The one-dimensional Euclidean preferences: Finitely many forbidden substructures are not enough

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Abstract

We show that one-dimensional Euclidean preference profiles cannot be characterized in terms of finitely many forbidden substructures. This result is in strong contrast to the case of single-peaked and single-crossing preference profiles, for which such finite characterizations have been derived in the literature.

1 Introduction

Single-peakedness, single-crossingness, and one-dimensional Euclideanness are popular domain restrictions that show up in a variety of models in the social sciences and in economics [4, 34, 24]. In many situations, these domain restrictions guarantee the existence of a desirable entity that would not exist without the restriction, as for instance a strategy-proof voting rule, or a Condorcet winner.

Preferences are *single-peaked* if there exists a linear order of the *alternatives* such that each voter's preference relation along this ordering is either always increasing, always decreasing, or first increasing and then decreasing. Single-peakedness goes back to the work of Black [4] and has been studied extensively over the years. Single-peakedness implies a number of nice properties, as for instance strategy-proofness of a family of voting rules [32] and transitivity of the majority relation [25] (also known as the Condorcet principle). Preferences are *single-crossing* if there exists a linear order of the *voters* such that each pair of alternatives separates this order into two sub-orders where in each sub-order, all voters agree on the relative order of this pair. Single-crossingness goes back to the work of Karlin [26] in applied mathematics and the papers of Mirrlees [31], Roberts [34] on income taxation. Diamond and Stiglitz [13] use it in the economics of uncertainty. Single-crossingness also plays a role in coalition formation [12, 10], income redistribution [30], local public goods distribution and stratification [35, 18], in the choice of constitutional voting rules [2], and in the analysis of the majority rule [22, 21].

Preferences are *one-dimensional Euclidean* if there exists a common embedding of voters and alternatives into the real numbers such that every voter prefers alternatives that are embedded close to him to alternatives that are embedded farther away from him. One-dimensional Euclidean preferences go back to Hotelling [24]. He argues that when selecting stores to go shopping, a customer usually prefers a store that is located closer to him than one that is farther away. Thus, his preferences over the stores are (not necessarily one-dimensional) Euclidean. One-dimensional Euclidean preferences have also been discussed by Coombs [11] under the name "unidimensional unfolding" representations. They combine all the properties of single-peaked and single-crossing preferences. Doignon and Falmagne [14] discussed Euclidean preferences in the context of behavioral sciences, and Brams et al. [6] discussed them in the context of coalition formation.

Forbidden substructures. The scientific literature contains many characterizations of combinatorial objects in terms of forbidden substructures. For instance, Kuratowski's theorem [28] characterizes planar graphs in terms of forbidden subgraphs: a graph is planar if and only if it does not contain a subdivision of the complete graph on five vertices, K_5, or the complete bipartite graph on two disjoint vertex sets of size three each, $K_{3,3}$. In a similar spirit, Lekkerkerker and Boland [29] characterized interval graphs through five (infinite) families of forbidden induced subgraphs, and Földes and Hammer [20] characterized split graphs in terms of three forbidden induced subgraphs. Hoffman et al. [23] characterized totally-balanced 0-1-matrices in terms of certain forbidden submatrices. The characterizations of split graphs and totally-balanced 0-1-matrices use a finite number of forbidden substructures, while the characterizations of planar graphs and interval graphs both involve infinitely many forbidden substructures.

In the area of social choice, Ballester and Haeringer [1] characterize single-peaked preferences and group-separable preferences in terms of a small finite number of forbidden substructures. Single-crossing preferences also allow a characterization by finitely many forbidden substructures [7]. Let us stress that every *monotone* property of profiles (that is, every property that is preserved under the removal of voters and/or alternatives) can be characterized by a set of forbidden substructures. For many monotone properties, however, the number of forbidden substructures may be infinite. The Condorcet principle, that is, the existence of a Condorcet winner, however, is a typical *non-monotone* example of a property in Social Choice that *cannot* be characterized at all through forbidden substructures; for instance, the existence of a Condorcet winner may fail after the deletion of some voter.

A characterization by finitely many forbidden substructures has many positive consequences. Whenever a family $\mathcal{F}$ of combinatorial objects allows such a finite characterization, this directly implies the existence of a polynomial-time algorithm for recognizing the members of $\mathcal{F}$: one may simply work through the forbidden substructures one by one, and check whether the object in question contains the substructure. By looking deeper into the combinatorial structure of such families $\mathcal{F}$, one usually manages to find recognition algorithms that are much faster than this simple approach. As an example, there are algorithms for recognizing single-peaked preference profiles that are due to Bartholdi III and Trick [3], Doignon and Falmagne [14], and Escoffier et al. [19]. Single-crossingness can also be recognized very efficiently; see [14], [17], and [7].

As another positive consequence, a characterization by finitely many forbidden substructures often helps in understanding the algorithmic and combinatorial behavior of family $\mathcal{F}$. For example, Bredereck et al. [8] investigate the problem of deciding whether a given preference profile is close to having a nice structure. The distance is measured by the number of voters or alternatives that have to be deleted from the given profile in order to reach a nicely structured profile. For the cases where ‘nicely structured means single-peaked or single-crossing, their proofs are heavily based on characterizations ([1] and [7]) by finitely many forbidden substructures. Elkind and Lackner [16] studied similar questions and derive approximation algorithms for the number of deleted voters or alternatives. All of their results are centered around preference profiles that can be characterized by a finite number of forbidden substructures.

Scope and contribution of this paper. As one-dimensional Euclidean profiles form a special case of single-peaked and single-crossing profiles, every forbidden substructure to single-peakedness and every forbidden substructure to single-crossingness will automatically also form a forbidden substructure to one-dimensional Euclideanness. Now the question arises: *“Are there any further forbidden substructures to one-dimensional Euclideanness?”* Coombs [11] answered this back in 1964: *“Yes, there are!”* (see Section 2.2 for more information). This immediately takes us to another question: *“Is there a characterization of one-dimensional Euclideanness in terms of finitely many forbidden substructures?”* The answer to this second question is negative, as we are going to show in this paper.

To this end, we construct an infinite sequence of preference profiles that have two crucial properties. First, none of these profiles is one-dimensional Euclidean. Secondly, every profile just *barely* violates one-dimensional Euclideanness, as the deletion of an arbitrary voter immediately makes the profile one-dimensional Euclidean. The second property implies that each profile in the sequence is on the edge of being Euclidean, and that the reason for its non-Euclideanness must lie in its overall structure. In other words, each of these infinitely many profiles yields a separate forbidden substructure for one-dimensional Euclideanness, and this is exactly what we want to establish.

The definition of the infinite profile sequence and the resulting analysis are quite involved. Ironically, the complicatedness of our proof is a consequence of the very statement we are going to prove. As part of our proof, we have to argue that deletion of an arbitrary voter from an arbitrary profile in the sequence yields a one-dimensional Euclidean profile. If there was a characterization of one-dimensional Euclideanness by finitely many forbidden substructures, then the following argument would be relatively easy to get through: we could simply analyze the preference profile and show that the deletion of any voter removes all forbidden substructures. But unfortunately, such a characterization does not exist. The only viable (and fairly tedious) approach is to explicitly specify the corresponding Euclidean representations (one representation per deleted voter!) and to prove by case distinctions that each such representation correctly encodes the preferences of all remaining voters.

Organization of the paper. In Section 2 we summarize the central definitions, state useful observations, and provide some examples. In Section 3 we formulate our main results in Theorems 1 and 2, and we show how Theorem 2 follows from Theorem 1. Section 3.1 defines an infinite sequence of profiles which are shown *not* to be one-dimensional Euclidean in Section 3.2. Section 3.3 gives, for each voter, an embedding for the profile resulting from the deletion of this voter. We obmit the long and technical proofs of the correctness of these embeddings are deferred to the appendix. Section 4 shows how to construct other infinite sequences of non-one-dimensional Euclidean profiles. Section 5 concludes with some discussion on future research work.

2 Definitions, notations, and examples

Let $A = \{1, \ldots, m\}$ be a set of m alternatives and let $v_1, \ldots, v_n$ be n voters. A *preference profile* specifies the *preference orders* of the voters, where voter v_i ranks the alternatives according to a strict linear order $\succ_i$. Since we study characterization of preference properties through forbidden substructures, it is not necessary to have two voters with the same preference order. Thus, throughout this paper we assume that *no* two voters have the same preference order. For alternatives a and b, the relation $a \succ_i b$ means that voter v_i strictly prefers a to b. If the meaning is clear from the context, we will sometimes simply write $\succ$ instead of $\succ_i$ and suppress the dependence on i.

In the following, we introduce the three major concepts used in this paper: single-peaked profiles, single-crossing profiles, and one-dimensional Euclidean profiles.

2.1 Single-peaked profiles and single-crossing profiles

A linear order L of the alternatives A is *single-peaked* with respect to a fixed voter v_i if for each three distinct alternatives $a, b, c \in A$ with $a \succ_L b \succ_L c$ it holds that $a \succ_L b \succ_L c$ implies that if $a \succ_i b$, then $b \succ_i c$. A preference profile is *single-peaked* if it allows an order (that is, a permutation) of the alternatives that is single-peaked with respect to every voter. The following proposition states a characterization of single-peakedness in terms of finitely many forbidden substructures. Hereby, for a set A of alternative, an alternative b with $b \notin A$, and a preference order, we say that a preference order $\succ^*$ *satisfies* the condition $A \succ b$ if for each alternative $a \in A$ it holds that $a \succ^* b$. We do this analogously for $b \succ A$.

Proposition 1. *(Ballester and Haeringer [1])*
A preference profile is single-peaked if and only if it avoids the following two substructures. The first forbidden substructure is a 3×3 profile with alternatives a, b, c, and three voters v_1, v_2, v_3 whose preference orders satisfy:

$$v_1\colon \{b, c\} \succ a, \qquad v_2\colon \{a, c\} \succ b, \qquad v_3\colon \{a, b\} \succ c.$$

The second forbidden substructure is a 2×4 profile with alternatives a, b, c, d, and two voters v_1 and v_2 whose preference orders satisfy:

$$v_1\colon \{a, d\} \succ b \succ c, \qquad v_2\colon \{c, d\} \succ b \succ a.$$

A linear order L of the voters is *single-crossing* with respect to a pair $\{a, b\}$ of alternatives if for each two voters v_i and v_j with $v_i \succ_L v_j$ it holds that if the first voter in L ranks $a \succ b$ but voter v_i ranks $b \succ a$, then voter v_j ranks $b \succ a$. A preference profile is *single-crossing* if it allows an order of the voters that is single-crossing with respect to every pair of alternatives. The following proposition states a characterization of single-crossingness in terms of finitely many forbidden substructures.

Proposition 2 (Bredereck et al. [7])**.** *A preference profile is single-crossing if and only if it avoids the following two forbidden substructures. The first forbidden substructure is a profile with three (not necessarily disjoint) pairs of alternatives $\{a, b\}$, $\{c, d\}$, and $\{e, f\}$, and three voters v_1, v_2, v_3 whose preference orders satisfy:*

$$v_1\colon b \succ a \text{ and } c \succ d \text{ and } e \succ f,$$
$$v_2\colon a \succ b \text{ and } d \succ c \text{ and } e \succ f,$$
$$v_3\colon a \succ b \text{ and } c \succ d \text{ and } f \succ e.$$

The second forbidden substructure is a profile with two (not necessarily disjoint) pairs of alternatives $\{a, b\}$ and $\{c, d\}$, and four voters v_1, v_2, v_3, v_4 whose preference orders satisfy:

$$v_1\colon a \succ b \text{ and } c \succ d, \qquad v_2\colon a \succ b \text{ and } d \succ c,$$
$$v_3\colon b \succ a \text{ and } c \succ d, \qquad v_4\colon b \succ a \text{ and } d \succ c.$$

2.2 One-dimensional Euclidean profiles

The one-dimensional Euclidean preferences describe the situation where voters rank alternatives according to their spatial perception. Consider a common embedding (E, F) of the voters and alternatives into the real number line, which assigns to every alternative a a real number $E[a]$, and to every voter v_i a real number $F[i]$. A preference profile is *one-dimensional Euclidean* or simply *Euclidean* if there is a common embedding of the voters and alternatives so that for every voter v_i and for every two alternatives a and b, $a \succ_i b$ holds if and only if the distance from $F[i]$ to $E[a]$ is strictly smaller than the distance from $F[i]$ to $E[b]$, that is,

$$|F[i] - E[a]| < |F[i] - E[b]|.$$

In other words, small spatial distances from the point $F[i]$ indicate strong preferences of voter v_i. We subsequently also call a Euclidean embedding (E, F) a *Euclidean representation.* It is known (and easy to see) that every Euclidean profile is single-peaked and single-crossing [11, 14]: the left-to-right order of the alternatives along the Euclidean representation is single-peaked, and the left-to-right order of the voters along the Euclidean representation is single-crossing. Thus, a Euclidean preference profile over m alternatives can have at most $m \cdot (m-1)/2 + 1$ voters as a single-crossing profile can have at most $m \cdot (m-1)/2 + 1$ voters (recall that we assume that voters have pairwise distinct preference orders). Figure 1 shows a possible representation for a Euclidean preference profile with $m = 3$ alternatives and $3 \cdot 2/2 + 1 = 4$ voters.

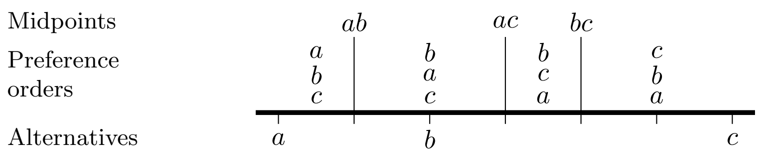

Figure 1: A one-dimensional Euclidean representation of three alternatives: a, b, c, and four voters with preference orders: $a \succ b \succ c$, $b \succ a \succ c$, $b \succ c \succ a$, $c \succ b \succ a$, respectively. The alternatives are embedded from left to right into the real line so that the midpoints of all possible pairs of alternatives divide the line into four intervals. In this way, each voter (represented by his preference order) must be embedded in exactly one of the intervals.

Coombs [11, page 91] discussed a profile with 16 voters and 6 alternatives which is both single-peaked and single-crossing, but fails to be one-dimensional Euclidean. The following example contains the smallest profile known to us that has these properties.[1]

Example 1. *Consider the following profile with six alternatives a, b, c, d, e, f and three voters v_1, v_2, v_3:*

$$v_1 : c \succ b \succ a \succ d \succ e \succ f, \quad v_2 : c \succ d \succ b \succ e \succ f \succ a, \quad v_3 : e \succ d \succ c \succ f \succ b \succ a.$$

This profile is single-peaked with respect to the order $a \succ b \succ c \succ d \succ e \succ f$, and it is single-crossing with respect to the order $v_1 \succ v_2 \succ v_3$. However, we will see that the profile is not one-dimensional Euclidean, but deleting any single voter makes the profile Euclidean.

As the profile in Example 1 is single-peaked and single-crossing, it does not contain any of the forbidden substructures listed in Propositions 1 and 2. Hence, there must be some other forbidden substructure contained in it which is responsible for its non-Euclideanness. Example 1 and the 16×6 profile by Coombs provide first indications that the forbidden substructures for one-dimensional Euclideanness might be complex and intricate to analyze.

The following two propositions state simple observations that will be used repeatedly in our reasoning; the proofs are deferred to Appendices A and B.

Proposition 3. *Let a and b be two alternatives in a Euclidean representation (E, F) of some profile with $E[a] < E[b]$. Then, voter v_i prefers a to b if and only if $F[i] < \frac{1}{2}(E[a] + E[b])$, and he prefers b to a if and only if $F[i] > \frac{1}{2}(E[a] + E[b])$.*

Proposition 3 describes how to embed the voters if we know the relative order of the alternatives in the embedding. For instance, in Figure 1, since $E(a) < E(b)$, a voter preferring $a \succ b$ must be embedded left to the midpoint of $E(a)$ and $E(b)$, that is, $\frac{1}{2}(E[a] + E[b])$. The next result is derived from the single-peakedness that every Euclidean profile must possess.

Proposition 4. *Let a, b, c be three alternatives in a Euclidean representation (E, F) of some preference profile with $E[a] < E[b] < E[c]$. If voter v_i prefers a to b, then he also prefers b to c. If voter v_i prefers c to b, then he also prefers b to a.*

Example 2. *We continue our discussion of Example 1. Suppose for the sake of contradiction that the profile in Example 1 is one-dimensional Euclidean. Let (E, F) be such a Euclidean representation. We can verify that the only single-peaked orders of the alternatives are $a \succ b \succ c \succ d \succ e \succ f$ and its reverse.*

By Proposition 3 and by the fact that voter v_1 ranks $c \succ b$ and $a \succ d$, it follows that $E[b] + E[c] < 2F[1] < E[a] + E[d]$. Again, by Proposition 3 and by the fact that voter v_2 ranks $f \succ a$ and $b \succ e$, it follows that $E[a] + E[f] < 2F[2] < E[b] + E[e]$. Finally, by

[1] Indeed, Bulteau and Chen [9] recently show that every single-peaked and single-crossing profile with up to two voters or with up to five alternatives is Euclidean.

Proposition 3 and by the fact that voter v_3 ranks $e \succ d$ and $c \succ f$, it follows that $E[d] + E[e] < 2F[3] < E[c] + E[f]$. If we add up the above three inequalities, then we obtain the contradiction $\sum_{x=a}^{f} E[x] < \sum_{x=a}^{f} E[x]$. Appendix C shows that deleting any voter makes the profile Euclidean.

Finally, we mention that the (mathematical) literature on one-dimensional Euclidean preference profiles is scarce. Doignon and Falmagne [14], Knoblauch [27], and Elkind and Faliszewski [15] designed polynomial-time algorithms for deciding whether a given preference profile is Euclidean. Their approaches are not purely combinatorial, as they are partially based on linear programming formulations. In the first phase, while the approaches of Doignon and Falmagne [14] and of Elkind and Faliszewski [15] first construct a single-crossing order and then a single-peaked order, the approach of Knoblauch [27] first constructs a single-peaked order and then uses this order to define the relative order of the voters that a Euclidean representation would obey. In the second phase, they use linear programming to find an embedding satisfying the order found in phase one, and the Euclideanness.

3 Main results

In this section we formulate the two closely related main results of this paper. The first result is technical and states the existence of infinitely many non-Euclidean profiles that are minimal with respect to voter deletion.

Theorem 1. *For each integer $k \geq 2$, there exists a preference profile $\mathcal{P}_k^*$ with $n = 2k$ voters and $m = 4k$ alternatives, such that the following holds.*

(a) Profile $\mathcal{P}_k^*$ is not one-dimensional Euclidean.

(b) Profile $\mathcal{P}_k^*$ is minimal in the following sense: the deletion of an arbitrary voter from $\mathcal{P}_k^*$ yields a one-dimensional Euclidean profile.

The proof of Theorem 1 is long and will fill most of the rest of this paper. As an immediate consequence of Theorem 1, we derive our second main result as stated in the following theorem.

Theorem 2. *One-dimensional Euclidean preference profiles cannot be characterized in terms of finitely many forbidden substructures.*

Proof. Suppose for the sake of contradiction that such a characterization with finitely many forbidden substructures would exist. Let t denote the largest number of voters in any forbidden substructure, and consider a profile $\mathcal{P}_k^*$ from Theorem 1 with $k \geq t$. As $\mathcal{P}_k^*$ is not one-dimensional Euclidean by Property (a), it must contain one of these finitely many forbidden substructures with at most t voters. As profile $\mathcal{P}_k^*$ contains $2k > t + 1$ voters, one of its voters is not part of this forbidden substructure. If we delete this voter, the resulting profile will still contain the forbidden substructure; hence it is not one-dimensional Euclidean, which contradicts Property (b). □

3.1 Definition of the profiles

In this section, we begin the proof of Theorem 1 by defining the underlying profiles $\mathcal{P}_k^*$. Properties (a) and (b) stated in Theorem 1 will be established in the next sections. We consider $n = 2k$ voters $v_1, v_2, \ldots, v_{2k}$ together with $m = 4k$ alternatives $1, 2, 3, \ldots, 4k$. The preference orders of the voters will be constructed from pieces X_i, Y_i, Z_i with $1 \le i \le k$:

$$\begin{aligned} X_i &:= 2k+2i-2 \succ 2k+2i-3 \succ 2k+2i-4 \succ \ldots \succ 2i+2, \\ Y_i &:= 2i-2 \succ 2i-3 \succ 2i-4 \succ \ldots \succ 1, \\ Z_i &:= 2k+2i+1 \succ 2k+2i+2 \succ 2k+2i+3 \succ \ldots \succ 4k. \end{aligned}$$

Note that for every $i = 1, \ldots, k$, the three pieces X_i, Y_i, Z_i cover contiguous intervals of $2k-3$, $2i-2$, and $2k-2i$ alternatives, respectively. Together, the three pieces cover $4k-5$ of the alternatives, and only the five alternatives in the set $U_i = \{2i-1,\ 2i,\ 2i+1\} \cup \{2k+2i-1,\ 2k+2i\}$ remain uncovered. Also note that the pieces Y_1 and Z_k are empty. The preference orders of the voters are defined as follows. The two voters v_{2i-1} and v_{2i} always form a couple with fairly similar preferences. For $1 \le i \le k-1$, these voters v_{2i-1} and v_{2i} have the following preferences:

$$v_{2i-1}\colon \quad X_i \succ 2i+1 \succ 2k+2i-1 \succ 2i \succ 2i-1 \succ 2k+2i \succ Y_i \succ Z_i, \tag{1a}$$

$$v_{2i}\colon \quad X_i \succ 2k+2i-1 \succ 2k+2i \succ 2i+1 \succ 2i \succ 2i-1 \succ Y_i \succ Z_i. \tag{1b}$$

Note that the voters v_{2i-1} and v_{2i} both rank the three alternatives $2i+1$, $2i$, $2i-1$ in U_i in the same decreasing order, with the two other alternatives $2k+2i-1$ and $2k+2i$ shuffled into that order. The last two voters v_{2k-1} and v_{2k} are defined separately:

$$v_{2k-1}\colon \quad X_k \succ 2k+1 \succ 4k-1 \succ 2k \succ 2k-1 \succ 4k \succ Y_k, \tag{2a}$$

$$v_{2k}\colon \quad X_k \succ 2k+1 \succ 2k \succ \ldots \succ 3 \succ 2 \succ 4k-1 \succ 4k \succ 1. \tag{2b}$$

Since piece Z_k is empty, the preferences of voter v_{2k-1} in (2a) are actually very similar to the preferences of the other odd-numbered voters v_{2i-1} with $1 \le i \le k-1$ in (1a). The last voter v_{2k}, however, behaves quite differently from the other even-numbered voters: on top of his preference order, there are the alternatives in X_k, followed by an intermingling of the alternatives in Y_k and U_k (the alternatives $2k+1, \ldots, 2$ in decreasing order, and then the three alternatives $4k-1$, $4k$, and 1).

Example 3. *For $k = 4$, the preference profile $\mathcal{P}_4^*$ has $n = 8$ voters and $m = 16$ alternatives and looks as follows (for the sake of readability, we omitted the preference symbol $\succ$ between the alternatives, and for each preference order, we list the alternatives from left to right starting with the most preferred alternative):*

v_1:	8	7	6	5	4	3	9	2	1	10	11	12	13	14	15	16
v_2:	8	7	6	5	4	9	10	3	2	1	11	12	13	14	15	16
v_3:	10	9	8	7	6	5	11	4	3	12	2	1	13	14	15	16
v_4:	10	9	8	7	6	11	12	5	4	3	2	1	13	14	15	16
v_5:	12	11	10	9	8	7	13	6	5	14	4	3	2	1	15	16
v_6:	12	11	10	9	8	13	14	7	6	5	4	3	2	1	15	16
v_7:	14	13	12	11	10	9	15	8	7	16	6	5	4	3	2	1
v_8:	14	13	12	11	10	9	8	7	6	5	4	3	2	15	16	1

The alternatives in the five leftmost columns form the pieces X_i. In the first seven rows, the five columns in the center correspond to the sets U_i, while the remaining six columns make up the pieces of Y_i and Z_i. The last row illustrates the unique behavior of the last voter v_8.

3.2 The profiles are not Euclidean

In this section, we discuss the single-crossing, single-peaked and one-dimensional Euclidean properties of the profiles $\mathcal{P}_k^*$. First, we can verify that every profile $\mathcal{P}_k^*$ with $k \geq 2$ is single-crossing with respect to the order $v_1 \succ v_2 \succ \ldots \succ v_{2k-2} \succ v_{2k} \succ v_{2k-1}$ of the voters (that is, the natural order of voters by increasing index, but with the last two voters v_{2k-1} and v_{2k} swapped). As this single-crossing property is of no relevance to our further considerations, the proof is omitted. Next, let us turn to single-peakedness, which is crucial for the construction of our embeddings in Section 3.3; the proof is deferred to Appendix D.

Lemma 1. *For $k \geq 2$, the profile $\mathcal{P}_k^*$ is single-peaked. Furthermore, the only two single-peaked orders of the alternatives are the increasing order $1, 2, 3, \ldots, 4k$ and the decreasing order $4k, \ldots, 3, 2, 1$.*

The following lemma shows that every profile $\mathcal{P}_k^*$ satisfies Property (a) of Theorem 1.

Lemma 2. *For $k \geq 2$, the profile $\mathcal{P}_k^*$ is not one-dimensional Euclidean.*

Proof. Suppose for the sake of contradiction that profile $\mathcal{P}_k^*$ has a Euclidean representation (E, F), $F[j]$ for the voters $v_j \in \{v_1, \ldots, v_{2k}\}$ and $E[i]$ for the alternatives $i \in \{1, \ldots, 4k\}$. As the Euclidean representation induces a single-peaked order of the alternatives, we assume by Lemma 1 that the alternatives are embedded in increasing order with

$$E[1] \ < \ E[2] \ < \ E[3] \ < \ \ldots \ < \ E[4k-1] \ < \ E[4k]. \tag{3}$$

Next, we claim that in any Euclidean representation under (3), the embedded alternatives satisfy the following inequalities:

$$E[2k+2i-1] + E[2i] \ < \ E[2k+2i] + E[2i-1] \qquad \text{for } 1 \leq i \leq k \tag{4a}$$

$$E[2k+2i] + E[2i+1] \ < \ E[2k+2i-1] + E[2i+2] \qquad \text{for } 1 \leq i \leq k-1 \tag{4b}$$

$$E[4k] + E[1] \ < \ E[4k-1] + E[2] \tag{4c}$$

The correctness of this claim can be seen as follows. For each $i = 1, \ldots, k$, voter v_{2i-1} ranks $2k + 2i - 1 \succ 2i$ and $2i - 1 \succ 2k + 2i$, which by Proposition 3 yields

$$\tfrac{1}{2}\left(E[2k+2i-1] + E[2i]\right) \ < \ F[2i-1] \ < \ \tfrac{1}{2}\left(E[2k+2i] + E[2i-1]\right),$$

which in turn implies (4a). Similarly, for $i = 1, \ldots, k-1$, voter v_{2i} ranks $2i + 2 \succ 2k + 2i - 1$ and $2k + 2i \succ 2i + 1$ which leads to (4b). Finally, voter v_{2k} ranks $2 \succ 4k - 1$ and $4k \succ 1$, which implies (4c). This shows the correctness of the inequalities (4a)–(4c). By adding up all the inequalities, we derive the contradiction $\sum_{x=1}^{4k} E[x] < \sum_{x=1}^{4k} E[x]$. ☐

Before we move on to the next section where we discuss the Euclidean embeddings, let us take a closer look at the cause of the non-Euclideanness of our constructed profile. First, we rearrange the terms of the inequalities (4a)–(4c) to obtain the following.

$$E[2i] - E[2i-1] \ < \ E[2k+2i] - E[2k+2i-1] \text{ for } 1 \leq i \leq k \tag{5a}$$

$$E[2k+2i] - E[2k+2i-1] \ < \ E[2i+2] - E[2i+1] \qquad \text{for } 1 \leq i \leq k-1 \tag{5b}$$

$$E[4k] - E[4k-1] \ < \ E[2] - E[1] \tag{5c}$$

By the chain of inequalities (3), the alternatives $1, 2, \ldots, 4k$ are embedded from left to right in increasing order. Thus, both the left-hand side and the right-hand side of the inequalities (5a)–(5c) refer to distances of two neighboring alternatives $2i - 1$ and $2i$ in the embedding. Thus, if we consider the "smaller than $(<)$" relation of these distances, then we find that this relation is cyclic, which is impossible for any Euclidean representation. See Figure 2 for an illustration. In fact, any preference profile with a cyclic relation is not Euclidean. We will discuss this issue in Section 4.

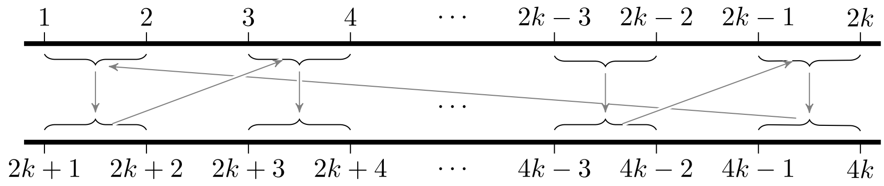

Figure 2: The "smaller" relation of the distances between two neighboring alternatives $2i+1$ and $2i+2$. Arrow "$\rightarrow$" indicates the "smaller than ($<$)" relation. For the sake of readability, the top line depicts the real line up to $2k$, and the bottom line depicts the real line starting from $2k+1$.

3.3 Definition of the Euclidean embeddings

We fix an integer s with $1 \leq s \leq 2k$ and construct Euclidean embeddings F_s and E_s of the voters and alternatives in profile $\mathcal{P}_k^*$. We show that F_s (minus voter v_s's embedding) and E_s together form a Euclidean representation of the profile obtained from $\mathcal{P}_k^*$ by deleting v_s.

We start by defining the Euclidean embedding E_s of the alternatives. We anchor the embedding by placing the first alternative at the position

$$E_s[1] \;=\; 0. \tag{6}$$

The remaining values $E_s[2], \ldots, E_s[4k]$ are defined recursively by the equations (7)–(12) below. For $1 \leq i \leq k-1$, we set

$$E_s[2i+1] - E_s[2i] \;=\; 2 \tag{7}$$

and for $1 \leq i \leq k$ we set

$$E_s[2i] - E_s[2i-1] \;=\; (4i - 2s - 3 \bmod 4k). \tag{8}$$

Note that the relations (6)–(8) define $E_s[x]$ for all $x \leq 2k$. For $1 \leq i \leq k-1$, we set

$$E_s[2k+2i-1] - E_s[2k+2i-2] = \begin{cases} E_s[2k+2i-3] - E_s[2i+1] + 2 & \text{if } s \neq 2i-1 \\ E_s[2k+2i-3] - E_s[2i+2] + 2 & \text{if } s = 2i-1. \end{cases} \tag{9}$$

For $1 \leq i \leq k-1$, we define

$$E_s[2k+2i] - E_s[2k+2i-1] \;=\; (4i - 2s - 1 \bmod 4k). \tag{10}$$

Note that the relations (9) and (10) define $E_s[x]$ for all x with $2k+1 \leq x \leq 4k-2$. Finally, we define the Euclidean embedding of the last two alternatives as

$$E_s[4k-1] - E_s[4k-2] \;=\; \begin{cases} E_s[4k-3] - E_s[2] + 2 & \text{if } s \neq 2k \\ E_s[4k-3] - E_s[2k+1] + 2 & \text{if } s = 2k \end{cases} \tag{11}$$

and

$$E_s[4k] - E_s[4k-1] \;=\; \begin{cases} E_s[2] - E_s[1] - 2 & \text{if } s \neq 2k \\ E_s[2k+1] - E_s[2k-1] & \text{if } s = 2k. \end{cases} \tag{12}$$

This completes the description of the Euclidean embedding E_s of the alternatives. Note that $E_s[x]$ is integer for all alternatives x.

Lemma 3. *The embedding E_s satisfies $E_s[x] < E_s[y]$ for all alternatives x and y with $1 \leq x < y \leq 4k$. In other words, E_s satisfies the inequalities in (3).*

	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}	d_{11}	d_{12}	d_{13}	d_{14}	d_{15}	d_{16}
E_1:	**15**	2	**3**	2	**7**	2	**11**	13	**1**	35	**5**	62	**9**	145	**13**
E_2:	**13**	2	**1**	2	**5**	2	**9**	12	**15**	30	**3**	68	**7**	151	**11**
E_3:	**11**	2	**15**	2	**3**	2	**7**	24	**13**	35	**1**	81	**5**	187	**9**
E_4:	**9**	2	**13**	2	**1**	2	**5**	20	**11**	30	**15**	68	**3**	171	**7**
E_5:	**7**	2	**11**	2	**15**	2	**3**	32	**9**	54	**13**	97	**1**	242	**5**
E_6:	**5**	2	**9**	2	**13**	2	**1**	28	**7**	46	**11**	84	**15**	207	**3**
E_7:	**3**	2	**7**	2	**11**	2	**15**	24	**5**	54	**9**	100	**13**	233	**1**
E_8:	**1**	2	**5**	2	**9**	2	**13**	20	**3**	46	**7**	84	**11**	142	**33**

Table 1: This table is discussed in Example 4 and illustrates the Euclidean embedding of the alternatives in profile $\mathcal{P}_4^*$. Every row is labeled with the corresponding embedding E_s. If a column is labeled d_i, then its entries show the Euclidean distances $E_s[i] - E_s[i-1]$ between the two consecutively embedded alternatives $i-1$ and i.

Example 4. *We continue our discussion of profile $\mathcal{P}_4^*$ from Example 3. For every embedding E_s with $1 \le s \le 8$, the corresponding row in Table 1 lists the distances $d_i = E_s[i] - E_s[i-1]$ between pairs of consecutive alternatives according to formulas (7)–(12). For instance, the intersection of row E_5 and column d_4 contains an entry with value 11; this means that in the embedding E_5, the distance $E_5[4] - E_5[3]$ between the embedded alternatives 3 and 4 equals 11. As $E_s[1] = 0$, we see that for $2 \le i \le 4k$, $E_s[i]$ equals $d_2+d_3+\cdots+d_i$. For instance in E_5, alternative 4 will be embedded at $E_5[4] = 7+2+11 = 20$.*

We note the periodic structure of part of the data in Table 1. For instance, every even-numbered column except the last one contains a circular shift of the eight numbers 15, 13, 11, 9, 7, 5, 3, 1 as shown in boldface, which results from equations (8) and (10). Furthermore, all entries in the three columns d_3, d_5, d_7 have the same value 2 because of (7). The values in other parts of the table look somewhat irregular, which is caused by formula (9). For us, the most convenient way of analyzing this data is via the recursive definitions (6)–(12).

Now we turn to the Euclidean embedding of the voters. The position $F_s[j]$ of every voter v_j is the average of exactly four embedded alternatives. For $1 \le i \le k-1$, we define

$$F_s[2i-1] \;=\; \frac{1}{4}\left(E_s[2i-1] + E_s[2i] + E_s[2k+2i-1] + E_s[2k+2i]\right). \tag{13}$$

Similarly, for $1 \le i \le k-1$, we define

$$F_s[2i] \;=\; \frac{1}{4}\left(E_s[2i+1] + E_s[2i+2] + E_s[2k+2i-1] + E_s[2k+2i]\right). \tag{14}$$

If $s \neq 2k$, then we embed voter v_{2k-1} according to (13), while for $s = 2k$, we embed it in a slightly different way. More precisely, we set

$$F_s[2k-1] \;=\begin{cases} \frac{1}{4}\left(E_s[2k-1] + E_s[2k] + E_s[4k-1] + E_s[4k]\right) & \text{if } s \neq 2k \\ \frac{1}{4}\left(E_s[2k-2] + E_s[2k+1] + E_s[4k-1] + E_s[4k]\right) & \text{if } s = 2k. \end{cases} \tag{15}$$

Finally, the very last voter v_{2k} is placed at

$$F_s[2k] \;=\; \frac{1}{4}\left(E_s[1] + E_s[2] + E_s[4k-1] + E_s[4k]\right). \tag{16}$$

Equations (13)–(16) define $F_s[j]$ for all voters v_j with $1 \le j \le 2k$. This completes the description of the Euclidean representation F_s of the voters.

We note that the location $F_s[s]$ of voter v_s has been specified, but will be irrelevant to our further arguments. We can show that F_s and E_s together constitute a correct Euclidean representation of the $2k-1$ voters in $\{v_1, \dots, v_{2k}\} \setminus \{v_s\}$ together with all $4k$ alternatives $1, 2, \dots, 4k$. In other words, the deletion of voter v_s from profile $\mathcal{P}_k^*$ yields a one-dimensional Euclidean profile, which completes the proof of Property (b) in Theorem 1. To this end, the following lemma will be established in Section G.

Lemma 4. *For all r and s with $1 \le r \ne s \le 2k$, the E_s and F_s together form a Euclidean representation of the preferences of voter v_r.*

The correctness of Lemma 4 for small profiles $\mathcal{P}_k^*$ with $k \in \{2, 3, 4\}$ can be easily verified by a computer program. Hence, from now on we assume that

$$k \ge 5. \tag{17}$$

Furthermore, note that the proof of our main result in Theorem 2 is not affected by this assumption, as it builds on the profiles $\mathcal{P}_k^*$ for which k is large and tends towards infinity.

4 Other profiles that are not Euclidean

We have just seen that one-dimensional Euclidean preferences cannot be characterized by finitely many forbidden substructures. We achieved this by providing an infinite sequence of $2k \times 4k$ profiles which fail to be Euclidean, but deleting any arbitrary voter makes them Euclidean. This naturally leads to the following question: *"Is this infinite sequence of profiles, together with the four forbidden substructures for the single-peakedness and single-crossingness, sufficient to characterize one-dimensional Euclideanness?"* The answer to this question is "no". For instance, the 3×6 profile shown in Example 1 is a forbidden substructure for Euclideanness, but it is not part of the sequence of profiles we constructed. In fact, any forbidden substructure with an odd number of voters cannot be constructed in that way. But how did we find the 3×6 profile? To answer this, let us take a closer look at the profiles constructed in Section 3.1. We observe that in each Euclidean embedding, the distances of two consecutive alternatives of such profiles display a so-called cyclic relation: (5a)–(5c), whose existence precludes Euclideanness. Indeed, we can construct an infinite sequence of non-Euclidean $n \times 2n$ profiles using the cyclic relations discussed at the end of Section 3.2. We consider n voters $v_1, v_2, \dots, v_n$ and $2n$ alternatives $1, 2, \dots, 2n$. We assume that the single-peaked permutations of the alternatives are $(1, 2, \dots, 2n)$ and its reverse. First, we observe that any cyclic relation of the distances between two neighboring alternatives $2i-1$ and $2i$, $1 \le i \le n$, in the single-peaked order can be considered as a permutation over the numbers $1, 2, \dots, n$. For instance, the cyclic relation of the 8×16 profile in Example 3 is

$$\begin{aligned} E[2]-E[1] < E[10]-E[9] < E[4]-E[3] < E[12]-E[11] < E[6]-E[5] \\ < E[14]-E[13] < E[8]-E[7] < E[16]-E[15] < E[2]-E[1]. \end{aligned}$$

It corresponds to the permutation $(1, 5, 2, 6, 3, 7, 4, 8)$. We have shown that this profile is not Euclidean (Lemma 2). In fact, we can show that every profile with a unique single-peaked order and with a cyclic relation with respect to this order is not Euclidean. However, how do we obtain this profile given a permutation that corresponds to a cyclic relation?

In the remainder of this section, we show how to construct a non-Euclidean profile for a given permutation $(\sigma(1), \sigma(2), \dots, \sigma(n))$ of the numbers $1, 2, \dots, n$. For each index i with $1 \le i \le n$, let $\min(i)$ and $\max(i)$ denote the minimum resp. maximum of the two numbers $\sigma(i)$ and $\sigma(i+1 \bmod n)$. Just as in Section 3.1, we define three preference pieces R_i, S_i, T_i.

$$\begin{aligned} R_i &:= 2\max(i)-2 \succ 2\max(i)-3 \succ \dots \succ 2\min(i)+1, \\ S_i &:= 2\min(i)-2 \succ 2\min(i)-3 \succ \dots \succ 1, \\ T_i &:= 2\max(i)+1 \succ 2\max(i)+2 \succ \dots \succ 2n. \end{aligned}$$

Note that for every $i = 1, \dots, n$, the pieces X_i, Y_i, Z_i cover contiguous intervals of respectively $2\max(i) - 2\min(i) - 2$, $2\min(i) - 2$, and $2n - 2\max(i)$ alternatives. Hence, they jointly cover $2n-4$ of the alternatives, and only the four alternatives $2\min(i)-1$, $2\min(i)$, $2\max(i)-1$, and $2\max(i)$ remain uncovered. We set

$$U_i := \begin{cases} 2\min(i) \succ 2\max(i)-1 \succ 2\max(i) \succ 2\min(i)-1 & \text{if } \min(i) \neq \sigma(i) \\ 2\max(i)-1 \succ 2\min(i) \succ 2\min(i)-1 \succ 2\max(i) & \text{if } \min(i) = \sigma(i). \end{cases}$$

As to the preference orders of the voters, for $1 \leq i \leq n$, voter v_i has the following preferences:

$$v_i: \quad R_i \succ U_i \succ S_i \succ T_i. \tag{18}$$

Using a similar reasoning as in the proof of Lemma 2, we can show that the profile with voters $v_1, v_2, \dots, v_{2n}$ whose preference orders are given by (18) is single-peaked and single-crossing, but not Euclidean. The 3×6 profile in Example 1 was constructed using this approach for alternatives $a = 1, b = 2, c = 3, d = 4, e = 5, f = 6$ and for the permutation $(1, 2, 3)$. Furthermore, the profiles in Section 3.1 were constructed using the permutation $(1, k+1, 2, k+2, \dots, k, 2k)$.

We conclude this section by conjecturing that the profile we constructed is minimal non-Euclidean with respect to voter deletion (see Property (b) of Theorem 1). One possible approach to proving this conjecture would require for each voter v_i, a Euclidean representation of the profile without this voter v_i; this requirement is the same as the one for the profiles constructed in Section 3.1. Such representations are, however, not easy to obtain.

5 Conclusions

We have shown that one-dimensional Euclidean preference profiles cannot be characterized in terms of finitely many forbidden substructures. This is similar to interval graphs, which also cannot be characterized by finitely many forbidden substructures. For interval graphs, however, we have a full understanding of all the forbidden substructures that are *minimal* with respect to vertex deletion [29]. In a similar vein, it would be interesting to determine all the (infinitely many) forbidden substructures for one-dimensional Euclidean preferences that are minimal with respect to deletion of voters or alternatives. At the moment, we only know how to construct non-Euclidean but single-peaked and single-crossing profiles using our cyclic construction (Section 4). We can neither show that such profiles are minimal with respect to voter deletion, nor can we show that such profiles are sufficient to characterize the one-dimensional Euclidean property.

As for general d-dimensional Euclidean preference profiles, we feel that the situation should be similar to the one-dimensional case: we conjecture that for any fixed value of $d \geq 2$, there will be no characterization of d-dimensional Euclidean profiles through finitely many forbidden substructures. Recently, Peters [33] confirms our conjecture for each fixed $d \geq 2$. We remind the reader that in a d-dimensional Euclidean preference profile, the voters and alternatives are embedded in d-dimensional Euclidean space, such that small distance corresponds to strong preference; see for instance the work of Bogomolnaia and Laslier [5].

Despite the negative result for higher-dimensional Euclideanness, Bulteau and Chen [9] study small non-Euclidean profiles for up to two dimension. In particular, every 3×7 profile, that is, a profile with three voters and up to seven alternatives, is 2-dimensional Euclidean. The corresponding proof is based on brute-force search through all possible profiles via computer programs. They also provide a non-2-dimensional Euclidean profile with three voters and 28 alternatives. It would be interesting to know the exact boundary between Euclideanness and non-Euclideanness.

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References

[1] M. Á. Ballester and G. Haeringer. A characterization of the single-peaked domain. *Social Choice and Welfare*, 36(2):305–322, 2011.

[2] S. Barberà and M. O. Jackson. Choosing how to choose: Self-stable majority rules and constitutions. *Quarterly Journal of Economics*, 119(3):1011–1048, 2004.

[3] J. J. Bartholdi III and M. Trick. Stable matching with preferences derived from a psychological model. *Operations Research Letters*, 5(4):165–169, 1986.

[4] D. Black. On the rationale of group decision making. *Journal of Political Economy*, 56 (1):23–34, 1948.

[5] A. Bogomolnaia and J.-F. Laslier. Euclidean preferences. *Journal of Mathematical Economics*, 43(2):87–98, 2007.

[6] S. J. Brams, M. A. Jones, and D. M. Kilgour. Single-peakedness and disconnected coalitions. *Journal of Theoretical Politics*, 14(3):359–383, 2002.

[7] R. Bredereck, J. Chen, and G. J. Woeginger. A characterization of the single-crossing domain. *Social Choice and Welfare*, 41(4):989–998, 2013.

[8] R. Bredereck, J. Chen, and G. J. Woeginger. Are there any nicely structured preference profiles nearby? *Mathematical Social Sciences*, 79:61–73, 2016.

[9] L. Bulteau and J. Chen. d-dimensional Euclidean preferences, 2016. working paper.

[10] F. chin Kung. An algorithm for stable and equitable coalition structures with public goods. *Journal of Public Economic Theory*, 8(3):345–355, 2006.

[11] C. H. Coombs. *A Theory of Data*. John Wiley and Sons, 1964.

[12] G. Demange. Intermediate preferences and stable coalition structures. *Journal of Mathematical Economics*, 23(1):45–58, 1994.

[13] P. A. Diamond and J. E. Stiglitz. Increases in risk and in risk aversion. *Journal of Economic Theory*, 8(3):337–360, 1974.

[14] J. Doignon and J. Falmagne. A polynomial time algorithm for unidimensional unfolding representations. *Journal of Algorithms*, 16(2):218–233, 1994.

[15] E. Elkind and P. Faliszewski. Recognizing 1-Euclidean preferences: An alternative approach. In *Proceedings of the 7th International Symposium on Algorithmic Game Theory (SAGT '14)*, volume 8768 of *Lecture Notes in Computer Science*, pages 146–157. Springer, 2014.

[16] E. Elkind and M. Lackner. On detecting nearly structured preference profiles. In *Proceedings of the 28th AAAI Conference on Artificial Intelligence (AAAI '14)*, pages 661–667. AAAI Press, 2014.

[17] E. Elkind, P. Faliszewski, and A. Slinko. Clone structures in voters' preferences. In *Proceedings of the 13th ACM Conference on Electronic Commerce (EC '12)*, pages 496–513. ACM Press, 2012.

[18] D. Epple and G. J. Platt. Equilibrium and local redistribution in an urban economy when households differ in both preferences and incomes. *Journal of Urban Economics*, 43(1):23–51, 1998.

[19] B. Escoffier, J. Lang, and M. Öztürk. Single-peaked consistency and its complexity. In *Proceedings of the 18th European Conference on Artificial Intelligence (ECAI '08)*, pages 366–370. IOS Press, 2008.

[20] S. Földes and P. L. Hammer. Split graphs. In *Proceedings of the Eighth Southeastern Conference on Combinatorics, Graph Theory and Computing*, pages 311–315, 1977.

[21] J. S. Gans and M. Smart. Majority voting with single-crossing preferences. *Journal of Public Economics*, 59(2):219–237, 1996.

[22] J.-M. Grandmont. Intermediate preferences and the majority rule. *Econometrica*, 46 (2):317–330, 1978.

[23] A. J. Hoffman, A. W. J. Kolen, and M. Sakarovitch. Totally-balanced and greedy matrices. *SIAM Journal on Algebraic and Discrete Methods*, 6(4):721–730, 1985.

[24] H. Hotelling. Stability in competition. *Economic Journal*, 39(153):41–57, 1929.

[25] K. ichi Inada. The simple majority decision rule. *Econometrica*, 37(3):490–506, 1969.

[26] S. Karlin. *Total Positivity*. Stanford University Press, 1968.

[27] V. Knoblauch. Recognizing one-dimensional Euclidean preference profiles. *Journal of Mathematical Economics*, 46(1):1–5, 2010.

[28] K. Kuratowski. Sur le problème des courbes gauches en topologie. *Fundamenta Mathematicae*, 15:271–283, 1930.

[29] C. G. Lekkerkerker and J. C. Boland. Representation of finite graphs by a set of intervals on the real line. *Fundamenta Mathematicae*, 51:45–64, 1962.

[30] A. H. Meltzer and S. F. Richard. A rational theory of the size of government. *Journal of Political Economy*, 89(5):914–927, 1981.

[31] J. A. Mirrlees. An exploration in the theory of optimal income taxation. *Review of Economic Studies*, 38:175–208, 1971.

[32] H. Moulin. On strategy-proofness and single peakedness. *Public Choice*, 35(4):437–455, 1980.

[33] D. Peters. Recognising multidimensional Euclidean preferences. In *Proceedings of the 6th International Workshop on Computational Social Choice (COMSOC '16)*, 2016.

[34] K. W. Roberts. Voting over income tax schedules. *Journal of Public Economics*, 8(3): 329–340, 1977.

[35] F. Westhoff. Existence of equilibria in economies with a local public good. *Journal of Economic Theory*, 14(1):84–112, 1977.

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Appendix

A Proof for Proposition 3

Let a and b be two alternatives in a Euclidean representation (E, F) of some profile with $E[a] < E[b]$. Then, voter v_i prefers a to b if and only if $F[i] < \frac{1}{2}(E[a]+E[b])$, and he prefers b to a if and only if $F[i] > \frac{1}{2}(E[a] + E[b])$.

Proof. By the definition of Euclideanness, voter v_i prefers a to b if and only if

$$|F[i] - E[a]| < |F[i] - E[b]| \quad (\star)$$

holds. Since $E[a] < E[b]$, $(\star)$ holds if and only if $F[i] < \frac{1}{2}(E[a] + E[b])$. The case of v_i preferring b to a can be shown using a symmetric reasoning. □

B Proof for Proposition 4

Let a, b, c be three alternatives in a Euclidean representation (E, F) of some preference profile with $E[a] < E[b] < E[c]$. If voter v_i prefers a to b, then he also prefers b to c. If voter v_i prefers c to b, then he also prefers b to a.

Proof. By the definition of the Euclideanness, the left-to-right order of the alternatives obtained from the embedding E is a single-peaked order. Then, the two statements follow immediately from the definition of single-peakedness. □

C Continued discussion of Example 2

We show that deleting any single voter from the 3×6 profile in Example 2 makes the profile one-dimensional Euclidean.

For the profile without voter v_1, we can verify that the following embedding is Euclidean:

$$E[a] = 0,\ E[b] = 5,\ E[c] = 7,\ E[d] = 8,\ E[e] = 10,\ E[f] = 13,\ F[v_2] = 7,\ F[v_3] = 9.5.$$

For the profile without voter v_2, we can verify that the following embedding is Euclidean:

$$E[a] = 0,\ E[b] = 1,\ E[c] = 3,\ E[d] = 6,\ E[e] = 7,\ E[f] = 12,\ F[v_1] = 2.5,\ F[v_3] = 7.$$

For the profile without voter v_3, we can verify that the following embedding is Euclidean:

$$E[a] = 0,\ E[b] = 3,\ E[c] = 5,\ E[d] = 10,\ E[e] = 12,\ E[f] = 13,\ F[v_1] = 4.5,\ F[v_3] = 7.$$

D Proof for Lemma 1

For $k \geq 2$, the profile $\mathcal{P}^*_k$ is single-peaked. Furthermore, the only two single-peaked orders of the alternatives are the increasing order $1, 2, 3, \ldots, 4k$ and the decreasing order $4k, \ldots, 3, 2, 1$.

Proof. Every voter v_{2i-1} and v_{2i} with $1 \le i \le k$ ranks $2k+2i-2$ in the first position. Furthermore, he ranks the small alternatives $1, 2, \ldots, 2k+2i-2$ in decreasing order, and he ranks the large alternatives $2k+2i-2, 2k+2i-1, \ldots, 4k$ in increasing order. Hence, $\mathcal{P}_k^*$ indeed is single-peaked with respect to the permutation $(1, 2, \ldots, 4k)$ and its reverse $(4k, \ldots, 2, 1)$.

Next, consider an arbitrary single-peaked permutation $(\pi(1), \pi(2), \ldots, \pi(4k))$ of the alternatives. Since $4k$ and 1 are the least preferred alternatives of voters v_1 and v_{2k}, these two alternatives must be extremal in the single-peaked order; by symmetry we assume that $\pi(1) = 1$ and $\pi(4k) = 4k$.

- Voter v_1 ranks $1 \succ 2k+2 \succ 2k+3 \succ \ldots \succ 4k$, without any other alternatives ranked in-between. This implies $\pi(x) = x$ for $2k+2 \le x \le 4k$.
- Voter v_{2k-1} ranks $2k+2 \succ 2k+1 \succ 2k \succ \ldots \succ 3 \succ 2 \succ 1$. This implies $\pi(x) = x$ for the remaining alternatives x with $1 \le x \le 2k+1$.

Summarizing, we have $\pi(x) = x$ for all x, which completes the proof. □

E Proof for Lemma 3

The embedding E_s satisfies $E_s[x] < E_s[y]$ for all alternatives x and y with $1 \le x < y \le 4k$. In other words, E_s satisfies the inequalities in (3).

Proof. The statement follows from (6)–(12) by an easy inductive argument. The right-hand sides in (7), (8), and (10) are all positive. The right-hand sides in (9) and (11) can be seen to be positive by induction. Finally for $i = 1$ and $s \ne 2k$, the right-hand side of (8) is a positive odd integer strictly greater than 1; this yields $E_s[2] \ge 3$ which implies that the right-hand side $E_s[2] - E_s[1] - 2$ in (12) is positive, too. □

F A collection of technical results

The embeddings we defined in Section 3.3 are compact but hard to comprehend. In order to show that they are indeed one-dimensional Euclidean, it suffices to know the distance between two alternatives instead of their exact positions. Thus, in this section, we state five technical lemmas which basically restate or summarize the distances of two alternatives in the embeddings. We will see that they are extensively used in Appendix G, where we show the correctness of the embeddings we defined.

We briefly explain the purpose of each of these five lemmas. Lemmas 5 and 6 summarize a number of useful identities which basically rewrite the distances between two consecutive alternatives in the embeddings. They serve as references in our later analysis. Lemmas 7 through 9 state important *inequalities* that will be central to our proofs of the Euclideanness of the embeddings defined in Section 3.3; the proofs are shown in Appendix G.

Recall that throughout we assume that $k \ge 5$ (see (17)) and that from Section 3.3, for each integer s, $1 \le s \le 2k$, we construct F_s and E_s for the profile obtained from $\mathcal{P}_k^*$ by deleting the voter v_s.

Lemma 5. *For $1 \le s \le 2k$, the Euclidean embedding E_s satisfies the following.*

$$E_s[2] \;=\; 4k - 2s + 1 \tag{19a}$$

$$E_s[3] \;=\; 4k - 2s + 3 \tag{19b}$$

$$E_s[4] \;=\; \begin{cases} 4k - 4s + 8 & \text{if } s \in \{1, 2\} \\ 8k - 4s + 8 & \text{if } s \ge 3. \end{cases} \tag{19c}$$

Furthermore, for $s \in \{1,2\}$, the embedding E_s satisfies the following.

$$E_s[2k-2] - E_s[2k-3] \;=\; 4k-2s-7 \tag{20a}$$

$$E_s[2k-4] - E_s[2k-5] \;=\; 4k-2s-11 \tag{20b}$$

Proof. These statements follow by straightforward calculations from (6)–(10). $\square$

Lemma 6. *If (a) $1 \le i \le k-1$ and $s \ne 2i-1$, or if (b) $i = k$ and $s \notin \{2k-1, 2k\}$, then the following holds:*

$$E_s[2k+2i] - E_s[2k+2i-1] \;=\; E_s[2i] - E_s[2i-1] + 2 \tag{21a}$$

If (c) $1 \le i \le k-1$ and $s \ne 2i$, the following holds:

$$E_s[2k+2i] - E_s[2k+2i-1] \;=\; E_s[2i+2] - E_s[2i+1] - 2 \tag{21b}$$

Proof. We distinguish five cases. The first case assumes $s = 2i-1$. In the setting of this lemma, this case can only occur under (c) with $1 \le i \le k-1$. Then (10) yields $E_s[2k+2i] - E_s[2k+2i-1] = 1$, while (8) yields $E_s[2i+2] - E_s[2i+1] = 3$. This implies the desired equality (21b) for this case.

The second case assumes $s = 2i$. In the setting of this lemma, this case can only occur under (a) with $1 \le i \le k-1$. Then (10) yields $E_s[2k+2i] - E_s[2k+2i-1] = 4k-1$, while (8) yields $E_s[2i] - E_s[2i-1] = 4k-3$. This implies the desired equality (21a).

The third case assumes $i = k$. In the setting of this lemma, this case can only occur under (b) with $1 \le s \le 2k-2$. Then (12) and (19a) yield $E_s[4k] - E_s[4k-1] = 4k-2s-1$, while (8) yields $E_s[2k] - E_s[2k-1] = 4k-2s-3$. This implies the desired equality (21a).

In the remaining cases we always have $s \notin \{2i-1, 2i\}$. The fourth case assumes that $1 \le i \le k-1$ and that $s = 2\ell - 1$ is odd, where $1 \le \ell \le k$ and $\ell \ne i$. In the setting of this lemma, this case can only occur under (a) and (b). Then (10) yields

$$E_s[2k+2i] - E_s[2k+2i-1] \;=\; 4(i-\ell)+1 \bmod 4k, \tag{22}$$

while (8) yields $E_s[2i] - E_s[2i-1] = 4(i-\ell)-1 \bmod 4k$. Since $i-\ell \ne 0$, these two equations together yield (21a). Furthermore, (8) yields $E_s[2i+2] - E_s[2i+1] = 4(i-\ell)+3 \bmod 4k$, which together with (22) gives (21b).

The fifth case assumes that $1 \le i \le k-1$ and that $s = 2\ell$ is even, where $1 \le \ell \le k$ and $\ell \ne i$. In the setting of this lemma, this case can only occur under (a) and (c). Then (10) yields

$$E_s[2k+2i] - E_s[2k+2i-1] \;=\; 4(i-\ell)-1 \bmod 4k, \tag{23}$$

while (8) yields $E_s[2i] - E_s[2i-1] = 4(i-\ell)-3 \bmod 4k$. Since $i-\ell \ne 0$, these two statements together imply (21a). Finally, (8) yields $E_s[2i+2] - E_s[2i+1] = 4(i-\ell)+1 \bmod 4k$. As $i-\ell \ne 0$, this inequality together with (23) yields (21b). This completes the proof. $\square$

Lemma 7. *For all alternatives x and y with $1 \le y \le x \le 4k$, the embedding E_s satisfies the inequality $E_s[x] - E_s[y] \ge x - y$.*

Proof. This follows from Lemma 3 and the integrality of E_s. $\square$

Lemma 8. *All i and s with $1 \le i \le 2k-1$ and $1 \le s \le 2k$ satisfy the following inequality.*

$$E_s[2i+1] - E_s[2i] \;\ge\; 2. \tag{24}$$

Proof. For $1 \le i \le k-1$, this follows directly from (7). For $k \le i \le 2k-1$, this follows from (9) and (11) in combination with Lemma 7. □

Lemma 9. *All i and s with $1 \le i \le k-1$ and $1 \le s \le 2k$ satisfy the following inequality.*

$$E_s[2k+2i-1] \;\ge\; E_s[2k] + E_s[2i] + 2. \tag{25}$$

Proof. We show the inequality by induction on $i = 1, \dots, k-1$. For the inductive base case $i = 1$ we distinguish between two cases for the value of s. The first case assumes $s \in \{1, 2\}$. Then (9) and $k \ge 5$, together with (7), (20a), (20b), and (19a) yield

$$\begin{aligned} E_s[2k+2i-1] - E_s[2k] \;&\ge\; E_s[2k-1] - E_s[4] + 2 \\ \ge\;\; & (E_s[2k-1] - E_s[2k-2]) + (E_s[2k-2] - E_s[2k-3]) \\ & +(E_s[2k-3] - E_s[2k-4]) + (E_s[2k-4] - E_s[2k-5]) + 2 \\ =\;\; & 2 + (4k-2s-7) + 2 + (4k-2s-11) + 2 \\ =\;\; & 8k - 4s - 12 \;>\; (4k-2s+1) + 2 \;=\; E_s[2] + 2. \end{aligned}$$

The second case assumes $s \ge 3$. Then, the first line of (9) together with $k \ge 5$, (19c), (19b) and (19a) yields

$$\begin{aligned} E_s[2k+2i-1] - E_s[2k] \;&=\; E_s[2k-1] - E_s[3] + 2 \\ \ge\;\; & E_s[4] - E_s[3] + 2 \;=\; (8k-4s+8) - (4k-2s+3) + 2 \\ =\;\; & 4k - 2s + 7 \;>\; E_s[2] + 2. \end{aligned}$$

Summarizing, in both cases we have established the desired inequality (25). This completes the analysis of the inductive base case $i = 1$. Next, let us state the inductive assumption as

$$E_s[2k+2i-3] \;\ge\; E_s[2k] + E_s[2i-2] + 2. \tag{26}$$

In the inductive step, we will use the following implication of (9):

$$E_s[2k+2i-1] - E_s[2k+2i-2] \;\ge\; E_s[2k+2i-3] - E_s[2i+2] + 2. \tag{27}$$

Furthermore, by (10), the left-hand side of the following inequality equals $(4i - 2s - 5 \bmod 4k)$, while by (8), its right-hand side equals $(4i - 2s - 3 \bmod 4k) - 2$. This implies

$$E_s[2k+2i-2] - E_s[2k+2i-3] \;\ge\; E_s[2i] - E_s[2i-1] - 2. \tag{28}$$

Adding up (26), (27) and (28), and rearranging and simplifying the resulting inequality yields

$$\begin{aligned} & E_s[2k+2i-1] - E_s[2k] - E_s[2i] - 2 \\ \ge\;\; & E_s[2k+2i-3] - E_s[2i+2] + E_s[2i-2] - E_s[2i-1] \\ \ge\;\; & (2k+2i-3) - (2i+2) - 2 \;=\; 2k-7 \;>\; 0. \end{aligned}$$

Here we use Lemma 7 to bound $E_s[2k+2i-3] - E_s[2i+2]$, and we use eq. (7) to get rid of $E_s[2i-2] - E_s[2i-1]$. As this implies eq. (25), the inductive argument is complete. □

G Correctness of the Euclidean embeddings

In this section, we prove Lemma 4, that is, we show that the embeddings defined in Section 3.3 are one-dimensional Euclidean. This directly implies Property (b) of Theorem 1, one of our main contribution.

Let v_r and v_s be two arbitrary voters with $r \neq s$. We recall that by Lemma 3 the Euclidean representation E_s embeds the alternatives $1, \ldots, 4k$ in increasing order from left to right. We show that any two alternatives x and y with $x \succ_r y$ which are consecutive in the preference order of voter v_r satisfy

$$2F_s[r] \;<\; E_s[x] + E_s[y] \qquad \text{whenever } x < y, \tag{29a}$$

$$2F_s[r] \;>\; E_s[x] + E_s[y] \qquad \text{whenever } x > y. \tag{29b}$$

By our construction, all preference orders in profile $\mathcal{P}_k^*$ contain long monotonously increasing or decreasing runs of alternatives. By Proposition 4, it is therefore sufficient to establish (29a) and (29b) at the few turning points where the preference order of voter v_r changes its monotonicity behavior. We emphasize that the first pair of alternatives in every preference order forms a turning point by default.

Our proof distinguishes between four cases which are handled separately in the following four subsections. Sections G.1 and G.2 deal with the cases with odd r, while Sections G.3 and G.4 deal with the cases where r is even.

G.1 The cases with odd r (with a single exception)

In this section we assume that $r = 2i-1$ for $1 \leq i \leq k$, implying that $s \neq 2i-1$ because v_s is the deleted voter. If $i = k$ (and hence $r = 2k-1$), then we additionally assume $s \neq 2k$; the remaining case with $i = k$ and $s = 2k$ will be dealt with in the next subsection. Note that under these assumptions, the value $F_s[2i-1]$ is given by (13). Furthermore (21a) in Lemma 6 yields

$$E_s[2k+2i] + E_s[2i-1] \;=\; E_s[2i] + E_s[2k+2i-1] + 2. \tag{30}$$

In order to prove (29a) and (29b) for the preference orders in (1a) and (2a), it is sufficient to establish the following six inequalities for the turning points.

$$2F_s[2i-1] \;>\; E_s[2k+2i-2] + E_s[2k+2i-3] \tag{31a}$$

$$2F_s[2i-1] \;<\; E_s[2i+1] + E_s[2k+2i-1] \tag{31b}$$

$$2F_s[2i-1] \;>\; E_s[2k+2i-1] + E_s[2i] \tag{31c}$$

$$2F_s[2i-1] \;<\; E_s[2i-1] + E_s[2k+2i] \tag{31d}$$

$$2F_s[2i-1] \;>\; E_s[2k+2i] + E_s[2i-2] \tag{31e}$$

$$2F_s[2i-1] \;<\; E_s[1] + E_s[2k+2i+1] \tag{31f}$$

Note that for $i = 1$, inequality (31e) vanishes as Y_1 is empty, and that for $i = k$ inequality (31f) vanishes as Z_k is empty. We use (13) or the first line of (15) together with (30), and rewrite the left-hand side of all inequalities (31a)–(31f) as

$$\begin{aligned} 2F_s[2i-1] = \;& \frac{1}{2}\left(E_s[2i-1] + E_s[2i] + E_s[2k+2i-1] + E_s[2k+2i]\right) \\ =\;& E_s[2i] + E_s[2k+2i-1] + 1 \;=\; E_s[2i-1] + E_s[2k+2i] - 1. \end{aligned} \tag{32}$$

For (31a), we distinguish between two subcases. The first subcase assumes $i \leq k-1$. We use (32), (9) with $s \neq 2i-1$, and (7) to obtain

$$\begin{aligned} & 2F_s[2i-1] - E_s[2k+2i-2] - E_s[2k+2i-3] \\ & = (E_s[2i] + E_s[2k+2i-1] + 1) - E_s[2k+2i-2] - E_s[2k+2i-3] \\ & = E_s[2i] + 1 - E_s[2i+1] + 2 = 1 > 0. \end{aligned}$$

The second subcase deals with the remaining case $i = k$. We use (32), the first line in (11), and Lemma 3 to obtain

$$\begin{aligned} & 2F_s[2i-1] - E_s[2k+2i-2] - E_s[2k+2i-3] \\ & = (E_s[2k] + E_s[4k-1] + 1) - E_s[4k-2] - E_s[4k-3] \\ & = (E_s[2k] + 1) + (-E_s[2] + 2) = (E_s[2k] - E_s[2]) + 3 > 0. \end{aligned}$$

For (31b), using (32) and (24) yields

$$\begin{aligned} & 2F_s[2i-1] - E_s[2i+1] - E_s[2k+2i-1] \\ & = (E_s[2i] + E_s[2k+2i-1] + 1) - E_s[2i+1] - E_s[2k+2i-1] < 0. \end{aligned}$$

For (31c), from (32) we obtain

$$\begin{aligned} & 2F_s[2i-1] - E_s[2k+2i-1] - E_s[2i] \\ & = (E_s[2i] + E_s[2k+2i-1] + 1) - E_s[2k+2i-1] - E_s[2i] = 1 > 0. \end{aligned}$$

For (31d), we use (32) and get

$$\begin{aligned} & 2F_s[2i-1] - E_s[2i-1] - E_s[2k+2i] \\ & = (E_s[2i-1] + E_s[2k+2i] - 1) - E_s[2i-1] - E_s[2k+2i] = -1 < 0. \end{aligned}$$

For (31e) with $i \geq 2$, using (32) and (7) we obtain

$$\begin{aligned} & 2F_s[2i-1] - E_s[2k+2i] - E_s[2i-2] \\ & = (E_s[2i-1] + E_s[2k+2i] - 1) - E_s[2k+2i] - E_s[2i-2] = 1 > 0. \end{aligned}$$

It remains to prove inequality (31f) which takes more effort. Since (31f) vanishes for $i = k$, we assume $i \leq k-1$. We first use (32) and (6) to derive

$$\begin{aligned} & 2F_s[2i-1] - E_s[1] - E_s[2k+2i+1] \\ & = E_s[2i-1] - 1 - (E_s[2k+2i+1] - E_s[2k+2i]). \end{aligned} \tag{33}$$

Our goal is to show that the value in (33) is strictly negative, and for this we distinguish between three subcases. The first subcase assumes $i \leq k-2$. We use (9), (25), and Lemma 3 to obtain

$$\begin{aligned} & E_s[2i-1] - 1 - (E_s[2k+2i+1] - E_s[2k+2i]) \\ & \leq E_s[2i-1] - 1 - (E_s[2k+2i-1] - E_s[2i+4] + 2) \\ & \leq E_s[2i-1] - 3 + E_s[2i+4] - (E_s[2k] + E_s[2i] + 2) \\ & = (E_s[2i+4] - E_s[2k]) + (E_s[2i-1] - E_s[2i]) - 5 < 0. \end{aligned}$$

The second subcase assumes $i = k - 1$ and $s \neq 2k$. We use (11), (25), and Lemma 3 to obtain

$$
\begin{aligned}
& E_s[2i-1] - 1 - (E_s[2k+2i+1] - E_s[2k+2i]) \\
&= E_s[2k-3] - 1 - (E_s[4k-1] - E_s[4k-2]) \\
&= E_s[2k-3] - 1 - (E_s[4k-3] - E_s[2] + 2) \\
&\leq E_s[2k-3] - 3 + E_s[2] - (E_s[2k] + E_s[2k-2] + 2) \\
&= (E_s[2k-3] - E_s[2k]) + (E_s[2] - E_s[2k-2]) - 5 \; < \; 0.
\end{aligned}
$$

The third and last subcase assumes $i = k - 1$ and $s = 2k$. We use the second line of (11), the first line of (9), inequality (25), equality (7), and Lemma 3 to obtain

$$
\begin{aligned}
& E_s[2i-1] - 1 - (E_s[2k+2i+1] - E_s[2k+2i]) \\
&= E_s[2k-3] - 1 - (E_s[4k-1] - E_s[4k-2]) \\
&= E_s[2k-3] - 1 - (E_s[4k-3] - E_s[2k+1] + 2) \\
&= E_s[2k-3] - 3 + E_s[2k+1] - (E_s[4k-4] + E_s[4k-5] - E_s[2k-1] + 2) \\
&\leq E_s[2k-3] - 5 + E_s[2k+1] - E_s[4k-4] \\
&\quad + E_s[2k-1] - (E_s[2k] + E_s[2k-4] + 2) \\
&= (E_s[2k+1] - E_s[4k-4]) + (E_s[2k-1] - E_s[2k]) - 5 \; < \; 0.
\end{aligned}
$$

As (33) is strictly negative in each of the three subcases, the proof of (31f) is complete. Together, E_s and F_s form an Euclidean representation of the preferences of voter v_r.

G.2 The exceptional case with odd r

In this section we consider the special case when $i = k$ (and hence $r = 2k - 1$) and $s = 2k$, which has been left open in the previous subsection. In this case, the embedding $F_s[2k-1]$ is given by the second option in formula (15). Furthermore, (12) and (7) yield

$$E_s[4k] - E_s[4k-1] \;=\; E_s[2k+1] - E_s[2k-2] - 2.$$

Altogether this leads to

$$
\begin{aligned}
2F_s[2k-1] &= \frac{1}{2}\left(E_s[2k-2] + E_s[2k+1] + E_s[4k-1] + E_s[4k]\right) \\
&= E_s[2k-2] + E_s[4k] + 1 \;=\; E_s[2k+1] + E_s[4k-1] - 1. \qquad (34)
\end{aligned}
$$

As inequality (31f) vanishes for $i = k$, our goal in this section is to establish the five inequalities (31a)–(31e) for $i = k$ and $s = 2k$. For (31a), we use (34) and (11) and get

$$
\begin{aligned}
& 2F_s[2k-1] - E_s[4k-2] - E_s[4k-3] \\
&= (E_s[2k+1] + E_s[4k-1] - 1) - E_s[4k-2] - E_s[4k-3] \\
&= E_s[2k+1] - E_s[4k-3] - 1 + (E_s[4k-3] - E_s[2k+1] + 2) \;=\; 1 \; > \; 0.
\end{aligned}
$$

For (31b), we use (34) and obtain

$$
\begin{aligned}
& 2F_s[2k-1] - E_s[2k+1] - E_s[4k-1] \\
&= (E_s[2k+1] + E_s[4k-1] - 1) - E_s[2k+1] - E_s[4k-1] \;=\; -1 \; < \; 0.
\end{aligned}
$$

For (31c), using (34) and (24) yields

$$\begin{aligned} &2F_s[2k-1] - E_s[4k-1] - E_s[2k] \\ &= \; (E_s[2k+1] + E_s[4k-1] - 1) - E_s[4k-1] - E_s[2k] \; > \; 0. \end{aligned}$$

For (31d), we use (34) and (7) and get

$$\begin{aligned} &2F_s[2k-1] - E_s[2k-1] - E_s[4k] \\ &= \; (E_s[2k-2] + E_s[4k] + 1) - E_s[2k-1] - E_s[4k] \; = \; -1 \; < \; 0. \end{aligned}$$

For (31e), from (34) we obtain

$$\begin{aligned} &2F_s[2k-1] - E_s[4k] - E_s[2k-2] \\ &= \; (E_s[2k-2] + E_s[4k] + 1) - E_s[4k] - E_s[2k-2] \; = \; 1 \; > \; 0. \end{aligned}$$

This completes the analysis of the special case with odd r. In this case as well, E_s and F_s together form a Euclidean representation of preferences of voter v_r.

G.3 The cases with even r (with a single exception)

In this section, we consider the cases of even $r = 2i$ for $1 \leq i \leq k-1$, and of $s \neq 2i$. We deal with the remaining case where $r = 2k$ in the next subsection. Note that in this case, the value $F_s[2i]$ is given by (14). Furthermore (21b) in Lemma 6 yields

$$E_s[2i+1] + E_s[2k+2i] \; = \; E_s[2i+2] + E_s[2k+2i-1] - 2. \tag{35}$$

In order to prove (29a) and (29b) for the preference orders in (1b), it is sufficient to show that the following four inequalities for the turning points hold.

$$2F_s[2i] \; > \; E_s[2k+2i-2] + E_s[2k+2i-3] \tag{36a}$$

$$2F_s[2i] \; < \; E_s[2i+2] + E_s[2k+2i-1] \tag{36b}$$

$$2F_s[2i] \; > \; E_s[2k+2i] + E_s[2i+1] \tag{36c}$$

$$2F_s[2i] \; < \; E_s[1] + E_s[2k+2i+1] \tag{36d}$$

We use the definition of $F_s[2i]$ in (14) together with (35) to rewrite the left-hand side of all inequalities (36a)–(36d) as

$$\begin{aligned} 2F_s[2i] = \; & \frac{1}{2}\left(E_s[2i+1] + E_s[2i+2] + E_s[2k+2i-1] + E_s[2k+2i]\right) \\ = \; & E_s[2i+1] + E_s[2k+2i] + 1 \; = \; E_s[2i+2] + E_s[2k+2i-1] - 1. \end{aligned} \tag{37}$$

For (36a), we use (37) and (9) to obtain

$$\begin{aligned} &2F_s[2i] - E_s[2k+2i-2] - E_s[2k+2i-3] \\ &= \; (E_s[2i+2] + E_s[2k+2i-1] - 1) - E_s[2k+2i-2] - E_s[2k+2i-3] \\ &\geq \; E_s[2i+2] - E_s[2k+2i-3] - 1 + (E_s[2k+2i-3] - E_s[2i+2] + 2) = 1 > 0. \end{aligned}$$

For (36b), from (37) we get

$$\begin{aligned} &2F_s[2i] - E_s[2i+2] - E_s[2k+2i-1] \\ &= \; (E_s[2i+2] + E_s[2k+2i-1] - 1) - E_s[2i+2] - E_s[2k+2i-1] = -1 < 0. \end{aligned}$$

For (36c), using (37) yields

$$\begin{aligned}
&2F_s[2i] - E_s[2k+2i] - E_s[2i+1] \\
&\quad = \quad (E_s[2i+1] + E_s[2k+2i] + 1) - E_s[2k+2i] - E_s[2i+1] = 1 > 0.
\end{aligned}$$

It remains to prove inequality (36d) which takes a considerable amount of work. We distinguish between three subcases. The first subcase assumes $1 \leq i \leq k-2$. Then Lemma 3 implies $E_s[2i+4] \leq E_s[2k]$. We use (37), (6), (9), (25) and (7) to derive

$$\begin{aligned}
&2F_s[2i] - E_s[1] - E_s[2k+2i+1] \\
&\quad = \quad (E_s[2i+1] + E_s[2k+2i] + 1) - E_s[2k+2i+1] \\
&\quad \leq \quad E_s[2i+1] + 1 - (E_s[2k+2i-1] - E_s[2i+4] + 2) \\
&\quad \leq \quad E_s[2i+1] + E_s[2i+4] - 1 - (E_s[2k] + E_s[2i] + 2) \\
&\quad = \quad E_s[2i+4] - E_s[2k] - 1 \ < \ 0.
\end{aligned}$$

The second subcase assumes $i = k-1$ and $s \neq 2k$. To prove (36d), we use (37), (6), (11), (25) and (7) and obtain

$$\begin{aligned}
&2F_s[2k-2] - E_s[1] - E_s[4k-1] \\
&\quad = \quad (E_s[2k-1] + E_s[4k-2] + 1) - E_s[4k-1] \\
&\quad = \quad E_s[2k-1] + 1 - (E_s[4k-3] - E_s[2] + 2) \\
&\quad \leq \quad E_s[2k-1] + E_s[2] - 1 - (E_s[2k] + E_s[2k-2] + 2) \\
&\quad = \quad E_s[2] - E_s[2k] - 1 \ < \ 0.
\end{aligned}$$

The third and last subcase assumes $i = k-1$ and $s = 2k$. We begin by deriving a number of auxiliary equations and inequalities. First, we obtain $E_s[3] = 3$ from (19b), and use (9) and (7) to get

$$\begin{aligned}
&E_s[2k+1] - E_s[2k-2] \\
&\quad = \quad (E_s[2k+1] - E_s[2k]) + (E_s[2k] - E_s[2k-1]) + (E_s[2k-1] - E_s[2k-2]) \\
&\quad = \quad (E_s[2k-1] - E_s[3] + 2) + (E_s[2k] - E_s[2k-1]) + 2 \quad = \quad E_s[2k] + 1. \qquad (38)
\end{aligned}$$

Next, (21b) gives us

$$E_s[2k-2] - E_s[2k-3] - 2 \quad = \quad E_s[4k-4] - E_s[4k-5]. \qquad (39)$$

We express $E_s[4k-3]$ once by the first line of (9) and once by the second line of (11), which by equating yields

$$\begin{aligned}
&E_s[4k-4] + E_s[4k-5] - E_s[2k-1] + 2 \\
&\quad = \quad E_s[2k+1] - 2 + E_s[4k-1] - E_s[4k-2]. \qquad (40)
\end{aligned}$$

Next, we add (38), (39), (40) and rearrange the result to obtain

$$\begin{aligned}
&E_s[2k-1] + E_s[4k-2] - E_s[4k-1] + 1 \\
&\quad = \quad 2E_s[2k-1] + E_s[2k] + E_s[2k-3] - 2E_s[4k-5]. \qquad (41)
\end{aligned}$$

We obtain $E_s[2k] - E_s[2k-1] = 4k-3$ and $E_s[2k-2] - E_s[2k-3] = 4k-7$ from (8), and use these together with (7) to get

$$\begin{aligned} & E_s[2k-1] - E_s[2k-4] + E_s[2k-1] - E_s[2k] \\ =\ & (E_s[2k-2]+2) - (E_s[2k-3]-2) - (E_s[2k] - E_s[2k-1]) \\ =\ & 2 + (4k-7) + 2 - (4k-3) \ = \ 0. \end{aligned} \tag{42}$$

In the third and last subcase, to finally prove (36d), we use (37), (6), (41), (25), (42) and (7) to obtain

$$\begin{aligned} & 2F_s[2k-2] - E_s[1] - E_s[4k-1] \\ =\ & (E_s[2k-1] + E_s[4k-2] + 1) - E_s[4k-1] \\ =\ & 2E_s[2k-1] + E_s[2k] + E_s[2k-3] - 2E_s[4k-5] \\ \leq\ & 2E_s[2k-1] + E_s[2k] + E_s[2k-3] - 2(E_s[2k] + E_s[2k-4] + 2) \\ =\ & E_s[2k-3] - E_s[2k-4] - 4 \ = \ -2 \ < \ 0. \end{aligned}$$

This completes the proof of inequality (36d). Summarizing, E_s and F_s together form a Euclidean representation of the preferences of v_r.

G.4 The exceptional case with even r

In this section, we consider the last remaining case of even r, where $r = 2k$ and $s \neq 2k$. In order to prove (29a) and (29b) for the preference orders in (2b), it is sufficient to establish the following three inequalities for the turning points.

$$2F_s[2k] \ > \ E_s[4k-2] + E_s[4k-3] \tag{43a}$$

$$2F_s[2k] \ < \ E_s[2] + E_s[4k-1] \tag{43b}$$

$$2F_s[2k] \ > \ E_s[4k] + E_s[1] \tag{43c}$$

For the common left-hand side of (43a)–(43c), the definition of $F_s[2k]$ in (16), and (12) with $s \neq 2k$ yield that

$$\begin{aligned} 2F_s[2k] =\ & \frac{1}{2}\,(E_s[1] + E_s[2] + E_s[4k-1] + E_s[4k]) \\ =\ & E_s[4k] + E_s[1] + 1 \ = \ E_s[4k-1] + E_s[2] - 1. \end{aligned} \tag{44}$$

For (43a), we use (44) and (11) with $s \neq 2k$ to obtain

$$\begin{aligned} & 2F_s[2k] - E_s[4k-2] - E_s[4k-3] \\ =\ & (E_s[4k-1] + E_s[2] - 1) - E_s[4k-2] - E_s[4k-3] \\ =\ & E_s[2] - E_s[4k-3] - 1 + (E_s[4k-3] - E_s[2] + 2) \ = \ 1 \ > \ 0. \end{aligned}$$

For (43b), using (44) we get

$$\begin{aligned} & 2F_s[2k] - E_s[2] - E_s[4k-1] \\ =\ & (E_s[4k-1] + E_s[2] - 1) - E_s[2] - E_s[4k-1] \ = \ -1 \ < \ 0. \end{aligned}$$

For (43c), using (44) we obtain

$$\begin{aligned} & 2F_s[2k] - E_s[4k] + E_s[1] \\ =\ & (E_s[4k] + E_s[1] + 1) - E_s[4k] - E_s[1] \ = \ 1 \ > \ 0. \end{aligned}$$

This settles the last case. The proof of Lemma 4 and with it the proof of Theorem 1 are finally completed.