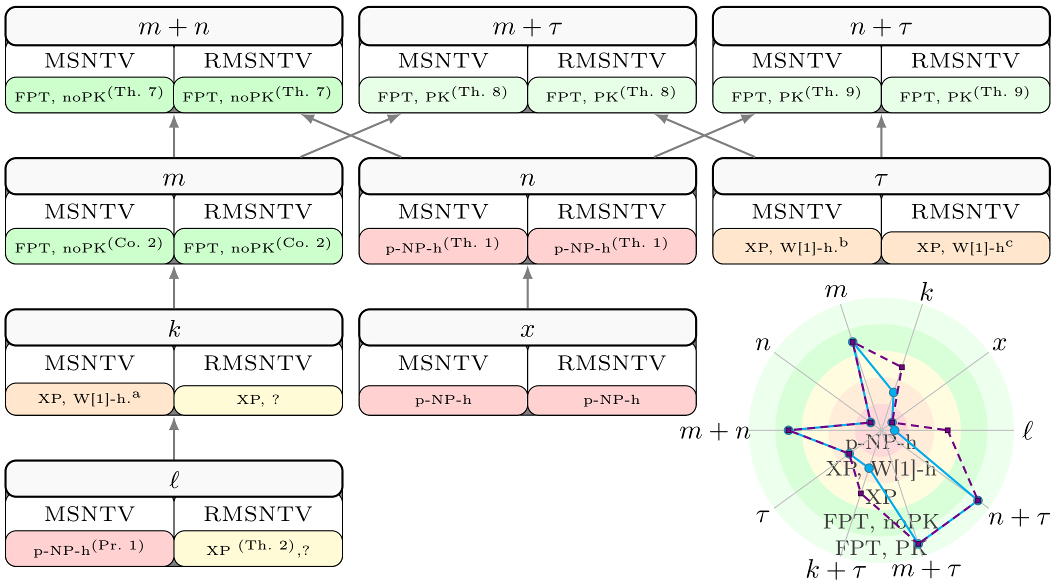

Figure 1: Overview of our results. PK, noPK, p-NP-h, and W[1]-h respectively abbreviate "polynomial kernel," "no polynomial kernel unless NP $\subseteq$ coNP / poly," para-NP-hard, and W[1]-hard. An arrow from one parameter p to another parameter p' indicates that p can be upper bounded by some function in p' (e.g., $\ell \leq 2k$, $k \leq m$, or $x \leq n$). "?" indicates that a classification as FPT or W-hard is unknown. On below right, a spiderweb diagram depicts our results (solid: conservative; dashed: revolutionary). [a](Th. 3 & 4) [b](Th. 4 & 6) [c](Th. 5 & 6)

Related work. Our model follows the recently proposed *multistage* model [12, 21], that led to several multistage problems [3, 6, 16, 17, 2, 9, 22, 5, 4] next to ours. Graph problems considered in the multistage model often study classic problems on temporal graphs (a sequence of graphs over the same vertex set). While all the multistage problems known from literature cover the variant we call "conservative," our revolutionary variant forms a novel submodel herein.

Although, to the best of our knowledge our model is novel, another aspects of selecting multiple (sub)committees have also been studied in (computational) social choice theory. The closest related to ours is a recent work of Bredereck et al. [8], who augment classic multiwinner elections with a time dimension. Accordingly, they consider selecting a sequence of committees. However, the major differences with our work are, first, that they do not allow agents (voters) to change their ballots over time and, second, that there are no explicit constraints on the differences between two successive committees. Freeman et al. [19], Lackner [23], and Parkes and Procaccia [24] allow this but they consider an online scenario (in contrary to our problem that is offline). Finally, Aziz and Lee [1] study a so-called subcommittee voting, where a final committee is a collection of several subcommittees. Their model, however, does not take into account time and requires that all subcommittees are mutually disjoint.

Our Contributions. We present the first work in the multistage model that studies a problem from computational social choice and that compares the two cases that we call conservative and revolutionary. We prove MSNTV and RMSNTV to be NP-complete, even for two agents. We present a full parameterized complexity analysis of the two problems (see Figure 1 for an overview of our results). MSNTV and RMSNTV are almost indistinguishable regarding their parameterized complexity, but when parameterized by ℓ, MSNTV is NP-hard and RMSNTV is contained in XP. Moreover, both problems are contained in XP and W[1]-hard regarding the parameter number τ of stages; Note that for many natural multistage problems (and even temporal graph problems), such a classification is unknown—τ usually leads to para-NP-hardness. Our results further indicate that efficient data reductions (see Section 6 for the definition) in terms of polynomial-size problem kernelizations require a combination with τ: While combining the number of agents with the number of candidates allows for no polynomial-size problem kernel (unless NP $\subseteq$ coNP / poly), combining any of the two with τ yields kernels of polynomial size.

Due to the space constraints, many details are deferred to an appendix (marked by ★).

2 Hardness even for few agents

We first settle the classical computational complexity of both MSNTV and RMSNTV. We prove that both problems are NP-complete, even for two agents (and hence for $x \leq 2$).

Theorem 1. MSNTV with $\ell = 0$ and RMSNTV with $\ell = 2k$ are NP-complete even for two agents.

We prove the statements of Theorem 1 separately, starting with a specific, computationally intractable case of MSNTV, which follows by a reduction from a Vertex Cover variant.[2]

Proposition 1 (★). MSNTV is NP-hard even for two agents, $\ell = 0$, $x = 1$, and $k = |C|/2$.

NP-hardness of RMSNTV follows from the fact that we can reduce the specific variant of MSNTV that we proved to be NP-hard in Proposition 1 to RMSNTV without increasing the number of agents.

Lemma 1 (★). There is an algorithm that, on every instance (A, C, U, k, ℓ, x) with $\ell = 0$ and $k = |C|/2$ of MSNTV, computes an equivalent instance $(A, C', U', k', \ell', x)$ of RMSNTV with $k' = |C'|/2$, $\ell' = 2k'$, and $|U'| = 2|U| + 1$ in polynomial time.

Combining Lemma 1 and Proposition 1, we get the following.

Corollary 1. RMSNTV is NP-hard even for two agents, $\ell = 2k$, $x = 1$, and $k = |C|/2$.

Theorem 1 now follows from Proposition 1 and Corollary 1. In fact, $\ell = 0$ (MSNTV) and $\ell = 2k$ (RMSNTV) are not the only intractable cases. We also get the following.

Proposition 2 (★). MSNTV with $\ell = 1$ and RMSNTV with $\ell = m - 1 = 2k - 2$ are NP-hard.

3 Role of ℓ

We investigate the role of ℓ for our two problem variants. To start with, we have the following easy observation.

Observation 1. MSNTV *is polynomial-time solvable if* $\ell = 2k$ *and* RMSNTV *is polynomial-time solvable if* $\ell = 0$.

Observation 1 suggests that while MSNTV is NP-hard even for constant ℓ (Theorem 1), this seems to be different for RMSNTV. In fact, next we prove that RMSNTV is contained in XP regarding ℓ.

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[2] We remark that $x = 1$ allows that most of the candidate can have no support. At least in the conservative setting, however, there can be situations with small ℓ where this is unavoidable. Moreover, we can modify the reduction to include at each stage for each candidate one (additional) supporter and increase x by k.

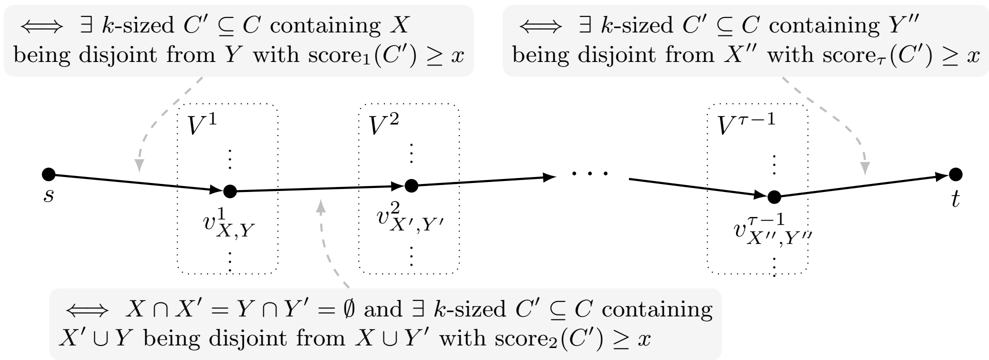

Figure 2: Illustration of an in-out graph from Definition 1.

Theorem 2. Every instance $I = (A, C, U, k, \ell, x)$ of RMSNTV with n agents, m candidates, and τ voting profiles can be decided in $\mathcal{O}(\tau \cdot m^{4\ell+1} \cdot n)$ time.

Note that we can assume that $\ell \leq 2k$. The crucial observation behind the XP-algorithm comes from the structure of every solution $\mathcal{C} = (C_1, \ldots, C_\tau)$: Since $|C_i \triangle C_{i+1}| \geq \ell$, there are $X \subseteq C_i \setminus C_{i+1}$ and $Y \subseteq C_{i+1} \setminus C_i$ such that $X \cap Y = \emptyset$ and $|X \cup Y| \geq \ell$. This allows us to build a directed graph D as follows (see Figure 2 for an illustration).

Definition 1. Given an instance $I = (A, C, U, k, \ell, x)$ of RMSNTV, the *in-out* graph of I is a directed graph D_I with vertex set $V = V^1 \cup \cdots \cup V^{\tau-1} \cup \{s, t\}$ where $V^i = \{v^i_{X,Y} \mid X, Y \subseteq C, |X|, |Y| \leq \ell, X \cap Y = \emptyset, |X \cup Y| \geq \ell\}$ for every $i \in \{1, \ldots, \tau - 1\}$, containing the arcs $(s, v^1_{X,Y})$ if and only if there is a k-sized committee at time step 1 containing X being disjoint from Y with score at least x, the arcs $(t, v^{\tau-1}_{X,Y})$ if and only if there is a k-sized committee at time step τ containing Y being disjoint from X with score at least x, and an arc $(v^i_{X,Y}, v^j_{X',Y'})$ if and only if (i) $j = i+1$, $X \cap X' = Y' \cap Y = \emptyset$, and (ii) there is a k-sized committee at time step $i+1$ containing $Y \cup X'$ being disjoint from $X \cup Y'$ with score at least x.

With an XP-running time (regarding ℓ), we can compute the in-out graph for every given instance.

Lemma 2 (★). Given an instance of RMSNTV with n agents, m candidates, and τ voting profiles, the in-out graph D_I of I can be computed in $\mathcal{O}(\tau \cdot m^{4\ell+1} n)$ time.

We prove that deciding an instance of RMSNTV can be done through deciding whether there is an s-t path in the instance's in-out graph.

Lemma 3 (★). Let I be an instance of RMSNTV and D_I its in-out graph. Then, there is an s-t path in D_I if and only if I is a `yes`-instance.

Given Lemmas 2 & 3, we are set to prove Theorem 2.

Proof of Theorem 2. Let $I = (A, C, U, k, \ell, x)$ be an instance of RMSNTV with n agents, m candidates, and τ voting profiles. First, construct the in-out graph D_I of I in $\mathcal{O}(\tau \cdot \ell^4 m^{4\ell+1} n)$ time (due to Lemma 2). Next, in time linear in the size of D_I check for an s-t path in D_I. Due to Lemma 3, if an s-t path is found, then report that I is a `yes`-instance, and otherwise, if no such s-t path is found, then report that I is a `no`-instance, □

We remark that the proof of Lemma 3 contains the description of how to make the algorithm constructive (that is, if one requires to return a solution).

4 Size of a Committee and of Their Sequence

In this section, we investigate the role of the committee size k and the parameter $k + \tau$ describing the size $k \cdot \tau$ of the solution—a sequence of committees. Firstly, we prove that MSNTV and RMSNTV are contained in XP when parameterized by k.

Theorem 3 (★). MSNTV and RMSNTV both admit an $\mathcal{O}(\tau \cdot m^{2k+1} \cdot n)$-time algorithm and hence are contained in XP when parameterized by k.

The proof of Theorem 3 is based on computing in XP-running time an auxiliary directed graph in which we then check for the existence of an s-t path witnessing a `yes`-instance. Clearly, since $k \leq m$, Theorem 3 proves both problem variants to be fixed-parameter tractable when parameterized by m.

Corollary 2. MSNTV and RMSNTV both are solvable in time $2^{\mathcal{O}(m \log(m))} \cdot \tau n$.

Given Corollary 2, the question arises whether Theorem 3 can be improved to show fixed-parameter tractability regarding k. Next we first prove that this seems unlikely for MSNTV and then discuss the question for RMSNTV.

We first state Theorem 4 and then prove it using several intermediate lemmas.

Theorem 4. MSNTV is W[1]-hard when parameterized by $k + \tau$, even if $\ell = 0$.

In the reduction behind the proof of Theorem 4, we employ *Sidon sets* defined subsequently.

Definition 2. A Sidon set is a set $S = \{s_1, s_2, \ldots, s_b\}$ of b natural numbers such that every pairwise sum of the elements in S is different.

Sidon sets can be computed efficiently.

Lemma 4. A Sidon set of size b can be computed in $\mathcal{O}(b)$ time if b is encoded in unary.

Proof. Suppose we aim at obtaining a Sidon set $S = \{s_1, \ldots, s_b\}$. For every $i \in \{1, \ldots, b\}$, we compute $s_i := 2\hat{b}i + (i^2 \bmod \hat{b})$, where $\hat{b}$ is the smallest prime number greater than b [13]. Thus, given $\hat{b}$, one can compute S in linear time.

It remains to show how to find $\hat{b}$ in linear time. Due to the Bertrand-Chebyshev [27] theorem, we have that $\hat{b} < 2b$. Searching all prime numbers smaller than $2b$ is doable in $\mathcal{O}(b)$ time (see, for example, an intuitive algorithm by Gries and Misra [20]). □

We are set to prove Theorem 4.

Proof of Theorem 4. We reduce from MULTICOLORED CLIQUE that is W[1]-complete when parameterized by the solution size [25, 15]. An instance $\hat{I}$ of MULTICOLORED CLIQUE consists of a q-partite graph $G = (V_1 \uplus V_2 \uplus \cdots \uplus V_q, E)$ and the task is to decide whether there is a set K of q pairwise connected vertices, each from a distinct part. For brevity, for some $i, j \in \{1, \ldots, q\}$, $i < j$, let E_i^j be a set of edges connecting vertices from parts V_i and V_j; thus, $E = \bigcup_{i,j \in \{1,\ldots,q\}, i<j} E_i^j$.

Construction. In the corresponding instance I of MSNTV, we let all vertices and edges in G be candidates. Then, we define three gadgets (see Figure 3 for an illustration): the *vertex selection* gadget, the *edge selection* gadget, and the *coherence* gadget. Further, we show how to use the gadgets to construct I. Instance I will be constructed in a way that its solution is a single committee of size exactly $q + \binom{q}{2}$ corresponding to vertices and edges of a clique witnessing a `yes`-instance of $\hat{I}$ (if one exists). To define the gadgets, we use a value x that we explicitly define at the end of the construction.

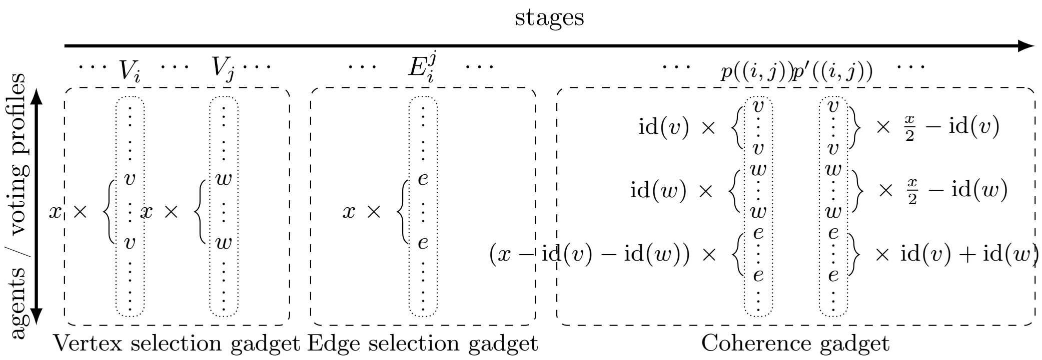

Figure 3: Illustration of the construction in the proof of Theorem 4, exemplified with edge $e = \{v, w\} \in E_i^j$ with $v \in V_i$ and $w \in V_j$. A column represents a stage (which in turn represents an vertex or edge selection gadget for some vertex or edge set, respectively, or a coherence gadget of a pair of colors) and a row represents an agent (approving either a vertex or an edge).

Vertex selection gadget: Fix some part V_i, $i \in \{1, \dots, q\}$. The vertex selection gadget for i ensures that exactly one vertex from V_i is selected. We construct the gadget by forming a preference profile $p(V_i)$ consisting of $x \cdot |V_i|$ agents such that each vertex $v \in V_i$ is approved by exactly x agents.

Edge selection gadget: Similarly, for each two parts V_i and V_j such that $i < j$, we construct the edge selection gadget that allows to select exactly one edge from E_i^j. Accordingly, we build a preference profile $p(E_i^j)$ consisting of $x \cdot |E_i^j|$ agents. Again, each edge in E_i^j is approved by exactly x agents.

Coherence gadget: For the construction of the coherence gadget, let $h := |\bigcup_{i \in \{1,\dots,q\}} V_i|$ and let $S = \{s_1, \dots, s_h\}$ be a Sidon set computed according to Lemma 4. We define a bijective function $\mathrm{id}\colon \bigcup_{i \in \{1,\dots,q\}} V_i \to S$ associating each vertex of G with its (unique) *id*. Now, the construction of the coherence gadget for some pair $\{V_i, V_j\}$ of parts such that $i < j$ goes as follows. We introduce two preference profiles $p((i,j))$ and $p'((i,j))$. In preference profile $p((i,j))$, (i) each candidate $v \in V_i \cup V_j$ is approved by exactly $\mathrm{id}(v)$ agents and (ii) each edge $e = \{v, v'\} \in E_i^j$ is approved by exactly $(x - \mathrm{id}(v) - \mathrm{id}(v'))$ agents. In preference profile $p'((i,j))$, (i) each candidate $v \in V_i \cup V_j$ is approved by exactly $\frac{x}{2} - \mathrm{id}(v)$ agents and (ii) each edge $e = \{v, v'\} \in E_i^j$ is approved by exactly $(\mathrm{id}(v) + \mathrm{id}(v'))$ agents.

Having all the gadgets defined it remains to use them to form the agents and the preference profiles of instance I; and to define x, ℓ, and k. Since we want to have a committee consisting of q vertices and $\binom{q}{2}$ edges, we let $k := q + \binom{q}{2}$. We aim at a single committee, thus we set $\ell = 0$, which enforces that the committee must stay the same over time. Further, we set $x = 2s_h$. Finally, to form the preference profiles of I we put together, in any order, vertex selection gadgets for every part V_i, $i \in \{1, \dots, q\}$ as well as edge selection gadgets and coherence gadgets for every pair $\{V_i, V_j\}$ of parts such that $i < j$. As for the agents of I, with each gadget $\mathcal{G}$ we add a separate set of agents needed to implement $\mathcal{G}$ making sure that all other agents introduced by all other gadgets are approving no candidate in their voting profiles occurring in $\mathcal{G}$.

The running time analysis and correctness proof are deferred to the appendix (★). $\square$

Interestingly, we can transfer the intractability result to RMSNTV only regarding τ.

Theorem 5. RMSNTV parameterized by τ is W[1]-hard.

To prove Theorem 5, we aim for employing Lemma 1, which forms a parameterized reduction regarding τ. To this end, we need firstly to reduce any instance with $\ell = 0$ of MSNTV to an equivalent instance of MSNTV with $k = |C|/2$ and $\ell = 0$ such that the number of resulting voting profiles only depends on τ.

Lemma 5 (★). There is an algorithm that, on every instance (A, C, U, k, ℓ, x) with $\ell = 0$ of MSNTV, computes in polynomial time an equivalent instance $(A', C', U', k', \ell', x')$ of MSNTV with $k' = |C'|/2$, $\ell' = 0$, and $|U'| = |U|$.

Theorem 5 now follows from Lemmas 1 & 5 and Theorem 4 (★). We leave open whether RMSNTV is fixed-parameter tractable when parameterized by k or even by ℓ.

5 Tractability with Few Stages

Now, we study the complexity regarding the parameter lifetime τ. Known results for multistage or temporal (graph) problems suggest that either the problems are NP-hard for constant lifetime or they are (trivially) fixed-parameter tractable regarding the lifetime. We prove that for MSNTV and RMSNTV, this is different: Both are W[1]-hard (Theorems 4 & 5), yet contained in XP due to the following.

Theorem 6 (★). When parameterized by τ, MSNTV and RMSNTV are contained in XP.

6 On Efficient Data Reduction

In this section, we investigate the complexity regarding the parameter combinations $m + n$ (Section 6.1), $m + \tau$ (Section 6.2), and $n + \tau$ (Section 6.3). Since we already know we have fixed-parameter tractability (Corollary 2) regarding m for both problem variants, we focus on efficient and provably effective data reduction, namely *problem kernelization*. A problem kernelization for a parameterized problem L is a polynomial-time algorithm that maps any instance $(x, p) \in \Sigma^* \times \mathbb{N}_0$ of L to an equivalent instance (x', p') of L (the problem kernel) such that $|x'| + p' \leq f(p)$ for some function f only depending on the parameter p. Preferably, we want f to be some polynomial, in which case we call the problem kernelization polynomial. We prove that $m + \tau$ allows for and $m + n$ disallows for polynomial problem kernelization (under some complexity theoretic assumption in the latter case). We further prove that $n + \tau$ allows for polynomial problem kernelization; surprisingly, since for each parameter alone we have para-NP-hardness (for n) and W[1]-hardness (for τ).

6.1 Number of candidates and agents combined

When parameterized by $m + n$ polynomial problem kernelization turns out to be unlikely, which may surprise, since one parameterizes by all dimensions of an input except for the time aspect (number of stages).

Theorem 7 (★). Each of MSNTV and RMSNTV admits no problem kernel of size polynomial in $m + n$ unless NP $\subseteq$ coNP / poly.

Proof (Sketch). To prove Theorem 7, we are going to prove that both, MSNTV and RMSNTV, when parameterized by $m+n$ are AND-compositional. That is, there is an algorithm taking p instance $I_1, \ldots, I_p$, each with the same number n of agents, m of candidates, and τ of profiles, and constructs in time polynomial in $\sum_{i=1}^{p} |I_i|$ an instance I such that the number of agents and candidates is in $(m+n)^{O(1)}$, and that I is a `yes`-instance if and only if each of $I_1, \ldots, I_p$ is `yes`-instance. When a parameterized problem admits an AND-composition, then it admits no polynomial kernel unless NP $\subseteq$ coNP / poly.

Construction 1 (MSNTV)**.** Consider p instances $I_1, \dots, I_p$ of MSNTV such that $k, x \in \mathbb{N}$ and $\ell = 1$, each with n agents $A^i = \{a_1^i, \dots, a_n^i\}$, m candidates $C^i = \{c_1^i, \dots, c_m^i\}$, and τ voting profiles. Add the agent set $A = \{a_1, \dots, a_n\} \cup \{\widehat{a}_1, \dots, \widehat{a}_n\}$ and candidate set $C = \{c_1, \dots, c_n\} \cup \{z\}$. Arrange the p voting profiles consecutively, where we identify each agent a_j^i with a_j and each candidate c_j^i with c_j. Moreover, each agent $\widehat{a}_j$ approves z. Next, between the last voting profile and the first voting profile of two consecutive instances, add $2k$ profiles where all agents approve z (we call them *transfer profiles*). Set $x' = x + n$, $k' = k + 1$, and $\ell' = 1$. Note that $\tau' = p\tau + 2k(p - 1)$. $\diamond$

The correctness proof and discussion of RMSNTV is deferred to the Appendix E.1. $\square$

6.2 Number of candidates and lifetime combined

In the previous section, we have seen that efficient data reduction for all dimensions but τ is unlikely. This indicates that the lifetime is crucial for efficient data reduction. In fact, considering $m+\tau$, we can first reduce our problem to a weighted version, shrink the weights, and then reduce it back to our problem, thereby obtaining a problem kernel of size polynomial in $m + \tau$.

Theorem 8. MSNTV and RMSNTV admit problem kernels of size polynomial in $m + \tau$.

The weighted version of each problem takes number m of candidates many agents, where a_i assigns a weight to candidate i (this corresponds to the number of agents approving candidate i in the unweighted variant). Formally, for MSNTV the weighted version is defined as follows:

Weighted MSNTV (W-MSNTV)
Input: A set W of weight vectors $w^1, \dots, w^\tau \in \mathbb{N}_0^m$, and three integers k, ℓ, x.
Question: Are there subsets $C_1, \dots, C_\tau$ of $\{1, \dots, m\}$ each of size at most $\leq k$ with (i) $\sum_{i \in C_t} w_i^t \geq x$ for all $t \in \{1, \dots, \tau\}$, and (ii) $|C_t \triangle C_{t+1}| \leq \ell$ for all $t \in \{1, \dots, \tau - 1\}$?

The reduction from MSNTV to W-MSNTV is obvious (the sum of weights equals n).

Observation 2 (★)**.** *There is a polynomial-time many-one reduction from* MSNTV *to* W-MSNTV *with τ weight vectors from $\{0, \dots, n\}^m$.*

Lemma 6 (★)**.** There is an algorithm that, given an instance $I = (W, k, \ell, x)$ of W-MSNTV, computes in polynomial time an instance $I' := (W' = \{\widehat{w}^1, \dots, \widehat{w}^\tau\}, k, \ell, \widehat{x})$ such that (i) $\|\widehat{w}^t\|_\infty, |\widehat{x}| \in 2^{\mathcal{O}(m^3\tau^3)}$ for all $t \in \{1, \dots, \tau\}$, and (ii) $\mathcal{C}$ is a solution to I if and only if it is a solution to I'.

It is not hard to see that W-MSNTV is NP-complete. We are set to prove our main result.

Proof of Theorem 8. Let $I = (A, C, k, \ell, x)$ be an instance of MSNTV. Compute in polynomial time an instance $J = (W, k, \ell, x)$ of W-MSNTV being equivalent to I. Next apply Lemma 6 to J to obtain an equivalent instance J' of W-MSNTV of encoding length in $\mathcal{O}((m\tau)^4)$. Finally, reduce J' back to an instance I' of MSNTV in polynomial time. Hence, the encoding length of I' is in $(m\tau)^{\mathcal{O}(1)}$, proving I' to be a polynomial problem kernel. For RMSNTV, the proof works analogously. $\square$

6.3 Number of agents and lifetime combined

In this section, we prove effective preprocessing regarding the parameter $n + \tau$ (and hence fixed-parameter tractability).

Theorem 9. MSNTV and RMSNTV admit problem kernels of size polynomial in $n + \tau$.

Firstly, we note that there are at most $n \cdot \tau$ approvals in any instance. Hence, we have the following.

Observation 3. *There are at least* $\max\{0, m - n \cdot \tau\}$ *candidates which are never approved.*

Upon Observation 3, we will next discuss deleting candidates which are never approved, in order to upper-bound the number m of candidates by some polynomial function in $n + \tau$. With the latter, together with Theorem 8 we will obtain the polynomial-size problem kernels. We treat each of MSNTV and RMSNTV separately.

Observation 3 allows us to reduce any instance of MSNTV to an equivalent instance with $m \leq n \cdot \tau$.

Proposition 3. There is a polynomial-time algorithm that computes for any given instance $I = (A, C, U, k, \ell, x)$ of MSNTV an equivalent instance $I' = (A, C', U, k, \ell, x)$ with $|C'| \leq |A| \cdot |U|$.

To obtain Proposition 3, we employ the following.

Reduction Rule 1 (★). If $m > n\tau$, delete a candidate which is never approved.

Intuitively, Reduction Rule 1 is correct because selecting a candidate which is never approved into a committee at some stage is not beneficial: it only increases the symmetric difference at the respective stage but not the committee's score.

We can safely apply Reduction Rule 1 exhaustively in polynomial-time, hence we get Proposition 3.

For RMSNTV, a similar approach applies. However, we need to carefully adjust ℓ. We first note that if $k < \ell/2$, then we are trivially facing a `no`-instance. Thus, in the following we assume that $k \geq \ell/2$. Additionally, we assume that $m > n\tau$; otherwise, the polynomial-time algorithm from Proposition 3 simply outputs the original instance.

We start with the following reduction rule that allows us to upper-bound the number of candidates by $n\tau$ or $k\tau$, depending on which of n and k is larger.

Reduction Rule 2 (★). If $m > \max\{n, k\} \cdot \tau$, then delete a candidate that is never approved.

Reduction Rule 2 can be applied in polynomial time; thus, in the case of $k \leq n$, applying this rule exhaustively already yields the result from Theorem 9. Yet, the case of $k > n$ still needs to be refined, which we do in the following Lemma 7.

Lemma 7. There is a polynomial-time algorithm that, given an instance of RMSNTV with $k > n$ and $m = k\tau$, computes an equivalent instance of RMSNTV with $m = n\tau$.

Proof. Let $I = (A, C, U, k, \ell, x)$ be an instance of RMSNTV with $m = k\tau$ candidates, $|A| = n$ agents and $k > n$. We give a polynomial-time reduction that transforms I into an equivalent instance $I' = (A, C', U, k', \ell', x)$ with $|C'| = n\tau$ candidates, where $k' = n$ and $\ell' = \ell - 2(k - n)$. Precisely, denoting the number of approved candidates by α (note that $\alpha \leq n\tau$), the set $C' \subset C$ consists of all approved candidates and $n\tau - \alpha$ arbitrary never-approved candidates.

($\Rightarrow$) Let I be a `yes`-instance with solution $\mathcal{C} = (C_1, C_2, \ldots, C_\tau)$. Without loss of generality, we assume that each committee of $\mathcal{C}$ has at least n members. If this is not the case, then we select an arbitrary committee with fewer than n candidates and we add arbitrarily chosen unused candidates to reach n members. We repeat this procedure for all committees that have fewer than n members. We can afford this because we always have enough unused candidates. Indeed, suppose that for some iteration of the procedure, we need $y \leq n$ candidates to fill up the corresponding committee $C^* \in \mathcal{C}$. Then, at most $k(\tau-1)$ candidates are used in other committees. Comparing the number of all candidates with the upper-bound on the number of used candidates as follows

$$|C| - (k(\tau-1)+n-y) = k\tau - (k(\tau-1)+n-y) = k-n+y > n-n+y = y$$

shows that we have even more than y unused candidates that the procedure can use to fill up committee C^*. The updates carried out by the procedure clearly do not lower the scores of the altered committees.
For each $i \in \{1,\dots,\tau\}$ we order C_i's members decreasingly with respect to the number of approvals the members get (breaking ties arbitrarily). Then, we select top n of them forming a set $\mathrm{trunc}(C_i)$. We claim that $\mathcal{C}' = (\mathrm{trunc}(C_1), \mathrm{trunc}(C_2), \dots, \mathrm{trunc}(C_\tau))$ is a solution to I'. Indeed, the committees of $\mathcal{C}'$ are of size n and their score is at least x. The latter follows from the fact that $x \leq n$ and thus n candidates is always enough to yield score at least x. In order to show that $|\mathrm{trunc}(C_i) \triangle \mathrm{trunc}(C_{i+1})| \geq \ell'$, for all $i \in \{1,\dots,\tau-1\}$, assume for contradiction this is not the case. That is, suppose $|\mathrm{trunc}(C_j) \triangle \mathrm{trunc}(C_{j+1})| < \ell'$ for some $j \in \{1,\dots,\tau-1\}$. Let $\bar{C}_i \coloneqq C_i \setminus \mathrm{trunc}(C_i)$, for $i \in \{j, j+1\}$. Observe that $|\bar{C}_j| \leq (k-n)$ and $|\bar{C}_{j+1}| \leq (k-n)$. Thus, $|\bar{C}_j \triangle \bar{C}_{j+1}| \leq 2(k-n)$. Recognizing that

$$|C_j \triangle C_{j+1}| = |\mathrm{trunc}(C_j) \triangle \mathrm{trunc}(C_{j+1})| + |\bar{C}_j \triangle \bar{C}_{j+1}| < \ell' + 2(k-n) = \ell$$

gives that $|C_j \triangle C_{j+1}| < \ell$, which contradicts the fact that $\mathcal{C}$ is a solution to I.

($\Leftarrow$) Let I' be a yes-instance, $\mathcal{C}' = (C'_1, C'_2, \dots, C'_\tau)$ be a solution to I', and $\bar{C} \coloneqq C \setminus C'$. A solution to instance I is then $\mathcal{C} = (C_1, C_2, \dots, C_\tau)$ where each of sets C_t is a copy of C'_t with $k-n$ unique candidates from $\bar{C}$ added; formally, for each $t \in \{1,\dots,\tau\}$, $C_t = C'_t \cup \bar{C}_t$ such that $|\bar{C}_t| = k-n$ and $\bigcap_{i=1}^{\tau} \bar{C}_i = \emptyset$. Since

$$\left| \bigcup_{t=1}^{\tau} \bar{C}_t \right| = (k-n)\tau = k\tau - n\tau = |C| - |C'| = |\bar{C}|,$$

such a solution $\mathcal{C}$ can be constructed. Naturally, it holds for all $t \in \{1,\dots,\tau\}$ that $\mathrm{score}_t(C_t) \geq \mathrm{score}_t(C'_t) \geq x$ and that $|C_t| \leq k$. Furthermore, for each $t \in \{1,\dots,\tau-1\}$, $|C_t \triangle C_{t+1}| \geq \ell$. This is due to the fact that

$$|C_t \triangle C_{t+1}| = |C'_t \triangle C'_{t+1}| + |\bar{C}_t \triangle \bar{C}_{t+1}| = |C'_t \triangle C'_{t+1}| + 2(k-n) \geq \ell' + 2(k-n) = \ell. \quad \square$$

Lemma 7 concludes the case of $k > n$ and so the proof of Theorem 9 for RMSNTV.

7 Conclusion

At first glance, our multivariate analysis with several hardness results may appear as overly negative. We emphasize, however, that we identified some practically relevant tractable cases. In natural applications, such as electing committees serving for only few days/events, the number τ of stages is usually small. Furthermore, in many elections the committee size k is not very large and either the number n of voters or the number m of candidates is expected to be small. Hence, our results for τ and for k (polynomial-time solvability for constant values) as well as for $\tau+n$, for $\tau+m$, or for m alone (fixed-parameter tractability) appear very promising. Moreover, the tractability result for τ seems generally insightful for the multistage community where problems usually remain computationally hard even a for constant number of stages. Additionally, we think that the revolutionary multistage model may be relevant on its own and that it paves the way for studying more new models where consecutive changes are both lower- and upper-bounded.

Although it is naturally justified to start with the simplest meaningful model variant, our focus on SNTV might look restrictive. We stress that most results transfer easily to general Approval profiles (see W-MSNTV): Replace each voter approving multiple candidates by multiple voters, each approving one candidate. Also basic scoring rules can be modeled: Create for each candidate c that receives score $s(c)$ exactly $s(c)$ votes for c. The main drawback is that now the parameter number n of voters corresponds to the total number of approvals or the total score sum, respectively; yet, most positive results still hold. It remains open whether a direct modeling of more complex preferences instead of blowing up in the number of voters can avoid blowing up the running time as well. Motivated by this, we consider investigations of even more complex voting rules promising.

As opposed to the single-stage case, conservative and revolutionary committee election over multiple stages are NP-complete problems, even for a constant number of agents. From a parameterized algorithmic point of view, computing a revolutionary multistage committee is easier than a conservative one: When asking for committees to change for all but constantly many candidates, RMSNTV is polynomial time solvable, yet when asking for committees to change for only constant many candidates, MSNTV remains NP-hard. Facing a constant number of stages, both MSNTV and RMSNTV become polynomial-time solvable. Moreover, for both MSNTV and RMSNTV, while efficient data reduction to size polynomial in the number of agents and the number of candidates we proved to be unlikely, combining any of the two parameters with the number of stages allows for efficient polynomial-sized data reduction. This underlines the importance of the time aspect for preprocessing and forms a positive message: If there are few stages—which seems likely in reality, where planning too far in advance usually increases uncertainty—then our problems might be efficiently solvable.

We left open whether RMSNTV is fixed-parameter tractable or W-hard regarding the parameters k and ℓ. Further future work may include studying approximate or randomized algorithms for MSNTV and RMSNTV. Moreover, further concepts and problems (e.g., bribery and manipulation) from computational social choice may be studied in the (conservative and revolutionary) multistage model. Note that, for instance, 2-Approval (each agent approves up to two candidates) in the conservative multistage setup is already NP-hard for one agent (follows from the proof of Proposition 1). Additionally, we think that the revolutionary multistage model may be relevant on its own.

Importantly, our model is applicable for offline scenarios, in which preferences are collected in advance, which nowadays is realized frequently by social media polls, Internet profiling, customer targeting, and the like. However, considering online scenarios is also an interesting research direction, yet requiring significant changes in our original models. To observe this, consider an online scenario of selecting two committees such that in the first profile all committees are scoring equally high and in the second profile there is exactly one such a committee. In the worst case, any algorithm returns a solution requiring exchanging all candidates between the two selected committees; thus, no reasonable guarantee concerning the number of changes is achievable. To avoid such trivial cases, when studying the online setting, one needs to carry out significant model modifications. One way to proceed (following approaches from the multistage literature [5, 21, 4]) is to introduce a goal function that takes into account the quality of the selected committees as well as the symmetric difference of each two consecutive committees. Another way to study online variants of our problems would be to restrict the differences of consecutive profiles (analogously to Parkes and Procaccia [24]). Such a correlation between consecutive profiles, which restricts the model significantly, provides necessary information usable to achieve some guarantees on the solution.

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Appendix

A Additional Material for Section 2

A.1 Proof of Proposition 1

In the following proof, we give a polynomial-time many-one reduction from a special variant Half Vertex Cover of the Vertex Cover problem, where r is set to half the number of vertices:

Vertex Cover
Input: An undirected graph G and an integer $r \in \mathbb{N}$.
Question: Is whether there is a set $X \subseteq V(G)$ of size at most r such that $G - X$ contains no edge.

It is not difficult to see that Half Vertex Cover is NP-complete: We can reduce any instance (G, r) of Vertex Cover to Half Vertex Cover by adding a clique on $|V(G)| - 2r + 2$ vertices to G if $r < |V(G)|/2$, or adding enough isolated vertices to G until $r = |V(G)|/2$.

Proof. Let $I = (G = (V, E))$ be an instance of Half Vertex Cover, and let $E = \{e_1, \ldots, e_m\}$ without loss of generality. We construct an instance $I' = (A, C, u, k, \ell, x)$ of MSNTV in polynomial time as follows.
Construction: Set the set C of candidates equal to V, and the set A of agents equal to $\{a_1, a_2\}$. Next, construct m voting profiles as follows. For profile u_t, $t \in \{1, \ldots, m\}$, set $u_t(a_1) = \{v\}$ and $u_t(a_2) = \{w\}$, where $e_t = \{v, w\}$. Finally, set $k = |V|/2$, $\ell = 0$, and $x = 1$. This finishes the construction.
Correctness: We claim that I is `yes`-instance if and only if I' is a `yes`-instance.

($\Rightarrow$) Let $X \subseteq V$ be a vertex cover of size at most $|V|/2$. We claim that the sequence $(C_1, \ldots, C_m)$ with $C_t = X$ for every $t \in \{1, \ldots, m\}$ is a solution to I'. Suppose towards a contradiction that this is not the case. Firstly, observe that $C_t \triangle C_{t+1} = \emptyset$ for every $t \in \{1, \ldots, m-1\}$, and that $|C_t| \leq k$ for every $t \in \{1, \ldots, m\}$. Hence, there is a $t \in \{1, \ldots, m\}$ such that $\text{score}_t(C_t) = 0$. This means that for edge $e_t = \{v, w\}$, both $v, w \notin X$, contradicting the fact that X is a vertex cover of G. Hence, $(C_1, \ldots, C_m)$ is a solution to I'.

($\Leftarrow$) Let $(C_1, \ldots, C_m)$ be a solution to I'. Note that since $\ell = 0$, $C_i = C_j$ for every $i, j \in \{1, \ldots, m\}$. We claim that $X \coloneqq C_1$ is a vertex cover of G (note that $|X| \leq k$). Let $t \in \{1, \ldots, m\}$ be arbitrary but fixed. Since $\text{score}_t(C_t) \geq 1$, we know that $C_t \cap e_t \neq \emptyset$. Hence, since $X = C_t$, edge e_t is covered by X. Since t was chosen arbitrarily, it follows that X is a vertex cover of G of size at most k. □

A.2 Proof of Lemma 1

Proof. Let $I = (A, C, U, k, \ell, x)$ with $\ell = 0$, $k = |C|/2$, and τ voting profiles be an instance of MSNTV. We construct instance $I' = (A, C', U', k', \ell', x)$ of RMSNTV with $\ell' = 2k'$ in polynomial time as follows.
Construction: Set the set of candidates $C' = C \uplus \{z, y\}$, where z, y are new candidates not in C. Next, construct $2 \cdot \tau + 1$ voting profiles as follows. For all $a \in A$ and all $t \in \{1, \ldots, \tau\}$ set $u'_{2t-1}(a) = u_{2t-1}(a)$ and $u'_{2t}(a) = \{y\}$. Moreover, set $u'_{2\tau+1}(a) = \{z\}$ for all $a \in A$. Finally, set $k' = k + 1$ and $\ell' = 2k' = |C'|$. This finishes the construction.
Correctness: We claim that I is a `yes`-instance if and only if I' is a `yes`-instance.

($\Rightarrow$) Let $(C_1, \ldots, C_\tau)$ be a solution to I such that $|C_1| = k$. Since $\ell = 0$, we have that $C_t = C_{t'}$ for every $t, t' \in \{1, \ldots, \tau\}$. Let $X \coloneqq C_1 \cup \{z\}$ and $Y \coloneqq C' \setminus X$. Note that $|X| = |Y| = |C'|/2$. We claim that $(C'_1, \ldots, C'_{2\tau+1})$ with $C'_{2t-1} = X$ for every $t \in \{1, \ldots, \tau + 1\}$

and $C'_{2t} = Y$ for every $t \in \{1, \dots, \tau\}$ is a solution to I'. Since $y \in Y$, $\text{score}_{2t}(Y) = |A| \geq x$ for every $t \in \{1, \dots, \tau\}$. Since $u'_{2t-1}(a) = u_{2t-1}(a)$ for every $a \in A$ and $t \in \{1, \dots, \tau\}$, and $(C_1, \dots, C_\tau)$ is a solution to I, we have $\text{score}_{2t-1}(X) \geq x$. Moreover, since $z \in X$, we have $\text{score}_{2\tau+1}(X) = |A| \geq x$. Lastly, as $C'_t \triangle C'_{t+1} = C'$ for every $t \in \{1, \dots, 2\tau\}$, the claim follows.

($\Leftarrow$) Let $(C_1, \dots, C_{2\tau+1})$ be a solution to I'. First observe that, due to $\ell = |C|$, we have that $C_t \triangle C_{t+1} = C$ for every $t \in \{1, \dots, 2\tau\}$. It follows that $C_{2t-1} = C_{2t'-1}$ and $z \in C_{2t-1}$ for every $t, t' \in \{1, \dots, \tau + 1\}$, and $C_{2t} = C_{2t'}$ and $y \in C_{2t}$ for every $t, t' \in \{1, \dots, \tau\}$. We claim that $(C'_1, C'_2, \dots, C'_\tau)$ with $C'_i \coloneqq C_1 \setminus \{z\}$ is a solution to I. By construction, $\text{score}_i(C'_i) \geq x$, and as $C'_i = C'_j$ and $|C'_i| \leq k$, the claim follows. $\square$

A.3 Proof of Proposition 2

Lemma 8. MSNTV is NP-hard for $\ell = 1$.

Proof. Let $I = (A, C, U, k, \ell, x)$ be an instance of MSNTV with $A = \{a_1, a_2\}$, $x = 1$, and $\ell = 0$. We construct an instance $I' = (A', C', U', k', \ell', x')$ of MSNTV with $A' = \{a_1, \dots, a_6\}$, $C' = C \cup \{v', v, w\}$, $k' = k + 2$, $\ell' = 1$, and $x' = x + 4$ in polynomial time as follows.

Construction: For each $1 \leq t \leq \tau \coloneqq \tau(U)$, let a_1, a_2 approve the same candidates as in u_t. If t is even, then let a_3, a_4 approve v' if t is divisible by four, and approve v otherwise. If t is odd, then let a_3, a_4 approve w. Agents a_5, a_6 approve always w. This finishes the construction.

Correctness: We claim that I is a `yes`-instance if and only if I' is a `yes`-instance.

($\Rightarrow$) Let $(C_1, \dots, C_\tau)$ be a solution to C. We claim that the sequence $(C'_1, \dots, C'_\tau)$ with $C_t \coloneqq C_t \cup \{w\}$ if t is odd, $C_t \coloneqq C_t \cup \{v', w\}$ if t is even and divisible by four, and $C_t \coloneqq C_t \cup \{v, w\}$ if t is even and not divisible by four. Observe that $|C_t| \leq k + 2$ and $C'_t \triangle C'_{t+1} = \{v'\}$ or $C'_t \triangle C'_{t+1} = \{v\}$. Moreover, since in time step t, at least one candidate of C_t is a approved by a_1, a_2, by construction we have $\text{score}_t(C'_t) \geq 5$. This proves the claim.

($\Leftarrow$) Let $(C'_1, \dots, C'_\tau)$ be a solution to I'. Observe that every solution to I', w must be in every committee, v' must be in every committee in an even time step divisible by four, and v must be in every committee in an even time step not divisible by four. Hence, $C'_t \triangle C'_{t+1} = \{v'\}$ or $C'_t \triangle C'_{t+1} = \{v\}$ for every $t \in \{1, \dots, \tau - 1\}$. We claim that $(C_1, \dots, C_\tau)$ with $C_t = C'_t \cap C$ is a solution to I. Observe that $|C_t| \leq k$ and that $C_t \triangle C_{t+1} = (C'_t \triangle C'_{t+1}) \cap C = \emptyset$. Moreover, since for every $t \in \{1, \dots, \tau\}$ we have that $\text{score}_t(C'_t) \geq 5$, at least one of a_1, a_2 must approve a candidate in $C'_t \cap C = C_t$. Thus, $\text{score}_t(C_t) \geq 1$, and the claim follows. $\square$

Similarly, having $\ell = 2k$ is not necessary for RMSNTV to be NP-hard.

Lemma 9. RMSNTV is NP-hard for $\ell = m - 1 = 2k - 2$.

Proof. Let $I = (A, C, U, k, \ell, x)$ be an instance of RMSNTV with $A = \{a_1, a_2\}$, $\ell = 2k$, $x = 1$, and $k = |C|/2$. We construct an instance $I' = (A', C', U', k', \ell', x')$ of MSNTV with $A' = \{a_1, \dots, a_4\}$, $C' = C \cup \{w\}$, $k' = k + 1$, $\ell' = 2k' - 2$, and $x' = x + 2$ in polynomial time as follows.

Construction: For each $1 \leq t \leq \tau \coloneqq \tau(U)$, let a_1, a_2 approve the same candidates as in u_t, and a_3, a_4 approve w. Let $u'_1, \dots, u'_\tau$ denote the obtained voting profiles.

Correctness: We claim that I is a `yes`-instance if and only if I' is a `yes`-instance.

($\Rightarrow$) Let $(C_1, \dots, C_\tau)$ be a solution to instance I. We claim that $(C'_1, \dots, C'_\tau)$ with $C'_t = C_t \cup \{w\}$ is a solution to I'. Note that $|C'_t| \leq k + 1$, and that $|C'_t \triangle C'_{t+1}| = |C_t \triangle C_{t+1}| \geq 2k = 2k' - 2 = \ell'$. Moreover, note that since $x' = 3$, for each $t \in \{1, \dots, \tau\}$ we have that a_1, a_2 approve a candidate from C'_t, which is different to w. Hence, by construction, $\text{score}_t(C_t) \geq 1$.

($\Leftarrow$) Let $(C'_1, \ldots, C'_\tau)$ be a solution to I'. Observe that for every solution to I', w must be in every committee. Moreover, since $\ell' = |C'| - 1 = |C|$, we have that $(C_t \cup C_{t+1}) \setminus \{w\} = C$. We claim that $(C_1, \ldots, C_\tau)$ with $C_t = C'_t \setminus \{w\}$ is a solution to I. Note that $|C_t| \leq k$, $|C_t \triangle C_{t+1}| = |C'_t \triangle C'_{t+1}| \geq \ell$, and that $\text{score}_t(C_t) = \text{score}\, C'_t - 2 \geq x$. Thus, the claim follows. $\square$

B Additional Material for Section 3

B.1 Proof of Lemma 2

Proof. We can compute each V^i in $\mathcal{O}(m^{2\ell})$ time by brute forcing every pair of candidate subsets each of size at most ℓ. Hence, we can compute V in $\mathcal{O}(\tau \cdot m^{2\ell})$ time. The arcs incident with s and t can be computed in $\mathcal{O}(n+m)$ time, same for conditions (i) and (ii), resulting in an overall running time in $\mathcal{O}(\tau \cdot m^{4\ell+1} n)$. $\square$

B.2 Proof of Lemma 3

Proof. Let $I = (A, C, U, k, \ell, x)$ be an instance of RMSNTV, and D_I the in-out graph for I.

($\Rightarrow$) By construction, we know that if there is an s-t path in D_I, then it is of the form $P = (s, v^1_{X_1,Y_1}, \ldots, v^{\tau-1}_{X_{\tau-1},Y_{\tau-1}}, t)$. Since arc $(s, v^1_{X_1,Y_1})$ exists, by construction there is a k-size committee C_1 containing X but disjoint from Y of score at least x (similar for C_τ). Since the arc $(v^i_{X_i,Y_i}, v^{i+1}_{X_{i+1},Y_{i+1}})$ exists, at time step $i+1$ there is a k-size committee C_{i+1} containing $Y_i \cup X_{i+1}$ but disjoint from $X_i \cup Y_{i+1}$ of score at least x. Observe that $C_i \triangle C_{i+1} \supseteq X_i \cup Y_i$ and thus, of size at least ℓ. Then we find in each time step a committee of score x, and consecutive committees differ in at least ℓ elements.

($\Leftarrow$) Assume there is a solution $\mathcal{C} = (C_1, \ldots, C_\tau)$. For every $i \in \{1, \ldots, \tau - 1\}$, let $X_i \subseteq C_i \setminus C_{i+1}$, $Y_i \subseteq C_{i+1} \setminus C_i$. Note that C_1 is a committee containing X but disjoint from Y of score at least x in time step one, and thus arc $(s, v^1_{X_1,Y_1})$ exists (analogously for C_τ and arc $(v^{\tau-1}_{X_{\tau-1},Y_{\tau-1}}, t)$). We claim that there is an arc from $v^i_{X_i,Y_i}$ to $v^{i+1}_{X_{i+1},Y_{i+1}}$ for every $i \in \{1, \ldots, \tau-2\}$. Note that since $X_i \cap C_{i+1} = \emptyset$, we have $X_i \cap X_{i+1} = \emptyset$ (analogously for Y). Moreover, C_{i+1} is a k-size committee with score at least x (since $\mathcal{C}$ is a solution), and it contains $Y_i \cup X_{i+1}$, and is disjoint from $X_i \cup Y_{i+1}$. Hence, the arc exists. It follows that $P = (s, v^1_{X_1,Y_1}, \ldots, v^{\tau-1}_{X_{\tau-1},Y_{\tau-1}}, t)$ forms an s-t path in D_I. $\square$

C Additional Material for Section 4

C.1 Proof of Theorem 3

Proof. Let $I = (A, C, U, k, \ell, x)$ be an instance of MSNTV (or RMSNTV) with n agents, m candidates, and τ voting profiles. We compute the following directed graph $D = (V, A)$ with vertex set $V = \{s, t\} \uplus V^1 \uplus \cdots \uplus V^\tau$, where $V^t = \{v^t_X \mid X \subseteq C, |X| \leq k, \text{score}_t(X) \geq x\}$, and arc set A composed of the sets $\{(s, v) \mid v \in V^1\}$, $\{(v, t) \mid v \in V^\tau\}$, and for each $t \in \{1, \ldots, \tau - 1\}$ the sets $\{(v^t_X, v^{t+1}_Y) \mid |X \triangle Y| \leq \ell\}$ in the conservative variant (and $\{(v^t_X, v^{t+1}_Y) \mid |X \triangle Y| \geq \ell\}$ in the revolutionary variant). Note that there are at most $\tau \cdot m^k + 2$ vertices and at most $(\tau - 1) \cdot (m^k)^2 + 2m^k$ arcs. Hence, D can be constructed in $\mathcal{O}(\tau \cdot k^2 m^{2k+1} n)$ time. We claim that I is a `yes`-instance if and only if D admits an s-t path. Note that we can check for an s-t path in D in time linear in the size of D.

($\Rightarrow$) Let $(C_1, \ldots, C_\tau)$ be a solution to I. We claim that $P = (s, v^1_{C_1}, \ldots, v^\tau_{C_\tau}, t)$ is an s-t path in D. Clearly, the arcs $(s, v^1_{C_1})$ and $(v^\tau_{C_\tau}, t)$ are in A. For each $t \in \{1, \ldots, \tau - 1\}$, the arc $(v^t_{C_t}, v^{t+1}_{C_{t+1}})$ exists since $|C_t \triangle C_{t+1}| \leq \ell$ ($\geq \ell$ in the revolutionary case). Hence, P is an s-t path in D.

($\Leftarrow$) Let $P = (s, v^1_{C_1}, \ldots, v^\tau_{C_\tau}, t)$ be an s-t path in D. We claim that $(C_1, \ldots, C_\tau)$ is a solution to I. First observe that $|C_t| \leq k$ and $\mathrm{score}_t(C_t) \geq x$. Moreover, we have that $|C_t \triangle C_{t+1}| \leq \ell$ ($\geq \ell$ in the revolutionary case) for each $t \in \{1, \ldots, \tau - 1\}$. The claim thus follows. □

C.2 Proof of Theorem 4

Running Time. The reduction builds a polynomial number (with respect to the input size) of copies of gadgets. However, the construction time of the coherence gadget heavily depends on the computation of Sidon set S (of size h) and the value of its largest element s_h. Since h is linearly bounded in the size of the input, due to Lemma 4, we get a polynomial running time (with respect to the input size) of computing S. As a by-product we get that the largest element of S is also polynomially upper-bounded (with respect to the input size).
Correctness. Naturally, if instance I of MSNTV is a `yes`-instance, then there is a committee that is witnessing this fact; otherwise, such a committee does not exist. Since the candidates are all vertices and edges of graph G, we refer to edges and vertices being part of some committee as, respectively, *selected* vertices and edges.

On the way to prove the correctness of the reduction, we state the following lemma about the coherence gadget.

Lemma 10. For a pair $\{V_i, V_j\}$ of parts such that $i < j$, let C be a committee selecting exactly one vertex from either of them, v, v' respectively, and exactly one edge $e \in E^j_i$ connecting some vertices of parts V_i and V_j. Then, the scores of C for the profiles of the coherence gadget for parts V_i and V_j are at least x if and only if $e = \{v, v'\}$.

Proof. Assume that a selected edge connects vertices u and u' from, respectively, part V_i and part V_j. Let us compute the scores of C for profiles $p((i,j))$ and $p'((i,j))$ of the coherence gadget for i and j (note that due to assumptions on C only candidates v, v', and e contribute to the scores):

$$\mathrm{score}_{p((i,j))}(C) = \mathrm{id}(v) + \mathrm{id}(v') + x - \mathrm{id}(u) - \mathrm{id}(u'), \tag{2}$$
$$\mathrm{score}_{p'((i,j))}(C) = \frac{x}{2} - \mathrm{id}(v) + \frac{x}{2} - \mathrm{id}(v') + \mathrm{id}(u) + \mathrm{id}(u'). \tag{3}$$

If both of the scores are at least x, after simplifying the equations we arrive at:

$$\begin{aligned} &\mathrm{id}(v) + \mathrm{id}(v') \geq \mathrm{id}(u) + \mathrm{id}(u') \\ \wedge\, &\mathrm{id}(u) + \mathrm{id}(u') \geq \mathrm{id}(v) + \mathrm{id}(v'). \end{aligned}$$

This, in turn, simply means that:

$$\mathrm{id}(u) + \mathrm{id}(u') = \mathrm{id}(v) + \mathrm{id}(v'). \tag{4}$$

Recall that the image of bijective function $\mathrm{id}(\cdot)$ is a Sidon set. Thus, by the definition of the Sidon set, Equation (4) is true if and only if $\{v, v'\} = \{u, u'\} = e$.

The opposite direction follows directly from the definition of Sidon sets because all $\mathrm{id}(\cdot)$-terms in Equation (2) and Equation (3) cancel out resulting in both scores being x. □

Analogously, we provide the following lemma that describes the role of the vertex and edge selection gadgets.

Lemma 11. Every committee witnessing a $\mathtt{yes}$-instance selects exactly one vertex from each part of G and exactly $\binom{q}{2}$ edges, one for each distinct pair $\{V_i, V_j\}$ of parts. Additionally, the scores of such a committee C in all vertex and edge selection gadgets are exactly x.

Proof. Let us fix a committee C witnessing a $\mathtt{yes}$-instance. Towards a contradiction, assume that there exists a part V_i a vertex of which is not selected. Then, since in profile $p(V_i)$ only vertices from part V_i are approved, the score of C in $p(V_i)$ is zero; the contradiction. Similarly, assume that there is a pair $\{V_i, V_j\}$ of parts such that no edge in E_i^j is selected by C. Again, by an analogous argument, if this is the case, then the scores of C in profiles $p(E_i^j)$ and $p'(E_j^i)$ are zero which gives a contradiction. From the fact that there are exactly q vertex selection gadgets and exactly $\binom{q}{2}$ edge selection gadgets, it follows that C selects exactly one vertex from each part of G and exactly one edge for a pair of parts. By the construction of the vertex and edge selection gadgets, indeed each of them gives score exactly x. $\square$

Having the above lemmas, we finally show the correctness of our reduction. In one direction, suppose K is a clique of size q in graph G of an instance $\hat{I}$ of MULTICOLORED CLIQUE. We construct a committee C selecting all vertices of K and all edges connecting the vertices in K. Indeed, the necessary condition from Lemma 11 is met because of the definition of a clique and the fact that K is a clique of size q. Again by the definition of a clique, for every two vertices from distinct parts, committee C selects the edge that connects them; thus, by Lemma 10, committee C is witnessing the positive answer.

For the opposite direction, suppose there is no clique in G of $\hat{I}$. We show that there is no committee C witnessing a $\mathtt{yes}$-instance. Due to Lemma 11 it follows that C selects a vertex for each part of G and an edge for each pair of parts. However, since there is no clique in G every committee C there exists at least one pair $\{v, v'\}$ of vertices from distinct parts that are not connected with an edge. Thus, due to Lemma 10 at least in one coherence gadget there is at least one profile for which the score of C is below x; which finishes the argument.

C.3 Proof of Lemma 5

Proof. Let $I = (A, C, U, k, \ell, x)$ with $\ell = 0$ be an instance of MSNTV. We construct an instance $I' = (A', C', U', k', \ell', x')$ of MSNTV with $k' = |C'|/2$, $\ell' = 0$, and $|U'| = |U|$, as follows.

Construction: Initially, we set I' to I. If $k = |C|/2$, we are done. If $k > |C|/2$, then we add $2k - |C|$ candidates to C, forming C'. If $k < |C|/2$, then we add $|C| - 2k$ candidates to C, say set C^*, forming $C' = C \cup C^*$. Moreover, we add $|A| \cdot (|C| - 2k)$ agents to A, say A^*, forming $A' = A \cup A^*$, where for each $c \in C^*$, in each stage, exactly $|A|$ of the new agents approve c (this forms U'). Note that $|U'| = |U|$. Set $k' = |C'|/2$, $\ell' = 0$, and $x' = x + |A|(|C| - 2k)$. Since correctness for the cases $k = |C|/2$ and $k > |C|/2$ is immediate, we prove in the following correctness for the case of $k < |C|/2$.

Correctness: In the case of $k < |C|/2$, we claim that I is a $\mathtt{yes}$-instance if and only if I' is a $\mathtt{yes}$-instance. $\mathtt{yes}$-instance ($\Rightarrow$) Let $\mathcal{C} = (C_1, \ldots, C_\tau)$ be a solution to I. We claim that $(C'_1, \ldots, C'_\tau)$ with $C'_i = C_i \cup C^*$ is a solution to I'. First note that $|C'_t| \leq |C_t| + |C| - 2k \leq |C| - k = |C'|/2$. Moreover, we have that $|C'_t \triangle C'_{t+1}| = |C_t \triangle C_{t+1}| = 0$. Finally, we have that $\mathrm{score}_t(C'_t) = \mathrm{score}_t(C_t) + |A|(|C| - 2k) \geq x + |A|(|C| - 2k) = x'$.

($\Leftarrow$) Let $\mathcal{C}' = (C'_1, \ldots, C'_\tau)$ be a solution to I'. First observe that $C^* \subseteq C'_t$ for all $i \in \{1, \ldots, \tau\}$. Suppose not, that is, there is a $t \in \{1, \ldots, \tau\}$ and $c \in C^*$ such that $c \notin C'_t$. Then the score of C'_t is at most $|A| + |A|(|C| - 2k - 1) = |A|(|C| - 2k) < x'$, a contradiction to the fact that $\mathcal{C}'$ is a solution. We claim that $(C_1, \ldots, C_\tau)$ with $C_t = C^*_t \setminus C^*$ a solution to I. Note that $|C_t| = |C'_t| - (|C| - 2k) \leq (2|C| - 2k)/2 - (|C| - 2k) = k$, and that $|C_t \triangle C_{t+1}| =$

$|C'_t \triangle C'_{t+1}| = 0$. Finally, note that $\text{score}_t(C_t) = \text{score}_t(C'_t) - |A|(|C| - 2k) \geq x + |A|(|C| - 2k) - |A|(|C| - 2k) = x$. □

C.4 Proof of Theorem 5

Proof of Theorem 5. Lemma 5 followed by Lemma 1 gives a parameterized reduction from MSNTV to RMSNTV regarding the parameter τ. Theorem 4 then finishes the proof. □

D Additional Material for Section 5

D.1 Proof of Theorem 6

Proof. We describe a dynamic programming algorithm using the key insight that there are only 2^τ possible subsets of committees for a single candidate to be part of. With τ being constant, we can effort to iterate through all these possibilities. Even more importantly, when building up all committees simultaneously in m phases, with increasing i, we consider in each phase i only committee members among the first i candidates. Herein, we keep track of the sizes, symmetric differences, and scores of all τ committees using only polynomially many table entries.

To implement the above idea, we define a boolean dynamic programming table $T[i, k_1, \dots, k_\tau, d_1, \dots, d_{\tau-1}, s_1, \dots, s_\tau]$ with $0 \leq i \leq m, 0 \leq k_t \leq k, 0 \leq d_t \leq 2k, 0 \leq s_t \leq n$ and the following interpretation. Let $T[i, k_1, \dots, k_\tau, d_1, \dots, d_{\tau-1}, s_1, \dots, s_\tau]$ be true if and only if there are committees $C_1, \dots, C_\tau \subseteq \{c_1, \dots, c_i\}$ with

- $|C_t| = k_t$ for each $t \in \{1, \dots, \tau\}$,
- $|C_t \triangle C_{t+1}| = d_t$ for each $t \in \{1, \dots, \tau - 1\}$, and
- $\text{score}_t(C_t) = s_t$ for each $t \in \{1, \dots, \tau\}$.

It can be easily verified (using the definition of the boolean table), that there exists a solution to our problem if and only if $T[m, k'_1, \dots, k'_\tau, d'_1, \dots, d'_{\tau-1}, s'_1, \dots, s'_\tau]$ is true for some combination of $k'_1, \dots, k'_\tau$, $d'_1, \dots, d'_{\tau-1}$, $s'_1, \dots, s'_\tau$ with $k'_t \leq k$ for each $t \in \{1, \dots, \tau\}$, $d'_t \leq \ell$ (respectively $d'_t \geq \ell$, in the revolutionary case) for each $t \in \{1, \dots, \tau - 1\}$, and $s'_t \geq x$ for each $t \in \{1, \dots, \tau\}$. The table is of size $\mathcal{O}(m \cdot k^{2\tau} \cdot n^\tau)$. It remains to show how to fill the table and that each table entry can be computed efficiently.

As initialization, we set $T[0, 0, \dots, 0]$ to true and all other table entries $T[0, \dots]$ to false. For each $i > 0$, we compute all entries $T[i, \dots]$ as follows, assuming that the entries $T[i-1, \dots]$ have been computed. For each *candidate fingerprint* $F \subseteq 2^{\{1,\dots,\tau\}}$ we set $T[i+1, k_1, \dots, k_\tau, d_1, \dots, d_{\tau-1}, s_1, \dots, s_\tau]$ to true if $T[i, k'_1, \dots, k'_\tau, d'_1, \dots, d'_{\tau-1}, s'_1, \dots, s'_\tau]$ is true where for each $t \in \{1, \dots, \tau\}$ we have

$$k'_t = \begin{cases} k_t - 1 & \text{if } t \in F, \\ k_t & \text{otherwise,} \end{cases}$$

$$d'_t = \begin{cases} d_t - 1 & \text{if } t \in F \text{ xor } t + 1 \in F, \\ d_t & \text{otherwise, and} \end{cases}$$

$$s'_t = \begin{cases} s_t - |u_t^{-1}(c_{i+1})| & \text{if } t \in F, \\ s_t & \text{otherwise.} \end{cases}$$

We set all other entries $T[i+1, \dots]$ to false.

Running time. Computing all table entries $T[i,\dots]$ for some i costs $\mathcal{O}(2^\tau)$ time steps for setting the "true"-entries and another $\mathcal{O}(k^{2\tau}\cdot n^\tau)$ time steps for setting the "false"-entries. Hence, the whole table can be computed in $\mathcal{O}(m\cdot k^{2\tau}\cdot n^\tau)$ time.

The correctness is deferred to the appendix ($\bigstar$). $\square$

E Additional Material for Section 6.1

E.1 Proof of Theorem 7

By construction, we have the following.

Observation 4. *Let I' from Construction 1 be a* `yes`*-instance of* MSNTV*. Then in every solution $(C_1,\dots,C_{\tau'})$ it holds true that $z\in C_t$ in all $t\in\{1,\dots,\tau'\}$.*

Proof. Let $\mathcal{C}=(C_1,\dots,C_{\tau'})$ be a solution to I'. Assume there is $i\in\{1,\dots,\tau'\}$ such that $z\notin C_i$. Then, $\mathrm{score}_i(C_i)\le|A|-n=n<n+x=x'$, contradicting that $\mathcal{C}$ is a solution. $\square$

Lemma 12. Let I' be the instance obtained from Construction 1 given $I_1,\dots,I_p$. Then I' is a `yes`-instance if and only if each instance of $I_1,\dots,I_p$ a `yes`-instance.

Proof. ($\Rightarrow$) Let $\mathcal{C}=(\mathcal{C}_1,\mathcal{C}_{1,2},\mathcal{C}_2,\dots,\mathcal{C}_p)$ be a solution for I', where $\mathcal{C}_i$ is sequence of committee for the voting profiles obtained from instance I_i, and $\mathcal{C}_{i,i+1}$ is a sequence of committees for the transfer voting profiles between the consecutive instances I_i and I_{i+1}. We claim that $\mathcal{C}_i$ without z forms a solution to instance I_i for every $i\in\{1,\dots,p\}$. Due to Observation 4, we have that z is contained in each committee in $\mathcal{C}_i$. Moreover, each committee admits a score of at least $x'=x+n$, and hence as z contributes n to the score, each committee admits a score of at least x in the respective voting profile in instance I_i.

($\Leftarrow$) Let each of $I_1,\dots,I_p$ be a `yes`-instance, and let $\mathcal{C}_q=(C_q^1,\dots,C_q^\tau)$ be a solution to I_q for each $q\in\{1,\dots,p\}$. Note that $\widehat{\mathcal{C}}_q=(\widehat{C}_q^1,\dots,\widehat{C}_q^\tau)$ where $\widehat{C}_q^t$ contains the set $\{c_j\mid c_j^q\in C_q^t\}$ and z is a sequence of committees that is a partial solution to I' on the voting profiles corresponding to I_q. It remains to construct a sequence $\mathcal{C}_{q,q+1}=(C_{q,q+1}^1,\dots,C_{q,q+1}^{2k})$ for each $q\in\{1,\dots,p-1\}$. Let $\widehat{C}_q^\tau=\{z\}\cup\{b_1,\dots,b_r\}$ and let $\widehat{C}_{q+1}^1=\{z\}\cup\{d_1,\dots,d_s\}$. Initially, set $C_{q,q+1}^t=\{z\}$ for all $t\in\{1,\dots,2k\}$. For $t\le r$, set $C_{q,q+1}^t=\widehat{C}_q^\tau\setminus\bigcup_{i=1}^t b_i$. For $t\ge 2k-s+1$, and $C_{q,q+1}^t=\widehat{C}_{q+1}^1\setminus\bigcup_{i=1}^{2k-t+1} d_i$. Clearly, $|C_{q,q+1}^t|\le k'$ and since each contains z, each admits a score of $2n$. Finally, by construction we have that $|C_{q,q+1}^t\triangle C_{q,q+1}^{t+1}|\le 1$ for every $t\in\{1,\dots,2k-1\}$, $|C_q^\tau\triangle C_{q,q+1}^1|\le 1$, and $|C_{q+1}^1\triangle C_{q,q+1}^{2k}|\le 1$. Altogether, it holds that $\mathcal{C}:=(\widehat{\mathcal{C}}_1,\mathcal{C}_{1,2},\widehat{\mathcal{C}}_2,\dots,\widehat{\mathcal{C}}_p)$ forms a solution to I'. $\square$

For RMSNTV, the construction is similar to Construction 1 in the sense of using transition profiles; However, this time, we only need three.

Construction 2. Consider p instances $I_1,\dots,I_p$ of RMSNTV such that $k,x\in\mathbb{N}$ and $\ell=2k$, each with n agents $A^i=\{a_1^i,\dots,a_n^i\}$, $m=\ell$ candidates $C^i=\{c_1^i,\dots,c_m^i\}$, and τ voting profiles. Add the agent set $A=\{a_1,\dots,a_n\}\cup\{\widehat{a}_j^i\mid 0\le i\le\ell,1\le j\le n\}$ and candidate set $C=\{c_1,\dots,c_n\}\cup\{z,y_1,\dots,y_\ell\}$. Arrange the p voting profiles consecutively, where we identify each agent a_j^i with a_j and each candidate c_j^i with c_j. Moreover, each agent $\widehat{a}_j^0$ approves z, and each $\widehat{a}_j^i$ approves y_i. Next, between the last voting profile and the first voting profile of two consecutive instances, add one profile where all agents approve z (we call it again *transfer profile*). Set $x'=x+n\cdot(\ell+1)$, $k'=k+\ell+1$, and $\ell'=\ell$. Note that $\tau'=p\tau+(p-1)$. $\diamond$

Again, similar to the case of MSNTV, we observe that z must be contained in each committee when we are facing a solution.

Observation 5. *Let I' from Construction 2 be a* `yes`*-instance of* RMSNTV*. Then in every solution $(C_1, \ldots, C_{\tau'})$ it holds true that $z \in C_t$ in all $t \in \{1, \ldots, \tau'\}$ and $\{y_1, \ldots, y_\ell\} \in C_t$ if C_t is no transition profile.*

Lemma 13. Let I' be the instance obtained from Construction 2 given $I_1, \ldots, I_p$. Then I' is a `yes`-instance if and only if each instance of $I_1, \ldots, I_p$ a `yes`-instance.

Proof. $(\Rightarrow)$ Let $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_{1,2}, \mathcal{C}_2, \ldots, \mathcal{C}_p)$ be a solution for I', where $\mathcal{C}_i$ is sequence of committee for the voting profiles obtained from instance I_i, and $\mathcal{C}_{i,i+1}$ is a sequence of committees for the transfer voting profiles between the consecutive instances I_i and I_{i+1}. We claim that $\mathcal{C}_i$ without z forms a solution to instance I_i for every $i \in \{1, \ldots, p\}$. Due to Observation 5, we have that z and $\{y_1, \ldots, y_\ell\}$ are contained in each committee in $\mathcal{C}_i$. Moreover, each committee admits a score of at least $x' = x + n \cdot (\ell + 1)$, and hence as each of $z, y_1, \ldots, y_\ell$ contributes n to the score, each committee admits a score of at least x in the respective voting profile in instance I_i.

$(\Leftarrow)$ Let each of $I_1, \ldots, I_p$ be a `yes`-instance, and let $\mathcal{C}_q = (C_q^1, \ldots, C_q^\tau)$ be a solution to I_q for each $q \in \{1, \ldots, p\}$. Note that $\widehat{\mathcal{C}}_q = (\widehat{C}_q^1, \ldots, \widehat{C}_q^\tau)$ where $\widehat{C}_q^t$ contains the set $\{c_j \mid c_j^q \in C_q^t\}$ and z is a sequence of committees that is a partial solution to I' on the voting profiles corresponding to I_q. It remains to construct the transition profiles $C_{q,q+1}$ for each $q \in \{1, \ldots, p-1\}$. Let $\widehat{C}_q^\tau = \{z\} \uplus \{y_1, \ldots, y_\ell\} \uplus \{b_1, \ldots, b_r\}$ and let $\widehat{C}_{q+1}^1 = \{z\} \uplus \{y_1, \ldots, y_\ell\} \uplus \{d_1, \ldots, d_s\}$. Set $C_{q,q+1} = \{z\}$. Clearly, $|C_{q,q+1}| \leq k'$, Moreover, since in the transition profiles all agents approve z, committee $C_{q,q+1}$ has score $A' \geq x'$. Finally, observe that $\{y_1, \ldots, y_\ell\}$ is a subset of each of $\widehat{C}_q^\tau \triangle C_{q,q+1}$ and of $C_{q,q+1} \triangle \widehat{C}_{q+1}^1$, and hence each symmetric difference is of size at least ℓ. Altogether, $\mathcal{C} \coloneqq (\widehat{\mathcal{C}}_1, C_{1,2}, \widehat{\mathcal{C}}_2, \ldots, \widehat{\mathcal{C}}_p)$ forms a solution to I'. ☐

F Additional Material for Section 6.2

F.1 Proof of Observation 2

Proof. For each $t \in \{1, \ldots, \tau\}$, set w_i^t equal to the number of approvals of candidate c_i in the tth utility function, that is, $w_i^t = |u_t^{-1}(c_i)|$. ☐

We can shrink the weights of the weighted version of each of our problems the following result due to Frank and Tardos [18].

Proposition 4 ([18, Section 3])**.** There is an algorithm that, on input $w \in \mathbb{Q}^d$ and integer N, computes in polynomial time a vector $\widehat{w} \in \mathbb{Z}^d$ with

(i) $\|\widehat{w}\|_\infty \leq 2^{4d^3} N^{d(d+2)}$ such that

(ii) $\operatorname{sign}(w^\top b) = \operatorname{sign}(\widehat{w}^\top b)$ for all $b \in \mathbb{Z}^d$ with $\|b\|_1 \leq N - 1$.

Proposition 4 now gives the following.

F.2 Proof of Lemma 6

Proof. Let $\omega \in \mathbb{N}_0^{m \cdot \tau}$ the concatenation of the weight vectors in W, and let $w \coloneqq (\omega, x)$. Apply Proposition 4 to w with $N = k + 2$ (note that $d = m \cdot \tau + 1$) to obtain a vector $\widehat{w} = (\widehat{\omega}, \widehat{x})$. Property (i) holds true by Proposition 4(i). Let $\mathcal{C} = (C^1, \ldots, C^\tau)$ be a sequence of subsets of $\{1, \ldots, m\}$. Let $b^t \in \{0, 1\}^m$ be the vector associated with C^t (that is, $b_i^t = 1$ if and only if $i \in C^t$). Then, by Proposition 4(i) (since $\|b^t\|_1 \leq k$), $(b^t, -1)^\top (w^t, x) \geq 0$ if and only if $(b^t, -1)^\top (\widehat{w}^t, \widehat{x}) \geq 0$. Hence, $\mathcal{C}$ is a solution to I if and only if it is to I', proving property (ii). □

G Additional Material for Section 6.3

G.1 Proof of Reduction Rule 1

Proof. Let I' be the instance obtained from I by Reduction Rule 1 deleting $z \in C$ which is never approved (its existence follows from Observation 3). If $\mathcal{C}'$ is a solution to I', then it is also to I. Conversely, if there is a solution that does not contain z, then this is a solution to I'. So, assume that every solution contains z.

Let $\mathcal{C} = (C_1, \dots, C_\tau)$ be a solution to I such that the first appearance of z in the sequence is the latest. Let $z \in C_{t_1}$, where t_1 is the smallest index with this property. Let $t_2 > t_1$ be the largest index such that $z \in C_{t_2}$ for all $t_1 \leq t \leq t_2$. We claim that deleting z from C_t for $t_1 \leq t \leq t_2$ constructs another solution with the first appearance of z being later, contradicting our choice of $\mathcal{C}$. It holds true that for all $t_1 \leq t \leq t_2$, we have $|C_t \setminus \{z\}| = |C_t| - 1 < k$ and, since z is never approved, we have that $\mathrm{score}_t(C_t \setminus \{z\}) \geq x$. Moreover, for each $|C_t \triangle C_{t+1}| \leq \ell$ for each $t_1 \leq t < t_2$. Finally, we have that $|C_{t_1-1} \triangle (C_{t_1} \setminus \{z\})| < |C_{t_1-1} \triangle C_{t_1}| \leq \ell$, since $z \notin C_{t_1-1}$. Similarly, $|(C_{t_2} \setminus \{z\}) \triangle C_{t_2+1}| < |C_{t_2} \triangle C_{t_2+1}| \leq \ell$, since $z \notin C_{t_2+1}$. Hence, we constructed a solution where the first appearance of z is later than for $\mathcal{C}$, contradicting the choice of $\mathcal{C}$. It follows that there is a solution not containing z, witnessing that I is a `yes`-instance. Thus, the correctness of the rule follows.

The claimed running time is immediate: in linear time, identify $m - n\tau$ candidates that are never approved and delete them from the candidate set. □

G.2 Proof of Reduction Rule 2

Proof. Let $I = (A, C, U, k, \ell, x)$ be an instance of RMSNTV such that $m > \max\{n, k\} \cdot \tau$, and let $I' = (A, C', U, k, \ell, x)$ be the instance of RMSNTV obtained from applying the reduction rule to I, where $C' = C \setminus \{z\}$ (z is a candidate that is never approved, which exists since $m > n\tau$). We claim that I is a `yes`-instance if and only if I' is a `yes`-instance.

($\Leftarrow$) Immediate.

($\Rightarrow$) Let I be a `yes`-instance. We claim that there is a solution $\mathcal{C}' = (C'_1, \dots, C'_\tau)$ such that $z \notin \bigcup_{t=1}^{\tau} C'_t$. Suppose not, and let $\mathcal{C} = (C_1, \dots, C_\tau)$ be a solution to I which contains z. Since $|C_t| \leq k$, there is $y \in C$ such that $y \notin \bigcup_{t=1}^{\tau} C_t$, as $m > k\tau$. We claim that $\widehat{\mathcal{C}} = (\widehat{C}_1, \dots, \widehat{C}_\tau)$ with

$$\widehat{C}_t = \begin{cases} C_t, & z \notin C_t \\ (C_t \setminus \{z\}) \cup \{y\}, & \text{otherwise,} \end{cases}$$

is a solution to I (roughly, $\widehat{\mathcal{C}}$ is obtained from $\mathcal{C}$ by replacing each occurrence of z by y). Clearly, $|\widehat{C}_t| \leq k$, $\mathrm{score}_t(\widehat{C}_t) \geq x$, and even $|\widehat{C}_t \triangle \widehat{C}_{t+1}| \geq \ell$, since whenever $z \in (C_t \triangle C_{t+1})$, we now have $y \in (\widehat{C}_t \triangle \widehat{C}_{t+1})$. It follows that $\widehat{\mathcal{C}}$ is a solution not containing z. Hence, $\mathcal{C}'$ exists and is a solution to I'. □