Worst Case in Voting and Bargaining

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Abstract

The guarantee of an anonymous mechanism is the worst case welfare an agent can secure against unanimously adversarial others. How high can such a guarantee be, and what type of mechanism achieves it?

We address the worst case design question in the n-person probabilistic voting/bargaining model with p deterministic outcomes. If $n \geq p$ the uniform lottery is the only maximal (unimprovable) guarantee; there are many more if $p > n$, in particular the ones inspired by the random dictator mechanism and by voting by veto.

If $n = 2$ the maximal set $\mathcal{M}(n,p)$ is a simple polytope where each vertex combines a round of vetoes with one of random dictatorship. For $p > n \geq 3$, writing $d = \lfloor \frac{p-1}{n} \rfloor$, we show that the dual veto and random dictator guarantees, together with the uniform one, are the building blocks of 2^d simplices of dimension d in $\mathcal{M}(n,p)$. Their vertices are guarantees easy to interpret and implement. The set $\mathcal{M}(n,p)$ may contain other guarantees as well; what we can say in full generality is that it is a finite union of polytopes, all sharing the uniform guarantee.

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1 Guarantees and protocols

Worst case analysis is a simple mechanism design question. Fix an arbitrary collective decision problem by its feasible outcomes – allocation of resources, public decision making, etc.. –, the domain of individual preferences and the number n of relevant agents. We evaluate a mechanism (game form) solving this problem by the *guarantee* it offers to the participants. This is the welfare level each one can secure in this game form without any prior knowledge of how others will play their part: the worst case assumption is that their moves are collectively adversarial (what I know or believe about their preferences, what I expect about their behaviour is irrelevant). My guarantee is the value of the two-person zero-sum game pitting me against the rest of the world.

Given the decision problem, what guarantees can any mechanism offer, and which mechanisms implement such guarantees? We are particularly interested in the *maximal* guarantees, those that cannot be improved: a higher guarantee is a better default option if I am clueless about other participants or unwilling to engage in risky strategic moves; it encourages acceptance of and participation in the mechanism.

These questions were first addressed by the cake-cutting literature ([4], [1], [3]). Two agents divide a cake over which their utilities are additive and non atomic. In the Divide and Choose mechanism (D&C for short) they each can guarantee a share worth 1/2 of the whole cake: the Divider must cut the cake in two parts of equal worth, any other move is at her own risk. This guarantee is not only maximal but also optimal (higher than any other feasible guarantee): when the two agents have identical preferences, their common guarantee cannot be worth more than 1/2 of the cake.

Contrast D&C with the simple Nash demand game: each agent claims a share of the cake, demands are met if they are compatible, otherwise nobody gets any cake. Against an adversarial player I will not get anything: this mechanism offers no guarantee at all. The appeal of D&C is that I cannot be tricked to accept a share worth less than 1/2.

Worst case analysis is related to the familiar implementation methodology in mechanism design, but only loosely. We speak of a mechanism *implementing* a certain guarantee – mapping my preferences to a certain welfare level – but we do not postulate that each agent behaves under the worst case assumption, nor do we ask what social choice function will then be realised, as some papers reviewed in section 2 did. Instead the guarantee of a game form is just one of its features, an important one for two reasons:

- an agent using a best reply to the other agents' strategies gets at least her guaranteed welfare (because she has a safe strategy achieving that level no matter what); so any Nash equilibrium of the game delivers at least the guaranteed welfare to everyone,
- if an agent plays the mechanism repeatedly with changing sets of participants, the safe strategy is always available when she happens to be clueless about the other agents' behaviour,

Many different mechanisms can implement the same guarantee, as the example below makes clear.[1] We call the whole class of game forms sharing a certain guarantee the *protocol* implementing it. The two concepts of guarantee and protocol and their relation is the object of worst case analysis.

We initiate this approach in the probabilistic voting model, where the protocols we identify can be interpreted as the guidelines for a partially informal bargaining process. There are finitely many pure (deterministic) outcomes and we must choose a convex compromise (probabilistic or otherwise) between these. For tractability, we maintain a symmetric treatment of agents (Anonymity) and of outcomes (Neutrality). We find that, depending on the number n of agents and p of pure outcomes the set of maximal guarantees and their protocols can be either very simple and dull (when $n \geq p$, see below) or dauntingly complex.

A good starting point is the simple case of deterministic voting over p outcomes with ordinal preferences. An anonymous and neutral guarantee is a rank k from 1 to p (where rank 1 is the worst): it is feasible if for any preference profile there is at least one outcome ranked k or above by each voter. Suppose first $n \geq p$: at a profile where each outcome is the worst for some agent, the rank k must be 1, so the guarantee idea has no bite. Now if $n < p$ we can give to each voter the right to veto up to $d = \lfloor \frac{p-1}{n} \rfloor$ outcomes: this is feasible because $nd < p$, so the rank $k = d + 1$ is a feasible guarantee, clearly the best possible one. Worst case analysis' simple advice to a committee smaller than the number of outcomes it chooses from, is to distribute d veto rights to its members, not at all what a standard voting rule à la Condorcet or Borda does. However the corresponding protocol contains, inter alia, the following mechanisms: ask everyone to pick independently d outcomes to veto, then use any voting rule to pick among the remaining free outcomes, often more than $p - nd$ of them.

[1] Similarly we can implement the optimal "one half of the whole cake" guarantee by D&C or by any one of Dubins and Spanier' moving knife procedures ([1])

We allow compromises between the pure outcomes, interpreted as lotteries, time shares, or the division of a budget. Distributing veto tokens is a natural way to achieve a high guarantee, but there are others. The familiar random dictator[2] mechanism ([2]) with two voters implements the guarantee putting a 1/2 probability on both my first and worst ranks. And the uniform lottery over all ranks is yet another guarantee implemented by any mechanism where each participant has the right, at some stage of the game that could depend upon the agent, the play of the game etc.., to force the decision by flipping a fair coin between all outcomes.

Example: three agents, six outcomes The uniform guarantee $UNI(6)$ is the lottery $\lambda^{uni} = (\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$, where each rank is equally probable.

Distributing one veto token to each agent implements $\lambda^1 = (0, 1, 0, 0, 0, 0)$ (recall the first coordinate is the worst rank), as in the deterministic case. But λ^1 is not maximal: it is improved by making the protocol a bit more precise. After the veto tokens have been used, we can pick one of the remaining outcomes uniformly, or give the option to force this random choice to each agent. Then the rank distribution cannot be worse for anyone than $\lambda^{vt} = (0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0)$, because my worst case after the vetoing phase is that the two other agents killed my two best outcomes. And λ^{vt} stochastically dominates λ^1. We will use the notation $VT(3, 6)$ instead of λ^{vt} when it is important to specify n and p.

The random dictator mechanism between our three agents delivers the guarantee $\lambda^2 = (\frac{2}{3}, 0, 0, 0, 0, \frac{1}{3})$: my worst case is that the two other agents pick my worst outcome. Again λ^2 is not maximal, and improved by the following protocol: agents report (one of) their top outcome(s); if they all agree on a we choose a; if they each choose a different outcome, we pick one of them with uniform probability; but if the choices are a, a, b we randomize uniformly between a, b *and an arbitrary third outcome* c. This implements the correct random dictator guarantee $RD(3, 6) : \lambda^{rd} = (\frac{1}{3}, \frac{1}{3}, 0, 0, 0, \frac{1}{3})$, that stochastically dominates λ^2.

It is easy to check directly that $UNI(6)$, $VT(3, 6)$ and $RD(3, 6)$ are all maximal. For instance this follows for λ^{rd} and λ^{vt} by inspecting respectively the left or right profile of strict ordinal preferences

$\prec_1$	a	b	x	y	z	c	$\prec_1$	a	x	y	z	b	c
$\prec_2$	b	c	y	z	x	a	$\prec_2$	b	y	z	x	c	a
$\prec_3$	c	a	z	x	y	b	$\prec_3$	c	z	x	y	a	b

[2] Each agent has an equal chance to choose the final outcome.

(where agent 1's worst is a and best is c). At the left profile, to give a $\frac{1}{3}$ chance of their best outcome to all agents a protocol implementing λ^{rd} must pick a, b or c, each with probability $\frac{1}{3}$: then each agent experiences exactly the distribution λ^{rd} over her ranked outcomes, and no other lottery λ stochastically dominating λ^{rd} is a feasible guarantee at this profile. Similarly at the right profile, implementing λ^{vt} implies zero probability on a, b, c, and at most (hence exactly) $\frac{1}{3}$ on each of x, y and z. The symmetry of these two arguments is not a coincidence: a critical duality relation connects λ^{vt} and λ^{rd} (section 4).

What other guarantees are maximal for $n = 3, p = 6$? Convex combinations preserve feasibility but not maximality: for instance an equal chance of the protocols implementing $VT(3,6)$ and $RD(3,6)$ delivers the feasible guarantee $\frac{1}{2}\lambda^{rd} + \frac{1}{2}\lambda^{vt} = (\frac{1}{6}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}, 0, \frac{1}{6})$ which is dominated by λ^{uni}. But lotteries between $UNI(6)$ and $VT(3,6)$, or between $UNI(6)$ and $RD(3,6)$ are in fact maximal. Moreover for this choice of n and p, the maximal guarantees cover exactly the two intervals $[\lambda^{uni}, \lambda^{vt}]$ and $[\lambda^{uni}, \lambda^{rd}]$ (Theorem 1 in section 5).

The choice facing the worst case designer in this example is sharp, and its resolution is context dependent: the veto guarantee is a good fit when bargaining is about choosing an expensive infrastructure project, or a person to hold a position for life; the random dictator approach makes sense if we are dividing time between different activities, or choosing a pair of roman consuls; the uniform guarantee stands out if we value a disagreement outcome revealing no information about individual preferences.

Critical to their practical application, the protocols implementing UNI, VT and RD above rely on ordinal preferences only, as do the agents' safe action when they report which outcome(s) they veto, or which ones they prefer among those still in play.

The punchline Our results cast a new light on two familiar collective decison mechanisms, random dictator and voting by veto. Together and in combination with the uniform guarantee, they generate all maximal guarantees if $n = 2$, and essentially all of them again if $3 \leq n < p \leq 2n$. In the general case they can be sequentially combined to produce a very large subset of maximal guarantees.

1.1 Contents of the paper

After a review of the literature in section 2 we define in section 3 the concept of guarantee in three related models. In the first one, agents have ordinal preferences over the pure outcomes, and incomplete preferences over lotteries by stochastic dominance. In the second they have von Neuman Morgenstern (vNM) utilities over lotteries. In the third they have quasi-linear utilities over outcomes and money, and lotteries are replaced by cash compensations. A guarantee is a convex combination of the ranks 1 to p where rank 1 is the worst. It is feasible if at each profile of preferences, there is a lottery over pure outcomes, or a pure outcome and a balanced set of cash compensations in the quasi-linear model, that everyone weakly prefers to her guaranteed utility.

Lemma 1 shows that the three definitions are equivalent and that feasible guarantees cover a canonical polytope $\mathcal{G}(n,p)$ in the simplex with p ranked vertices. Its Corollary gives a compact though abstract characterisation of $\mathcal{G}(n,p)$.

Section 4 focuses on the subset $\mathcal{M}(n,p)$ of maximal guarantees, starting with a complete characterisation in two easy cases (section 4.1):

If $n \geq p$ the unique maximal guarantee is $UNI(p)$, dominating every other feasible guarantee (Proposition 1), so the worst case viewpoint tells us to allow each agent to force this canonical anonymous and neutral disagreement outcome. In every other case there are many more options.

If $n = 2 < p$ a guarantee λ is maximal if and only if it is symmetric with respect to the middle rank (Proposition 2). For instance $\mathcal{M}(2,6)$ is the convex hull of $\lambda^{rd} = (\frac{1}{2},0,0,0,0,\frac{1}{2})$ $(RD(2,6))$, $\lambda^{mix} = (0,\frac{1}{2},0,0,\frac{1}{2},0)$, and $\lambda^{vt} = (0,0,\frac{1}{2},\frac{1}{2},0,0)$ (two veto tokens per person). Here is the protocol for λ^{mix}: the agents veto one outcome each, then choose randomly a dictator (equivalently, we randomly give one veto token to one agent and four tokens to the other). Note that $UNI(p)$ is the center of the polytope $\mathcal{M}(2,p)$.

When $3 \leq n < p$, the structure of $\mathcal{M}(n,p)$ is much more complicated. Section 4.2 describes a critical duality property inside $\mathcal{M}(n,p)$, relating $VT(n,p)$ and $RD(n,p)$, while $UNI(p)$ is self-dual: Proposition 3. We define in section 4.3 the large set $\mathcal{C}(n,p)$ of *canonical guarantees*: for three or more agents they are a rich family of vertices of $\mathcal{M}(n,p)$. Their protocols combine up to d successive rounds (recall $d = \lfloor \frac{p-1}{n} \rfloor$) of either veto (one token each) or a (partial) random dictator.

Our first main result Theorem 1 in section 5.1, gives a fairly complete picture of all maximal guarantees with three or more agents and at most twice as many pure outcomes ($p \leq 2n \iff d = 1$). As long as $p \neq 2n-1$ and $(n,p) \neq (4,8),(5,10)$, they cover exactly the two intervals $[UNI(p), VT(n,p)]$ and $[UNI(p), RD(n,p)]$, as in the numerical example above. There are additional maximal guarantees when $p = 2n-1$ or $(n,p) = (4,8),(5,10)$, some of them described after the Theorem (Proposition 4).

In section 5.2 we turn to the general case $3 \leq n < p$ with no restrictions on d. The set $\mathcal{M}(n,p)$ is a union of polytopes (faces of $\mathcal{G}(n,p)$), of which $UNI(p)$ is always a vertex. Theorem 2 uses the canonical guarantees in $\mathcal{C}(n,p)$ to construct 2^d simplices of dimension d, one for each sequence Γ of length d in $\{VT, RD\}$: the vertices of such a simplex are lotteries in $\mathcal{C}(n,p)$ obtained from the d initial subsequences of Γ, plus $UNI(p)$. For instance the sequence $\Gamma = (VT, RD)$ gives the triangle in $\mathcal{C}(3,7)$ with vertices $UNI(p)$, $VT(3,7)$, and $\lambda = (0, \frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}, 0, 0)$ denoted $VT \otimes RD$; the latter is implemented by a first round of one veto each, followed by $RD(3,q)$ over the remaining q outcomes (four or more). This construction does not cover the entire set $\mathcal{M}(n,p)$ but delivers a large subset built from simple combinations of veto and random dictator steps.

Section 6 gathers some open questions and potential research directions. Several proofs are gathered in the Appendix, section 7.

References

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