The Single-Peaked Domain Revisited: A Simple Global Characterization[*]

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Abstract *It is proved that, among all restricted preference domains that guarantee consistency (i.e. transitivity) of pairwise majority voting, the single-peaked domain is the only minimally rich and connected domain that contains two completely reversed strict preference orders. It is argued that this result explains the predominant role of single-peakedness as a domain restriction in models of political economy and elsewhere. The main result has a number of corollaries, among them a dual characterization of the single-dipped domain; it also implies that a single-crossing ('order-restricted') domain can be minimally rich only if it is a subdomain of a single-peaked domain. The conclusions are robust as the results apply both to domains of strict and of weak preference orders, respectively.*

JEL Classification D71, C72

Keywords: Social choice, restricted domains, Condorcet domains, single-peakedness, single-dippedness, majority voting, single-crossing property.

1 Introduction

A subset of preference orders on a finite set of alternatives is called *single-peaked* if there exists a left-to-right arrangement of alternatives such that all upper contour sets are connected ('convex') with respect to the given left-to-right arrangement of alternatives. The celebrated median voter theorem of Black [1948] and Arrow [1951] states that the domain of all single-peaked linear orders with respect to a fixed underlying spectrum of alternatives form a 'Condorcet domain,' i.e. pairwise majority voting with an odd number of individuals each of whom has preferences from the given domain induces a transitive relation. Moreover, the domain of all single-peaked preferences is *minimally rich* in the sense that every alternative is on top of at least one preference ordering; it is *connected* in the sense that every two single-peaked orders can be obtained from each other by a sequence of transpositions of neighboring alternatives such that the resulting order remains single-peaked at each step; and it contains two *completely reversed* orders (namely, the order that has the left-most alternative at the top and the order that has the right-most alternative at the top).

This paper's main result (Theorem 1) shows that, conversely, every minimally rich and connected Condorcet domain which contains at least one pair of completely reversed orders must be single-peaked.[1] As is easily seen, any single-peaked domain contains at most *one* pair of completely reversed orders. We thus obtain as a corollary that, for any given pair of completely reversed orders, there is a unique *maximal* Condorcet domain that contains them and is minimally rich as well as connected: the domain of *all* orders that are single-peaked with respect to either one of the given pair of completely reversed orders (Corollary 1).

This result is remarkable in particular in view of the fact that quite a number of non-single-peaked Condorcet domains have been identified in the literature, among others the domains satisfying Sen's 'value restriction' condition (Sen [1966]) with the 'single-dipped' domain (Inada [1964]) as a special case, the domains satisfying the so-called 'intermediateness' property (Grandmont [1978], Demange [2012]), and the 'order-restricted' domains identified by Rothstein [1990]; the latter domains are sometimes also referred to as the domains with the *single-crossing property* (Gans and Smart [1996], Saporiti [2009], Puppe and Slinko [2015]). Our analysis shows that none of these domains can jointly satisfy the three conditions of minimal richness, connectedness and the inclusion of a pair of completely reversed orders unless it is *also* single-peaked. In particular, a single-crossing domain can be minimally rich only if it is at the same time single-peaked (Corollary 3), see also Elkind et al. [2014].

The purpose of the present analysis is not to justify the assumption of single-peakedness *per se* and, in fact, the empirical evidence for single-peakedness is mixed, see the review of the literature below. The main argument put forward here is that, among all domains that guarantee consistency of pairwise majority voting, the single-peaked domain is distinguished by a remarkably simple set of additional requirements: connectedness, minimal richness and the existence of two completely reversed orders. The main conclusion to be drawn from the present analysis is therefore that, *if* a modeler wishes to guarantee transitivity of the majority relation for any possible profile of agents' preferences, *then* the assumption of single-peakedness follows very naturally. In this sense, the present study may thus be interpreted as a *conditional defense* of single-peakedness.[2]

The results presented here are robust as they generalize with some additional work, but using the same underlying logic, to the case of weak preference orders, i.e. to the case in which individual preferences may display indifferences. In this case, the domain of *all* (weakly) single-peaked weak orders does not form a Condorcet domain in our sense since, even with an odd number of voters, the indifference relation corresponding to pairwise majority voting may be intransitive. Moreover, the notion of connectedness has to be suitably adapted since two 'neighboring' weak orders may differ in the ranking of more than one pair of alternatives if one of these orders displays an indifference class with more than two elements. Nevertheless, we still obtain that any connected, minimally rich Condorcet domain that contains (at least) two completely reversed strict orders must be single-peaked with respect to either one of the pair

[1] In fact, as detailed in Section 2 below, the condition of connectedness can be substantially relaxed in this result to the condition that there exist *one* path that connects a pair of completely reversed orders.

[2] Of course, by contraposition, the same argument transforms potential doubts about the validity of single-peakedness in specific contexts into corresponding doubts on the existence of consistent pairwise majorities at all in these contexts.

of completely reversed orders. The details of how this is achieved are, however, left to the full paper version (see http://micro.econ.kit.edu/downloads/Charact-SP.pdf).

The above characterization result of the single-peaked domain implies a 'dual' characterization result for the *single-dipped* domain in a straightforward way, both in the case of linear and and in the case of weak orders. Specifically, any connected Condorcet domain containing two completely reversed orders such that every alternative is the least preferred alternative for some order in the domain must be single-dipped (see Theorem 2).

Relation to the literature

The literature on single-peaked preferences is abundant both in economics and political science. Their application ranges from the Hotelling-Downs model of political competition to models of local public good provision (for a modern treatment see, e.g., Austen-Smith and Banks [1998]). It is well-known that the assumption of single-peakedness enables possibility results both in the theory of preference aggregation (Black [1948], Arrow [1951]) and in the theory of strategy-proof social choice (Moulin [1980]). Moreover, it has frequently been argued that the assumption of single-peakedness is reasonable in contexts in which alternatives are naturally arranged according to an exogenous one-dimensional scale, e.g., in terms of political views on a left-to-right spectrum, or in terms objective distance, temperature, etc. The empirical evidence on single-peakedness is mixed. Some authors have argued that a tendency towards single-peakedness may be assumed under certain circumstances, in particular when there is repeated interaction and/or 'deliberation' (see, e.g., Spector [2000], DeMarzo et al. [2003], List et al. [2013]). Others have cast doubt on the applicability of single-peakedness, in particular in cases where compromises are difficult to reach and a search for them threatens to lead to a deadlock (Egan [2014]).

The paper in the literature that is closest to the present analysis is Ballester and Haeringer [2011]. These authors also provide an axiomatic characterization of the single-peaked domain (though only in the case of linear orders).[3] However, the conditions employed by these authors are very different in character from the ones used here. Specifically, Ballester and Haeringer [2011] use two families of conditions. The first is the condition that among every triple of alternatives their should be at least one that is never the worst of the three for any voter. This is one of Sen's family of 'value restrictions' (Sen [1966]), and it evidently amounts to assuming single-peakedness on all triples, hence it directly implies transitivity of the majority relation. However, it is also known that single-peakedness on all triples ('local single-peakedness') is not sufficient to guarantee single-peakedness globally (cf. Inada [1964]). Therefore, additional conditions are needed to characterize the single-peaked domain. The important contribution of Ballester and Haeringer [2011] is to show that the absence of a certain preference constellation on all quadruples of alternatives does the job. By contrast, the present analysis only assumes transitivity of the majority relation corresponding to every profile with an odd number of voters, and derives the single-peakedness of the domain from the three conditions of connectedness, minimal richness and the existence of two completely reversed orders (either of which then represents the underlying left-to-right spectrum). The latter three conditions are *global* properties of a domain, hence the title of this paper. By contrast, all conditions used by Ballester and Haeringer [2011] are 'local' conditions as they apply simultaneously either to all triples or to all quadruples of alternatives. While the conditions of the present analysis can be suitably adapted to yield a corresponding characterization of the weakly single-peaked domain, it is not obvious how to appropriately formulate Ballester's and Haeringer's local conditions in the case of weak orders.

The present study also informs the literature that aims at identifying 'large' Condorcet domains, see the excellent survey Monjardet [2009] and the more recent work on this topic by Danilov et al. [2012], Danilov and Koshevoy [2013]. Indeed, our main result suggests that Condorcet domains with the maximal *number* of elements on a given set of alternatives, the so-called *maximum* Condorcet domains, are most likely not minimally rich. This may seem particularly surprising, as it is known that that the cardinality of a maximum Condorcet domain exceeds the cardinality of the domain of all single-peaked domain considerably; for instance, with n alternatives a single-peaked domain has at most 2^{n-1} elements,

[3] Strictly speaking, Ballester and Haeringer [2011] characterize the domain of all single-peaked *profiles* which is a slightly different task.

while the cardinalities of the maximum Condorcet domains are 9, 20, 45, for $n = 4, 5, 6$, respectively, and these numbers are all attained by connected Condorcet domains containing a pair of completely reversed orders. The general structure and precise cardinality of the maximum Condorcet domains for larger n is unknown. However, it is known that the largest cardinality of a connected Condorcet domain on n alternatives that contains two completely reversed orders always exceeds 2^{n-1}, see Fishburn [1997]. By this paper's main result, the corresponding domains can never be minimally rich.

We finally emphasize the implications of our analysis for the literature on the single-crossing property. In the present finite framework, a domain is said to have the single-crossing property if the agents can be arranged in a fixed linear order such that, for every pair of alternatives, if two voters prefer one alternative to the other, then so do all agents that are between them in the given linear order of voters.[4] It has frequently been noted that single-peakedness and the single-crossing property in this sense are logically independent conditions, see, e.g., Saporiti [2009]. However, since every single-crossing domain can be extended to a connected single-crossing domain containing two completely reversed orders, our main result yields as a corollary that all minimally rich single-crossing domains must *also* be single-peaked, i.e. a subset of the domain of all single-peaked orders. Since for $n > 3$, only proper subsets of the domain of *all* single-peaked preferences can have the single-crossing property, this has the further interesting consequence that no minimally rich single-crossing domain can be a *maximal* Condorcet domain.

2 Characterizing the single-peaked domain: The case of strict preference orders

2.1 Statement of main result

Consider a finite set of alternatives X and the set $\mathcal{P}(X)$ of all *linear (strict) orders* (i.e., complete, transitive and antisymmetric binary relations) on X. A subset $\mathcal{D} \subseteq \mathcal{P}(X)$ will be called a *domain of preferences* or simply a *domain*. A *profile* $\pi = (P_1, \ldots, P_n)$ *on* $\mathcal{D}$ is an element of the Cartesian product $\mathcal{D}^n$ for some number $n \in \mathbb{N}$ of 'voters,' where the linear order P_i represents the preferences of the ith voter over the alternatives from X. A profile with an odd number of voters will simply be referred to as an *odd profile*. Frequently, we will denote linear orders simply by listing the alternatives in descending order, e.g. the linear order that ranks a first, b second, c third, etc., is denoted by $abc\ldots$.

The *majority relation* associated with a profile π is the binary relation P_π^{maj} on X such that $xP_\pi^{\text{maj}}y$ if and only if more than half of the voters rank x above y. Note that, according to this definition, the majority relation is asymmetric and for any odd profile π and any two distinct alternatives $x, y \in X$, we have either $xP_\pi^{\text{maj}}y$ or $xP_\pi^{\text{maj}}y$. The class of domains $\mathcal{D} \subseteq \mathcal{P}(X)$ such that, for all odd n, the majority relation associated with any profile $\pi \in \mathcal{D}^n$ is transitive has received significant attention in the literature, see the excellent survey of Monjardet [2009] and the references therein. In the following, we will refer to any such domain as a *Condorcet domain*. A domain $\mathcal{D}$ is called a *maximal Condorcet domain* if every Condorcet domain (on the same set of alternatives) that contains $\mathcal{D}$ as a subset must in fact coincide with $\mathcal{D}$. It is well-known that any maximal Condorcet domain $\mathcal{D}$ is *closed* in the sense that the majority relation of any odd profile from $\mathcal{D}$ is again an element of $\mathcal{D}$ (and not only of $\mathcal{P}(X)$), cf. [Puppe and Slinko, 2015, Lemma 2.1].

A domain $\mathcal{D}$ is *single-peaked with respect to the linear order* $>$ on X if, for all $P \in \mathcal{D}$ and all $w \in X$, the upper contour sets $U_P(w) := \{y \in X : yPw\}$ are connected ('convex') in the order $>$, i.e. $\{x, z\} \subseteq U_P(w)$ and $x < y < z$ jointly imply $y \in U_P(w)$. A domain $\mathcal{D}$ is simply called *single-peaked* if there exists *some* linear order $>$ such that $\mathcal{D}$ is single-peaked with respect to $>$. The domain of *all* orders that are single-peaked with respect to the fixed order $>$ on X is denoted by $\mathcal{SP}_>(X)$. If a domain is single-peaked with respect to $>$, we will often call the linear order $>$ the *spectrum* underlying the single-peaked domain.

A *path* in $\mathcal{P}(X)$ is subset $\{P_1, \ldots, P_m\} \subseteq \mathcal{P}(X)$ with $m \geq 2$ such that for all $j = 1, \ldots, m-1$, the two consecutive orders P_j and P_{j+1} differ in the ranking of exactly one pair x, y of (distinct) alternatives;

[4] This condition is related to but *prima facie* different from the well-known Spence-Mirrlees 'single-crossing' condition which requires that agents' types are unambiguously ordered according to their marginal rate of substitution uniformly across the good space.

note in that case x and y must be adjacent alternatives in both orders P_j and P_{j+1}. A pair of orders which differ in the ranking of exactly one (adjacent) pair of alternatives will be called *neighbors*. A domain $\mathcal{D}$ will be called *connected* if, for every pair $P, P' \in \mathcal{D}$ of distinct orders in $\mathcal{D}$, there exists a path $\{P_1, \ldots, P_m\}$ that connects P and P' (i.e. $P_1 = P$ and $P_m = P'$) and that lies entirely in $\mathcal{D}$ (i.e. $P_j \in \mathcal{D}$ for all $j = 1, \ldots, m$).

Two orders P and P^{inv} are called *completely reversed* if P and P^{inv} rank the alternatives in X in exactly the opposite order, i.e. for all distinct x and y, $xPy \Leftrightarrow \text{not}(xP^{\text{inv}}y)$. Note that by the completeness assumption, two orders $P, P^{\text{inv}} \in \mathcal{P}(X)$ are completely reversed if and only if $xPy \Leftrightarrow yP^{\text{inv}}x$. A domain is said to have *maximal width* if it contains at least one pair of completely reversed orders.[5] The following property may look artificial at first, but turns out to be conceptually very natural. Say that a domain $\mathcal{D} \subseteq \mathcal{P}(X)$ is *semi-connected* if it contains two completely reversed orders P and P^{inv} and an entire path connecting them (cf. [Danilov et al., 2012, p.938]). Evidently, semi-connectedness implies maximal width, and is implied by, but logically weaker than, the conjunction of connectedness and maximal width. Finally, a domain $\mathcal{D}$ will be called *minimally rich* if, for every alternative $x \in X$, there exists an order $P \in \mathcal{D}$ such that P has x as the top alternative.

The following characterization of the single-peaked domain is this paper's main result.

Theorem 1. **a)** *For every linear order $>$ on X, the domain $\mathcal{SP}_>(X)$ of all single-peaked orders with respect to $>$ is a connected and minimally rich Condorcet domain with maximal width. (In particular, $\mathcal{SP}_>(X)$ is semi-connected.)*

b) *Conversely, let $\mathcal{D} \subseteq \mathcal{P}(X)$ be a semi-connected and minimally rich Condorcet domain. Then, $\mathcal{D}$ is single-peaked.*

Except perhaps for the connectedness, the properties of the domain of all single-peaked orders stated in part a) are straightforward to verify. Clearly, the two completely reversed orders are $>$ itself and its reverse. Note that a single-peaked domain can contain at most one pair of completely reversed orders,[6] therefore such a pair uniquely determines a corresponding maximal single-peaked domain, and we have the following corollary.

Corollary 1. *Let $\mathcal{D} \subseteq \mathcal{P}(X)$ be a maximal Condorcet domain that is (semi-)connected, minimally rich and contains the pair P, P^{inv} of completely reversed orders, then $\mathcal{D} = \mathcal{SP}_>(X)$ where the spectrum $>$ is given by either P or P^{inv}.*

Figure 1 below depicts the (unique) maximal single-peaked domain containing the pair *abcd* and *dcba* of completely reversed orders on the set $X = \{a, b, c, d\}$ (the domain consists of the orders marked in red color).

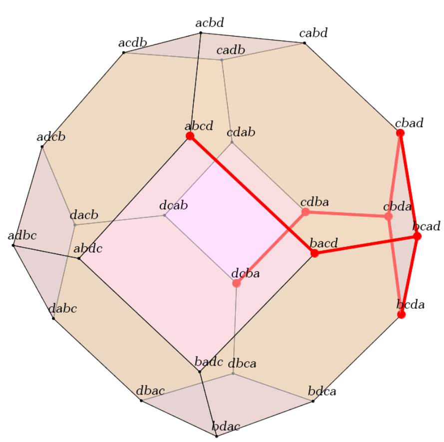

Fig. 1: A maximal single-peaked domain on $X = \{a, b, c, d\}$.

[5]Domains with that property are called 'normal' in Danilov and Koshevoy [2013].

[6]This follows, e.g., immediately from Fact 2.2 below.

Note that in the sufficiency part b) it is *not* asserted that $\mathcal{D}$ must contain *all* orders that are single-peaked with respect to some given linear order (and this does in fact not follow); on the other hand, due to the semi-connectedness, any domain satisfying the conditions of Theorem 1b) must contain at least $\#X \cdot (\#X - 1)/2 + 1$ elements (because every ordered pair of alternatives has to be switched at least once on any path connecting two completely reversed orders).

We discuss the meaning and significance of the characterizing conditions in Theorem 1 in the next subsection. We will also show by means of concrete examples that the characterization of the single-peaked domain provided by Theorem 1 is *tight* in the sense that each condition in part b) is indeed necessary to obtain the conclusion.

2.2 Discussion

The conditions imposed on preference domains in Theorem 1 will now be discussed. In particular, we demonstrate by means of examples that each condition in part b) is necessary to obtain the single-peakedness of the domain. A secondary purpose of this subsection is to illustrate the great diversity of the class of (maximal) Condorcet domains.

2.2.1 Consistency of majority voting

The condition of consistency of pairwise majority voting lies of at the heart of the present analysis and 'defines' the problem. Transitivity of the majority relation (for odd profiles) is certainly a strong requirement but, as already noted in the introduction, the goal here is not to justify it but to study its implications. Clearly, the condition that $\mathcal{D}$ be a Condorcet domain is necessary for the conclusion that $\mathcal{D}$ is single-peaked. For instance, the universal domain $\mathcal{P}(X)$ evidently satisfies all other conditions in Theorem 1b), but the universal domain is clearly not single-peaked.

2.2.2 Maximal width: The existence of two completely reversed orders

One may think of a domain as the description of a 'society.' Under this interpretation, the existence of two orders in the domain that are completely reverses of each other is a condition of maximal 'width' of opinions. The underlying society is required to admit the most extreme opinions with respect to at least one dimension. As with the other conditions in Theorem 1, the maximal width condition describes a substantial requirement.[7] On the other hand, it is not evident whether there are natural classes of maximal Condorcet domains that violate the maximal width condition. Figure 2 illustrates the necessity of the maximal width condition in Theorem 1b) by displaying a Condorcet domain of $\mathcal{P}(X)$ on $X = \{a, b, c, d\}$ that is not single-peaked but connected as well as minimally rich. The connectedness and minimal richness of the depicted domain (the red marked orders) is evident. To verify that it is not single-peaked, note first that a and d are the only two alternatives that occur at the bottom of each marked order. Since $abcd$ is a member, this implies that if the domain is to be single-peaked with respect to $>$, we must have either $a > b > c > d$ or $d > c > b > a$. However, in either case the contained order $bdca$, for instance, would not qualify as single-peaked.

The fact that the domain depicted in Fig. 2 is indeed a Condorcet domain can be easily inferred from part iv) of the following well-known result.[8]

Fact 2.1. *Let $\mathcal{D} \subseteq \mathcal{P}(X)$ with X finite. The following statements are equivalent.*

- **i)** *$\mathcal{D}$ is a Condorcet domain, i.e. the majority relation corresponding to every odd profile on $\mathcal{D}$ is an element of $\mathcal{P}(X)$.*

[7] Also mathematically, it has significant consequences. Indeed, it is well-known that, together with the consistency of majority voting, maximal width implies that the domain can be embedded in a distributive lattice (cf. Abello [1991], Chameni-Nembua [1989], Monjardet [2009], Danilov and Koshevoy [2013], Puppe and Slinko [2015]).

[8] See, e.g. [Monjardet, 2009, p. 142]). Condition iii) is Sen's [1966] 'value restriction' and condition iv) has been introduced by Ward [1965] as the 'absence of a Latin square' (in other terminology, it requires the absence of a 'Condorcet cycle'; cf. Condorcet, 1785). In light of this condition, Condorcet domains of (linear) orders are sometimes referred to as *'acyclic sets of linear orders'* (e.g. by Fishburn [1997]).

ii) *The majority relation corresponding to every profile on $\mathcal{D}$ is acyclic.*[9]

iii) *In any triple $x, y, z \in X$ of pairwise distinct alternatives, there exists one element that either never has rank 1, or never has rank 2, or never has rank 3 in the restrictions of the orders in $\mathcal{D}$ to the set $\{x, y, z\}$.*

iv) *For no triple $P_1, P_2, P_3 \in \mathcal{D}$, and no triple $x, y, z \in X$ of pairwise distinct alternatives one has xP_1yP_1z, yP_2zP_2x and zP_3xP_3y.*

We finally note that the domain depicted in Fig. 2 is in fact a *maximal* Condorcet domain.[10]

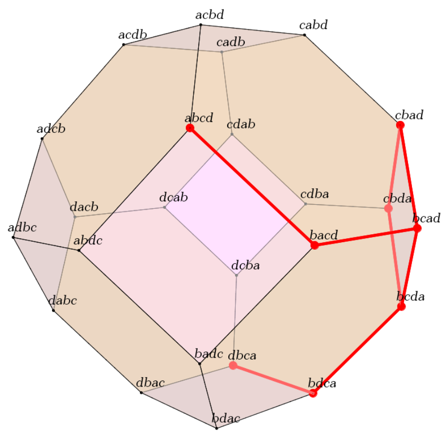

Fig. 2: A connected and minimally rich Condorcet domain without a pair of two completely reversed orders.

2.2.3 (Semi-)Connectedness

Continuing with the metaphor of a domain representing a society, connectedness has a clear meaning as well: it must be possible to reach any admissible opinion from any other admissible opinion by a series of minimal changes in the corresponding rankings while staying in the domain at each step. This may be viewed as a 'homogeneity' condition which forbids that opinions are clustered around a few 'representative' opinions. Figure 3 illustrates this; it depicts a maximal Condorcet domain on $X = \{a, b, c, d\}$ that satisfies all conditions of Theorem 1b) except semi-connectedness.

The 'society' corresponding to this domain is clustered around the opinion that the pair of alternatives $\{a, b\}$ dominates the pair $\{c, d\}$ (the 4-cycle in the front) and the opposite opinion that the pair $\{c, d\}$ dominates the pair $\{a, b\}$ (the 4-cycle in the back). That the depicted domain is a maximal Condorcet domain follows again easily using Fact 2.1; that it is not single-peaked follows at once from the fact that it violates the following simple (and well-known) necessary condition for single-peakedness.

Fact 2.2. *Suppose that $\mathcal{D} \subseteq \mathcal{P}(X)$ is single-peaked. Then there are at most two alternatives in X which can occur at the bottom of any order in $\mathcal{D}$.*[11]

The conditions of connectedness and also its weakening to semi-connectedness are arguably the most substantial and restrictive conditions used in Theorem 1b) (on top of the consistency of majority voting). Indeed, the minimally rich Condorcet domain displayed in Fig. 3 is only one instance of a general procedure that yields 'large' Condorcet domains that are neither connected nor even semi-connected, Fishburn's so-called *replacement scheme* (Fishburn [1997]). The scheme takes two Condorcet domains $\mathcal{D}_1 \subseteq \mathcal{P}(X_1)$

[9] An asymmetric binary relation P is *acyclic* if there does not exist a subset $\{x_1, \ldots, x_m\} \subseteq X$ such that x_1Px_2, $x_2Px_3, \ldots, x_{m-1}Px_m$ and x_mPx_1.

[10] The verification of this statement is straightforward if somewhat tedious.

[11] Note that this condition is clearly not sufficient for single-peakedness as the domain depicted in Fig. 2 shows.

and $\mathcal{D}_2 \subseteq \mathcal{P}(X_2)$ on two disjoint sets of alternatives and replaces one alternative, say $x \in X_1$, in each of the orders in $\mathcal{D}_1$ by each of the orders in $\mathcal{D}_2$ to obtain a new Condorcet domain $\mathcal{D}_1 * \mathcal{D}_2$ on the set $(X_1 \setminus \{x\}) \cup X_2$. It is easily verified that the domain $\mathcal{D}_1 * \mathcal{D}_2$ is not semi-connected. On the other hand, $\mathcal{D}_1 * \mathcal{D}_2$ evidently is minimally rich whenever both $\mathcal{D}_1$ and $\mathcal{D}_2$ are, and it contains two completely reversed orders whenever both $\mathcal{D}_1$ and $\mathcal{D}_2$ do. Whether the replacement scheme is important for economic applications is open to debate.

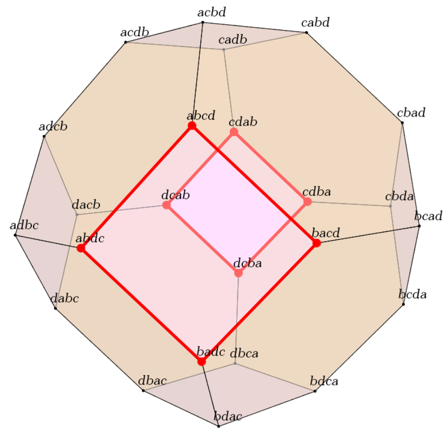

Fig. 3: A non-(semi-)connected, minimally rich Condorcet domain containing pairs of completely reversed orders.

2.2.4 Minimal richness

Minimal richness has a straightforward interpretation as well: no alternative should a priori be ruled out as the individually most desired choice. The condition is termed *'minimal'* here because in the literature much stronger 'richness' conditions have been imposed.[12] Note that the domain of all single-peaked preferences with respect to some fixed linear order $>$ on X in fact also satisfies a stronger richness condition, namely that each alternative occurs not only sometimes as the best but also as the second-best alternative.

Despite its innocuous appearance, the minimal richness condition has quite some bite as well, as illustrated by the two domains depicted in Figure 4. Both domains are maximal, connected Condorcet domains and contain the pair *abcd* and *dcba* of completely reversed orders. Evidently, neither domain is minimally rich, and by Fact 2.2 above, neither domain is single-peaked. That the depicted domains are indeed Condorcet domains follows again from Fact 2.1, and their respective maximality can be verified in a straightforward way.

Interestingly, among all maximal connected Condorcet domains on $X = \{a, b, c, d\}$ with maximal width, the domain on the left hand side of Fig. 4 has the *minimal* number of elements (in this case, $\#X \cdot (\#X - 1)/2 + 1 = 4 \cdot 3/2 + 1 = 7$) and the domain on the right hand side has the *maximal* number of elements. In fact, the domain depicted on the r.h.s. of Fig. 4 has the maximal number of elements among *all* Condorcet domains on a set of four alternatives, namely 9 (Monjardet [2009]). A Condorcet domain with the maximal *number* of elements is sometimes referred to as a *maximum Condorcet domain*. It is known that for $\#X \leq 6$, the maximal number of elements of a Condorcet domain is attained by connected domains with maximal width,[13] and that the maximal number of elements of such domains

[12] Our terminology follows Aswal et al. [2003] and Chatterji and Sen [2011]; the latter work also discusses domains violating the minimal richness requirement. Stronger richness conditions have been used, e.g., in Chatterji et al. [2013], Nehring and Puppe [2007]).

[13] For $\#X = 3, 4, 5, 6$, the maximum Condorcet domains are connected and have 4, 9, 20, 45 elements, respectively (Fishburn [1997, 2002]). For $\#X = 7$, the maximal number of elements of a connected Condorcet domain is 100 (Galambos and Reiner [2008]), but it is not known whether this is also the maximal number of elements among *all* Condorcet domains on a set with 7 elements.

always exceeds the number of elements of any single-peaked domain (Monjardet [2009]). Theorem 1b) thus implies that these domains are never minimally rich, and that the maximal cardinality of a semi-connected and minimally rich Condorcet domain is $2^{\#X-1}$, the number of all single-peaked orders with respect to some fixed linear order of X.

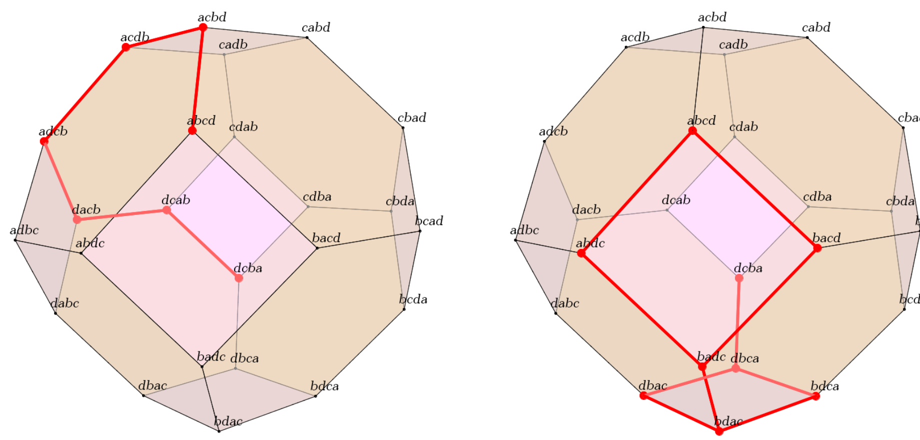

Fig. 4: Two connected but not minimally rich Condorcet domains containing a pair of completely reversed orders.

The strength of the minimal richness condition, at least when imposed jointly with semi-connectedness, can also be inferred from the following immediate corollary of Theorem 1b) and Fact 2.2.

Corollary 2. *Let $\#X \geq 3$. There does not exist a semi-connected and minimally rich Condorcet domain on X such that every alternative in X is worst for some order in $\mathcal{D}$.*

Note that the domain depicted in Fig. 3 above satisfies all conditions in Corollary 2 except for the semi-connectedness.

2.3 A dual characterization of the single-dipped domain

Theorem 1 above entails a 'dual' characterization of the single-dipped domain in a straightforward way, as follows.

Theorem 2. **a)** *For every linear order $>$ on X, the domain $\mathcal{SD}_>$ is a connected Condorcet domain with maximal width such that every alternative in X is the worst alternative for some order.*

b) *Conversely, let $\mathcal{D} \subseteq \mathcal{P}(X)$ be a semi-connected Condorcet domain such that every alternative in X is the worst alternative for some order in $\mathcal{D}$. Then, $\mathcal{D}$ is single-dipped.*

2.4 A corollary for single-crossing domains

Our analysis has an important implication for a class of domains known as 'single-crossing' domains. A domain $\mathcal{D} \subseteq \mathcal{P}(X)$ is a *single-crossing domain* if it can be written in the form $\mathcal{D} = \{P_1, ..., P_m\}$ such that, for all ordered pairs $(x, y) \in X \times X$, the set $\{P \in \mathcal{D} : xPy\}$ is 'connected' in $\{1, ..., m\}$, i.e., for all $x, y \in X$, xP_jy and xP_ly with $j < l$ implies xP_ky for all $k \in \{j, \ldots, l\}$, and yP_jx and yP_lx with $j < l$ implies yP_kx for all $k \in \{j, \ldots, l\}$. This property has been introduced in the literature by Rothstein [1990] under the name of 'order-restriction.' It underlies the analysis in Roberts [1977], Gans and Smart [1996], and is employed frequently under the name of 'single-crossing property' (see, e.g., Saporiti [2009]). It is well-known that all single-crossing domains are Condorcet domains (Rothstein [1990, 1991]).

Corollary 3. *Let $\mathcal{D} \subseteq \mathcal{P}(X)$ be a single-crossing domain. If $\mathcal{D}$ is minimally rich, then it is single-peaked. If every alternative in X is worst in at least one order in $\mathcal{D}$, then $\mathcal{D}$ is single-dipped.*

Proof. Every single-crossing domain $\mathcal{D}$ can be extended to a semi-connected single-crossing domain $\mathcal{D}^* \supseteq \mathcal{D}$ (note that we are *not* asserting that $\mathcal{D}^*$ is a maximal Condorcet domain). Indeed, if $\mathcal{D} = \{P_1, ..., P_k\}$ has the single-crossing property, then so does the domain $\{P_1, ..., P_k, P_{k+1}\}$ where $P_{k+1} = P_1^{\text{inv}}$ (we allow that $P_{k+1} = P_k$). We can now fill possible 'gaps' in the sequence $P_1, ..., P_k, P_{k+1}$ as follows. If P_1 and P_2 differ in the ranking of more than one pair of alternatives, at least one of these pairs must be an adjacent pair in P_1. Then, we can add the order P_1' that switches exactly this pair and agrees with P_1 in the ranking of all other pairs, and consider the domain $\{P_1, P_1', P_2, ...\}$. Continuing in this fashion, we obtain a semi-connected and single-crossing domain $\mathcal{D}^* \supseteq \mathcal{D}$. Evidently, if $\mathcal{D}$ is minimally rich, so is $\mathcal{D}^*$; and if every alternative is worst in at least one order in $\mathcal{D}$, then the same property holds for $\mathcal{D}^*$. Since every single-crossing domain is a Condorcet domain, we thus obtain by Theorem 1b) that $\mathcal{D}^*$, and hence also $\mathcal{D}$, is single-peaked if $\mathcal{D}$ is minimally rich. Similarly, if $\mathcal{D}$ has every alternative at the bottom of some order, then $\mathcal{D}^*$, and hence also $\mathcal{D}$, is single-dipped by Theorem 2b). □

Remark. The semi-connected domain $\mathcal{D}^*$ constructed in the proof of Corollary 3 is in fact a *maximal* single-crossing domain in the sense that no proper superdomain of $\mathcal{D}^*$ can be single-crossing. The maximal single-crossing domains are sometimes referred to as *maximal chains* in the literature, since they indeed correspond to the maximal chains in the so-called 'weak Bruhat order' (Abello [1991], Chameni-Nembua [1989], Galambos and Reiner [2008], Monjardet [2009]). Note that the maximal single-crossing domains are in general not maximal as Condorcet domains. For instance, all paths connecting the orders *abcd* and *dcba* in the single-peaked domain in Fig. 1, as well as in the maximum domain on the right hand side of Fig. 4 correspond to maximal single-crossing domains; but evidently, these maximal paths do not form maximal Condorcet domains. On the other hand, the maximal single-crossing domain on the left hand side of Fig. 4 is also maximal as Condorcet domain; a simple necessary and sufficient condition for the maximality (as Condorcet domain) of a maximal single-crossing domain is given in [Monjardet, 2007, p. 79] and Puppe and Slinko [2015]. The fact that every single-crossing *profile* of linear orders in which every alternative is at the top of at least one voter must be single-peaked has also been observed by Elkind et al. [2014].

Conclusion

How restrictive is the assumption of single-peakedness as a domain restriction? In this paper it is argued that single-peakedness follows very naturally from transitivity of the majority relation for all odd profiles under a few simple and reasonable conditions of richness and connectedness. But transitivity of the majority relation for every odd profile is clearly a very demanding condition. As is well-known, its full strength is not needed in order to derive possibility results in social choice theory. For instance, the existence of non-dictatorial Arrovian aggregators and/or strategy-proof social choice functions can be demonstrated under much weaker domain restrictions (Kalai and Muller [1977]). Also in this context, richness and/or connectedness assumptions have frequently been imposed, and variants of the single-peakedness condition have been found to play an important role in the derivation of possibility results (Nehring and Puppe [2007], Chatterji et al. [2013], Chatterji and Massó [2015]). It seems a worthwhile task for future research to explore whether, and how, the present methodology can contribute to our understanding of the weaker domain restrictions that still enable consistent preference aggregation and/or non-dictatorial strategy-proof social choice. One lesson that can already be drawn from the present analysis is that each of the conditions of (semi-)connectedness, minimal richness, and the presence of two completely reversed orders substantially restrict the combinatorial space of possibilities. Even if these conditions are justifiable by the specific context or application at hand, they cannot be considered mere 'technical' requirements.

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