On the Indecisiveness of Kelly-Strategyproof Social Choice Functions

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Abstract

Social choice functions (SCFs) map the preferences of a group of agents over some set of alternatives to a non-empty subset of alternatives. The Gibbard-Satterthwaite theorem has shown that only extremely unattractive single-valued SCFs are strategyproof when there are more than two alternatives. For set-valued SCFs, or so-called social choice correspondences, the situation is less clear. There are miscellaneous—mostly negative—results using a variety of strategyproofness notions and additional requirements. The simple and intuitive notion of Kelly-strategyproofness has turned out to be particularly compelling because it is weak enough to still allow for positive results. For example, the Pareto rule is strategyproof even when preferences are weak, and a number of attractive SCFs (such as the top cycle, the uncovered set, and the essential set) are strategyproof for strict preferences. In this paper, we show that, for weak preferences, only indecisive SCFs can satisfy strategyproofness. In particular, *(i)* every strategyproof rank-based SCF violates Pareto-optimality, *(ii)* every strategyproof support-based SCF (which generalize Fishburn's C2 SCFs) that satisfies Pareto-optimality returns at least one most preferred alternative of every voter, and *(iii)* every strategyproof non-imposing SCF returns a Condorcet loser in at least one profile.

1 Introduction

Whenever a group of multiple agents aims at reaching a joint decision in a fair and principled way, they need to aggregate their individual preferences using a social choice function (SCF). SCFs are traditionally studied by economists and mathematicians, but have also come under increasing scrutiny from computer scientists who are interested in their computational properties or want to utilize them in computational multiagent systems (see, e.g., Brandt et al., 2016b; Endriss, 2017).

An important phenomenon in social choice is that agents misrepresent their preferences in order to obtain a more preferred outcome. An SCF that is immune to strategic misrepresentation of preferences is called strategyproof. Gibbard (1973) and Satterthwaite (1975) have shown that only extremely restricted single-valued SCFs are strategyproof: either the range of the SCF is restricted to only two outcomes or the SCF always returns the most preferred alternative of the same voter. Perhaps the most controversial assumption of the Gibbard-Satterthwaite theorem is that the SCF must always return a single alternative (see, e.g., Gärdenfors, 1976; Kelly, 1977; Barberà, 1977b; Duggan and Schwartz, 2000; Nehring, 2000; Barberà et al., 2001; Ching and Zhou, 2002; Taylor, 2005). This assumption is at variance with elementary fairness conditions such as anonymity and neutrality. For instance, consider an election with two alternatives and two voters such that each alternative is favored by a different voter. Clearly, both alternatives are equally acceptable, but single-valuedness forces us to pick a single alternative based on the preferences only.

We therefore study the manipulability of *set-valued* SCFs (or so-called *social choice correspondences*). When SCFs return sets of alternatives, there are various notions of strategyproofness, depending on the circumstances under which one set is considered to be preferred to another. When the underlying notion of strategyproofness is sufficiently strong, the negative consequences of the Gibbard-Satterthwaite theorem remain largely intact (see, e.g., Duggan and Schwartz, 2000; Barberà et al., 2001; Ching and Zhou, 2002; Benoît, 2002; Sato, 2014).[1] In this paper, we are concerned with a rather weak—but natural and intuitive—notion of strategyproofness attributed to Kelly (1977). Several attractive SCFs have been shown to be strategyproof for this notion when preferences are strict (Brandt, 2015; Brandt et al., 2016a). These include the top cycle, the uncovered set, the minimal covering set, and the essential set. However, when preferences are weak, these results break down and strategyproofness is not well understood in general.

Feldman (1979) has shown that the Pareto rule is strategyproof according to Kelly's definition, even when preferences are weak. Moreover, the omninomination rule and the intersection of the Pareto rule and the omninomination rule are strategyproof as well (Brandt et al., 2021, Remark 1). These results are encouraging because they rule out impossibilities using Pareto-optimality and other weak properties.[2] In the context of strategic abstention (i.e., manipulation by deliberately abstaining from an election), even more positive results can be obtained. Brandl et al. (2019) have shown that all of the above mentioned SCFs that are strategyproof for strict preferences are immune to strategic abstention even when preferences are weak.

A number of negative results were shown for severely restricted classes of SCFs. Kelly (1977) and Barberà (1977b) have shown independently that there is no strategyproof SCF that satisfies quasi-transitive rationalizability. However, this result suffers from the fact that quasi-transitive rationalizability is almost prohibitive on its own (see, e.g., Mas-Colell and Sonnenschein, 1972).[3] In subsequent work by MacIntyre and Pattanaik (1981) and Bandyopadhyay (1983), quasi-transitive rationalizability has been replaced with weaker conditions such as minimal binariness or quasi-binariness, which are still very demanding and violated by most SCFs. Barberà (1977a) has shown that positively responsive SCFs fail to be strategyproof under mild assumptions. However, positively responsive SCFs are almost always single-valued and of all commonly considered SCFs only Borda's rule and Black's rule satisfy this criterion. Taylor (2005, Th. 8.1.2) has shown that every SCF that returns the set of all weak Condorcet winners whenever this set is non-empty fails to be strategyproof. This result was strengthened by Brandt (2015), who showed that every SCF that returns a (strict) Condorcet winner whenever one exists fails to be strategyproof. More recently, Brandt et al. (2021) have shown with the help of computers that every Pareto-optimal SCF whose outcome only depends on the pairwise majority margins can be manipulated.

In this paper, we study strategyproofness in three broad classes of SCFs. These classes are *rank-based* SCFs (which include all scoring rules), *support-based* SCFs (which generalize Fishburn's C2 SCFs), and *non-imposing* SCFs (which return every alternative as the unique winner for some preference profile). An overview of the three classes and typical examples of SCFs belonging to these classes are given in Figure 1. The classes are unrelated in a set-theoretic sense: for any pair of classes, their intersection is non-empty, and Borda's rule is contained in all three classes. Taken together, they cover virtually all SCFs commonly considered in the literature.

For rank-based and support-based SCFs, we show that Pareto-optimality and strategyproofness imply that every voter is a nominator, i.e., the resulting choice sets contain at least one most preferred alternative of every voter. In the case of ranked-based SCFs, this entails an impossibility whereas for support-based SCFs it demonstrates a high degree of indecisiveness. For non-imposing SCFs, we show that strategyproofness implies that a

[1] We refer to Barberà (2010) and Brandt et al. (2021) for a more detailed overview over this extensive stream of research.

[2] For example, Brandt et al. (2021) have shown that Pareto-optimality is incompatible with anonymity and a notion of strategyproofness that is slightly stronger than Kelly's.

[3] This is acknowledged by Kelly (1977) who writes that "one plausible interpretation of such a theorem is that, rather than demonstrating the impossibility of reasonable strategy-proof social choice functions, it is part of a critique of the regularity [rationalizability] conditions."

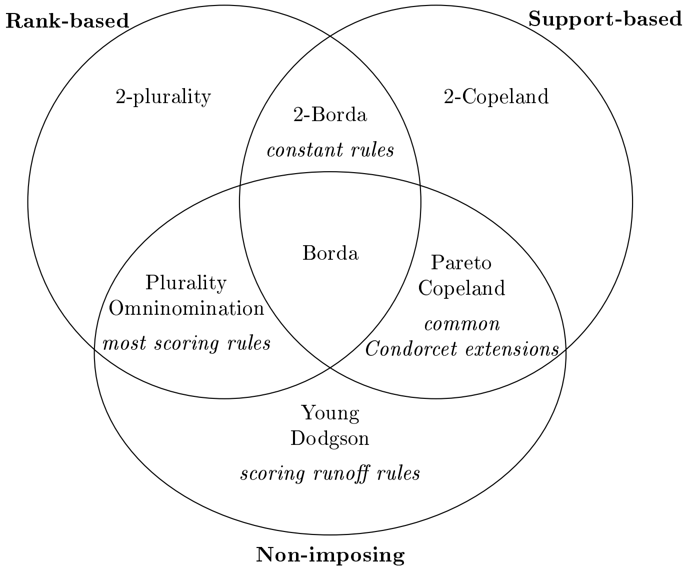

Figure 1: The classes of rank-based, support-based, and non-imposing SCFs and typical examples. 2-plurality, 2-Copeland, and 2-Borda return all alternatives whose respective score is at least as large as the second-highest score. All scoring rules except Borda's rule are rank-based, non-imposing, but not support-based. Common Condorcet extensions include the top cycle, the uncovered set, the minimal covering set, the essential set, the Simpson-Kramer rule, Nanson's rule, Schulze's rule, and Kemeny's rule.

Condorcet loser has to be returned in at least one preference profile. The latter result remarkably holds without imposing fairness conditions such as anonymity or neutrality and can again be phrased in terms of indecisiveness: every strategyproof SCF that satisfies the Condorcet loser property will never return certain alternatives as unique winners. Even though these results are rather negative, they are important to improve our understanding of strategyproof SCFs. Much more positive results are obtained by making minuscule adjustments to the assumptions such as restricting the domain of preferences to strict preferences, weakening the underlying notion of strategyproofness, or replacing strategic manipulation with strategic abstention (see, e.g., Nehring, 2000; Brandt, 2015; Brandl et al., 2019). In all of these cases, a small number of support-based Condorcet extensions such as the top cycle, the uncovered set, the minimal covering set, and the essential set constitute appealing positive examples.

Our results can also be interpreted in the context of randomized social choice. When transferred to this setting, Kelly-strategyproofness is weaker than weak *SD*-strategyproofness and we thus obtain three strong impossibilities.

2 The Model

Let $N = \{1, \dots, n\}$ denote a finite set of voters and let $A = \{a, b, \dots\}$ denote a finite set of m alternatives. Moreover, let $[x \dots y] = \{i \in N \colon x \leq i \leq y\}$ denote the subset of voters from x to y and note that $[x \dots y]$ is empty if $x > y$. Every voter $i \in N$ is equipped with a *weak preference relation* $\succsim_i$, i.e., a complete, transitive, and reflexive binary relation on A. We denote the strict part of $\succsim_i$ by $\succ_i$, i.e., $x \succ_i y$ if and only if $x \succsim_i y$ and $y \not\succsim_i x$, and the indifference part by $\sim_i$, i.e., if $x \sim_i y$ if and only if $x \succsim_i y$ and $y \succsim_i x$. We compactly represent a preference relation as a comma-separated list, where sets of alternatives express indifferences. For example, $x \succ y \sim z$ is represented by $x, \{y, z\}$. Furthermore, we call a preference relation $\succsim$ strict if its irreflexive part is equal to its strict part $\succ$. The set of all weak preference relations on A is called $\mathcal{R}$. A *preference profile* $R \in \mathcal{R}^n$ is an n-tuple containing the preference relation of every voter $i \in N$. When defining preference profiles, we specify a set of voters who share the same preference relation by writing the set directly before the preference relation. For instance, $[x \dots y]$: a, b, c means that all voters $i \in [x \dots y]$ prefer a to b and b to c. We omit the brackets for singleton sets. For two alternatives $x, y \in A$, the pairwise *support* of x over y is defined as the number of voters who strictly prefer x to y, i.e., $s_{xy}(R) = |\{i \in N \colon x \succ_i y\}|$. Our central objects of study are *social choice functions (SCFs)*, or so-called social choice correspondences, which map a preference profile to a non-empty set of alternatives, i.e., functions of the form $f : \mathcal{R}^n \mapsto 2^A \setminus \emptyset$. The mere mathematical description of SCFs is so general that it allows for rather undesirable functions. We now introduce a number of axioms in order to narrow down the set of SCFs. The most basic fairness condition is anonymity, which requires that all voters are treated equally: an SCF f is *anonymous* if $f(R) = f(R')$ for all preference profiles R, R' for which there is a permutation $\pi : N \to N$ such that $R_i = R'_{\pi(i)}$ for all $i \in N$.

Perhaps one of the most prominent axioms in economic theory is Pareto-optimality, which is based on the notion of Pareto-dominance: an alternative x *Pareto-dominates* another alternative y if $x \succsim_i y$ for all $i \in N$ and there is a voter $j \in N$ with $x \succ_j y$. An alternative is *Pareto-optimal* if it is not Pareto-dominated by any other alternative. This idea leads to the *Pareto rule* which returns all Pareto-optimal alternatives. An SCF f is Pareto-optimal if it never returns Pareto-dominated alternatives. An axiom that is closely related to Pareto-optimality is *near unanimity*, as introduced by Benoît (2002). Near unanimity requires that $f(R) = \{x\}$ for all alternatives $x \in A$ and preference profiles R in which at least $n-1$ voters uniquely top-rank x. The more voters there are, the more compelling near unanimity is. A natural weakening of these axioms is *non-imposition* which requires that for every alternative $x \in A$, there is a profile R such that $f(R) = \{x\}$. For single-valued SCFs, non-imposition is almost imperative because it merely requires that the SCF is onto. For set-valued SCFs, as considered in this paper, this is not necessarily the case. For example, every SCF that always returns at least two alternatives fails non-imposition (see, for example, 2-plurality, 2-Borda, and 2-Copeland in Figure 1).

An influential concept in social choice theory is that of a Condorcet winner, which is an alternative $a \in A$ that wins all pairwise majority comparisons, i.e., $s_{ax}(R) > s_{xa}(R)$ for all $x \in A \setminus \{a\}$. An SCF is *Condorcet-consistent* or a so-called *Condorcet extension* if it uniquely returns a Condorcet winner whenever one exists. Analogously, one can define a Condorcet loser by requiring that $s_{xa}(R) > s_{ax}(R)$ for all $x \in A \setminus \{a\}$. An SCF f satisfies the *Condorcet loser property* if $x \notin f(R)$ whenever x is a Condorcet loser in R. While there are Condorcet extensions that violate the Condorcet loser property (e.g., the Simpson-Kramer rule) and SCFs that satisfy the Condorcet loser property but fail Condorcet-consistency (e.g., Borda's rule), the Condorcet loser property "feels" weaker. This could be justified by arguing that both properties affect exactly the same number of preference profiles, but the Condorcet loser property only excludes a single alternative (and leaves otherwise a lot of freedom) whereas Condorcet-consistency completely determines the (singleton) choice set.

While the axioms so far make reference to the entire preference profile, there are also concepts that only refer to the preferences of a single voter. One such concept that is particularly important in our context is that of a nominator. A voter is a *nominator* if $f(R)$ always contains at least one of his most preferred alternatives. A nominator is a weak dictator in the sense that he can always force an alternative into the choice set by declaring it his uniquely most preferred one.

2.1 Rank-Basedness and Support-Basedness

In this section, we introduce two classes of anonymous SCFs that capture many of the SCFs commonly studied in the literature: rank-based and support-based SCFs. The basic idea of rank-basedness is that voters assign ranks to the alternatives and that an SCF should only depend on the ranks of the alternatives, but not on which voter assigns which rank to an alternative. In order to formalize this idea, we first need to define the rank of an alternative. In the case of strict preferences, this is straightforward. The rank of alternative x according to $\succsim_i$ is $\bar{r}(\succsim_i, x) = |\{y \in A \colon y \succsim_i x\}|$ (Laslier, 1996). In contrast, there are multiple possibilities how to define the rank in the presence of ties. We define a very weak notion of ranked-basedness for weak preferences, making our results only stronger. To this end, define the *rank tuple* of x with respect to $\succsim_i$ as

$$\begin{aligned} r(\succsim_i, x) &= (\bar{r}(\succ_i, x), \bar{r}(\sim_i, x)) \\ &= (|\{y \in A \colon y \succ_i x\}|, |\{y \in A \colon y \sim_i x\}|). \end{aligned}$$

The rank tuple contains more information than many other generalizations of the rank and therefore, it leads to a more general definition of rank-basedness. Next, we define the *rank vector* of an alternative a which contains the rank tuple of a with respect to every voter in increasing lexicographic order, i.e., $r^*(R, x) = (r(\succsim_{i_1}, x), r(\succsim_{i_2}, x), \dots, r(\succsim_{i_n}, x))$ where $\bar{r}(\succ_{i_j}, x) \leq \bar{r}(\succ_{i_{j+1}}, x)$ and if $\bar{r}(\succ_{i_j}, x) = \bar{r}(\succ_{i_{j+1}}, x)$, then $\bar{r}(\sim_{i_j}, x) \leq \bar{r}(\sim_{i_{j+1}}, x)$. Finally, the *rank matrix* $r^*(R)$ of the preference profile R contains the rank vectors as rows. An SCF f is called *rank-based* if $f(R) = f(R')$ for all preference profiles $R, R' \in \mathcal{R}^n$ with $r^*(R) = r^*(R')$. The class of rank-based SCFs contains many popular SCFs such as all scoring rules or the omninomination rule, which returns all top-ranked alternatives.

A similar line of thought leads to support-basedness, which is based on the pairwise support of an alternative x against another one y. We define the *support matrix* $s^*(R) = (s_{xy}(R))_{x,y \in A}$ which contains the supports for all pairs of alternatives. Finally, an SCF f is *support-based* if it yields $f(R) = f(R')$ for all preference profiles $R, R' \in \mathcal{R}^n$ with $s^*(R) = s^*(R')$. Note that support-basedness generalizes Fishburn's C2 to weak preferences (Fishburn, 1977). Hence, many well-known SCFs such as Borda's rule, Kemeny's rule, the Simpson-Kramer rule, Nanson's rule, Schulze's rule, and the essential set are support-based. Support-basedness is less restrictive than *pairwiseness*, which requires that $f(R) = f(R')$ for all preference profiles $R, R' \in \mathcal{R}^n$ with $s_{ab}(R) - s_{ba}(R) = s_{ab}(R') - s_{ba}(R')$ for all $a, b \in A$ (see, e.g., Brandt et al., 2021). For example, the Pareto rule is support-based, but fails to be pairwise.

2.2 Strategyproofness

One of the central problems in social choice theory is manipulation, i.e., voters may lie about their true preferences to obtain a more preferred outcome. For single-valued SCFs, it is clear what constitutes a more preferred outcome. In the case of set-valued SCFs, there are various ways to define manipulation depending on what is assumed about the voters' preferences over sets of alternatives. Here, we make a simple and natural assumption: a voter i weakly prefers a set X to another set Y, denoted by $X \succsim_i Y$, if and only if $x \succsim_i y$ for all $x \in X, y \in Y$. Thus, the strict part of this preference extension is

$$\begin{aligned} X \succ_i Y \text{ if and only if for all } & x \in X, y \in Y, x \succsim_i y \text{ and} \\ & \text{there are } x' \in X, y' \in Y \text{ with } x' \succ_i y'. \end{aligned}$$

An SCF is manipulable if a voter can improve his outcome by lying about his preferences. Formally, an SCF f is *manipulable* if there are a voter $i \in N$ and preference profiles R, R'

such that $\succsim_j = \succsim'_j$ for all $j \in N \setminus \{i\}$ and $f(R') \succ_i f(R)$. Moreover, f is *strategyproof* if it is not manipulable.

These assumptions can, for example, be justified by considering a randomized tie-breaking procedure (a so-called lottery) that is used to select a single alternative from every set of alternatives returned by the SCF. We then have that $X \succ_i Y$ if and only if all lotteries with support X yield strictly more expected utility than all lotteries with support Y for all utility functions that are ordinally consistent with $\succsim_i$ (see, e.g., Gärdenfors, 1979; Brandt et al., 2021).

3 Results

The unifying theme of our results is that strategyproofness requires a large degree of indecisiveness. In more detail, we show that every voter is a nominator for all ranked-based and support-based SCFs that satisfy Pareto-optimality and strategyproofness. For the very broad class of non-imposing SCFs, we show that every strategyproof SCF violates the Condorcet loser property. Due to space restrictions, we defer the proofs of all lemmas, theorems, and non-trivial claims in the remarks to the appendix. Instead, we give some intuition for the proofs by discussing proof sketches.

In order to prove the claim for rank-based and support-based SCFs, we focus on its contrapositive, i.e., we assume that there is a rank-based or support-based SCF f that satisfies Pareto-optimality and strategyproofness and for which a voter $i \in N$ is not a nominator. Our first lemma shows that these assumptions imply that f satisfies near unanimity.

Lemma 1. *Let f be an anonymous, Pareto-optimal, and strategyproof SCF that is defined for $m \geq 3$ alternatives and $n \geq 2$ voters. If a voter is not a nominator for f, then f satisfies near unanimity.*

Proof sketch. Consider an arbitrary SCF f that satisfies all required axioms and a voter i who is not a nominator for f. The last assumption means that there is a profile R such that $f(R)$ does not contain any of voter i's most preferred alternatives. As first step, we construct a profile R' in which $f(R') = \{a\}$ but a is not among the most preferred alternatives of voter i. This is helpful because strategyproofness becomes much more restrictive when there is only a single winner. Based on the profile R', we derive then that $n-1$ voters can ensure that a is the unique winner by submitting it as a uniquely most preferred alternative. Finally, we show that f satisfies near unanimity by generalizing this observation from a single alternative to all alternatives. □

Lemma 1 can be interpreted in various appealing ways. For instance, one can see it as a push-down lemma that allows a single voter to weaken the unique winner in his preference relation. Moreover, this lemma shows that, under the given assumptions, indecisiveness for a single preference profile of a particularly simple type entails a large degree of indecisiveness for the entire domain of preference profiles. More precisely, if an alternative is not chosen uniquely even if $n-1$ voters prefer it uniquely the most, then all voters are nominators.

Remark 1. There is also a variant of Lemma 1 without anonymity. Then, an alternative is the unique winner if all voters but the non-nominator prefer it uniquely the most. Thus, requiring the absence of nominators for a strategyproof and Pareto-optimal SCF implies near unanimity.

Remark 2. Remarkably, many impossibility results rule out that every voter is a nominator. For instance, Duggan and Schwartz (2000), Benoît (2002), and Sato (2008) invoke axioms prohibiting that every voter is a nominator. Moreover, a crucial step in the computer-generated proofs of Theorem 3.1 by Brandl et al. (2018) and Theorem 1 by Brandt et al.

(2021) is to show that there is a voter who is not a nominator. Lemma 1 gives intuition about why these assumptions and observations are important.

3.1 Rank-Based SCFs

In this section, we discuss our first impossibility theorem: no rank-based SCF satisfies Pareto-optimality and strategyproofness. This result follows from the observation that Pareto-optimality, strategyproofness, and rank-basedness require that every voter is a nominator, but Pareto-optimality and rank-basedness do not allow for such SCFs.

Theorem 1. *There is no rank-based SCF that satisfies Pareto-optimality and strategyproofness if $m \geq 4$ and $n \geq 3$.*

Proof sketch. The proof of this theorem works by contradiction, i.e., we assume that there is a rank-based SCF for $m \geq 4$ alternatives and $n \geq 3$ voters that satisfies Pareto-optimality and strategyproofness and prove two conflicting implications. In particular, we show the following two claims, which clearly contradict each other. Thus, no rank-based SCF can satisfy both Pareto-optimality and strategyproofness.

Claim 1: Not every voter is a nominator for a rank-based SCF that satisfies Pareto-optimality if $m \geq 4$ and $n \geq 3$.

Claim 2: Every voter is a nominator for a rank-based SCF that satisfies Pareto-optimality and strategyproofness if $m \geq 4$ and $n \geq 3$.

For proving Claim 1, we construct three profiles with the same rank matrix but different Pareto-dominated alternatives. As a consequence, a rank-based and Pareto-optimal SCF can only choose a single alternative, even though the voters do not agree on a best alternative. Hence, it follows that there is a voter who is not a nominator.

Claim 2 follows from another proof by contradiction: we assume that there is a rank-based SCF f that satisfies Pareto-optimality and strategyproofness and for which a voter is no nominator. As a consequence of the last assumption, Lemma 1 applies and shows that f is nearly unanimous, i.e., $f(R) = \{a\}$ if all voters but one report a as uniquely best alternative in R. Starting from such a profile R, we show that the voters who uniquely top-rank a can also report a and b as their most preferred alternatives without affecting the outcome. By repeatedly applying this argument, we arrive at a profile R' for which $f(R') = \{a\}$, $n-1$ voter report a and b as their best alternatives, and the last voter reports a as his uniquely least preferred alternative. Hence, a is Pareto-dominated by b in R' and f violates therefore Pareto-optimality. $\square$

Remark 3. The axioms used in Theorem 1 are independent: the Pareto rule satisfies all axioms except rank-basedness, the trivial SCF which always returns all alternatives only violates Pareto-optimality, and Borda's rule only violates strategyproofness.[4] Furthermore, the Pareto rule is rank-based if $m \leq 3$, and if $m = 4$ and $n \leq 2$, which entails that the bounds on m and n are tight if considered simultaneously. By contrast, the theorem is also true if $m \geq 5$ and $n = 2$. More details can be found in the appendix.

Remark 4. Theorem 1 is only an impossibility because of the lack of compatibility of rank-basedness and Pareto-optimality in Claim 1, independently of strategyproofness. In contrast, the main consequence of strategyproofness is indecisiveness as captured in Claim

[4] We define Borda's rule as the SCF that chooses all alternatives that maximize $m \cdot n - \sum_{i \in N} \bar{r}(\succ_i, a)$. This definition agrees with the standard notation used in literature on the strict domain and generalizes it to the weak domain.

2. This follows as Theorem 1 breaks down once we weaken Pareto-optimality to weak Pareto-optimality (which only excludes alternatives for which another alternative is strictly preferred by every voter) as the omninomination rule satisfies then all required axioms (Brandt et al., 2021, Remark 6). In contrast, the proof of Claim 2 shows that more decisive rank-based SCFs violate strategyproofness as near unanimity is already sufficient for a contradiction.

Remark 5. Theorem 1 holds also under weaker versions of rank-basedness. First, it uses rank-basedness only in very specific situations, namely when two voters rename exactly two alternatives. Moreover, the only real restriction on the rank function r is independence of the naming of other alternatives, i.e., $r(\succsim_i, a) = r(\succsim_i', a)$ for all preferences $\succsim_i$, $\succsim_i'$ that only differ in the naming of alternatives in $A \setminus \{a\}$. Hence, we may also define rank-basedness based on a rank function other than the rank tuple and the result still holds.

Remark 6. It is possible to show Theorem 1—as well as Theorem 2—by induction proofs where completely indifferent voters and universally bottom-ranked alternatives are used to generalize the statement to arbitrarily many voters and alternatives (see, e.g., Brandl et al., 2018, 2019; Brandt et al., 2021). However, these constructions seem artificial and thus, we prefer to give universal proofs for any numbers of voters and alternatives to stress the robustness of the respective constructions. As a consequence, our proofs often hold when restricting the domain of admissible profiles by prohibiting artificial constructs such as completely indifferent voters.

Remark 7. Theorem 1 does not hold when preferences are strict. For instance, the omninomination rule satisfies all required axioms for arbitrary numbers of voters and alternatives for strict preferences. It can even be shown that Claim 2 of the proof no longer holds for strict preferences as a variant of the 2-plurality rule, which chooses the two alternatives that are top-ranked by the most voters, is rank-based, Pareto-optimal, and strategyproof. However, no voter is a nominator for this rule. A formal definition of this SCF and proofs for its properties can be found in the appendix.

3.2 Support-Based SCFs

It is not possible to replace rank-basedness with support-basedness in Theorem 1 since the Pareto rule is strategyproof, Pareto-optimal, and support-based. Note that the Pareto rule always chooses one of the most preferred alternatives of every voter. Consequently, Claim 1 in the proof of Theorem 1 cannot be true in general for support-based SCFs. Nevertheless, we show next that Claim 2 remains true for such SCFs, i.e., every voter is a nominator for every support-based SCF that satisfies Pareto-optimality and strategyproofness.

Theorem 2. *In every support-based SCF that satisfies Pareto-optimality and strategyproofness, every voter is a nominator if* $m \geq 3$.

Proof sketch. We prove this theorem again by contradiction and assume therefore that there is a support-based SCF f for $m \geq 3$ alternatives that satisfies Pareto-optimality and strategyproofness and that a voter is no nominator for f. As a consequence of Lemma 1, f is nearly unanimous. Note that near unanimity itself results immediately in a contradiction if $n \leq 2$ and thus, we focus on the case that there are at least three voters. The central argument in this case is the following claim that we prove by induction over $k \in \{1, \dots, n-1\}$: for all preference profiles R and alternatives $a \in A$, if $n-k$ voters report a as their uniquely most preferred alternative, then $f(R) = \{a\}$. For $k \geq n/2$, we derive that less than half of the voters can enforce that $f(R) = \{a\}$ by uniquely top-ranking a. However, if half of the voters report a as their best alternative and the other half reports b as their best alternative,

$f(R) = \{a\}$ and $f(R) = \{b\}$ must be simultaneously true, a contradiction. Hence, it follows that every voter is a nominator for every support-based SCF that satisfies Pareto-optimality and strategyproofness if $m \geq 3$. $\square$

Remark 8. All axioms used in Theorem 2 are required as the following SCFs show. Every constant SCF satisfies support-basedness and strategyproofness, and violates Pareto-optimality and that every voter is a nominator. The SCF that chooses the lexicographic smallest Pareto-optimal alternative satisfies Pareto-optimality and support-basedness but violates strategyproofness and that every voter is a nominator. For defining an SCF that satisfies Pareto-optimality and strategyproofness but violates support-basedness and that every voter is a nominator, we define a transitive dominance relation by slightly strengthening Pareto-dominance by allowing additionally that an alternative a that is among the most preferred alternatives of $n-1$ voters can dominate another alternative b, even if a single voter prefers b strictly to a. Therefore, we say that an alternative a *dominates* alternative b if a Pareto-dominates b or $n-1$ voters prefer a the most while $s_{ab}(R) \geq 2$ and $s_{ba}(R) \leq 1$. It should be stressed that it is not required that a is uniquely top-ranked by $n-1$ voters, but only that it is among their best alternatives. The SCF f^* that chooses all maximal elements with respect to this dominance relation satisfies all required properties (see the appendix for more details). Also the bound on m is tight as the majority rule satisfies all axioms if $m = 2$ but no voter is a nominator for this SCF.

Remark 9. Brandt et al. (2021, Th. 5.4) have shown that there is no pairwise, Pareto-optimal and strategyproof SCF if $m \geq 3$ and $n \geq 3$. This result immediately follows from Theorem 2 by observing that strategyproofness, pairwiseness, and Pareto-optimality rule out that every voter is a nominator. For this, it suffices to find a preference profile in which a Pareto-dominates b and another profile with the same majority margins where b is uniquely top-ranked by a voter.

Remark 10. Just as in the proof of Theorem 1, we make only very restricted use of support-basedness in the proof of Theorem 2. It suffices if two voters are allowed to exchange their preferences over two alternatives. This technical restriction is significantly weaker than support-basedness, which allows any number of voters to change their preferences.

Remark 11. An important subclass of support-based SCFs are majority-based SCFs, which Fishburn (1977) calls C1 functions. They only rely on the majority relation $R_M = \{(a,b) \in A^2 \colon s_{ab}(R) \geq s_{ba}(R)\}$ of the input profile R to compute the choice set. For majority-based SCFs, an even more severe impossibility holds: there is no majority-based SCF that satisfies non-imposition and strategyproofness if $m \geq 3$ and $n \geq 3$. Even though this statement does not require Pareto-optimality and therefore cannot use Lemma 1, the result follows from a proof similar to the one of Theorem 2. See the appendix for more details.

Remark 12. If preferences are required to be strict, Theorem 2 does not hold. Several SCFs including the uncovered set, the minimal covering set, and the essential set are strategyproof, Pareto-optimal and support-based, but no voter is a nominator for them (see, e.g, (Brandt et al., 2016a) for more details).

Remark 13. Theorems 1 and 2 raise the question whether all voters must be nominators for every anonymous, Pareto-optimal, and strategyproof social choice function. This is not the case because the SCF f^*, as defined in Remark 8, satisfies near unanimity and therefore represents a counterexample.

3.3 Non-Imposing SCFs

Finally, we consider the class of non-imposing SCFs. Recall that an SCF is non-imposing if every alternative is returned as the unique winner in some profile. Among the SCFs typically studied in social choice theory, there are only very few that fail to be non-imposing, e.g., SCFs that never return certain alternatives (such as constant SCFs) or SCFs that never return singletons.

We will show a rather strong consequence of strategyproofness for non-imposing SCFs: every such function has to return a Condorcet loser in at least one preference profile and thus violate the Condorcet loser property. In the presence of neutrality (symmetry among alternatives), non-imposition can be seen as a decisiveness requirement. Accordingly, the theorem identifies a tradeoff between decisiveness and the undesirable property of selecting Condorcet losers.

Similarly to Theorem 1 and Theorem 2, we start with a lemma that allows a voter to push down the unique winner. Since we do not require Pareto-optimality in this section, we cannot use Lemma 1 and therefore propose a new lemma that uses non-imposition instead.

Lemma 2. *Let f denote a strategyproof SCF for $n \geq 3$ voters that satisfies non-imposition and the Condorcet loser property. Then, $f(R) = \{a\}$ for every preference profile R and every alternative $a \in A$ such that more than half of the voters in R report a as their unique top choice.*

Proof sketch. As first step of the proof of this lemma, we show that an alternative is the unique winner for such an SCF f if it is the unique top choice of every voter. Next, we prove that a voter i can make his uniquely best alternative a into his uniquely worst one while ensuring that a remains the single winner if more than half of the voters still report a as their unique best alternative after this modification. This step relies significantly on the fact that the voters who top-rank a can turn every other alternative b into the Condorcet loser without affecting the choice set. As a consequence, we can repeatedly choose an alternative x and reinforce it against a in voter i's preference while ensuring that x is not chosen as it is the Condorcet loser. Applying these steps repeatedly to multiple voters leads to a profile R' such that $f(R') = \{a\}$, $\lceil \frac{n+1}{2} \rceil$ voters prefer a uniquely the most, and the remaining voters prefer a uniquely the least. Finally, we can apply strategyproofness to turn this profile into any other profile in which a is uniquely top-ranked by a majority of the voters without changing the choice set. □

Note that the Condorcet loser property allows for a significantly stronger push-down lemma than Pareto-optimality, even though it only requires that a single alternative may not be chosen. The reason is that an absolute majority of voters can exclude every alternative from the choice set. We use Lemma 2 to show that there is no strategyproof SCF that satisfies the Condorcet loser property and non-imposition.

Theorem 3. *There is no strategyproof SCF that satisfies the Condorcet loser property and non-imposition if $m \geq 3$ and $n \geq 4$.*

Proof sketch. We prove this result by contradiction, i.e., we assume that there is a non-imposing SCF f for $m \geq 3$ alternatives and $n \geq 4$ voters that satisfies the Condorcet loser property and strategyproofness. Since the Condorcet loser property and Lemma 2 depend on the parity of the number of voters, we proceed with a case distinction with respect to n.

First consider the case that $n \geq 5$ is odd. In this case, the profile R^3, where $X = A \setminus \{a, b, c\}$, plays a central role in the proof. In particular, we can show with two simple deductions that there is no valid choice set for R^3.

R^3: 1: a, b, X, c [$2 \dots l{-}1$]: a, c, X, b l: c, a, X, b [$l{+}1 \dots n$]: b, c, X, a

Next, we infer restrictions on $f(R^3)$ from the profiles R^1 and R^2.

R^1: 1: a, b, X, c [$2 \dots l$]: a, c, X, b [$l{+}1 \dots n$]: $b, X, \{a, c\}$

R^2: [$1 \dots l{-}1$]: a, c, X, b l: c, a, X, b [$l{+}1 \dots n$]: c, b, X, a

First, we consider the derivation of R^3 from R^1 and note that $f(R^1) = \{a\}$ because of Lemma 2. Moreover, c is the Condorcet loser in R^1 regardless of voter l's preference. Therefore, we replace the preference of this voter with c, a, X, b, and the Condorcet loser property and strategyproofness entail that a is still the unique winner. Finally, we let the voters $i \in [l{+}1 \dots n]$ change their preference to b, c, X, a. Before these modifications, a, one of the least preferred alternatives of these voters, was the unique winner, and thus it follows from strategyproofness that $f(R^3) \subseteq \{a, c\}$. Moreover, $f(R^3) \neq \{c\}$ because otherwise, voter 1 can manipulate by reporting b as his best choice. After this modification, more than half of the voters prefer b uniquely the most, which entails that b is the unique winner because of Lemma 2. Hence $f(R^3) \in \{\{a\}, \{a, c\}\}$.

Next, we discuss the derivation of R^3 from R^2. Once again, Lemma 2 applies for R^2 and entails that $f(R^2) = \{c\}$. Moreover, note that b is the Condorcet loser, regardless of the preferences of the voters $i \in [l{+}1 \dots n]$. Hence, these voters can swap b and c one after another, and strategyproofness and the Condorcet loser property entail that c is still the unique winner afterwards. Finally, we derive R^3 by replacing the preference of voter 1 with a, b, X, c. Since c is the unique winner before the manipulation, $f(R^3) \notin \{\{a\}, \{a, c\}\}$ because voter 1 prefers these sets to $\{c\}$. This is a contradiction with the previous observation.

Finally, we have to consider the case that $n \geq 4$ is even. In this case, we provide first an involved argument showing that no non-imposing SCF can satisfy strategyproofness and the Condorcet loser property if $n = 4$. Next, we generalize this impossibility to larger values of n by adding pairs of voters with inverse preferences. This is possible since these voters do not affect any of the required properties. □

Remark 14. A more general version of Lemma 2 can be proven for a non-imposing and strategyproof SCF f: if there is a group of voters $S \subsetneq N$ such that $a \notin f(R)$ for every alternative a and profile R in which all voters in S reports a as their uniquely least preferred alternative, then $f(R') = \{b\}$ for all preference profiles R' and alternatives b such that all voters in S agree on b as their uniquely best alternative. In particular, this means that vetoers, i.e, voters whose uniquely least preferred alternative is never chosen, are dictators for f.

Remark 15. The axioms used in Theorem 3 are independent of each other. An SCF that only violates the Condorcet loser property is the Pareto rule. The SCF that returns all alternatives except the Condorcet loser only violates non-imposition. The SCF that returns all Pareto-optimal alternatives except the Condorcet loser only violates strategyproofness. The bounds on n and m are also tight. The majority rule satisfies all axioms if $m = 2$, the Pareto rule satisfies all axioms if $n \leq 2$, and a rather technical SCF based on a case distinction on the maximal plurality score of an alternative satisfies all axioms if $n = 3$, $m \geq 3$.

Remark 16. Brandt (2015, Th. 2) has shown that no Condorcet extension can be strategyproof if $m \geq 3$ and $n \geq 3m$. By replacing the Condorcet loser property and non-imposition with Condorcet-consistency, careful inspection of the proof of Theorem 3 reveals that Condorcet-consistency and strategyproofness are already incompatible if $m \geq 3$ and $n \geq 4$.

4 Conclusion and Discussion

We have studied which SCFs satisfy strategyproofness according to Kelly's preference extension and obtained results for three broad classes of SCFs. A common theme of our results is that strategyproofness entails that potentially "bad" alternatives need to be chosen. In particular, we have shown that *(i)* every strategyproof rank-based SCF returns a Pareto-dominated alternative in at least one profile, *(ii)* every strategyproof support-based SCF that satisfies Pareto-optimality returns at least one most preferred alternative of every voter, and *(iii)* every strategyproof non-imposing SCF returns a Condorcet loser in at least one profile. These results only leave room for rather indecisive strategyproof SCFs such as the Pareto rule, the omninomination rule, the SCF that returns all top-ranked alternatives that are Pareto-optimal, or the SCF that returns all alternatives except Condorcet losers.

Our results also have consequences for so-called *social decision schemes (SDSs)*, which map a preference profile to a *lottery* over alternatives. Since the notions of ranked-basedness and support-basedness are independent of the type of the output of the function and merely define an equivalence relation over preference profiles, they can be straightforwardly extended to SDSs. When extended to the support of lotteries, Kelly-strategyproofness is weaker than the well-studied notion of (weak) *SD*-strategyproofness (Brandt, 2017). Hence, Theorem 1 implies that there is no rank-based SDS that satisfies Pareto-optimality and *SD*-strategyproofness. Furthermore, Theorem 2 implies that every support-based SDS that satisfies Pareto-optimality and *SD*-strategyproofness puts positive probability on at least one most preferred alternative of every voter, a property that is known as *positive share* in the context of dichotomous preferences (Bogomolnaia et al., 2005). Finally, we can also define the Condorcet loser property for SDSs by requiring that Condorcet losers should always receive probability 0, and non-imposition by demanding that for every alternative, there is a profile such that this alternative receives probability 1. Then, Theorem 3 implies that there is no SDS that satisfies the Condorcet loser property, non-imposition, and *SD*-strategyproofness.

In comparison to other results on the strategyproofness of set-valued SCFs, we employ a very weak notion of strategyproofness. In particular, our notion of strategyproofness is weaker than those used by Duggan and Schwartz (2000), Barberà et al. (2001), Ching and Zhou (2002), Rodríguez-Álvarez (2007), and Sato (2008). This is possible because we consider the more general domain of weak preferences, which explicitly allows for ties. Interestingly, all proofs except that of Claim 1 in Theorem 1 can be transferred to the domain of strict preferences by carefully breaking ties and replacing Kelly-strategyproofness with the significantly stronger strategyproofness notion introduced by Duggan and Schwartz (2000). While the resulting theorems are covered by the Duggan-Schwartz impossibility, this raises intriguing questions concerning the relationship between strategyproofness results for weak and strict preferences.

In contrast to previous impossibilities for Kelly's preference extension (Brandl et al., 2019; Brandt et al., 2021), our proofs do not rely on the availability of artificial voters who are completely indifferent between all alternatives. Moreover, the results are tight in the sense that they cease to hold if we remove an axiom, reduce the number of alternatives or voters, weaken the notion of strategyproofness, or require strict preferences. For example, the essential set (Dutta and Laslier, 1999; Laslier, 2000) and a handful of other support-based Condorcet extensions satisfy strategyproofness if preferences are strict and participation for unrestricted preferences (Brandt, 2015; Brandl et al., 2019). Our results thus provide important insights on when and why strategyproofness can be attained.

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References

T. Bandyopadhyay. 1983. Coalitional manipulation and the Pareto rule. *Journal of Economic Theory* 29 (2), 359–363.

S. Barberà. 1977a. The Manipulation of Social Choice Mechanisms That Do Not Leave "Too Much" to Chance. *Econometrica* 45 (7), 1573–1588.

S. Barberà. 1977b. Manipulation of Social Decision Functions. *Journal of Economic Theory* 15 (2), 266–278.

S. Barberà. 2010. Strategy-proof social choice. In *Handbook of Social Choice and Welfare*, K. J. Arrow, A. K. Sen, and K. Suzumura (Eds.). Vol. 2. Elsevier, Chapter 25, 731–832.

S. Barberà, B. Dutta, and A. Sen. 2001. Strategy-proof social choice correspondences. *Journal of Economic Theory* 101 (2), 374–394.

J.-P. Benoît. 2002. Strategic Manipulation in Voting Games When Lotteries and Ties Are Permitted. *Journal of Economic Theory* 102 (2), 421–436.

A. Bogomolnaia, H. Moulin, and R. Stong. 2005. Collective choice under dichotomous preferences. *Journal of Economic Theory* 122 (2), 165–184.

F. Brandl, F. Brandt, M. Eberl, and C. Geist. 2018. Proving the Incompatibility of Efficiency and Strategyproofness via SMT Solving. *J. ACM* 65 (2), 1–28. Preliminary results appeared in the Proceedings of IJCAI-2016.

F. Brandl, F. Brandt, C. Geist, and J. Hofbauer. 2019. Strategic Abstention based on Preference Extensions: Positive Results and Computer-Generated Impossibilities. *Journal of Artificial Intelligence Research* 66, 1031–1056. Preliminary results appeared in the Proceedings of IJCAI-2015.

F. Brandt. 2015. Set-Monotonicity Implies Kelly-Strategyproofness. *Social Choice and Welfare* 45 (4), 793–804. Preliminary results appeared in the Proceedings of IJCAI-2011.

F. Brandt. 2017. Rolling the Dice: Recent Results in Probabilistic Social Choice. In *Trends in Computational Social Choice*, U. Endriss (Ed.). AI Access, Chapter 1, 3–26.

F. Brandt, M. Brill, and P. Harrenstein. 2016a. Tournament Solutions. In *Handbook of Computational Social Choice*, F. Brandt, V. Conitzer, U. Endriss, J. Lang, and A. D. Procaccia (Eds.). Cambridge University Press, Chapter 3.

F. Brandt, V. Conitzer, U. Endriss, J. Lang, and A. Procaccia (Eds.). 2016b. *Handbook of Computational Social Choice*. Cambridge University Press.

F. Brandt, C. Saile, and C. Stricker. 2021. Strategyproof Social Choice When Preferences and Outcomes May Contain Ties. Working Paper. Preliminary results appeared in the Proceedings of AAMAS-2018.

S. Ching and L. Zhou. 2002. Multi-valued strategy-proof social choice rules. *Social Choice and Welfare* 19 (3), 569–580.

J. Duggan and T. Schwartz. 2000. Strategic Manipulability without Resoluteness or Shared Beliefs: Gibbard-Satterthwaite Generalized. *Social Choice and Welfare* 17 (1), 85–93.

B. Dutta and J.-F. Laslier. 1999. Comparison Functions and Choice Correspondences. *Social Choice and Welfare* 16 (4), 513–532.

U. Endriss (Ed.). 2017. *Trends in Computational Social Choice*. AI Access.

A. Feldman. 1979. Manipulation and the Pareto Rule. *Journal of Economic Theory* 21, 473–482.

P. C. Fishburn. 1977. Condorcet Social Choice Functions. *SIAM J. Appl. Math.* 33 (3), 469–489.

P. Gärdenfors. 1976. Manipulation of Social Choice Functions. *Journal of Economic Theory* 13 (2), 217–228.

P. Gärdenfors. 1979. On definitions of manipulation of social choice functions. In *Aggregation and Revelation of Preferences*, J. J. Laffont (Ed.). North-Holland.

A. Gibbard. 1973. Manipulation of Voting Schemes: A General Result. *Econometrica* 41 (4), 587–601.

J. S. Kelly. 1977. Strategy-Proofness and Social Choice Functions Without Single-Valuedness. *Econometrica* 45 (2), 439–446.

J.-F. Laslier. 1996. Rank-Based Choice Correspondences. *Economics Letters* 52 (3), 279–286.

J.-F. Laslier. 2000. Interpretation of electoral mixed strategies. *Social Choice and Welfare* 17, 283–292.

I. MacIntyre and P. K. Pattanaik. 1981. Strategic voting under minimally binary group decision functions. *Journal of Economic Theory* 25 (3), 338–352.

A. Mas-Colell and H. Sonnenschein. 1972. General Possibility Theorems for Group Decisions. *Review of Economic Studies* 39 (2), 185–192.

K. Nehring. 2000. Monotonicity implies generalized strategy-proofness for correspondences. *Social Choice and Welfare* 17 (2), 367–375.

C. Rodríguez-Álvarez. 2007. On the manipulation of social choice correspondences. *Social Choice and Welfare* 29 (2), 175–199.

S. Sato. 2008. On strategy-proof social choice correspondences. *Social Choice and Welfare* 31, 331–343.

S. Sato. 2014. A fundamental structure of strategy-proof social choice correspondences with restricted preferences over alternatives. *Social Choice and Welfare* 42 (4), 831–851.

M. A. Satterthwaite. 1975. Strategy-Proofness and Arrow's Conditions: Existence and Correspondence Theorems for Voting Procedures and Social Welfare Functions. *Journal of Economic Theory* 10 (2), 187–217.

A. K. Sen. 1986. Social Choice Theory. In *Handbook of Mathematical Economics*, K. J. Arrow and M. D. Intriligator (Eds.). Vol. 3. Elsevier, Chapter 22, 1073–1181.

A. D. Taylor. 2005. *Social Choice and the Mathematics of Manipulation*. Cambridge University Press.

Appendix A: Proofs of Auxiliary Lemmas

Here, we discuss the missing proofs of Lemma 1 and Lemma 2 and some examples to give a better understanding of the applied constructions. Before stating the proofs, we first discuss two frequently used implications of strategyproofness. For this, consider an arbitrary strategyproof SCF f, a voter $i \in N$, and two preference profiles R and R' with $\succsim_j = \succsim'_j$ for all $j \in N \setminus \{i\}$. Moreover, let $X_j^+(R)$ denote the set of voter j's most preferred alternatives in R and $X_j^-(R)$ denote the set of voter j's least preferred alternatives in R. The following two implications are true:

i) If $f(R) \subseteq X_i^-(R)$, then $f(R') \subseteq X_i^-(R)$.

ii) If $f(R) \subseteq X_i^+(R')$, then $f(R') \subseteq X_i^+(R')$.

Both implications follow directly from strategyproofness: *i)* states that if a subset of voter i's least preferred alternatives is the choice set for R, then no better alternative can be chosen if he lies about his preferences; otherwise, voter i can manipulate. *ii)* states that making the current choice set $f(R)$ into a subset of voter i's most preferred alternatives in R' results into a choice set $f(R')$ that is also a subset of $X_i^+(R')$. If this was not true, voter i can manipulate by switching from R' to R.

Lemma 1. *Let f be an anonymous, Pareto-optimal, and strategyproof SCF that is defined for $m \geq 3$ alternatives and $n \geq 2$ voters. If a voter is not a nominator for f, then f satisfies near unanimity.*

Proof. Let f denote a Pareto-optimal and strategyproof SCF and assume that voter i is not a nominator for f. Thus, there is a profile R^0 such that $f(R^0)$ does not contain any of voter i's most preferred alternatives. We derive the lemma by modifying this profile in three steps. Firstly, we deduce a profile R' such that $f(R') \cap X_i^+(R') = \emptyset$ and $f(R') = \{x\}$ for some alternative $x \in f(R^0)$. Secondly, we infer from this profile that $f(R) = \{x\}$ for all preferences profiles R such that $n-1$ voters prefer x uniquely the most. Finally, we generalize this observation from a single alternative to all alternatives. This is reminiscent of the so-called *field expansion lemma* in proofs of Arrow's theorem (see, e.g., Sen, 1986).

Step 1: As a first step, we let every voter $j \in N \setminus \{i\}$ replace his preference in R^0 sequentially such that they prefer the alternatives in $f(R^0)$ the most. This leads to the preference profile R^1 and it follows from a repeated application of *ii)* that $f(R^1) \subseteq f(R^0)$. Next, let a denote one of voter i's most preferred alternatives in $f(R^0)$, i.e., $a \succsim_i b$ for all $b \in f(R^0)$. We replace the current preference of voter i in R^1 with a preference where all alternatives in $X_i^+(R^1)$ are preferred to a, which, in turn, is preferred to all alternatives in $A \setminus (X_i^+(R^1) \cup \{a\})$. This leads to the preference profile R^2 for which $f(R^2) = \{a\}$. This claim is true as all alternatives in $A \setminus (X_i^+(R^1) \cup \{a\})$ are Pareto-dominated by a and no alternative in $X_i^+(R^1)$ can be chosen; otherwise, voter i can manipulate by switching from R^1 to R^2. Thus, we have derived a profile $R' = R^2$ with $f(R') = \{a\}$ and $f(R') \cap X_i^+(R') = \emptyset$.

Step 2: Given the preference profile R' from the last step, we show that $f(R) = \{a\}$ for all preferences profiles R such that $n-1$ voters prefer a uniquely the most. We deduce this result by modifying and analyzing the profile R'. First, we sequentially replace the preference of every voter $j \in N \setminus \{i\}$ in R' with a new preference in which he prefers a uniquely the most and an alternative $b \in X_i^+(R')$ uniquely the second most. This leads to a profile R^3 for which $f(R^3) = \{a\}$ follows from a repeated application of *ii)*. Furthermore, every alternative in $A \setminus \{a, b\}$ is Pareto-dominated by b in R^3. We use this observation to replace voter i's current preference with a preference in which b is his uniquely most preferred alternative and a is his uniquely least preferred alternative. In this new profile R^4, all alternatives in $A \setminus \{a,b\}$ are still Pareto-dominated by b and therefore, $f(R^4) \subseteq \{a,b\}$. Furthermore, if $b \in f(R^4)$, voter i can manipulate by switching from R^3 to R^4. Hence, $f(R^4) = \{a\}$, and it follows from *i)* that $f(R'') = \{a\}$ for all preference profiles R'' with $\succsim_j'' = \succsim_j^4$ for all $j \in N \setminus \{i\}$ and from *ii)* that a is the unique winner if all voters in $N \setminus \{i\}$ prefer a uniquely the most. Thus, $f(R'') = \{a\}$ for all profiles R'' in which all voters in $N \setminus \{i\}$ prefer a uniquely the most. Finally, anonymity implies that a is chosen if $n-1$ voters agree that it is the uniquely most preferred alternative.

Step 3: It only remains to show that if $n-1$ voters can make a win uniquely by uniquely top-ranking it, they can make every alternative win uniquely by uniquely top-ranking it. Thus, consider the preference profile R^5 in which $n-1$ voters prefer a uniquely the most, and the remaining voter i prefers c uniquely the most, b uniquely second most and a uniquely the least. It follows from our previous observations that $f(R^5) = \{a\}$. Next, let the voters $j \in N \setminus \{i\}$ change their preferences sequentially such that they prefer a and b the most. This leads to a new preference profile R^6 with $f(R^6) = \{b\}$ because *ii)* implies that $f(R^6) \subseteq \{a,b\}$ and b Pareto-dominates a. Thereafter, we replace the preference of every voter $j \in N \setminus \{i\}$ with a new preference in which he prefers b uniquely the most. This step results in a new preference profile R^7 and the repeated application of *ii)* shows that $f(R^7) = \{b\}$. As voter i does not top-rank b, we can apply the constructions discussed in step 2 to deduce that b is uniquely chosen if $n-1$ voters prefer it uniquely the most. □

For a better understanding of the proof, we provide an example. Therefore, let f denote an anonymous SCF that satisfies Pareto-optimality and strategyproofness. Furthermore, assume that $f(R^0) = \{a,d\}$ for the profile R^0 shown in the sequel and note that voter 1 is therefore no nominator for f.

R^0: 1: $b, \{a,d\}, c$ 2: $d, a, \{b,c\}$ 3: $\{a,c\}, \{b,d\}$

As a first step, we let both voter 2 and 3 change their preferences such that $\{a,d\}$ is the set of their most preferred alternatives, which leads to the profile R^1.

R^1: 1: $b, \{a,d\}, c$ 2: $\{a,d\}, \{b,c\}$ 3: $\{a,d\}, \{b,c\}$

As consequence of *ii)*, it follows that $f(R^1) \subseteq \{a,d\}$. Moreover, every alternative that is strictly less preferred than a by voter 1 is Pareto-dominated. We use this observation to break the tie in the preference order of voter 1, which results in R^2.

R^2: 1: b, a, d, c 2: $\{a,d\}, \{b,c\}$ 3: $\{a,d\}, \{b,c\}$

Pareto-optimality implies that $f(R^2) \subseteq \{a,b\}$ and strategyproofness requires that $f(R^2) \notin \{\{a,b\}, \{b\}\}$. Consequently, $f(R^2) = \{a\}$. Next, we use again *ii)* to change the preference of voter 2 and 3 such that a is their uniquely most preferred alternative and b is their uniquely second best alternative. This results in the profile R^3 for which $f(R^3) = \{a\}$.

R^3: 1: b, a, d, c 2: $a, b, \{c,d\}$ 3: $a, b, \{c,d\}$

Note that b Pareto-dominates every alternative but a in R^3. We use this observation to replace the preference of voter 1 with b, d, c, a while ensuring that a is still the unique winner.

R^4: 1: b, d, c, a 2: $a, b, \{c,d\}$ 3: $a, b, \{c,d\}$

Observe that b also Pareto-dominates every alternative but a in R^4 and therefore $f(R^4) \subseteq \{a,b\}$. Moreover, b cannot be chosen as otherwise, voter 1 can manipulate by switching from R^3 to R^4. Thus, it follows that $f(R^4) = \{a\}$. Based on this profile, it follows from *i)*, *ii)* and anonymity that a is the unique winner if $n-1$ voters prefer it uniquely the most.

Next, we show that $n-1$ voters can also make another alternative b win uniquely by ranking it first. Therefore, consider the profile R^5 displayed in the sequel.

R^5: 1: c, b, d, a 2: $a, b, \{c, d\}$ 3: $a, b, \{c, d\}$

It follows from our previous observations that $f(R^5) = \{a\}$. Next, we let voter 2 and 3 change their preference such that they prefer both a and b uniquely the most, which leads to the profile R^6.

R^6: 1: c, b, d, a 2: $\{a, b\}, \{c, d\}$ 3: $\{a, b\}, \{c, d\}$

It holds that $f(R^6) = \{b\}$ as *ii)* implies that $f(R^6) \subseteq \{a, b\}$ and b Pareto-dominates a. Therefore, we let voter 2 and 3 replace their preferences to derive the profile R^7 and use again *ii)* to deduce that $f(R^7) = \{b\}$.

R^7: 1: c, b, d, a 2: $b, c, \{a, d\}$ 3: $b, c, \{a, d\}$

Finally, note that the profile R^7 is equivalent to R^3 up to renaming alternatives. Therefore, we can use the previously discussed steps to derive that $f(R) = \{b\}$ if $n-1$ voters prefer b uniquely the most. This concludes the example.

Lemma 2. *Let f denote a strategyproof SCF for $n \geq 3$ voters that satisfies non-imposition and the Condorcet loser property. Then, $f(R) = \{a\}$ for every preference profile R and every alternative $a \in A$ such that more than half of the voters in R report a as their unique top choice.*

Proof. First, note that the lemma is trivial if $m = 1$ as the single alternative is always the unique winner and if $m = 2$ as an alternative b is the Condorcet loser if more than half of the voters prefer the other alternative a uniquely the most. Thus, we focus in the sequel on the case $m \geq 3$ and consider a strategyproof SCF f that satisfies non-imposition and the Condorcet loser property.

Let N_a denote an arbitrary subset of voters such that $|N_a| = \lceil \frac{n+1}{2} \rceil$, and let R denote a profile in which every voter $i \in N_a$ prefers a uniquely the most and every voter $i \in N \setminus N_a$ prefers a uniquely the least. We show in the sequel that $f(R) = \{a\}$. This implies that $f(R') = \{a\}$ for every profile R' in which the voters $i \in N_a$ prefer a uniquely the most since *ii)* allows voters who uniquely top-rank the unique winner to reorder all other alternatives without affecting the choice set and *i)* allows voters who uniquely bottom-rank the unique winner to reorder all alternatives without affecting the choice set. Thus, we can transform R into any profile R' without changing the choice set if all voters in N_a prefer a uniquely the most in R'. Moreover, as N_a is arbitrarily chosen, the lemma follows by showing that $f(R) = a$. In the following, we assume for simplicity that $N_a = \{1, \dots, \lceil \frac{n+1}{2} \rceil\}$, i.e., the first $\lceil \frac{n+1}{2} \rceil$ voters in R prefer a uniquely the most and the remaining voters prefer a uniquely the least. This assumption is feasible as all of our arguments are closed under renaming voters and alternatives.

The proof of $f(R) = \{a\}$ proceeds by an induction over $k \in \{0, \dots, n - \lceil \frac{n+1}{2} \rceil\}$. Thus, let R^k denote a profile in which all voters in $\{1, \dots, n-k\}$ prefer a uniquely the most and the remaining voters prefer a uniquely the least. We show in the sequel that $f(R^k) = \{a\}$ for all $k \in \{0, \dots, n - \lceil \frac{n+1}{2} \rceil\}$, which implies that $f(R) = \{a\}$ since we can derive R from $R^{\lceil \frac{n+1}{2} \rceil}$ by applying *i)* and *ii)* without affecting the choice set since a is in both profiles either uniquely top-ranked or uniquely bottom-ranked. It should be mentioned that this argument can be often used, in which case we ignore all preferences between alternatives in $A \setminus \{a\}$. In particular, this is also the reason why we do not specify the preferences between these alternatives for the profiles R^k.

First, we consider the base case $k = 0$, i.e., we show that a is the unique winner if all voters agree that a is the uniquely best alternative. This claim follows as f is non-imposing and strategyproof: non-imposition implies that there is a profile R^* with $f(R^*) = \{a\}$. Then, we use *ii)* to subsequently change the preference of every voter into one in which he prefers a uniquely the most. This step results in the profile R^0 in which every voter prefers a uniquely the most and *ii)* implies that $f(R^0) = \{a\}$. Thus, the base case is proven.

Next, we focus on the induction step, i.e., we assume that $f(R^k) = \{a\}$ for some $k \in \{0, \dots, n - \lceil \frac{n+1}{2} \rceil - 1\}$ and show that $f(R^{k+1}) = \{a\}$. Let $i^* = n - k$ denote the voter who prefers a uniquely the most in R^k but not in R^{k+1}. In order to show why voter i^* can make a into his least preferred alternative, we require an auxiliary claim which we prove at the end: it holds for $a_l \in \{a_1, \dots, a_{m-1}\} = A \setminus \{a\}$ that $f(R^{k,l}) = \{a\}$. The profiles $R^{k,l}$ are defined in the sequel, where the $*$ symbol indicates that the preferences on the missing alternatives are arbitrary but extend the preferences shown. In particular, $x, *, y$ means that x is strictly preferred to every other alternative and every alternative is strictly preferred to y. Note that, even though the preferences of voters $j \in N \setminus \{i^*\}$ over alternatives in $A \setminus \{a\}$ can be arbitrary, we assume that $R_j^{k,l} = R_j^{k,l'}$ for all voters $j \in N \setminus \{i^*\}$ and indices $l, l' \in \{1, \dots, m-1\}$.

$R^{k,l}$: $[1 \dots i^*{-}1]$: $a, *$ i^*: $a_l, A \setminus \{a_l\}$ $[i^*{+}1 \dots m]$: $*, a$

Assume for now that the auxiliary claim is true. We proceed by replacing the preference of voter i^* in each of these profiles with $a_1, a_2, \dots, a_{m-1}, a$. As the profiles $R^{k,l}$ only differ in the preference of this voter, this leads always to the same profile $\tilde{R}^k$. Moreover, strategyproofness implies for all $l \in \{1, \dots, m-1\}$ that $a_l \notin f(\tilde{R}^k)$ as otherwise, voter i^* can manipulate by switching from $R^{k,l}$ to $\tilde{R}^k$. Thus, we derive that $f(\tilde{R}^k) = \{a\}$. Finally, we can transform $\tilde{R}^k$ into R^{k+1} without changing the choice set by repeatedly applying *i)* and *ii)* since these profiles can only differ on the preferences of voters in $A \setminus \{i^*\}$. The induction step is therefore shown.

Finally, it remains to prove the auxiliary claim, i.e., that $f(R^{k,l}) = \{a\}$ for all $l \in \{1, \dots, m-1\}$. For this, consider the profiles $R^{k,l,X}$ for $l \in \{1, \dots, m-1\}$ and $X \subseteq A \setminus \{a, a_l\}$.

$R^{k,l,X}$: $[1 \dots i^*{-}1]$: $a, *$ i^*: $a_l, \{a\} \cup X, A \setminus (X \cup \{a, a_l\})$ $[i^*{+}1 \dots m]$: $*, a$

In particular, voter i^*'s preference is $a_l, a, A \setminus \{a, a_l\}$ in $R^{k,l,\emptyset}$ and $a_l, A \setminus \{a_l\}$ in $R^{k,l,A \setminus \{a,a_l\}}$, i.e., $R^{k,l,A \setminus \{a,a_l\}} = R^{k,l}$.

In the sequel, we show by an induction on $z = |X|$ that $f(R^{k,l,X}) = \{a\}$ for all $l \in \{1, \dots, m-1\}$ and $X \subseteq A \setminus \{a, a_l\}$. This entails the auxiliary claim since $R^{k,l,A \setminus \{a,a_l\}} = R^{k,l}$ for all $l \in \{1, \dots, m-1\}$. Two observations are central for the subsequent argument: firstly, our argument is closed under renaming alternatives in $A \setminus \{a\}$. This means that if we can show that $f(R^{k,l,X}) = \{a\}$ for some $l \in \{1, \dots, m-1\}$ and $X \subseteq A \setminus \{a, a_l\}$ with $|X| = z$, this result holds for all $l' \in \{1, \dots, m-1\}$ and subsets of $A \setminus \{a, a_{l'}\}$ with size z. Secondly, we can turn any alternative $a_j \in \{a_1, \dots, a_{m-1}\}$ into the Condorcet loser by letting the voters $i \in [1 \dots i^*{-}1]$ uniquely bottom-rank a_j since this set contains more than half of the voters. Even more, this step does not change the choice set if a is the unique winner because of *ii)*.

First, we prove the base case $z = 0$, i.e., we show that $f(R^{k,l,\emptyset}) = \{a\}$ for all $l \in \{1, \dots, m-1\}$. Recall for this that it suffices to show that $f(R^{k,1,\emptyset}) = \{a\}$ due to the neutrality of the argument. By assumption, $f(R^k) = \{a\}$ and thus, a repeated application of *i)* and *ii)* entails that $f(\hat{R}^{k,1,\emptyset}) = \{a\}$, where the profile $\hat{R}^{k,1,\emptyset}$ is shown subsequently. This is true as the voters $i \in [1 \dots i^*]$ uniquely top-rank the unique winner a and the voters $i \in [i^*{+}1 \dots n]$ uniquely bottom-rank the unique winner before and after the modification.

$\hat{R}^{k,1,\emptyset}$: $[1 \dots i^*{-}1]$: $a, *, a_1$ i^*: $a, a_1, A \setminus \{a, a_1\}$ $[i^*{+}1 \dots m]$: $*, a$

Note that a_1 is the Condorcet loser in $\hat{R}^k$, regardless of the preference of voter i^*. Thus, a_1 is also not in the choice set if voter i^* swaps a and a_1 because of the Condorcet loser property. This step results in the profile $\tilde{R}^{k,1,\emptyset}$ displayed in the following. Moreover, as $a_1 \notin f(\hat{R}^{k,1,\emptyset})$, strategyproofness implies that $f(\tilde{R}^{k,1,\emptyset}) = \{a\}$. Otherwise, voter i^* can manipulate by switching from $\tilde{R}^{k,1,\emptyset}$ back to $\hat{R}^{k,1,\emptyset}$.

$\tilde{R}^{k,1,\emptyset}$: $[1\dots i^*-1]$: $a,*,a_1$ i^*: $a_1, a, A\setminus\{a_1,a\}$ $[i^*+1\dots m]$: $*,a$

Finally, we note that $\tilde{R}^{k,1,\emptyset}$ can only differ from $R^{k,1,\emptyset}$ in the preferences of the voters in $N\setminus\{i^*\}$ on the alternatives $A\setminus\{a\}$. As *i)* and *ii)* allow us to reorder the preferences of these voters on $A\setminus\{a\}$ arbitrarily without affecting the choice set, we derive that $f(R^{k,1,\emptyset})=\{a\}$, and hence, $f(R^{k,j,\emptyset})=\{a\}$ for all $j\in\{1,\dots,m-1\}$.

Next, we focus on the induction step, i.e., we assume that $f(R^{k,j,X})=\{a\}$ for all $j\in\{1,\dots,m-1\}$ and all $X\subseteq A\setminus\{a,a_j\}$ with $|X|=z-1$ and show that the same is true for all such X' of size z. Recall for this that the derivation of $f(R^{k,j,X'})=\{a\}$ is independent of the naming of the alternatives in $A\setminus\{a\}$, and thus, it suffices to show that $f(R^{k,z+1,\{a_1,\dots,a_z\}})=\{a\}$. For this, let $Z=\{a_1,\dots,a_z\}$, $Z_{+a}=Z\cup\{a\}$, and $Z_{-l}=Z\setminus\{a_l\}$ for every $l\in\{1,\dots,z\}$, and consider the profiles $R^{k,l,Z_{-l}}$. Our induction hypothesis implies for all these profiles that $f(R^{k,l,Z_{-l}})=\{a\}$ since $|Z_{-l}|=z-1$.

$R^{k,l,Z_{-l}}$: $[1\dots i^*-1]$: $a,*$ i^*: $a_l, Z_{-l}\cup\{a\}, A\setminus Z_{+a}$ $[i^*+1\dots m]$: $*,a$

For all of these profiles, we can repeatedly apply *i)* and *ii)* such that every voter $i\in[1\dots i^*-1]$ prefers alternative a_{z+1} uniquely the least and $R_j^{k,l,Z_{-l}}=R_j^{k,l',Z_{-l'}}$ for all voters $j\in N\setminus\{i^*\}$ and indices $l,l'\in\{1,\dots,z\}$. This step results in the profiles $\hat{R}^{k,l,Z_{-l}}$, for which *i)* and *ii)* imply that $f(\hat{R}^{k,l,Z_{-l}})=\{a\}$.

$\hat{R}^{k,l,Z_{-l}}$: $[1\dots i^*-1]$: $a,*,a_{z+1}$ i^*: $a_l, Z_{-l}\cup\{a\}, A\setminus Z_{+a}$ $[i^*+1\dots m]$: $*,a$

Note that a_{z+1} is the Condorcet loser in all of these profiles, regardless of the preference of voter i^*, and all voters but i^* have the same preference in all of these profiles. We use these observations to derive the profile $\tilde{R}^{k,z+1,Z}$ by replacing the preference of voter i^* in all these profiles with $a_{z+1}, Z_{+a}, A\setminus(Z_{+a}\cup\{a_{z+1}\}$. Formally, the profile $\tilde{R}^{k,z+1,Z}$ is defined as follows.

$\tilde{R}^{k,z+1,Z}$: $[1\dots i^*-1]$: $a,*,a_{z+1}$ i^*: $a_{z+1}, Z_{+a}, A\setminus(Z\cup\{a,a_{z+1}\})$
$[i^*+1\dots m]$: $*,a$

As mentioned, the Condorcet loser property implies that $a_{z+1}\notin f(\tilde{R}^{k,z+1,Z})$. Thus, strategyproofness from $\tilde{R}^{k,z+1,Z}$ to $\hat{R}^{k,l,Z_{-l}}$ implies that $f(\tilde{R}^{k,z+1,Z})\subseteq Z$ as otherwise, voter i^* can manipulate by reverting back to a profile $\hat{R}^{k,l,Z_{-l}}$. This is true because voter i^* prefers $f(\hat{R}^{k,l,Z_{-l}})=\{a\}$ to every set $Y\subseteq A\setminus\{a_{z+1}\}$ with $Y\setminus Z\neq\emptyset$. Finally, strategyproofness from $\hat{R}^{k,l,Z_{-l}}$ to $\tilde{R}^{k,z+1,Z}$ implies for every $l\in\{1,\dots,z\}$ that $a_l\notin f(\tilde{R}^{k,j,Z})$. Otherwise, voter i^* can manipulate as he prefers any set $Y\subseteq Z$ with $a_l\in Y$ to $\{a\}$ according to his preference in $\hat{R}^{k,l,Z_{-l}}$. Hence, we conclude that $f(\tilde{R}^{k,z+1,Z})=\{a\}$. By applying again *i)* and *ii)* to reorder the preferences of the voters $i\in N\setminus\{i^*\}$, it follows that $f(R^{k,z+1,Z})=\{a\}$. As the argument is closed under renaming alternatives in $A\setminus\{a\}$, the induction step is proven. Hence, it follows that $f(R^{k,l,X})=\{a\}$ for all $l\in\{1,\dots m-1\}$ and $X\subseteq A\setminus\{a,a_l\}$. In particular, this means that $f(R^{k,l,A\setminus\{a,a_l\}})=f(R^{k,l})=\{a\}$ for all $l\in\{1,\dots,m-1\}$, i.e., the auxiliary claim is proven. □

Just as for Lemma 1, we provide an example of the constructions used in the proof of Lemma 2. In particular, we show how a voter can turn his best alternative a into his worst one without changing the choice set if $f(R)=\{a\}$.

Therefore, let f denote a strategyproof SCF that satisfies non-imposition and the Condorcet loser property, and consider the profile R^1 defined for four voters and four alternatives $A=\{a,b,c,d\}$.

R^1: 1: a,b,c,d 2: a,b,c,d 3: a,b,c,d 4: a,b,c,d

As f satisfies non-imposition and strategyproofness, it follows that $f(R^1)=\{a\}$. Next, we let all voters $i\in\{1,2,3\}$ make b into the worst alternative such that it becomes the Condorcet loser. This results in the profile R^2 and *ii)* implies that $f(R^2)=\{a\}$.

R^2:	1: a, c, d, b	2: a, c, d, b	3: a, c, d, b	4: a, b, c, d

Note that b is the Condorcet loser in R^2, regardless of the preference of voter 4. Thus, we let this voter swap a and b next to derive the profile R^3. The Condorcet loser property implies that $b \notin f(R^3)$ and then, strategyproofness entails that $f(R^3) = \{a\}$.

R^3:	1: a, c, d, b	2: a, c, d, b	3: a, c, d, b	4: b, a, c, d

Thereafter, voters 1 to 3 can modify their preferences on the alternatives $A \setminus \{a\}$ again arbitrarily without affecting the choice set because of *ii)*. Hence, it follows that $f(R^4) = \{a\}$ where the profile R^4 is shown in the sequel.

R^4:	1: a, d, b, c	2: a, d, b, c	3: a, d, b, c	4: b, a, c, d

In the profile R^4, c is the Condorcet loser even if voter 4 changes his preferences. We use this observation to replace his current preference with $c, \{a, b\}, d$. This step results in the profile R^5 and $c \notin f(R^5)$ due to the Condorcet loser property. Then, strategyproofness from R^5 to R^4 implies that $f(R^5) \subseteq \{a, b\}$ and strategyproofness in the inverse direction implies that $f(R^5) \neq \{a, b\}$ and $f(R^5) \neq \{b\}$. Hence, $f(R^5) = \{a\}$ is the only valid option.

R^5:	1: a, d, b, c	2: a, d, b, c	3: a, d, b, c	4: $c, \{a, b\}, d$

Note that the voters $i \in \{1, 2, 3\}$ can now reorder again all alternatives in $A \setminus \{a\}$ without affecting the choice set. Moreover, a symmetric argument shows that voter 4 can also rename b and c in his preference and a is still the unique winner. In summary, we obtain that $f(R^6) = f(R^7) = \{a\}$ where the profiles R^6 and R^7 are shown in the sequel.

R^6:	1: a, b, c, d	2: a, b, c, d	3: a, b, c, d	4: $c, \{a, b\}, d$
R^7:	1: a, b, c, d	2: a, b, c, d	3: a, b, c, d	4: $b, \{a, c\}, d$

Alternative d is the Condorcet loser in both R^6 and R^7, regardless of the preference of voter 4. Hence, we replace voter 4's preference in both R^6 and R^7 with $d, \{a, b, c\}$ to derive the profile R^8. Since d is the Condorcet loser in R^8, $d \notin f(R^8)$. Moreover, strategyproofness from R^6 to R^8 implies that $c \notin f(R^8)$ and strategyproofness from R^7 to R^8 implies that $b \notin f(R^8)$. Hence, we conclude that $f(R^8) = \{a\}$.

R^8:	1: a, b, c, d	2: a, b, c, d	3: a, b, c, d	4: $d, \{a, b, c\}$

Again, the voters $i \in \{1, 2, 3\}$ can reorder the alternatives in $A \setminus \{a\}$ arbitrarily without affecting the choice set and symmetric arguments can be used to rename alternatives in the preference of the fourth voter. Hence, it follows that $f(R^9) = f(R^{10}) = f(R^{11}) = \{a\}$ for the profiles R^9, R^{10}, and R^{11} shown in the sequel.

R^9:	1: a, c, b, d	2: a, c, b, d	3: a, c, b, d	4: $b, \{a, c, d\}$
R^{10}:	1: a, c, b, d	2: a, c, b, d	3: a, c, b, d	4: $c, \{a, b, d\}$
R^{11}:	1: a, c, b, d	2: a, c, b, d	3: a, c, b, d	4: $d, \{a, b, c\}$

Finally, we replace the preference of voter 4 in all three profiles with b, c, d, a to derive the profile R^{12}. Strategyproofness from R^9 implies that $b \notin f(R^{12})$, from R^{10} that $c \notin f(R^{12})$, and from R^{11} that $d \notin f(R^{12})$. Consequently, $f(R^{12}) = \{a\}$ is the only option. Since all voters can reorder all alternatives in $A \setminus \{a\}$ without changing the choice set due to *i)* and *ii)*, it follows that a is also the unique winner regardless of the exact preferences of the voters.

R^{12}:	1: a, c, b, d	2: a, c, b, d	3: a, c, b, d	4: b, c, d, a

This concludes the example.

Appendix B: Proofs of the Main Theorems

In this section, we prove the main results stated in Theorems 1 to 3. Note that proof sketches for all of these results have been discussed in the main body, and that these results rely on Lemma 1 and Lemma 2.

Theorem 1. *There is no rank-based SCF that satisfies Pareto-optimality and strategyproofness if $m \geq 4$ and $n \geq 3$.*

Proof. Assume for contradiction that there is a rank-based SCF f that satisfies strategyproofness and Pareto-optimality and that is defined for fixed numbers of voters $n \geq 3$ and alternatives $m \geq 4$. We derive a contradiction to this assumption by proving two claims: on the one hand, there is a voter who is not a nominator for f. On the other hand, the assumptions on the SCF require that every voter is a nominator. These two claims contradict each other and therefore f cannot exist.

Claim 1: Not every voter is a nominator for f.

First, we prove that not every voter is a nominator for f. Consider therefore the following three profiles in which $X = A \setminus \{a, b, c, d\}$.

R^1:	1: $\{a, b\}, X, \{c, d\}$	2: $\{c, d\}, X, \{a, b\}$	$[3 \dots n]$: $a, \{b, c, d\}, X$
R^2:	1: $\{a, c\}, X, \{b, d\}$	2: $\{b, d\}, X, \{a, c\}$	$[3 \dots n]$: $a, \{b, c, d\}, X$
R^3:	1: $\{a, d\}, X, \{b, c\}$	2: $\{b, c\}, X, \{a, d\}$	$[3 \dots n]$: $a, \{b, c, d\}, X$

It can be easily verified that $r^*(R^1) = r^*(R^2) = r^*(R^3)$ and that a Pareto-dominates b in R^1, c in R^2, and d in R^3. This means that $f(R^1) = f(R^2) = f(R^3) \subseteq \{a\} \cup X$ because of rank-basedness and Pareto-optimality. Consequently, voter 2 is not a nominator for f.

Claim 2: Every voter is a nominator for f.

Assume for contradiction that a voter is no nominator for f and consider the profiles $R^{k,1}$ and $R^{k,2}$ for $k \in \{1, \dots, n\}$.

$R^{k,1}$:	1: $\{c, d\}, X, b, a$	$[2 \dots k]$: $\{a, b\}, X, c, d$	$[k{+}1 \dots n]$: a, X, b, c, d
$R^{k,2}$:	1: $\{b, d\}, X, c, a$	$[2 \dots k]$: $\{a, b\}, X, c, d$	$[k{+}1 \dots n]$: a, X, b, c, d

We prove by induction on $k \in \{1, \dots, n\}$ that $f(R^{k,1}) = f(R^{k,2}) = \{a\}$. The case $k = n$ yields a contradiction to Pareto-optimality as a is Pareto-dominated by b in $R^{n,1}$.

The base case $k = 1$ follows because $n - 1$ voters prefer a uniquely the most in both $R^{1,1}$ and $R^{1,2}$. Therefore, Lemma 1 implies that $f(R^{1,1}) = f(R^{1,2}) = \{a\}$. Assume now that the claim is true for some fixed $k \in \{1, \dots, n-1\}$.

By induction and strategyproofness, $f(R^{k+1,1}) \subseteq \{a, b\}$ since otherwise voter $k + 1$ can manipulate by switching back to $R^{k,1}$. Next, we derive the profile $R^{k,3}$ from $R^{k,2}$ by assigning voter $k + 1$ the preference $\{a, c\}, X, b, d$. Formally, $R^{k,3}$ is defined as follows.

$R^{k,3}$:	1: $\{b, d\}, X, c, a$	$[2 \dots k]$: $\{a, b\}, X, c, d$
	$k{+}1$: $\{a, c\}, X, b, d$	$[k{+}2 \dots n]$: a, X, b, c, d

The induction hypothesis entails that $f(R^{k,2}) = \{a\}$ and therefore, strategyproofness implies that $f(R^{k,3}) \subseteq \{a, c\}$; otherwise, voter $k + 1$ could manipulate by switching back to $R^{k,2}$. Next, we apply rank-basedness to conclude that $f(R^{k+1,1}) = \{a\}$ as $r^*(R^{k+1,1}) = r^*(R^{k,3})$. Finally, $R^{k+1,2}$ evolves from $R^{k+1,1}$ by having voter 1 change his preferences. As a is the uniquely least preferred alternative of this voter, strategyproofness implies that $f(R^{k+1,2}) = \{a\}$ as any other outcome benefits voter 1. $\square$

Theorem 2. *In every support-based SCF that satisfies Pareto-optimality and strategyproofness, every voter is a nominator if $m \geq 3$.*

Proof. Let f be a support-based SCF satisfying Pareto-optimality and strategyproofness for fixed numbers of voters $n \geq 1$ and alternatives $m \geq 3$. For $n = 1$, the theorem follows immediately from Pareto-optimality as only the most preferred alternatives of the single voter are Pareto-optimal. Moreover, Lemma 1 proves the theorem for $n = 2$. Indeed, if a voter is not a nominator, a single voter can determine the choice set. However, this means that $f(R) = \{a\}$ and $f(R) = \{b\}$ are simultaneously true if voter 1 prefers a uniquely the most and voter 2 prefers b uniquely the most.

Therefore, we focus on the case $n \geq 3$ and assume for contradiction that a voter is no nominator for f. We derive from this assumption by an induction on $k \in \{1, \dots, n-1\}$ that $n - k$ voters can determine a unique winner by uniquely top-ranking it. This results in a contradiction when $k \geq n/2$ because then, two alternatives can be simultaneously top-ranked by $n - k \leq n/2$ voters, and both of them must be the unique winner.

The induction basis $k = 1$ follows directly from Lemma 1 as this lemma states that f satisfies near unanimity. Next, we assume that our claim holds for a fixed $k \in \{1, \dots n-2\}$ and prove that also $n-(k+1)$ voters can determine the winner uniquely. For this, we focus only on three alternatives a, b, c and on a certain partition of the voters. This is possible as the induction hypothesis allows us to exchange the roles of the alternatives without affecting the proof and support-basedness allows us to reorder the voters. Thus, consider the profile $R^{k,1}$, in which $X = A \setminus \{a, b, c\}$, and note that $f(R^{k,1}) = \{a\}$ because of near unanimity.

$R^{k,1}$: $[1 \dots k]$: a, X, c, b $\quad$ $k{+}1$: c, X, b, a $\quad$ $[k{+}2 \dots n]$: a, b, X, c

Next, we aim to reverse the preferences of the voters $i \in [k{+}2 \dots n]$ over a and b. This is achieved by the repeated application of the following steps explained for voter $k + 2$. First, voter $k + 2$ changes his preference to $\{a, b\}, c, X$ to derive the profile $R^{k,2}$. Since a subset of $\{a, b\}$ was chosen before this step, strategyproofness implies that $f(R^{k,2}) \subseteq \{a, b\}$ as otherwise, voter $k+2$ can manipulate by reverting this modification. Next, we use support-basedness to exchange the preferences of voter $k + 1$ and $k + 2$ over a and b. This leads to the profile $R^{k,3}$ and support-basedness implies that $f(R^{k,3}) = f(R^{k,2}) \subseteq \{a, b\}$. As a subset of the least preferred alternatives of voter $k+1$ is chosen for $R^{k,3}$, strategyproofness implies that this voter cannot make another alternative win by manipulating. Thus, he can switch back to his original preference to derive $R^{k,4}$ and the fact that $f(R^{k,4}) \subseteq \{a, b\}$.

$R^{k,2}$: $[1 \dots k]$: a, X, c, b $\quad$ $k{+}1$: c, X, b, a $\quad$ $k{+}2$: $\{a, b\}, X, c$ $\quad$ $[k{+}3 \dots n]$: a, b, X, c

$R^{k,3}$: $[1 \dots k]$: a, X, c, b $\quad$ $k{+}1$: $c, X, \{a, b\}$ $\quad$ $k{+}2$: b, a, X, c $\quad$ $[k{+}3 \dots n]$: a, b, X, c

$R^{k,4}$: $[1 \dots k]$: a, X, c, b $\quad$ $k{+}1$: c, X, b, a $\quad$ $k{+}2$: b, a, X, c $\quad$ $[k{+}3 \dots n]$: a, b, X, c

It is easy to see that we can repeat these steps for every voter $i \in [k{+}3 \dots n]$. This process results in the profile $R^{k,5}$ and shows that $f(R^{k,5}) \subseteq \{a, b\}$. Moreover, consider the profile $R^{k,6}$ derived from $R^{k,5}$ by letting voter $k + 1$ make b his best alternative. As $n - k$ voters prefer b uniquely the most in $R^{k,6}$, the induction hypothesis entails that $f(R^{k,6}) = \{b\}$. This means that voter $k+1$ can manipulate by switching from $R^{k,5}$ to $R^{k,6}$ if $f(R^{k,5}) = \{a\}$ or $f(R^{k,5}) = \{a, b\}$. Consequently, $f(R^{k,5}) = \{b\}$ is the only valid choice set for $R^{k,5}$.

$R^{k,5}$: $[1 \dots k]$: a, X, c, b $\quad$ $k{+}1$: c, X, b, a $\quad$ $[k{+}2 \dots n]$: b, a, X, c

$R^{k,6}$: $[1 \dots k]$: a, X, c, b $\quad$ $k{+}1$: b, X, a, c $\quad$ $[k{+}2 \dots n]$: b, a, X, c

So far, we have found a profile in which b is uniquely chosen when only $n-(k+1)$ voters prefer it uniquely the most. Next, we show that b is always the unique winner if the voters $i \in [k{+}2 \dots n]$ prefer it uniquely the most. Therefore, consider the profile $R^{k,7}$ which is derived from $R^{k,5}$ by letting the voters $i \in [1 \dots k]$ subsequently change their preference to c, X, a, b. As $f(R^{k,5}) = \{b\}$ and b is the worst alternative for these voters, strategyproofness implies that $f(R^{k,7}) = \{b\}$.

$R^{k,7}$: $[1\dots k]$: c, X, a, b $\quad$ $k{+}1$: c, X, b, a $\quad$ $[k{+}2\dots n]$: b, a, X, c

As last step, we change the preferences of voter $k+1$ such that b is his least preferred alternative. For this, we first let all voters $i \in [k{+}2\dots n]$ subsequently change their preference to b, X, c, a. This modification results in the profile $R^{k,8}$ and strategyproofness implies that $f(R^{k,8}) = \{b\}$. Moreover, observe that alternative a is Pareto-dominated by c in $R^{k,8}$. Therefore, voter $k+1$ can now swap a and b to derive the profile $R^{k,9}$ and Pareto-optimality implies that $a \notin f(R^{k,9})$. Then, strategyproofness implies that $f(R^{k,9}) = \{b\}$ as any other subset of $A \setminus \{a\}$ is a manipulation for voter $k+1$.

$R^{k,8}$: $[1\dots k]$: c, X, a, b $\quad$ $k{+}1$: c, X, b, a $\quad$ $[k{+}2\dots n]$: b, X, c, a

$R^{k,9}$: $[1\dots k]$: c, X, a, b $\quad$ $k{+}1$: c, X, a, b $\quad$ $[k{+}2\dots n]$: b, X, c, a

Finally, observe that the voters $i \in [1\dots k{+}1]$ can change their preferences in $R^{k,9}$ arbitrarily without affecting the choice set, and the voters $i \in [k{+}2\dots n]$ can reorder all alternatives in $A \setminus \{b\}$ without affecting the choice set because of strategyproofness. Thus, b is always the unique winner if all voters $i \in [k{+}2\dots n]$ prefer b uniquely the most. Moreover, interchanging the roles of alternatives and reordering the voters shows that every alternative is chosen if it is uniquely top-ranked by $n - (k+1)$ voters. This completes the induction step and consequently, we derive that every voter is a nominator for a support-based SCF that satisfies strategyproofness and Pareto-optimality. ☐

Theorem 3. *There is no strategyproof SCF that satisfies the Condorcet loser property and non-imposition if* $m \geq 3$ *and* $n \geq 4$.

Proof. Assume for contradiction that there is a non-imposing SCF f that satisfies the Condorcet loser property and strategyproofness for $n \geq 5$ and $m \geq 3$ alternatives. Since the Condorcet loser property and Lemma 2 strongly depend on the parity of the number of voters, we proceed with a case distinction on n.

Case 1: n is odd

First, assume that f is defined for an odd number of voters $n \geq 4$ and recall that f chooses an alternative as unique winner if at least $l = \frac{n+1}{2}$ voters prefer it uniquely the most because of Lemma 2. Next, let $X = A \setminus \{a, b, c\}$ and consider profile R^1 shown in the sequel. It follows from Lemma 2 that $f(R^1) = \{a\}$ because the first l voters prefer a uniquely the most.

R^1: 1: a, b, X, c $\quad$ $[2\dots l]$: a, c, X, b $\quad$ $[l{+}1\dots n]$: $b, X, \{a, c\}$

Moreover, observe that c is the Condorcet loser in R^1 because every voter prefers a weakly to c and all voters in $[l{+}1\dots n]$ and voter 1 prefer all alternatives in $A \setminus \{a, c\}$ strictly to c. In addition, c remains Condorcet loser if voter l swaps a and c. Let R^2 denote the profile derived from this swap and observe that $c \notin f(R^2)$ because of the Condorcet loser property. Moreover, strategyproofness implies that $f(R^2) = \{a\}$ if $c \notin f(R^1)$; otherwise, voter l can manipulate by reverting back to R^1 as he prefers $\{a\}$ to every other subset of $A \setminus \{c\}$.

R^3: 1: a, b, X, c $\quad$ $[2\dots l{-}1]$: a, c, X, b $\quad$ l: c, a, X, b $\quad$ $[l{+}1\dots n]$: $b, X, \{a, c\}$

Next, we subsequently replace the preference of the voters $i \in [l{+}1\dots n]$ with b, c, X, a. Strategyproofness implies for each of these steps that a subset of $\{a, c\}$ is chosen if it has been chosen before the step. The reason for this is that every other set contains a strictly preferred alternative and obtaining it is therefore a manipulation. Hence, we deduce that this process results in a profile R^3 with $f(R^3) \subseteq \{a, c\}$ because $f(R^2) = \{a\}$. Moreover, $f(R^3) \neq \{c\}$ as otherwise voter 1 can manipulate by swapping a and b: after this step, b is uniquely top-ranked by more than half of the voters and therefore Lemma 2 implies that b

is the unique winner. As voter 1 prefers $\{b\}$ to $\{c\}$, it follows from strategyproofness that $f(R^3) \in \{\{a\}, \{a,c\}\}$.

R^3: 1: a, b, X, c $\quad$ $[2 \dots l{-}1]$: a, c, X, b $\quad$ l: c, a, X, b $\quad$ $[l{+}1 \dots n]$: b, c, X, a

Next, we provide another derivation for the choice set of R^3 which conflicts with $f(R^3) \in \{\{a\}, \{a,c\}\}$. Therefore, consider the profile R^4 shown in the sequel and note that $f(R^4) = \{c\}$ because of Lemma 2.

R^4: $[1 \dots l{-}1]$: a, c, X, b $\quad$ l: c, a, X, b $\quad$ $[l{+}1 \dots n]$: c, b, X, a

Note that b is the Condorcet loser in R^4 as it is uniquely bottom-ranked by the voters $i \in [1 \dots l]$. This even holds if the voters in $i \in [l{+}1 \dots n]$ change their preference. Thus, we let these voters swap b and c, and the Condorcet loser property always implies for the resulting profile that b is not chosen. Just as for R^3, strategyproofness implies then that c remains the unique winner after every change because otherwise, a voter can manipulate by reverting this modification. Thus, this process results in the profile R^5 with $f(R^5) = \{c\}$.

R^5: $[1 \dots l{-}1]$: a, c, X, b $\quad$ l: c, a, X, b $\quad$ $[l{+}1 \dots n]$: b, c, X, a

Finally, we derive the profile R^3 from R^5 by replacing the preference of voter 1 with a, b, X, c. Strategyproofness from R^5 to R^3 implies that $f(R^3) \neq \{a\}$ and $f(R^3) \neq \{a, c\}$ as otherwise, voter 1 can manipulate by switching from R^5 to R^3. This is in conflict with our previous observation. Hence, there is no strategyproof SCF for odd $n \geq 5$ that satisfies non-imposition and the Condorcet loser property.

Case 2: n is even

As second case, we assume that f is defined for an even number of voters $n \geq 4$. In this case, we first prove the statement for $n = 4$ and generalize this result then to a larger number of voters.

Case 2.1: $n = 4$

Assume for contradiction that f is a strategyproof SCF for $n = 4$ voters and $m \geq 3$ alternatives that satisfies the Condorcet loser property and non-imposition. Consider the profile R^1 shown in the sequel. By Lemma 2, $f(R^1) = \{a\}$.

R^1: 1: a, c, X, b $\quad$ 2: a, b, X, c $\quad$ 3: a, b, X, c $\quad$ 4: b, X, c, a

Moreover, c is the Condorcet loser in R^1, even if voter 1 is indifferent between a and c. Thus, we replace the preference of voter 1 with $\{a, c\}, X + b$, where $X + b = X \cup \{b\}$, to derive the profile R^2. As consequence, $c \notin f(R^2)$ due to the Condorcet loser property, and strategyproofness implies that $f(R) \subseteq \{a, c\}$. Otherwise, an alternative in $X + b$ is chosen and voter 1 can manipulate by reverting back to R^1. Hence, we deduce that $f(R^2) = \{a\}$.

R^2: 1: $\{a, c\}, X{+}b$ $\quad$ 2: a, b, X, c $\quad$ 3: a, b, X, c $\quad$ 4: b, X, c, a

As next step, we let voter 2 change his preference to a, c, X, b and voter 4 change his preference to $c, X, \{a, b\}$ in order to make b the Condorcet loser. By applying these modifications subsequently, it follows from strategyproofness that the choice set does not change: otherwise, voter 2 can manipulate by undoing this step since a is his best alternative after the modification, or voter 4 can manipulate by applying the modification since a is his least preferred alternative in R^2. Hence, these steps result in the profile R^3 with $f(R^3) = \{a\}$.

R^3: 1: $\{a, c\}, X{+}b$ $\quad$ 2: a, c, X, b $\quad$ 3: a, b, X, c $\quad$ 4: $c, X, \{a, b\}$

Note that b is the Condorcet loser, even if voter 3 swaps a and b. Hence, we derive the profile R^4 with $b \notin f(R^4)$ and by strategyproofness, $f(R^4) = \{a\}$.

R^4: 1: $\{a, c\}, X{+}b$ $\quad$ 2: a, c, X, b $\quad$ 3: b, a, X, c $\quad$ 4: $c, X, \{a, b\}$

Now, we let voter 4 change his preference to c, b, X, a to derive the profile R^5. As $f(R^4) = \{a\}$ and a is among the least preferred alternatives of voter 4, it follows that $f(R^5) \subseteq \{a, b\}$. Otherwise, voter 4 can manipulate by applying this modification.

R^5:	1: $\{a, c\}, X{+}b$	2: a, c, X, b	3: b, a, X, c	4: c, b, X, a

We can apply the same steps for profiles symmetric with respect to the voters or alternatives. Thus, we can infer the choice sets for the profiles R^6, R^7, and R^8 as $f(R^6) \subseteq \{a, c\}$, $f(R^7) \subseteq \{a, b\}$, and $f(R^8) \subseteq \{b, c\}$.

R^6:	1: $\{b, c\}, X{+}a$	2: a, c, X, b	3: b, a, X, c	4: c, b, X, a
R^7:	1: a, b, X, c	2: c, a, X, b	3: $\{b, c\}, X{+}a$	4: b, c, X, a
R^8:	1: a, b, X, c	2: c, a, X, b	3: $\{a, c\}, X{+}b$	4: b, c, X, a

Note that if $b \in f(R^5)$, then voter 1 can manipulate by switching to R^6 as $f(R^6) \subseteq \{a, c\}$. Hence, we derive that $f(R^5) = \{a\}$. By a symmetric argument for R^7 and R^8, it follows that $f(R^7) = \{b\}$.

Finally, consider the profile R^9 shown in the sequel.

R^9:	1: a, b, X, c	2: $a, b, X, c,$	3: b, a, X, c	4: b, a, X, c

We can derive the profile R^9 from R^5 and R^7 by replacing the preferences of some voters. In more detail, we obtain R^9 from R^5 by replacing the preference of voters 1 and 2 with a, b, X, c and the preference of voter 4 with b, a, X, c. If we apply these steps one after another, strategyproofness entails that the choice set is not allowed to change. Hence, $f(R^9) = \{a\}$. Moreover, we obtain R^9 from R^7 by replacing the preferences of voters 3 and 4 with b, a, X, c and the preference of voter 2 with a, b, X, c and obtain $f(R^9) = \{b\}$, a contradiction. Therefore, f cannot exist and there is no strategyproof SCF that satisfies non-imposition and the Condorcet loser property if $n = 4$ and $m \geq 3$.

Case 2.2: $n > 4$

Our goal is to reduce the case with $n > 4$ voters to the case with $n = 4$ voters. Hence, assume that there is a strategyproof SCF f for $n > 4$ voters, n even, and $m \geq 3$ alternatives that satisfies the Condorcet loser property and non-imposition. We use this SCF f to define another SCF g for $n = 4$ voters as follows: given a profile R on 4 voters, g adds $(n-4)/2$ voters with preference c, X, b, a and $(n-4)/2$ voters with preference a, b, X, c, where $X = A \setminus \{a, b, c\}$. Then, g returns the choice set of f on the resulting profile R', i.e., $g(R) = f(R')$. Subsequently, we prove that g satisfies all criteria required for case 2.1. As a consequence, g cannot exist, which implies that f also violates one of the required axioms. It should be mentioned here that the proof for $n = 4$ also works with slightly weaker properties: instead of the full power of Lemma 2, it suffices that this lemma applies for three alternatives a, b, c.

First, note that g is trivially strategyproof as f is strategyproof. The reason for this is that any manipulation for g would also be a manipulation for f because g always adds the same voters before calling f. Moreover, g cannot return a Condorcet loser because a Condorcet loser in a profile R on 4 voters is also a Condorcet loser in the profile R' that is obtained after g adds the $n-4$ extra voters. The reason for this is that the preference of the first half of these $n-4$ voters is inverse to the other half. In more detail, adding these $n-4$ voters increases every support $s_{xy}(R)$ by $(n-4)/2$ if $x \in \{a, b, c\}$ or $y \in \{a, b, c\}$ and the supports $s_{xy}(R)$ with $x, y \in X$ do not change at all. Consequently, the Condorcet loser does not change and g inherits the Condorcet loser property from f.

Finally, we need to show that g returns a, b, and c uniquely if three out of the four voters uniquely top-rank one of these alternatives. For a and c, this follows from Lemma 2 because g adds $(n-4)/2$ voters with preference a, b, X, c and $(n-4)/2$ voters with preference c, X, b, a to derive the input profile R' for f. Hence, if at least three voters agree that $x \in \{a, c\}$ is the uniquely best choice in g's input profile R, $n/2+1$ voters name this alternative as uniquely best choice in R'. As f satisfies all requirements of Lemma 2, it holds that $f(R') = \{x\}$ and thus $g(R) = \{x\}$ by definition.

A slightly more complicated argument is required for showing that g returns b if three voters prefer it uniquely the most. In more detail, consider the profile R shown in the sequel and note that Lemma 2 implies that $f(R) = \{b\}$. It should be mentioned here that the exact ordering of the first four voters is not important for the argument, i.e., we can apply the constructions regardless of which voters in $[1 \dots 4]$ prefer b the most.

R: $[1\dots3]$: b, X, c, a 4: a, c, X, b $[5\dots2{+}n/2]$: b, a, X, c $[3{+}n/2\dots n]$: c, X, b, a

Note that a is uniquely bottom-ranked by all voters in $[1\dots3]\cup[3{+}n/2\dots n]$ and thus, a is the Condorcet loser. This is also true if the voters $i \in [5\dots2{+}n/2]$ swap a and b one after another. Hence, the Condorcet loser property implies that a is not chosen after these swaps and strategyproofness entails then that b is still the unique winner since all voters in $[5\dots2{+}n/2]$ prefer $\{b\}$ to every other subset of $A \setminus \{a\}$. This means that $f(R') = \{b\}$, where the profile R' is displayed in the sequel.

R': $[1\dots3]$: b, X, c, a 4: a, c, X, b $[5\dots2{+}n/2]$: a, b, X, c $[3{+}n/2\dots n]$: c, X, b, a

In the profile R', the preferences of the last $n{-}4$ voters are equal to those used by g to pad up profiles of size 4 to size n. Hence, $g(R'') = f(R') = \{b\}$, where $R'' = (R'_1, R'_2, R'_3, R'_4)$. Moreover, the voters $i \in [1\dots3]$ can reorder all alternatives in $A \setminus \{b\}$ arbitrarily without affecting the choice set and voter 4 can reorder all alternatives arbitrarily without affecting the choice set because of strategyproofness. As the argument also holds if we reorder the first four voters, b is indeed the unique winner of g if three voters uniquely top-rank it.

Hence, g satisfies all axioms required by the case that $n = 4$, which means that g cannot exist. On the other hand, we have shown that the existence of a strategyproof SCF for an even number of voters $n > 4$ that satisfies the Condorcet loser property and non-imposition entails the existence of g. By the contraposition of this implication, the impossibility generalizes to every even number of voters $n > 4$. □

Appendix C: Additional Results

In this appendix, we discuss some of the claims mentioned in the remarks of the main part. We start with Remark 3 concerning rank-based SCFs and show that Theorem 1 also holds if $m \geq 5$ and $n = 2$.

Theorem 4. *There is no rank-based SCF that satisfies Pareto-optimality and strategyproofness if $m \geq 5$ and $n = 2$.*

Proof. Assume for contradiction that there is a rank-based SDS f that satisfies Pareto-optimality and strategyproofness if $m \geq 5$ and $n = 2$. Moreover, consider the profiles R^1, R^2, and R^3, where $X = A \setminus \{a, b, c, d, e\}$.

R^1: 1: $\{a, b\}, e, \{c, d\}, X$ 2: $\{c, d\}, a, \{b, e\}, X$

R^2: 1: $\{a, c\}, e, \{b, d\}, X$ 2: $\{b, d\}, a, \{c, e\}, X$

R^3: 1: $\{a, d\}, e, \{b, c\}, X$ 2: $\{b, c\}, a, \{d, e\}, X$

Note that $r^*(R^1) = r^*(R^2) = r^*(R^3)$ and thus, rank-basedness requires that $f(R^1) = f(R^2) = f(R^3)$. Moreover, a Pareto-dominates b in R^1, c in R^2, d in R^3, and all other alternatives in all three profiles. It follows that $f(R^1) = f(R^2) = f(R^3) = \{a\}$ and voter 2 is therefore no nominator for f. Hence, Lemma 1 entails that a single voter can decide the outcome by top-ranking a single alternative. However, this is a contradiction because, if voter 1 top-ranks a uniquely and voter b top-ranks b uniquely, both alternatives must be the unique winner. Hence, we have derived a contraction. $\square$

Next, we consider the claim in Remark 11 stating that majority-based SCFs cannot satisfy strategyproofness and non-imposition simultaneously. Recall that an SCF is majority-based if $f(R) = f(R')$ for all profiles R, R' with the same majority relation $R_M = \{(x, y) \in A^2 : n_{xy}(R) \geq n_{yx}(R)\}$.

Theorem 5. *There is no majority-based SCF that satisfies non-imposition and strategyproofness if $m \geq 3$ and $n \geq 3$.*

Proof. Assume for contradiction that there is a majority-based SCF f that satisfies non-imposition and strategyproofness for $m \geq 3$ and $n \geq 3$. As first step, we show that f satisfies Condorcet-consistency. Hence, choose an arbitrary alternative and consider a profile R such that $f(R) = \{a\}$. Note that such a profile exists by non-imposition. Next, consider the sequence of profiles R^0 to R^n such that $R^0 = R$ and R^i differs from R^{i-1} only in the fact that voter i uniquely top-ranks a in R^i. Strategyproofness implies that $f(R^i) = \{a\}$ if $f(R^{i-1}) = \{a\}$ as otherwise, voter i can manipulate by switching from R^i to R^{i-1}. Hence, it follows that $f(R^n) = \{a\}$. Finally, strategyproofness implies that all voters can reorder all alternatives in $A \setminus \{a\}$ in R^n without affecting the choice set. Since f is majority-based, it follows that a is the unique winner under all majority relations having a as Condorcet winner. Hence, f is Condorcet-consistent.

As a consequence of this observation, it follows that $l = \lceil \frac{n+1}{2} \rceil$ voters can ensure that $f(R) = \{x\}$ if they all top-rank x in R. We use this fact to derive that $f(R^1) = \{a\}$ for the profile R^1, where $X = A \setminus \{a, b, c\}$.

R^1: 1: c, X, b, a $[2 \dots l]$: a, b, c, X $[l{+}1 \dots n]$: a, c, X, b

As next step, we let every voter $i \in [2 \dots l]$ change their preference sequentially to $\{a, b\}, c, X$. This leads to the profile R^2 and strategyproofness implies that $f(R^2) \subseteq \{a, b\}$; otherwise there is a voter who can manipulate by reversing this modification.

R^2: 1: c, X, b, a $[2 \dots l]$: $\{a, b\}, c, X$ $[l{+}1 \dots n]$: a, c, X, b

In the sequel, we use the same idea as in the proof of Theorem 2: we let voter 1 and 2 change their preferences over a and b. This results in the profile R^3 and majority-basedness implies that $f(R^3) = f(R^2) \subseteq \{a, b\}$ as this step does not affect the majority relation.

R^3: 1: $c, X, \{a, b\}$ 2: b, a, c, X $[3 \dots l]$: $\{a, b\}, c, X$ $[l{+}1 \dots n]$: a, c, X, b

Note that $f(R^3)$ is a subset of the least preferred alternatives of voter 1. Thus, strategyproofness implies that voter 1 cannot make any other alternative but a and b win by lying about his preference as he could manipulate otherwise. As consequence, $f(R^4) \subseteq \{a, b\}$, where R^4 is shown in the sequel.

R^4: 1: c, X, b, a 2: b, a, c, X $[3 \dots l]$: $\{a, b\}, c, X$ $[l{+}1 \dots n]$: a, c, X, b

As voter 1 prefers b to a in R^4 after these steps, we can repeat them with every voter in $[3 \dots l]$. This leads to the profile R^5 and the fact that $f(R^5) \subseteq \{a, b\}$.

R^5: 1: c, X, b, a $[2 \dots l]$: b, a, c, X $[l{+}1 \dots n]$: a, c, X, b

Finally, consider the profile R^6 which differs from R^5 only in the fact that voter 1 reports b, c, X, a. It follows from Condorcet-consistency that $f(R^6) = \{b\}$.

R^6: 1: b, c, X, a $[2 \dots l]$: b, a, c, X $[l{+}1 \dots n]$: a, c, X, b

Observe that $f(R^6) = \{b\}$ and $f(R^5) \subseteq \{a, b\}$ imply that $f(R^5) = \{b\}$ as switching from R^5 to R^6 is a manipulation for voter 1, otherwise. This means that the voters $i \in [l{+}1 \dots n]$ receive their worst possible outcome for R^5, and hence they cannot affect the outcome by deviating because they prefer any other outcome. Moreover, the best alternatives of the voters $i \in [2 \dots l]$ is the unique winner and therefore, these voters can reorder all alternatives in $A \setminus \{b\}$ without affecting the outcome. This implies that $f(R^7) = \{b\}$ for the profile R^7 shown in the sequel.

R^7: 1: c, X, b, a $[2 \dots l]$: b, X, c, a $[l{+}1 \dots n]$: c, X, b, a

Finally, note that $f(R^7) = \{b\}$ is a contradiction. If n is odd, then c is the Condorcet winner in $f(R^7)$ and thus, Condorcet-consistency requires that $f(R^7) = \{c\}$. Moreover, if n is even, we can exchange the roles of b and c in the derivation of R^7 to derive that $f(R^7) = \{c\}$ must also be true. This is possible as b and c are symmetric to each other in R^7. As $f(R^7) = \{b\}$ and $f(R^7) = \{c\}$ cannot be true simultaneously, we also have a contradiction for even n. Hence, the initial assumption is wrong and no majority-based SCF exists that satisfies strategyproofness and non-imposition if $m \geq 3$ and $n \geq 3$. □

In the remainder of the section, we review some of the specific SCFs that we considered in our analysis of the tightness of the axioms in our results.

First, we deal with rank-basedness under strict preferences. Therefore, we consider the variant of the 2-plurality rule in Remark 7, which we call *2^*-plurality*. For introducing this rule, we define the plurality score $PL(a, R)$ of an alternative a in profile R as the number of voters that top-rank alternative a in the profile R. Given a profile R, let a_R denote the alternative with the second highest plurality score. Then, the 2^*-plurality rule, abbreviated by 2^*-$PL(R)$, chooses precisely all alternatives x with $PL(x, R) \geq PL(a_R, R)$ and $PL(x, R) > 0$, i.e., $2^*\text{-}PL(R) = \{x \in A \colon PL(x, R) \geq PL(a_R, R) \wedge PL(x, R) > 0\}$.

Proposition 1. *For strict preferences, the 2^*-plurality rule is rank-based, Pareto-optimal, and strategyproof, but no voter is a nominator if $m \geq 3$ and $n \geq 5$.*

Proof. First, note that 2^*-plurality is by definition rank-based and it satisfies Pareto-optimality as it only returns alternatives that are top-ranked by some voters. This criterion entails Pareto-optimality as we assume strict preferences. Moreover, no voter is a nominator for 2^*-plurality if there are at least 5 voters because the top-ranked alternative c of a voter can have plurality score 1 and two other alternatives may have plurality score 2. Hence, it only remains to show that it is strategyproof. We assume for contradiction that this is not the case, i.e., that there are preference profiles R and R' and a voter i such that $R_j = R'_j$ for all $j \in N \setminus \{i\}$ and $2^*\text{-}PL(R') \succ_i 2^*\text{-}PL(R)$. We proceed with a case distinction on whether voter i's most preferred alternative in R, denoted by a, is chosen.

First, assume that $a \in 2^*\text{-}PL(R)$. This means that voter i can only manipulate if $2^*\text{-}PL(R') = \{a\}$ as otherwise, there is an alternative $x \in 2^*\text{-}PL(R')$ with $a \succ_i x$. Moreover, if $2^*\text{-}PL(R) = \{a\}$, voter i can also not manipulate as his best alternative is the unique winner. Hence, another alternative b is chosen by 2^*-plurality, which implies that another voter reports b as his most preferred alternative in R. As consequence, $PL(b, R') > 0$ and therefore, $2^*\text{-}PL(R') \neq \{a\}$ as 2^*-plurality only returns a single winner if all voters report it as their best choice. Hence, no manipulation is possible in this case.

Next, assume that $a \notin 2^*\text{-}PL(R)$ and let b denote voter i's best alternative in R'. We proceed with another case distinction with respect to the plurality score of b in R. First, assume that b has the largest plurality score in R, i.e, $PL(b, R) \geq PL(c, R)$. This means that $PL(b, R') > PL(x, R')$ and consequently $b \neq a_{R'}$. Moreover, $a \neq a_{R'}$ because $a \notin 2^*\text{-}PL(R')$, which means that we can choose a_R such that $a_R = a_{R'}$. This implies that all $2^*\text{-}PL(R) = 2^*\text{-}PL(R')$ as all alternatives in R' have a larger plurality score than $a_{R'}$ if and only if they have already a larger plurality score than a_R in R.

As second case, assume that $PL(b, R) = PL(a_R, R)$ and that there is an alternative c with $PL(c, R) > PL(a_R, R)$. This assumption entails that $\{b, c\} \subseteq 2^*\text{-}PL(R)$. Moreover, we can derive that $PL(b, R') = PL(b, R) + 1 > PL(x, R')$ for all $x \in A \setminus \{b, c\}$ because

$PL(b, R) \geq PL(a_R, R) PL(x, R)$ for all these alternatives and voter i increases the plurality score of b. Furthermore, $PL(c, R') \geq PL(b, R')$ since c had a stricty larger plurality score than b in R. Thus, we deduce that $2^*\text{-}PL(R') = \{b, c\} \subseteq 2^*\text{-}PL(R)$. However, this is no manipulation for voter i as he is not indifferent between b and c, i.e., we can find alternatives $x \in 2^*\text{-}PL(R)$, $y \in 2^*\text{-}PL(R')$ such that $x \succ_i y$.

Finally, assume that $PL(b, R) < PL(a_R, R)$ and note that this assumption entails that there are at least two alternatives with a higher plurality score than a and b, i.e., $PL(a_R, R) > PL(a, R)$ and $PL(a_R, R) > PL(b, R)$. Hence, $PL(b, R') = PL(b, R) + 1 \leq PL(a_R, R)$. This means that $PL(a_R, R) = PL(a_{R'}, R')$ as b has a plurality score of at most $PL(a_R, R)$ and the plurality scores of all alternatives x with $PL(x, R) \geq PL(a_R, R)$ have not been affected by the manipulation. Consequently, it follows that every alternative chosen in $2^*\text{-}PL(R)$ is also chosen after the manipulation, i.e, $2^*\text{-}PL(R) \subseteq 2^*\text{-}PL(R')$. As $|2^*\text{-}PL(R)| \geq 2$ (because $|2^*\text{-}PL(R)| = 1$ only if an alternative is unanimously top-ranked), switching from R to R' is therefore no manipulation because we can find alternatives $x \in 2^*\text{-}PL(R)$, $y \in 2^*\text{-}PL(R) \subseteq \text{-}PL(R')$ such that $x \succ_i y$. Hence, no case allows for a manipulation, which means that 2^*-plurality is strategyproof for strict preferences. □

Next, we consider Remark 3 in which we claim that the bounds on n and m in Theorem 1 are tight as the Pareto rule is rank-based for small values of n and m. We prove this statement in the sequel.

Proposition 2. *The Pareto rule is rank-based, Pareto-optimal, and strategyproof if $m \leq 3$, or if $m \leq 4$ and $n \leq 2$.*

Proof. The Pareto rule is known to satisfy Pareto-optimality and strategyproofness, regardless of the number of alternatives or voters (see, e.g., Brandt et al., 2021). Hence, it only remains to show that it also satisfies rank-basedness under the restrictions on n and m. For $m = 1$, rank-basedness is obviously no restriction, and if $m = 2$, the rank vector of the single alternative determines all preferences except that we do not know which voter submits which alternatives. If an alternative a is uniquely top-ranked by a voter, its rank vector contains a $(0, 1)$ entry, and thus, the rank vector of the other alternative b must contain a $(1, 1)$ entry. Similarly, if a has a $(0, 2)$ entry, a voter is indifferent between both alternatives and thus, b has also a $(0, 2)$. Finally, we can apply a symmetric argument to the first case if the rank vector a contains a $(1, 1)$ entry, and thus, we can reconstruct a unique profile (up to renaming the voters) given a rank matrix. Hence, the Pareto rule is rank-based if $m = 2$.

Next, we focus on the case that $m = 3$ and consider an arbitrary rank matrix Q. First note that Q can only have the following entries: $(0, 3)$, $(0, 2)$, $(1, 2)$, $(0, 1)$, $(1, 1)$, and $(2, 1)$. Moreover, many of these entries specify the preferences of the voters. For instance, the $(0, 3)$ entry entails that a voter is completely indifferent between all alternatives. Consequently, we can focus on the rank vector of an alternative a, add a completely indifferent vector for every $(0, 3)$ entry and remove all these entries from Q afterwards. Also, the $(0, 2)$ entries in the rank vector of a specify a lot of information: there must be a voter who top-ranks a and another alternative x and bottom-ranks the last alternative y uniquely. We use this observation to formulate a system of linear equations. Let therefore n_a, n_b, and n_c denote the number of $(0, 2)$ entries in the rank vector of the respective alternatives. Moreover, let x_{ab}, x_{ac}, and x_{bc} denote the number of voters who top-rank both alternatives in the index. The following equations must hold for every profile R with $r^*(R) = Q$.

$$n_a = x_{ab} + x_{ac}$$
$$n_b = x_{ab} + x_{bc}$$
$$n_c = x_{bc} + x_{ac}$$

It can easily be checked that the unique solution of this system of equations is $x_{ab} = \frac{n_a+n_b-n_c}{2}$, $x_{bc} = \frac{n_b+n_c-n_a}{2}$, and $x_{ac} = \frac{n_a+n_c-n_b}{2}$. As this solution is unique, these entries determine some preferences uniquely. Moreover, note that a symmetric argument holds for all $(1,2)$ entries. Hence, we can now remove these entries from Q, as well as the corresponding $(2,1)$ and $(0,1)$ entries, to find a simpler rank matrix.

After the last step, Q only consists of $(0,1)$, $(1,1)$, and $(2,1)$ entries, which means that all remaining preferences are strict. If there are no such entries, we can check Pareto dominance with the derived preference profiles. Otherwise, these entries do not necessarily entail a unique profile, but we can use all our observations so far to check for an arbitrary pair of alternatives a and b, whether alternative a Pareto-dominates alternative b. For this, we first construct the preferences involving ties as explained before and check whether one of the voters thus far prefers b strictly to a. If this is the case, a cannot Pareto-dominate b and we are done. Otherwise, we consider the remaining entries in Q. We claim that a Pareto-dominates b in these preferences if and only if a has no $(2,1)$ entry and b has no $(0,1)$ entry. If a has an $(2,1)$ entry or b has an $(0,1)$ entry, then a is uniquely last-ranked or b is uniquely top-ranked by some voter. Conversely, if none of these entries exist, then b has to be last-ranked whenever a is second-ranked, and we derive Pareto dominance of b by a (note that there is at least one strict comparison because we assumed that the profile was not determined, yet). Since a and b were chosen arbitrary, we can check whether an alternative Pareto-dominates another alternative only based on the rank matrix if $m = 3$, i.e., the Pareto rule is rank-based in this case.

Finally, we show that the Pareto rule is also rank-based if $m = 4$ and $n = 2$. In this case, we use a different proof strategy by showing that if an alternative a is Pareto-dominated in a profile R, it is Pareto-dominated in every Profile R' with $r^*(R) = r^*(R')$. Given a rank matrix Q, we can therefore compute the Pareto rule on an arbitrary profile R with $r^*(R) = Q$ as the outcome is independent of the exact choice of R. Next, we consider the profile R in detail: as a is Pareto-dominated, there is an alternative b such that $b \succsim_i a$ for all $i \in \{1,2\}$ and this preference is strict for at least one of the voters. Let $(s_{xi}, t_{xi}) = r(R_i, x)$ denote the rank tuple of alternative x in the preference of voter i. We assume in the sequel without loss of generality that $s_{b1} \leq s_{b2}$; otherwise just exchange the voters' preferences. Note that if $\min_{i\in\{1,2\}} s_{ai} \geq s_{b2}$, then b Pareto-dominates a in all profiles R' with $r^*(R') = r^*(R)$. The reason for this is that if $s_{xi} \leq x_{yi}$, then $x \succsim_i y$. Hence, we focus on the case that $s_{a1} < s_{b2}$ ($s_{a2} < s_{b2}$ is impossible because b Pareto-dominates a). Next, consider a profile R' with $r^*(R') = r^*(R)$ such that $a \succ'_i b$ for some voter $i \in \{1,2\}$. If no such profile R' exists, we are done as b Pareto-dominates a in every profile R' with $r^*(R) = r^*(R')$. Moreover, we assume without loss of generality that $r(R_1, b) = r(R'_1, b)$ and $r(R_2, b) = r(R'_2, b)$ because we can just reorder the voters otherwise. It follows from this assumption that $a \succ'_2 b$ because $s_{b1} = \min(s_{b1}, s_{b2}, s_{a1}, s_{a2})$. Next note that a cannot be uniquely top-ranked in R since it is Pareto-dominated, which means that there must be another alternative c with $c \succsim'_2 a$ in R'. This implies that voter 2 prefers two alternatives strictly to b. Because of $r(R_2, b) = r(R'_2, b)$, voter 2 also prefers two alternatives strictly to b in R and, as b Pareto-dominates a and $m = 4$, a is among the least preferred alternatives of voter 2 in R. Hence, we derive from $r^*(R) = r^*(R')$ that a is among the least preferred alternatives of voter 1 in R'. Finally, note that at least one voter prefers b strictly to a in R. Hence, either a is uniquely bottom-ranked by voter 2 in R or it is not among the most preferred alternatives of voter 1. This means that, in every preference profile R' with $r^*(R') = r^*(R)$, a is Pareto-dominated by an alternative c that is among the most preferred alternatives of the voter who does not bottom-rank a. As consequence, if an alternative is Pareto-dominated in a profile R, it is also Pareto-dominated in every profile R' with $r^*(R) = r^*(R')$ and thus, we can compute the Pareto rule based on the rank matrix Q. $\square$

As last result, we discuss the SCF f^* that satisfies Pareto-optimality and strategyproofness but violates support-basedness and that every voter is a nominator. As described in Remark 8, this SCF chooses the maximal alternatives of a transitive dominance relation which slightly strengthens Pareto-dominance. In more detail, we say that an alternative a *dominates* alternative b in a profile R if a Pareto-dominates b or $n-1$ voters prefer a the most while $s_{ab}(R) \geq 2$ and $s_{ba}(R) = 1$. It should be stressed that it is not required that a is uniquely top-ranked by $n-1$ voters, but only that it is among their best alternatives. Subsequently, we show that f^* satisfies all axioms that we claim.

Proposition 3. *The SCF f^* satisfies Pareto-optimality and strategyproofness but violates support-basedness and that every voter is a nominator if $n \geq 3$.*

Proof. Before discussing the axioms, we first show that f^* is a well-defined SCFs by proving that it chooses the maximal elements of a transitive dominance relation. Hence, consider an arbitrary profile R and three alternatives a, b, c and assume that a dominates b and b dominates c. As there are two possibilities on how an alternative dominates another one (i.e., a either Pareto-dominates b, or $s_{ab}(R) \geq 2$, $s_{ba}(R) = 1$, and $n-1$ voter top-rank a), we proceed with a case distinction with respect to the dominance relations between a and b and between b and c. First, consider the case that a Pareto-dominates b and b Pareto-dominates c. Then, a Pareto-dominates c as the Pareto-dominance relation is transitive and thus, transitivity is in this case satisfied.

Next, consider the case that a Pareto-dominates b and b dominates c because $s_{bc}(R) \geq 2$, $s_{cb}(R) = 1$, and $n-1$ voter top-rank b. As every voter prefers a (weakly) to b, it follows that $s_{ac}(R) \geq s_{bc}(R) \geq 2$, $s_{ca}(R) \leq s_{cb}(R) = 1$ and that $n-1$ voters top-rank a. Hence, a either Pareto-dominates c if $s_{ca}(R) = 0$ or satisfies the second dominance criterion if $s_{ca}(R) = 1$. This means that the dominance relation is also in this case transitive.

As third case, assume that b Pareto-dominates c, and that $s_{ab}(R) \geq 2$, $s_{ba}(R) = 1$, and $n-1$ voters top-rank a. As b Pareto-dominates c, it follows that $s_{ac}(R) \geq s_{ab}(R) \geq 2$ and $s_{ca}(R) \leq s_{ba}(R) = 1$. Hence, transitivity is also in this case satisfied.

Finally, assume that neither a Pareto-dominates b nor b Pareto-dominates c, but a dominates b and b dominates c. Consequently, we derive that both a and b are top-ranked by $n-1$ voters. However, this means that at most a single voter prefers a strictly to b and thus, $s_{ab}(R) \leq 1$. This contradicts that a dominates b and thus, this case cannot occur. Hence, the resulting dominance relation is transitive and f^* is a well-defined SCF.

Next, note that f^* satisfies Pareto-optimality as it is defined by a dominance relation that refines Pareto-dominance. Moreover, no voter is a nominator for f^* because $f^*(R) = \{a\}$ for all profiles R in which $n-1$ voters report a as their uniquely best alternative. The SCF f^* is also not support-based. To this end, consider the profiles R^1 and R^2, where $X = A \setminus \{a, b, c\}$, and note that $f(R^1) = \{a\} \neq \{a, b, c\} = f^*(R^2)$ even though $s^*(R^1) = s^*(R^2)$.

R^1:	1: c, b, a, X	2: a, b, c, X	$[3 \dots n]$: a, b, c, X
R^2:	1: c, a, b, X	2: b, a, c, X	$[3 \dots n]$: a, b, c, X

Finally, it remains to show that f^* is strategyproof. Assume for contradiction that this is not the case, i.e., there are preference profiles R and R' and a voter i such that $R_j = R'_j$ for all $j \in N \setminus \{i\}$ and $f^*(R') \succ_i f^*(R)$. Moreover, let $X_i^+(R)$ denote voter i's best alternatives in R. We proceed with a case distinction with respect to whether $X_i^+(R) \cap f^*(R)$ is empty or not. First, assume that $X_i^+(R) \cap f^*(R)$ is non-empty. This means that voter i can only manipulate by switching to R' if $f^*(R') \subseteq X_i^+(R)$ and $f^*(R) \not\subseteq X_i^+(R)$. As the dominance relation defining f^* is transitive, it follows that there are alternatives $x \in X_i^+(R)$, $y \in f^*(R) \setminus X_i^+(R)$ such that x dominates y in R' but not in R. However, this is not possible. If x does not Pareto-dominate y in R, there is a voter $j \neq i$ with $y \succ_j x$ and thus, x cannot Pareto-dominate y in R'. Furthermore, since $x \succ_i y$, it follows that $s_{xy}(R) \geq s_{xy}(R')$ and

$s_{yx}(R) \leq s_{yx}(R')$, and since $x \in X_i^+(R)$, voter i can also not increase the number of voters who top-rank x. Consequently, since x does not dominate y in R, it does not dominate y in R'. Hence, it follows from the transitivity of the dominance relation defining f^* that $f^*(R')$ cannot be a subset of $X_i^+(R)$ if $f(R) \not\subseteq X_i^+(R)$, which means that no manipulation is possible in this case.

Next, assume that $X_i^+(R) \cap f^*(R) = \emptyset$, i.e., none of voter i's best alternatives are chosen. As at least one of voter i's best alternatives is Pareto-optimal, it follows that there is a non-empty set of alternatives B such that all voters $j \in N \setminus \{i\}$ top-rank all alternatives in B. Moreover, let a denote one of voter i's most preferred alternatives in $f^*(R)$ and let b denote one of voter i's most preferred alternatives in B. Observe that all alternatives x with $b \succ_i x$ are Pareto-dominated by b because all voters but i top-rank b and thus, these alternatives are not in $f^*(R)$. Moreover, it holds that $b \in f^*(R)$. Indeed, it could only be Pareto-dominated by alternatives in B, but it is voter i's best alternative among these. Moreover, $s_{yb}(R) \leq 1$ for all $y \in A$ because $n-1$ voters top-rank b and hence, it is not dominated.

As next step, we show that for all alternatives $y \in A$ with $y \succ_i a$, there is an alternative $z \in B$ such that $s_{zy}(R) \geq 2$ and $s_{yz}(R) \leq 1$. Assume that this is not the case and let $c \in A \setminus f^*(R)$ with $c \succ_i a$ such that $s_{xc}(R) \leq 1$ for all $x \in B$ ($s_{cx}(R) \leq 1$ must be true for all $x \in B$ since $n-1$ voters top-rank these alternatives). As $c \notin f^*(R)$, it is Pareto-dominated by an alternative d because $s_{xc}(R) \leq 1$ for all $x \in B$ by assumption and none of voter i's best alternatives is top-ranked by $n-1$ voters as otherwise $f^*(R) \cap X_i^+(R) \neq \emptyset$. As consequence, we derive that $s_{xd}(R) \leq s_{xc}(R) \leq 1$ for all $x \in B$. Moreover, Pareto-dominance entails that $d \succsim_i c \succ_i a$ and thus, $d \notin f^*(R)$. Hence, d must also be Pareto-dominated, and we can repeat the argument. As the Pareto-dominance relation is transitive, this process results eventually at a Pareto-optimal alternative e with $s_{xe}(R) \leq 1$ for all $x \in B$ and $e \succ_i a$. However, this means that $e \in f^*(R)$ contradicting that a is one of voter i's best alternative in $f^*(R)$. Hence, we derived a contradiction and it holds for all alternatives $y \in A$ with $y \succ_i a$ that there is an alternative $z \in B$ such that $s_{zy}(R) \geq 2$ and $s_{yz}(R) \leq 1$. As voter i prefers all such alternatives y to all alternatives $z \in B$ because $y \succ_i a \succsim_i b \succsim_i z$, it follows that $s_{zy}(R') \geq 2$ and $s_{yz}(R') \leq 1$. Since all voters in $N \setminus \{i\}$ top-rank all alternatives in B also in R', we derive that no alternative y with $y \succ_i a$ can be element of $f^*(R')$.

As consequence, voter i can only manipulate by switching from R to R' if $x \sim_i a$ for all $x \in f^*(R')$ and there is an alternative $y \in f^*(R)$ with $a \succ_i y$. The latter observation implies that $a \succ_i b$ (because otherwise $a \sim_i b$, but alternatives y with $b \succ_i y$ are Pareto-dominated and cannot be in $f^*(R)$). By the choice of b, $a \succ_i x$ for all $x \in B$. Finally, we show that $B \cap f^*(R') \neq \emptyset$, which entails that voter i cannot manipulate. Note for this that all alternatives in B are also in R' top-ranked by $n-1$ voters and thus $s_{xy}(R') \leq 1$ for all $x \in A$, $y \in B$. This means that an alternative $x \in B$ is only not chosen in $f^*(R')$ if it is Pareto-dominated. However, an alternative $x \in B$ can only be Pareto-dominated by another alternative in B as for any alternative $y \in A \setminus B$, there is a voter $j \in N \setminus \{i\}$ such that $x \succ_i y$; otherwise, all voters in $N \setminus \{i\}$ top-rank y implying that $y \in B$. Finally, as the Pareto-dominance relation is transitive, it follows that there is a Pareto-optimal alternative in B, and thus, $B \cap f^*(R') \neq \emptyset$. Altogether, f^* is strategyproof. □

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