Agendas with Priority

Sean Horan

Abstract

I characterize the social choice rules implemented by *sophisticated voting* on the broad class of *priority agendas*. The result provides key insights into the kinds of strategic voting outcomes that can arise in the context of legislative voting.

I. Introduction

Agenda voting is ubiquitous in legislative decision-making. While a wide variety of agendas are used in practice (Farquharson [1969]; Miller [1995]; Ordeshook and Schwartz [1987]; Riker [1958]), the literature has only considered *sophisticated voting* behavior (Farquharson [1969]) for a few specific agendas, notably the *Euro-Latin* and *Anglo-American* agendas (Apesteguia et al. [2014]; Banks [1985]; Miller [1977, 1980, 1995]; Sheplse and Weingast [1984]).[1]

In this paper, I introduce a much broader class of agendas, called *priority agendas*, and characterize sophisticated voting for these agendas. For priority agendas, alternatives are added one at a time using a *priority order* and an *amendment rule*. While the former determines *when* a given alternative is to be added to the agenda, the latter determines *how* it is to be added. The simple recursive structure of these agendas reflects the inherently incremental nature of the legislative process. Since they account for this reality, priority agendas possess two features associated with almost every agenda used in practice (see Ordeshook and Schwartz [1987]; Miller [1995]): every vote eliminates some alternatives from consideration; and, alternatives are contested until they are either eliminated or ultimately selected.[2] In other words, priority agendas are *non-repetitive* and *continuous*.

At the same time, priority agendas depart only minimally from Euro-Latin and Anglo-American agendas. Effectively, they only expand the scope of possible proposals. To elaborate, recall that agendas induce "binary" extensive-form games with majority voting in every stage. The figure below illustrates the Euro-Latin and Anglo-American games for three alternatives:

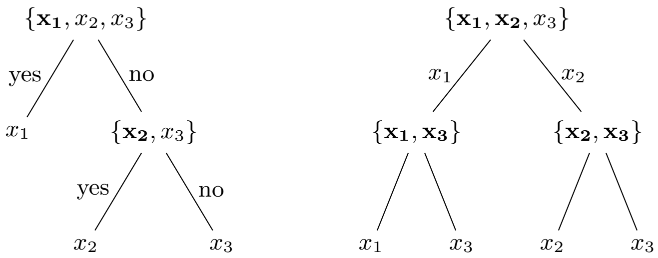

Figure 1: Euro-Latin (*left*) and Anglo-American (*right*) agendas on three alternatives

For the Euro-Latin agenda, voting is by *sequential majority approval*. In every stage, voters consider one alternative for approval (in bold). The selection from the agenda is the first alter-

[1] While a variety of names have been used for these two agendas, I follow the nomenclature of Schwartz [2008].
[2] Notably, the two-stage amendment agendas studied by Banks [1989] do not possess the second feature.

native approved by majority. For the *Anglo-American* agenda, voting is by *sequential majority comparison.* In every stage, voters compare two alternatives (in bold) – with the "loser" being eliminated and "winner" moving on to the next stage. The selection is the only alternative not eliminated by the end of this process.

It is straightforward to extend both agendas by proposing new alternatives as "amendments" to items already on the agenda.[3] For a Euro-Latin agenda, a new proposal amends the *last* alternative proposed. For an Anglo-American agenda, a new proposal amends *every* alternative proposed before it. To illustrate:

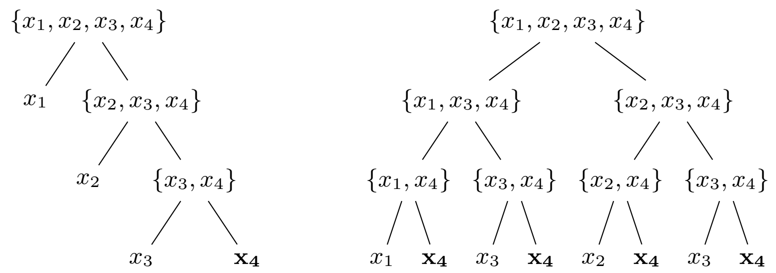

Figure 2: Extending the Euro-Latin (*left*) and Anglo-American (*right*) agendas in Figure 1

Intuitively, the force of an amendment is to confront voters with an additional decision. For a *Euro-Latin amendment* (*left*), voters only face such a decision when they would otherwise select the previously last alternative on the agenda (x_3). For an *Anglo-American amendment* (*right*), voters face an additional decision regardless of which alternative they would otherwise select.

The more flexible structure of priority agendas allows the agenda-setter to mix and match these two kinds of amendments: Euro-Latin agendas can be extended by Anglo-American amendment; and, Anglo-American agendas can be extended by Euro-Latin amendment. To illustrate:

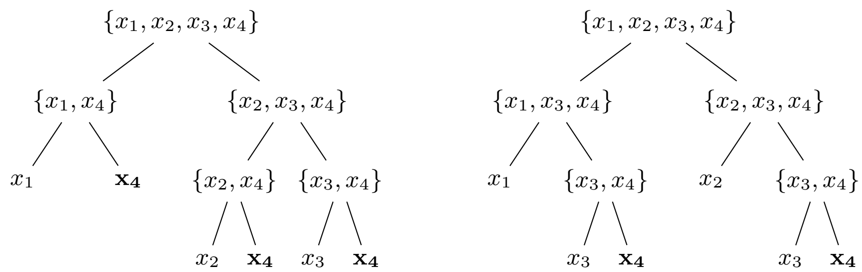

Figure 3: Priority agendas that extend Euro-Latin (*left*) and Anglo-American (*right*) agendas

More broadly, the amendment rules of priority agendas allow a new addition y to amend any alternative x already on the agenda—subject only to the natural restriction that y also amend every alternative added to the agenda after x. Intuitively, an alternative that "takes issue" with a particular alternative on the agenda must also take issue with the other additions that took issue with the same alternative.

[3] I use the term "amendment" out of convenience. Whether the proposal associated with a given alternative is technically an "amendment", a "motion", or a "substitute bill" will depend on how it is added to the agenda.

Related Literature: At one extreme, the literature on sophisticated voting considers only a narrow class of agendas. Besides the cited work on Euro-Latin and Anglo-American agendas, Banks [1989] studies sophisticated voting for *two-stage amendment agendas*; and, a few papers examine agendas implementing outcomes in the *Iterated Banks Set* (Coughlan and Le Breton [1999]) or outcomes with "high" *Copeland scores* (Fischer et al. [2011]; Iglesias et al. [2014]).

At the other, the literature focuses on necessary and sufficient conditions for implementation (Horan [2013]; McKelvey and Niemi [1978]; Moulin [1986]; Srivastava and Trick [1996]). While this work clearly delimits what can be implemented by sophisticated agenda voting *in general*, it does not help clarify what can be implemented by any *specific* agenda. In part, this lacuna is related to the fact that the results rely on proof techniques that are partially non-constructive.

The current paper bridges the gap between these disparate strands of the literature. By characterizing sophisticated voting for a wide range of agendas used in practice, it sheds light on the voting outcomes that can be implemented by decision-making procedures used in practice.

II. Basic Definitions

In this section, I briefly review the basic definitions and concepts used in the paper.

The environment consists of an odd number of voters with linear order preferences over the alternatives in a finite set X. A *preference profile* of voters is denoted by P and the collection of all profiles by $\mathbf{P}$. A *decision problem* (P, A) consists of a profile P and a set of alternatives (known as an *issue*) $A \subseteq X$. Where $\mathbf{X}$ denotes the collection of non-empty issues, a *decision rule* defines a mapping $v : \mathbf{P} \times \mathbf{X} \to X$ which selects a single social outcome $v(P, A) \in A$ for every decision problem $(P, A) \in \mathbf{P} \times \mathbf{X}$.

I study the implementation of decision rules by agenda. To formalize the notion of an agenda:

Definition 1 An **agenda** $\mathcal{T}_X$ on a set of alternatives X is a rooted binary tree such that:

(1) every terminal node is labeled by (a set consisting of) one alternative in X;[4]
(2) every alternative in X labels one or more terminal nodes; and,
(3) every non-terminal node is labeled by the set of alternatives that label its two successors.[5]

Figures 1-3 of the Introduction clearly illustrate these three features.

For an issue $A \subset X$, one can "prune" the agenda $\mathcal{T}_X$ by deleting the terminal nodes labeled by alternatives in $X \setminus A$. This operation is considered in a number of other papers (Bossert and Sprumont [2013]; Horan [2011]; and, Xu and Zhou [2007]). Like an elimination-style tournament in sports, the infeasible alternatives "forfeit" without changing the structure of the agenda.

Definition 2 Given an agenda $\mathcal{T}_X$, the **pruned agenda** $\mathcal{T}_{X|A}$ for an issue $A \subset X$ is defined as follows:

(1) delete every terminal node of $\mathcal{T}_X$ labeled by an alternative $x \in X \setminus A$; then,
(2) delete every non-terminal node with a unique successor, connecting its successor and predecessor;
(3) and, finally, relabel every non-terminal node of the resulting tree to conform with Definition 1.

To illustrate, consider the agenda $\mathcal{T}_X$ below and the pruned agenda for the issue $\{b, c, x\}$:

[4]Wherever it causes no confusion, I abuse set notation by omitting the brackets for singleton sets.

[5]Property (3) follows the "Farquharson-Miller" definition of agendas rather than the "Ordeshook-Schwartz" definition (see Schwartz [2008]). Since the interest is sophisticated voting rather than *sincere* voting, this is without loss of generality.

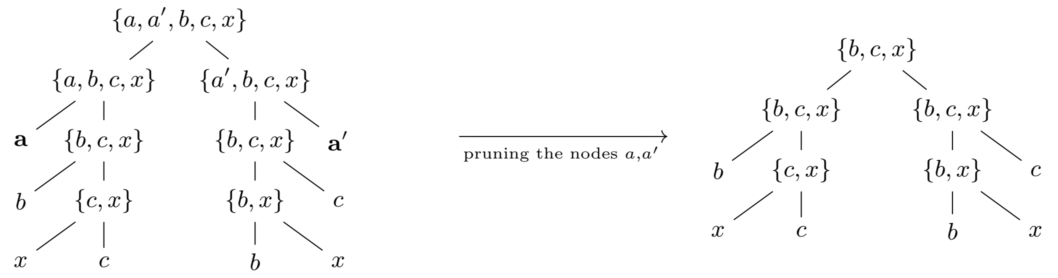

Figure 4: An agenda $\mathcal{T}_X$ (*left*) and its associated pruned agenda $\mathcal{T}_{X|\{b,c,x\}}$ (*right*)

Every agenda $\mathcal{T}_{X|A}$ defines an extensive game form where the outcomes are the terminal nodes and the stage games (or *decision nodes*) consist of majority voting between two subgames. Given a decision problem (P, A), the pair $(\mathcal{T}_{X|A}; P)$ describes a complete information extensive-form game on the pruned agenda $\mathcal{T}_{X|A}$. Every such game is *dominance solvable* (Moulin [1979]). In other words, $(\mathcal{T}_{X|A}; P)$ has a unique *undominated Nash equilibrium* outcome, denoted by $UNE[\mathcal{T}_{X|A}; P]$, for all $A \subseteq X$.

This solution concept corresponds to Farquharson's [1969] notion of *sophisticated voting.* The idea is that sophisticated voters anticipate the outcome of voting in later stages. Since they have a dominant strategy to endorse their preferred candidate in any terminal subgame, voters discount alternatives that lose at this stage. By using this "backward induction" reasoning to roll back the agenda to the root, one obtains the undominated Nash equilibrium outcome (McKelvey and Niemi [1978]).

To formalize the notion of implementation considered in the paper:

Definition 3 *A decision rule v is* ***implementable by agenda*** *if there exists an agenda $\mathcal{T}_X$ such that*

$$v(P, A) = UNE[\mathcal{T}_{X|A}; P]$$

for every decision problem (P, A). In that case, the agenda $\mathcal{T}_X$ is said to ***implement*** *v.*

Despite superficial appearances, this notion of implementation is no more (and no less) general than the standard notion of implementation where the issue is fixed (i.e. does not vary from X).

Remark 1 *If a decision rule v is implementable by agenda, then $v(P, A) = v(P^A, X)$ for any profile P^A that coincides with P on A but "demotes" all $x \in X \setminus A$ to the bottom of every voter preference.*

This shows that the sub-issues $A \subset X$ formally contribute nothing to the difficulty of the implementation problem. Once the agenda-setter has determined what to implement for X, the outcomes for all sub-issues are determined. Having said this, there are compelling reasons to think about the sub-issues explicitly. For one, some problems of economic interest, such as the issue of *strategic candidacy*, depend on how the outcomes change when some alternatives become unavailable (Dutta et al. [2002]). No less compelling, the sub-issues simplify the statement of the conditions for implementation as well as their interpretation.

III. Implementation by Simple Agenda

Before turning to priority agendas in Section IV, I first consider the more general class of *simple agendas.* After formally defining these agendas in part (a), I identify two conditions that are sufficient for implementation by simple agenda in part (b). Finally, I provide in part (c) a "recipe" for constructing a simple agenda to implement any rule which satisfies these conditions.

(a) Definition

Given a non-terminal node A of an agenda $\mathcal{T}_X$, the alternative $x \in A$ is said to be *contested* at A if $x \in B \setminus C$ where B and C denote the successors of A. For a simple agenda, every non-terminal node involves a "contest" between two alternatives that continue to be contested until they are either eliminated or selected as the outcome. To formalize:

Definition 4 *A* ***simple agenda*** $\mathcal{S}_X$ *on* X *is an agenda such that*
(i) there exists an alternative $b \in B \setminus C$ *that labels exactly one terminal node below* B *and,*
(ii) there exists an alternative $c \in C \setminus B$ *that labels exactly one terminal node below* C
for every non-terminal node A *of* $\mathcal{S}_X$ *whose successors are* B *and* C.[6]

Equivalently, an agenda is *simple* if it is at once *non-repetitive* and *continuous*.

Non-repetitiveness refers to the fact that every stage of voting in the agenda eliminates some alternatives *regardless* of which subgame the voters actually select. Formally:

Definition 5 *An agenda* $\mathcal{T}_X$ *is* ***non-repetitive*** *if*

$$B, C \subset A \text{ for every non-terminal node } A \text{ of } \mathcal{T}_X \text{ whose successors are labeled } B \text{ and } C.$$

Non-repetitive agendas have the appealing feature that the outcome is determined by relatively few votes. Since every subgame contains alternatives unavailable at its "sibling" subgame, the height of a non-repetitive agenda on X is $|X| - 1$ and the number of votes is limited by X.

Continuity, in turn, refers to the fact that some alternatives contested at any stage continue to be contested until they are either eliminated or selected as the outcome. Formally:

Definition 6 *An agenda* $\mathcal{T}_X$ *is* ***continuous*** *if, for every non-terminal node* A *of* $\mathcal{T}_X$ *whose successors are labeled* B *and* C*, some alternative* $x \in B$ *contested at* A *labels exactly one terminal node below* B.[7]

For some alternative x contested at the node A, there is a unique path starting at A that leads to the selection of x. On this path, x is contested at every node. Intuitively, this means that every stage of voting on path may be interpreted as a choice between continuing to entertain the possibility of x and rejecting this alternative once and for all.

To help illustrate these two properties, consider the following pair of agendas:

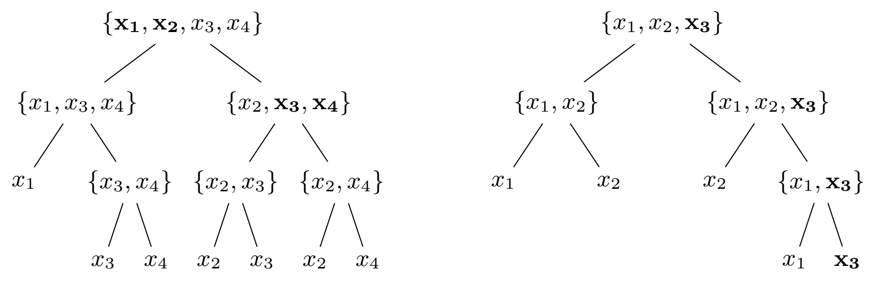

Figure 5: A non-repetitive non-continuous agenda (*left*) and a repetitive continuous agenda (*right*)

[6]A simple agenda on X will always be denoted by $\mathcal{S}_X$ to help distinguish it from a generic agenda $\mathcal{T}_X$.
[7]Continuity was originally defined only for "Ordeshook-Schwartz" agendas (footnote 5). The definition adapts the concept to "Farquharson-Miller" in a way that addresses the concerns of Groseclose and Krebhiel [1993].

The left-hand agenda is non-repetitive: the two successors of each node contain a strict subset of the alternatives. However, it is non-continuous: while x_2 is contested at the root, it is not contested at the successor $\{x_2, x_3, x_4\}$.[8] Conversely, the right-hand agenda is repetitive: the right successor of the root contains the same alternatives as the root. At the same time, it is continuous: the only alternative contested at the root (x_3) appears at a single terminal node.

(b) Sufficient Conditions

The sufficient conditions for implementation by simple agenda are related to Plott's [1973] *Path Independence* (PI) and Arrow's [1950] *Independence of Irrelevant Alternatives* (IIA).

The first condition, which weakens PI, states that the outcome for every issue can be determined by splitting it into simpler sub-issues. For a decision rule v, an issue $A \subseteq X$ can be *split* if there exists a pair of issues (B, C), called a *splitting*, such that: (i) $B \cap C \neq B, C$ (i.e. B and C are *distinctive*) and $B \cup C = A$ (i.e. B and C *cover* A); and, (ii) $v(P, A) = v(P, \{v(P, B), v(P, C)\})$ for every profile P. To formalize the splitting condition:

Issue Splitting **(IS)** *For v, every issue can be split into sub-issues.*

By comparison, PI imposes the stronger requirement that the identity in (ii) must hold for *all* pairs of sub-issues (B, C) that cover A, *regardless* of whether these issues are distinctive.[9]

In the spirit of IIA, the second condition states that outcomes are not affected by alternatives that *never* appeal to a majority.[10] To formalize, an alternative $a \in A$ is the *Condorcet loser* for the decision problem (P, A) if, for all $x \in A \setminus a$, the majority of voters in P prefer x to a. Then, a decision rule v is *independent of the losers* for the issue A if $v(P, A) = v(P, A \setminus a)$ for every profile P where a is the Condorcet loser on (P, A). To formalize the independence condition:

Independence of the Losing Alternatives **(ILA)** *For every issue, v is independent of the losers.*

Theorem 1 shows that every decision rule v satisfying these two conditions is implementable by a unique simple agenda $\mathcal{S}_X^v$. As discussed in section (b) below, the structure of $\mathcal{S}_X^v$ is straightforward to determine from the outcomes of v for decision problems of three alternatives, called *Condorcet triples*, where the pairwise majority preference forms a cycle:

Theorem 1 *If a decision rule v satisfies IS and ILA, then it is implementable by a unique simple agenda $\mathcal{S}_X^v$ whose structure is determined by the outcomes on Condorcet triples.*

It is worth commenting on the necessity of the two conditions. Clearly, ILA is necessary for implementation by simple agenda. Indeed, it is necessary for implementation by *any* kind of agenda. Since the Condorcet loser cannot win a majority vote in any terminal subgame of an agenda, sophisticated voters disregard it when deciding how to vote in earlier stages.

In contrast, IS is not even necessary for implementation by simple agenda. To see this, consider the decision rule implemented by the simple agenda $\mathcal{T}_X$ in Figure 4. Let P_{xbc} denote the Condorcet triple with cycle orientation xbc (i.e. x is preferred by majority to b, b to c, and c to x); and, let P_{xcb} denote the triple with the reverse orientation. "Backward induction" shows that $\mathcal{T}_{X|\{b,c,x\}}$ implements b for P_{xbc} and c for P_{xcb}. In other words, $\mathcal{T}_{X|\{b,c,x\}}$ selects the majority preferred alternative between b and c. However, there is no way to do this by splitting $\{b, c, x\}$ (as shown in Table 1 below).[11]

[8] The other alternative contested at the root (x_1) does appear at a single terminal node below $\{x_1, x_3, x_4\}$.

[9] IS also weakens the *Division Consistency* (DC) condition of Apesteguia et al. [2014]. For one, it does not require the sub-issues (B, C) to be disjoint. And, it does not impose any consistency between the splitting of A and its sub-issues. That is, IS does not require $v(P, D) = v(P, \{v(P, B \cap D), v(P, C \cap D)\})$ for any $D \subset A$.

[10] Apesteguia et al. [2014] call this condition *Condorcet Loser Consistency.*

[11] The outcome for P_{xbc} (resp. P_{xcb}) implies x cannot be paired with b (resp. c). So, the only potential splitting is $(x, \{b, c\})$. Since this requires x as the outcome for both P_{xbc} and P_{xcb} however, there is no way to split $\{b, c, x\}$.

(c) A Recipe

It is straightforward to define the agenda $\mathcal{S}^v_X$ from Theorem 1 by recursion. First, define a root node and label it X. Then, for any existing node whose label is a non-singleton issue A, construct two successors nodes, using the unique[12] splitting (B_A, C_A) of A to label them B_A and C_A, respectively. To illustrate:

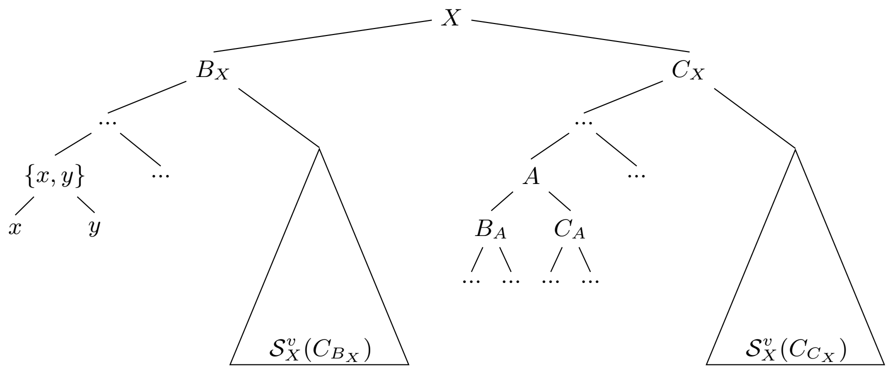

Figure 6: The recursive construction of $\mathcal{S}^v_X$

In Figure 6, the leftmost nodes below B_X illustrate the construction for $|A| = 2$ while the leftmost nodes below C_X illustrate it for $|A| > 2$. In turn, the two triangles represent the subgames starting from the nodes labeled C_{B_X} and C_{C_X} while the ellipses indicate where some details have been omitted.[13]

As Theorem 1 indicates, Condorcet triples may be used to describe $\mathcal{S}^v_X$ more explicitly. For the issue $\{x, b, c\}$, there are three splittings where b and c appear in separate sub-issues:

$$(\{b, x\}, \{c, x\}) \quad (b, \{c, x\}) \quad (\{b, x\}, c)$$

Each of these corresponds to the initial stage game of a different simple agenda on $\{x, b, c\}$:

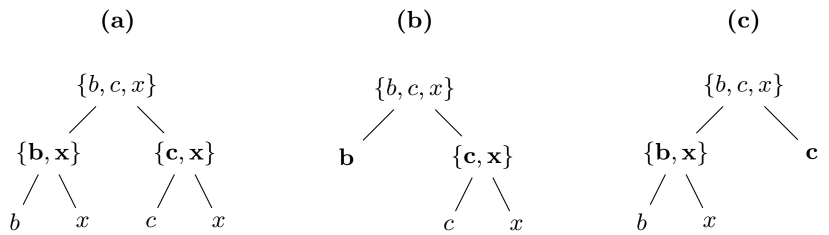

Figure 7: Simple agendas on three alternatives $\{x, b, c\}$

These three agendas implement different combinations of outcomes for the triples P_{xbc} and P_{xcb}:

[12] IS and ILA ensure that there is unique way to split every issue (see Claim 9 of the Appendix).

[13] It is important not to confuse $\mathcal{S}_X(A)$ and $\mathcal{S}_{X|A}$ (which may be quite different – see Figure 4). While the former refers to the subgame at node A in $\mathcal{S}_X$, the latter refers to the agenda obtained by pruning away $X \setminus A$.

Profile\Agenda	**(a)**	**(b)**	**(c)**
P_{xbc}	c	b	c
P_{xcb}	b	b	c
Outcomes	Majority loser between b and c	Outcome b for both triples	Outcome c for both triples

Table 1: Outcomes implemented by agendas (a)-(c) on $\{x, b, c\}$

By construction, there exist alternatives b_A and c_A that appear only on opposite sides of the agenda $\mathcal{S}_X^v(A)$ starting at any node A. For these alternatives, the outcomes on any issue $\{x, b_A, c_A\}$ involving an $x \in A$ must coincide with one of the possibilities in Table 1. After using this observation to locate some $b_A, c_A \in A$, one can use Table 1 to describe the splitting (B_A, C_A) of A in terms of Condorcet triples:

$$B_A \equiv \{b_A\} \cup \{x \in A : \text{type-(a) or type-(c) outcomes on } \{x, b_A, c_A\}\}$$
$$C_A \equiv \{c_A\} \cup \{x \in A : \text{type-(a) or type-(b) outcomes on } \{x, b_A, c_A\}\}$$

Intuitively, type-(a) outcomes reveal that x appears in both sub-issues of (B_A, C_A) while type-(c) outcomes (type-(b) outcomes) reveal that x appears only in the same sub-issue as b_A (c_A).

While the goal was to define a simple agenda for the "grand" issue X, the same approach also defines a simple agenda $\mathcal{S}_A^v$ that implements v for any issue $A \subset X$. This is a straightforward consequence of IS and ILA. This observation highlights a practical feature of decision rules satisfying these conditions. Instead of using the pruned agenda $\mathcal{S}_{X|A}^v$ to implement the *social choice function* $v(\,\cdot\,, A) : \mathbf{P} \to A$, one can use $\mathcal{S}_A^v$. The advantage is that the latter requires less voting. By virtue of its simplicity, every node in $\mathcal{S}_A^v$ must affect the outcome for some profile. In contrast, $\mathcal{S}_{X|A}^v$ may include redundant nodes that cannot affect the outcome for any profile.

To summarize, IS ensures that it is possible to "simplify" the agenda implementing a decision rule on X for every issue $A \subseteq X$. In other words:

Theorem 1* *If a decision rule v is implementable by agenda, then it satisfies IS if and only if the social choice function $v(\,\cdot\,, A)$ is implementable by simple agenda for every issue A.*

IV. Implementation by Priority Agenda

I define priority agendas in part (a) and characterize the rules that they implement in part (b).

(a) Definition

A priority agenda on X is defined by a pair $(\succsim, \alpha)$ consisting of a *weak priority* $\succsim$ and an *amendment rule* α. Intuitively, $(\succsim, \alpha)$ provides a way to construct the agenda by progressively adding alternatives. While $\succsim$ determines *when* each alternative should be added, α determines *how* each should be added.

Formally, $\succsim$ is a weak order on X whose indifference classes contain *at most* two alternatives. When $x \succ y$, the idea is that x has higher priority and is added to the agenda before y. When $x \sim y$, the two alternatives have *equal* priority and may be added to the agenda in either order.

To formalize the amendment rule α, let X_j denote the j^{th} highest indifference class of $\succsim$ and let $\widetilde{X} \equiv \{\{x\} : x \in X\} \cup \{X_j : |X_j| \neq 1\}$ denote the collection of singletons and equal priority pairs in X. Using this notation, the amendment rule is a mapping $\alpha : X \setminus X_1 \to \widetilde{X}$. An alternative z is said to *amend* another alternative x if (i) $x \in \alpha(z)$ or (ii) $y \succ x \succ z$ for some $y \in \alpha(z)$. The interpretation is that, when z is added to the agenda, it amends $y \in \alpha(z)$ and every alternative x already on the agenda that has strictly lower priority than y. To match this interpretation, α must satisfy the following additional restrictions: (i) every new addition to the agenda must amend some alternative already on the agenda; (ii) alternatives with the same priority must amend the same alternatives; and, (iii) every alternative whose priority is immediately below two alternatives with the same priority must amend both.[14]

The Euro-Latin and Anglo-American agendas are easy to describe in terms of this notation:

	$\succsim$	α
Euro-Latin	$x_1 \succ ... \succ x_{n-1} \sim x_n$	$\alpha(x_i) = \begin{cases} \{x_{i-1}\} & \text{for } i \neq n \\ \{x_{n-2}\} & \text{for } i = n \end{cases}$
Anglo-American	$x_1 \sim x_2 \succ ... \succ x_n$	$\alpha(x_i) = \{x_1, x_2\}$

Table 2: $(\succsim, \alpha)$ for Euro-Latin and Anglo-American agendas of n alternatives

To reconstruct the agendas from the $(\succsim, \alpha)$ pairs in this table, one simply adds the alternatives in decreasing order of priority $\succsim$ using the amendment rule α. Indeed, the same type of construction defines an agenda for every priority-amendment pair $(\succsim, \alpha)$. To formalize:

Definition 7 For any pair $(\succsim, \alpha)$ on X, the **priority agenda** $\mathcal{P}_{(\succsim,\alpha)}$ is defined recursively as follows:

(1) Define $\mathcal{P}^1_{(\succsim,\alpha)}$ to be the simple agenda $\mathcal{S}_{X_1}$ on the highest indifference class X_1.

(2) Define $\mathcal{P}^{j+1}_{(\succsim,\alpha)}$ by adding the alternatives $x_{j+1} \in X_{j+1}$ to $\mathcal{P}^{j}_{(\succsim,\alpha)}$ as follows:

(i) Replace every terminal node of $\mathcal{P}^{j}_{(\succsim,\alpha)}$ labeled by:

- $x_k \in \alpha(x_{j+1})$ with the simple agenda $\mathcal{S}_{\{x_k, X_{j+1}\}}$; and,
- $x_{k'} \in X_{k'}$ for k' s.t. $k < k' \leq j$ with the simple agenda $\mathcal{S}_{\{x_{k'}, X_{j+1}\}}$.

(ii) If $|X_{j+1}| \neq 1$, replace every node X_{j+1} in the agenda resulting from (i) with $\mathcal{S}_{X_{j+1}}$.

(3) Define $\mathcal{P}_{(\succsim,\alpha)}$ to be $\mathcal{P}^{K}_{(\succsim,\alpha)}$ where K is the number of indifference classes in $\succsim$.

Clearly, this recursive construction defines a simple agenda on X.[15] At any stage $j \leq K$, the simple agenda $\mathcal{P}^{j}_{(\succsim,\alpha)}$ is extended into a longer simple agenda $\mathcal{P}^{j+1}_{(\succsim,\alpha)}$ by appending new simple agendas (of two or three alternatives) to the terminal nodes. The figure below serves to illustrate:

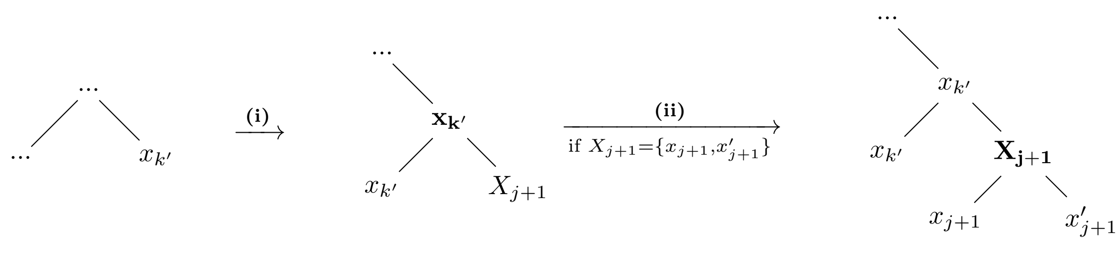

Figure 8: Detail at the terminal node x'_k in stage $j \leq K$ of the construction.

[14]These three requirements can be formalized as follows: (i) $x \in \alpha(z) \Rightarrow x \succ z$; (ii) $x \sim y \Rightarrow \alpha(x) = \alpha(y)$; and, (iii) $[x \sim y \succ z$ and no $z' \in X$ s.t. $x \sim y \succ z' \succ z] \Rightarrow [x \in \alpha(z)$ or $w \succ x$ for all $w \in \alpha(z)]$.

[15]Technically, one must relabel the non-terminal nodes to conform with Definition 1(3). Since this is straightforward but cumbersome, it has been omitted to preserve clarity. For the details, see Claim 1 of the Appendix.

Notice that Definition 7 effectively uses the class of priority agendas on m alternatives to define the class of priority agendas on $m+1$ alternatives. The following example serves to illustrate:

Example 1 *There are three consistent ways of proposing a new alternative x_4 to extend the (i) Euro-Latin and (ii) Anglo-American agendas in Figure 1 into priority agendas on $\{x_1, x_2, x_3, x_4\}$.*

(1) By Euro-Latin amendment – *x_4 amends only the last alternative x_3 in $\{x_1, x_2, x_3\}$, which leads to (i) the Euro-Latin agenda in Figure 2 and (ii) the right-hand agenda in Figure 3;*

(2) By Anglo-American amendment – *x_4 amends every alternative in $\{x_1, x_2, x_3\}$, which leads to (i) the left-hand agenda in Figure 3 and (ii) the Anglo-American agenda in Figure 2; and,*

(3) By Intermediate amendment – *x_4 amends x_2 and x_3, which leads to the two agendas below.*

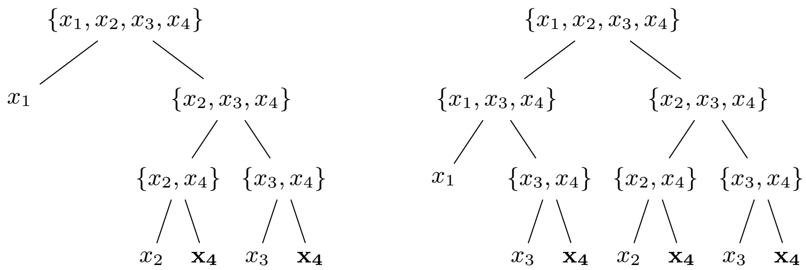

Figure 9: Priority agendas that extend Euro-Latin (*left*) and Anglo-American (*right*) agendas

Since the Euro-Latin and Anglo-American agendas are the only priority agendas on three alternatives, these six agendas constitute the entire class of priority agendas on four alternatives (up to permutation).

(b) Necessary and Sufficient Conditions

Euro-Latin and Anglo-American agendas are structured so that the lowest priority alternatives are the sophisticated voting outcomes only when they *always* appeal to a majority (Miller [1977]; Moulin [1991, Exercise 9.5]). To formalize, an alternative $a \in A$ is the *Condorcet winner* for the decision problem (P, A) if, for all $x \in A \setminus a$, the majority of voters in P prefer a to x. Then, an alternative $a^* \in A$ is said to be *marginal* for the issue A when $v(P, A) = a^*$ only if P is a profile where a^* is the Condorcet winner for (P, A). To state the property:

Weak Marginalization **(WM)** *For every issue, v has a marginal alternative.*

It turns out that the same property is satisfied by sophisticated voting on *every* priority agenda:

Proposition 1 *Every decision rule v implementable by priority agenda satisfies Weak Marginalization.*

Indeed, it is the distinguishing feature of rules implementable by priority agenda:

Theorem 2 *A decision rule v satisfies IS, ILA, and WM if and only if it is implementable by a priority agenda $\mathcal{P}_X^v$. For any decision rule v satisfying these three conditions, $\mathcal{P}_X^v$ is unique and the pair $(\succsim_v, \alpha_v)$ that defines this agenda is uniquely determined by the outcomes on Condorcet triples.*

To establish the sufficiency of the axioms, the key is to determine the priority structure imposed by WM. To accomplish this task, the proof relies on the familiar tool of revealed preference:

Definition 8 *Given a decision rule v, define the binary relations $\succ_v$ and $\sim_v$ on X by:*

- *$y \succ_v z$ if there exists an issue $A \supseteq \{y, z\}$ where z is marginal but y is not; and,*
- *$y \sim_v z$ if y is marginal for every issue A where z is marginal and vice versa.*

Using $\succ_v$ and $\sim_v$, define the binary relation $\succsim_v$ on X by $y \succsim_v z$ if $y \succ_v z$ or $y \sim_v z$.

For the Anglo-American and Euro-Latin agendas, $\succsim_v$ reflects the underlying weak priority. For every issue A, the first marginalizes the lowest ranked alternative in A according to $\succsim_v$ while the second marginalizes the *two* lowest ranked alternatives.[16] In fact, $\succsim_v$ defines a weak priority with similar features for any decision rule that satisfies IS, ILA, and WM:

Lemma 1 *If v satisfies IS, ILA, and WM, then: **(i)** $\succsim_v$ is a weak priority; and; **(ii)** for every issue A, v marginalizes either the lowest **or** two lowest alternatives in A according to $\succsim_v$.*

As indicated in Theorem 2, it is possible to re-formulate the revealed priority $\succsim_v$ in terms of Condorcet triples. For any two alternatives y and z, Table 1 shows that there are six combinations of outcomes for the triples P_{xyz} and P_{xzy}. Of these, four directly reveal $y \succ_v z$ or $z \succ_v y$ while two combinations are consistent with every possible priority ranking of y and z. By varying the alternative x, it is possible to resolve these ambiguous cases (see Corollary 2 of the Appendix).

Table 1 also shows how to define the amendment rule α_v in terms of Condorcet triples. Intuitively, x is "revealed to amend" a higher priority alternative b if the Condorcet triples for every alternative c with intermediate priority yields type-(a) or type-(c) outcomes—namely the outcomes where x appears in the same sub-issue as b in Figure 7. Then, $\alpha_v(x)$ can be defined as the highest priority alternative(s) that x is revealed to amend (see Definition 9 of the Appendix).

In light of Theorem 1, the sufficiency of the axioms in Theorem 2 follows by showing that the simple agenda $\mathcal{S}_X^v$ "branches" in the same way as the priority agenda $\mathcal{P}_X^v$ defined by $(\succsim_v, \alpha_v)$.

V. Concluding Discussion

To conclude, I highlight how priority agendas extend our understanding of legislative voting beyond Euro-Latin and Anglo-American agendas. In part (a), I first describe two distinctive features that every priority agenda shares with Euro-Latin and Anglo-American agendas. In part (b), I then highlight a key difference between these agendas and other priority agendas.

(a) Similarities

The voting literature emphasizes three features of sophisticated voting on Euro-Latin and Anglo-American agendas. The first is the Weak Marginalization property studied in Section IV. The other two features are monotone comparative statics.

The first relates to changes in voter preferences (Moulin [1986]). Given a profile P, let P^x denote a profile where all voter preferences are identical to P except for *one* voter, whose preference between x and the immediately preferred alternative are reversed.[17] That is, P^x differs from P only by improving x in the eyes of one voter. Then, a decision rule v is said to be *preference monotonic* if, for every decision problem (P, A) such that $v(P, A) = x$, $v(P^x, A) = x$ for every profile P^x where x improves in some voter's preference.

Like Euro-Latin and Anglo-American agendas, every priority agenda is preference monotonic:

Proposition 2 *Every decision rule v implementable by priority agenda is preference monotonic.*

[16] Eliaz et al. [2011] characterize choice behavior that is consistent with selecting the "lowest two" alternatives.
[17] For the voter in question, the preference "... $\succ_i y \succ_i x \succ_i$ ..." in P becomes "... $\succ_i^x x \succ_i^x y \succ_i^x$ ..." in P^x.

The second property relates to changes in priority (Jung [1990]; Moulin [1991, Exercise 9.5]). Given a priority agenda $\mathcal{P}_X$ defined by $(\succsim, \alpha)$, let $(\succsim_x, \alpha_x)$ denote a pair identical to $(\succsim, \alpha)$ except for *one* alternative x, whose priority and amendment features are swapped with an alternative y such that: (i) $y \succ z \succ x$ for no alternative $z \in X$; or, (ii) $x \sim y$ and x is amended by every alternative that amends y. Let $\mathcal{P}_X^x$ denote the priority agenda defined by $(\succsim_x, \alpha_x)$. Intuitively, x weakly improves in terms of priority in $\mathcal{P}_X^x$ by "swapping places" with y in the agenda.[18] Then, $\mathcal{P}_X$ is said to be *priority monotonic* if, for every decision problem (P, A) such that $UNE[\mathcal{P}_{X|A}, P] = x$, $UNE[\mathcal{P}^x_{X|A}, P] = x$ for $\mathcal{P}_X^x$.

Like Euro-Latin and Anglo-American agendas, every priority agenda is priority monotonic:

Proposition 3 *Every priority agenda $\mathcal{P}_X$ is priority monotonic.*

(b) Differences

Unlike other rules implemented by priority agenda, the Euro-Latin and Anglo-American agendas treat the alternatives neutrally after taking the priorities into account. Intuitively, the outcomes may depend on the "structure" of the profile and the priorities but *not* the "names" of the alternatives. Perhaps the simplest (if not the weakest) implication of this neutrality is that issues of the same size must have the same number of marginal alternatives. Where two issues A, A' such that $|A| = |A'|$ are understood to be *similar*, this neutrality property can be stated more formally as follows:

Neutral Priority **(NP)** *For similar issues, v has the same number of marginal alternatives.*

The next result shows that, besides the Euro-Latin and Anglo-American procedures, no other decision rule implementable by priority agenda satisfies even this weak form of neutrality:

Proposition 4 *The Euro-Latin and Anglo-American procedures both satisfy NP. In fact, they are the only decision rules implementable by priority agenda that satisfy this property.*

Whereas the Euro-Latin agenda *always* marginalizes two alternatives, the Anglo-American agenda *always* marginalizes one. No other rule that satisfies IS, ILA, and WM marginalizes the same number of alternatives, even for similar issues. In recent work, Apesteguia et al. [2014] provide a different characterization using the following properties:

Condorcet Priority **(CP)** *Every issue A has a* ***prioritarian alternative*** *$p^* \in A$ such that $v(P_{p^*xy}, \{p^*, x, y\}) = p^*$ for any Condorcet triple P_{p^*xy} involving alternatives $x, y \in A$.*

Condorcet Anti-Priority **(CA)** *Every issue A has an* ***anti-prioritarian alternative*** *$p_* \in A$ such that $v(P_{p_*xy}, \{p_*, x, y\}) = y$ for any Condorcet triple P_{p_*xy} involving alternatives $x, y \in A$.*

To characterize the Euro-Latin procedure, they use CP along with ILA and *Division Consistency* (DC) (see footnote 9). To characterze the Anglo-American procedure, they use CA along with ILA and a property called *Elimination Consistency* (EC).

Theorem 2 shows that one can replace DC and EC in these characterizations by IS:

Corollary 1 *A decision rule v is:*

(i) a Euro-Latin procedure if and only if it satisfies IS, ILA, and Condorcet Priority.
(ii) an Anglo-American procedure if and only if it satisfies IS, ILA, and Condorcet Anti-Priority.

Using $\succsim_v$ (particularly as characterized in Corollary 2 of the Appendix), it is easy to see that CP marginalizes two alternatives for every issue while CA marginalizes one. This shows that the key difference between the Euro-Latin and Anglo-American procedures is the structure of the amendments associated with each.

[18] Formally, $\mathcal{P}_X^x$ can be obtained by permuting the labels of the terminal nodes in $\mathcal{P}_X$ marked x and y.

VII. References

Altman, Procaccia, and Tennenholtz. 2009. "Nonmanipulable Selections from a Tournament." *IJCAI* 27–32.

Apesteguia, Ballester, and Masatlioglu. 2014. "A Foundation for Strategic Agenda Voting." forthcoming *Game Econ. Behav.*

Arrow. 1950. "A Difficulty in the Concept of Social Welfare." *J. Polit. Econ.* 58 328–346.

Banks. 1985. "Sophisticated Voting Outcomes and Agenda Control." *Soc. Choice Welfare* 1 295–306.

Banks. 1989. "Equilibrium Outcomes in Two-Stage Amendment Procedures." *Am. J. Polit. Sci.* 33 25–43.

Bossert and Sprumont. 2013. "Every Choice Function is Backwards-Induction Rationalizable." *Econometrica* 81 2521–2534.

Coughlan and Le Breton. 1999. "A Social Choice Function Implemented via Backward Induction with Values in the Ultimate Uncovered Set." *Rev. Econ. Design* 4 153–160.

Dutta, Jackson, and Le Breton. 2002. "Voting by Successive Elimination and Strategic Candidacy." *J. Econ. Theory* 103 190–218.

Eliaz, Richter, and Rubinstein. 2011. "Choosing the Two Finalists." *Econ. Theory* 46 211–219.

Farquharson. 1969. *Theory of Voting.* New Haven: Yale University Press.

Fishburn. 1982. "Monotonicity Paradoxes in the Theory of Elections." *Discrete Appl. Math.* 4 119-134.

Fischer, Procaccia, and Samorodnitsky. 2011. "A New Perspective on Implementation by Voting Trees." *Random Struct. Algor.* 39 59–82.

Groseclose and Krehbiel. 1993. "On the Pervasiveness of Sophisticated Sincerity" in Barnett, Hinich, Rosenthal, and Schofield (eds.), *Political Economy: Institutions, Information, Competition, and Representation.* Cambridge: Cambridge University Press.

Horan. 2011. "Choice by Tournament." *mimeo.*

Horan. 2013. "Implementation of Majority Voting Rules." *mimeo.*

Iglesias, Ince, and Loh. 2014. "Computing with Voting Trees." *SIAM J. Discrete Math* 28 673–684.

Jung. 1990. "Black and Farquharson on Order-of-Voting Effects: An Extension." *Soc. Choice Welfare* 7 319–329.

McKelvey and Niemi. 1978. "A Multistage Game Representation of Sophisticated Voting for Binary Procedures." *J. Econ. Theory* 18 1–22.

Miller. 1977. "Graph-Theoretical Approaches to the Theory of Voting." *Am. J. Polit. Sci.* 21 769–803.

Miller. 1980. "A New Solution Set for Tournaments and Majority Voting: Further Graph-Theoretical Approaches to the Theory of Voting." *Am. J. Polit. Sci.* 24 68–96.

Miller. 1995. *Committees, Agendas, and Voting.* Reading: Harwood Academic.

Moulin. 1986. "Choosing from a Tournament." *Soc. Choice Welfare* 3 271–291.

Moulin. 1991. *Axioms of Cooperative Decision Making.* New York: Cambridge University Press.

Moore. 1992. "Implementation, Contracts, and Renegotiation in Environments with Complete Information", in J-J Laffont (Ed.), *Advances in Economic Theory: Sixth World Congress Vol. 1*, New York: Cambridge University Press.

Ordeshook and Schwartz. 1987. "Agendas and the Control of Political Outcomes." *Am. Polit. Sci. Rev.* 81 179–200.

Plott. 1973. "Path Independence, Rationality, and Social Choice." *Econometrica* 41 1075–1091.

Riker. 1958. "The Paradox of Voting and Congressional Rules for Voting on Amendments." *Am. Polit. Sci. Rev.* 52 349–366.

Sanver and Zwicker. 2009. "One-way Monotonicity as a Form of Strategy-proofness." *Int. J. Game Theory* 38 553–574.

Shepsle and Weingast. 1984. "Uncovered Sets and Sophisticated Voting Outcomes with Implications for Agenda Institutions." *Am. J. Polit. Sci.* 28 49–74.

Schwartz. 2008. "Parliamentary Procedure: Principal Forms and Political Effects." *Public Choice* 136 353–377.

Srivastava and Trick. 1996. "Sophisticated Voting Rules: The Case of Two Tournaments." *Soc. Choice Welfare* 13 275–289.

Trick. 2006. "Small Binary Voting Trees." *mimeo.*

Xu and Zhou. 2007. "Rationalizability of Choice Functions by Game Trees." *J. Econ. Theory* 134 548–556.

Sean Horan
Département de sciences économiques
Université de Montréal
Montréal, QC, Canada
Email: `smhoran@gmail.com`

VIII. Appendix – Proofs

NOTE: Except as indicated otherwise, the claims in sections (c)-(f) below suppose that v satisfies IS and ILA.

(a) Proof of Remark 1

To formalize the remark, some notation is required. Given a profile P, let P^A denote the profile that coincides with P on A but places the alternatives in $X \setminus A$ at the bottom of each voter preference (in a fixed order).

Proof of Remark 1. Suppose v is implemented by $\mathcal{T}_X$. To establish $v(P, A) = v(P^A, X)$, note that "backward induction" determines the UNE on any agenda (McKelvey and Niemi [1978]). In any terminal subgame, it selects the Condorcet winner. One can then delete the Condorcet loser and repeat the argument on the resulting (smaller) agenda. From this, it follows that $UNE[\mathcal{T}_X; P^A] = UNE[\mathcal{T}_{X|A}; P^A]$. Since P^A and P coincide on A, $UNE[\mathcal{T}_{X|A}; P^A] = UNE[\mathcal{T}_{X|A}; P]$ as well. So, $UNE[\mathcal{T}_X; P^A] = UNE[\mathcal{T}_{X|A}; P]$. ■

(b) Proof of Propositions 1, 2, and 3

Some additional notation is required for these results. Given a weak order $\succsim$, let $L_{\succsim}(a) \equiv \{x \in X : a \succ x\}$ denote the strict lower contour set of $a \in X$. And, let $\succ^*$ denote any strict order such that $y \succ z$ implies $y \succ^* z$ for all $y, z \in X$. Given an issue $A = \{a_1, ..., a_K\}$ labeled according to $\succ^*$, let A_j^k denote the alternatives in A between a_j and a_k. Formally, let $A_j^k \equiv \{a_j, ..., a_k\}$ if $j \leq k \leq K$; and, let $A_j^k \equiv \emptyset$ otherwise.

Claim 1 *Given a priority agenda $\mathcal{P}_{(\succsim,\alpha)}$, the two successor nodes of any node A are $a_1 \cup A_j^K$ and A_2^K for some j s.t. $j \in \{3, ..., K+1\}$ where $A = \{a_1, ..., a_K\}$ is labeled according to $\succ^*$. Moreover:*

(i) *$A_2^K = X_{m-K+2}^m$ where $X = \{x_1, ..., x_m\}$ is labeled according to $\succ^*$; and,*

(ii) *$a_2 \succ a_j$ and a_j is a highest priority alternative in A_3^K that amends a_1 (if such an alternative exists).*

Proof. Let B and C denote the labels attached to the two successors of node A. Since $\succsim$ is a weak priority: **(1)** $\max_{\succsim} A = \{a_1, a_2\}$; or, **(2)** $\max_{\succsim} A = \{a_1\}$. Since the claim is trivial when $|A| = 2$, suppose $|A| \geq 3$.

(1) From the construction (step (ii) in Definition 7), a_1 and a_2 must have been added at A. Since they must be added to different successors, $a_1 \in B \setminus C$ and $a_2 \in C \setminus B$. By definition of α, the next highest priority alternative(s) in X are added under the successors identified with a_1 and a_2. So, $B = a_1 \cup L_{\succsim}(a_1)$ and $C = a_2 \cup L_{\succsim}(a_1)$ by a straightforward induction argument. This shows that $B = a_1 \cup A_3^K$, $C = A_2^K = X_{m-K+2}^m$, and $a_2 \succ a_3$.

(2) From the construction (step (i) in Definition 7), a_1 and the next highest priority alternative(s) at A must be added to different successors. Defining $A_{-1} \equiv \max_{\succsim}(A \setminus a_1)$, $a_1 \in B \setminus C$ and $A_{-1} \subseteq C \setminus B$. So, $C = A_{-1} \cup L_{\succsim}(a_2) = X_{m-K+2}^m$ by the same argument as (1). To add an $a_j \notin A_{-1}$ under the successor identified with a_1, the construction requires that a_j amends a_1. Then, by the same argument as (1), $B = a_1 \cup A_j^K$ where a_j is a highest priority in $C = X_{m-K+2}^m \equiv A_2^K$ that amends a_1 besides $a_2 \succ a_j$ (if such an alternative exists). ■

Claim 2 *Fix a priority agenda $\mathcal{P}_{(\succsim,\alpha)}$ on $X = \{x_1, ..., x_m\}$ as labeled according to $\succ^*$ and an alternative x_j that amends x_1 according to α. Then, every path from the root X to a terminal node in $\mathcal{P}_{(\succsim,\alpha)}$ passes through a non-terminal node $x_i \cup X_j^m$ whose successors are X_j^m and $x_i \cup X_k^m$ for some $i < j < k \leq m+1$.*

Proof. The proof is by strong induction on m. The base cases $m = 2, 3$ are straightforward. For the induction step $m = n + 1$, consider the two successors of the root X. By Claim 1, these are X_2^m and $x_1 \cup X_k^m$ for $2 < k \leq m+1$. If $j = 2$, then the root X is the desired node (since every path to a terminal node goes through this node). If $j > 2$, then $x_j \in X_k^m$ and $x_j \in X_3^m$. Moreover, x_j amends x_2 by definition of α. Since the agendas starting at X_2^m and $x_1 \cup X_k^m$ are priority agendas on n or fewer alternatives, the induction hypothesis implies that every path to a terminal node starting from X_2^m or $x_1 \cup X_k^m$ passes through a non-terminal node with the desired characteristics. And, since every path from the root X passes through X_2^m or $x_1 \cup X_k^m$, the result follows. ■

The next results require some additional definitions. First, two agendas $\mathcal{T}_X$ and $\mathcal{T}'_Y$ s.t. $Y \subset X$ are said to be *outcome-equivalent* on Y if $UNE[\mathcal{T}_{X|A}, P] = UNE[\mathcal{T}'_{Y|A}, P]$ for every decision problem (P, A) such that $A \subseteq Y$.

Second, for any priority agenda $\mathcal{P}_X$ defined by $(\succsim, \alpha)$ and any alternative $x \in X$, let $\mathcal{P}_{X\setminus x}$ denote the *deleted priority agenda* that "deletes" from the agenda $\mathcal{P}_X$ every subgame where x has highest priority as follows:

(1) locate the nodes A_x of $\mathcal{P}_X$ s.t. $x \in \max_{\succsim} A_x$ and $x \notin \max_{\succsim} A^p_x$ for the predecessor A^p_x of A_x;
(2) for all such A_x, delete the agenda starting at the (necessarily unique) successor A^s_x s.t. $x \in A^s_x$;
(3) delete every non-terminal node with a unique successor, connecting its successor and predecessor;
(4) and, finally, relabel every non-terminal node of the resulting tree to conform with Definition 1.

To see that $\mathcal{P}_{X\setminus x}$ defines a priority agenda, note that the only potential change to the amendment rule α involves an alternative y with *immediately* lower priority than x (i.e. $x \succ y$ and $x \succ z \succ y$ for no $z \in X$). If $\alpha(y) = \{x\}$, the deletion has the effect of changing $\alpha(y)$ to $\alpha(x)$. Formally, define the priority rule $\alpha_{-x}(y)$ by:

$$\alpha_{-x}(y) \equiv \begin{cases} \alpha(x) & \text{if } y \text{ has immediately lower priority than } x \text{ and } \alpha(y) = \{x\} \\ \alpha(y) & \text{otherwise} \end{cases}$$

By construction, α_{-x} is an amendment rule consistent with the priority $\succsim$. As such, $(\succsim, \alpha_{-x})$ defines a priority agenda. It is straightforward to see that this agenda is the deleted priority agenda $\mathcal{P}_{X\setminus x}$ described above.

It is worth noting that this agenda is formally distinct from the pruned priority agenda $\mathcal{P}_{X|X\setminus x}$. Intuitively, $\mathcal{P}_{X\setminus x}$ prunes "higher up" the agenda than $\mathcal{P}_{X|X\setminus x}$ (unless x has lowest priority according to $\succsim$). As a result, it prunes away a larger portion of $\mathcal{P}_X$. However, the two agendas are outcome-equivalent:

Claim 3 *For any priority agenda $\mathcal{P}_X$ and any $x \in X$, $\mathcal{P}_X$ is outcome-equivalent to $\mathcal{P}_{X\setminus x}$ on $X \setminus x$.*

Proof. Suppose $X = \{x_1, ..., x_m\}$ where X is labeled according to $\succ^*$. The proof is by strong induction on m. The base cases $m = 2, 3$ are trivial. For the induction step $m = n + 1$, note that the successors at the root X of $\mathcal{P}_X$ are $(x_1 \cup X^m_j, X^m_2)$ by Claim 1. Consider the three cases: (i) $j = m + 1$; (ii) $j = 3$; and, (iii) $3 < j \leq m$.

The argument is the same in all cases if $x \neq x_1, x_2$. In fact, the same argument also works for $x = x_2$ in cases (i) and (iii). Fix any $x = x_i$ s.t. $i \geq 2$ (where $i \neq 2$ if case (ii) is the relevant case). By the induction hypothesis, the priority agenda $\mathcal{P}_{X^m_2} = \mathcal{P}_X(X^m_2)$ starting at the "right" successor X^m_2 of the root is outcome-equivalent on $X^m_2 \setminus x_i$ to $\mathcal{P}_{X^m_2\setminus x_i}$. Similarly, the priority agenda $\mathcal{P}_{x_1\cup X^m_j} = \mathcal{P}_X(x_1 \cup X^m_j)$ starting at the "left" successor $x_1 \cup X^m_j$ is outcome-equivalent on $(x_1 \cup X^m_j) \setminus x_i$ to $\mathcal{P}_{(x_1\cup X^m_j)\setminus x_i}$. Then, "backward induction" shows that the outcome on $\mathcal{P}_X$ for any issue $A \subseteq X \setminus x_i$ does not change if one replaces the agendas at the root by their outcome-equivalents. Since the resulting agenda is $\mathcal{P}_{X\setminus x_i}$ by definition, the claim follows. The following diagram serves to illustrate:

To complete the proof, it suffices to establish the result for $x = x_1$ in cases (i)-(iii) and $x = x_2$ in case (ii).

(i) The claim is trivial since $\mathcal{P}_{X|(X\setminus x_1)} = \mathcal{P}_{X\setminus x_1}$ by definition.

(ii)-(iii) It suffices to show the claim for $x = x_1$. (In case (ii), the same reasoning works for $x = x_2$ as well.) By the induction hypothesis, $\mathcal{P}_{x_1\cup X^m_j}$ is outcome-equivalent on X^m_j to $\mathcal{P}_{X^m_j}$. Let $\mathcal{T}_{X\setminus x_1}$ denote the agenda obtained from $\mathcal{P}_X$ by replacing the agenda $\mathcal{P}_{x_1\cup X^m_j}$ at $x_1 \cup X^m_j$ with $\mathcal{P}_{X^m_j}$. To complete the proof, I show that $\mathcal{T}_{X\setminus x_1}$ is outcome-equivalent to the priority agenda $\mathcal{P}_{X^m_2}$ starting at X^m_2.

By way of contradiction, suppose there is some profile P s.t. $UNE[\mathcal{T}_{X\setminus x_1}, P] \equiv x \neq y \equiv UNE[\mathcal{P}_{X_2^m}, P]$. Since x is the outcome at the root X, "backward induction" shows that $UNE[\mathcal{P}_{X_j^m}, P] = x$. By Claims 1 and 2, every path down the agenda from X_2^m reaches a node $x_i \cup X_j^m$ where: (i) the "left" successor is $x_i \cup X_k^m$ for $i < j < k$; and, (ii) the "right" successor is X_j^m. Since $UNE[\mathcal{P}_{X_j^m}, P] = x$, y must be the outcome at some "left" successor(s) $x_i \cup X_k^m$. Since $UNE[\mathcal{P}_{X_j^m}, P] = x$ however, y must be eliminated at the predecessor $x_i \cup X_j^m$ of any such "left" successor. So, $UNE[\mathcal{P}_{X_2^m}, P] \neq y$, a contradiction which shows that $\mathcal{T}_{X\setminus x_1}$ is outcome-equivalent to $\mathcal{P}_{X_2^m}$. ■

Proof of Proposition 1. Suppose v is implementable by a priority agenda $\mathcal{P}_X$ defined by $(\succsim, \alpha)$. I show that x_m is marginal for $X = \{x_1, ..., x_m\}$ as labeled according to $\succ^*$. It then follows that x_m is marginal for any issue A s.t. $x_m \in A$. Since $\mathcal{P}_{X|\{x_1,...,x_{m-1}\}}$ is a priority agenda on $X \setminus x_m$ s.t. $x_j \succsim x_{m-1}$, the same argument shows that x_{m-1} is marginal for any issue $A \subseteq X \setminus x_m$ s.t. $x_{m-1} \in A$. The result follows by extending this reasoning.

The proof that x_m is marginal for X is by strong induction on m. The base cases $m = 2, 3$ are trivial. For the induction step $m = n + 1$, consider the successors of the root X. By Claim 1, these are X_2^m and $x_1 \cup X_k^m$ for $2 < k \leq m + 1$. There are three possibilities: (i) $k = m + 1$; (ii) $k = m$; and, (iii) $2 < k < m$.

(i) In this case, the root splits $\mathcal{P}_X$ into x_1 and a priority agenda $\mathcal{P}_{X_2^m}$ on n alternatives. Since $x_i \succsim x_m$ for all $x_i \in X_2^m$, the induction hypothesis implies that x_m is the outcome on $\mathcal{P}_{X_2^m}$ only if it is the Condorcet winner on X_2^m. Rolling back the agenda to the root node X, x_m is the outcome on $\mathcal{P}_X$ only if it is the Condorcet winner on $\{x_1, x_m\}$ as well. Combining the last two observations gives the result.

(ii) In this case, x_m amends x_1 by Claim 1. Then, by construction of $\mathcal{P}_X$, every terminal subgame pairs x_m against another alternative in X. So, any alternative majority defeated by x_m is eliminated at the last stage (i.e. only x_m and alternatives that beat x_m can be selected as outcomes). Rolling back the agenda to the root node, it then follows that x_m is the outcome on $\mathcal{P}_X$ only if it is the Condorcet winner on X.[19]

(iii) In this case, the agendas $\mathcal{P}_{x_1 \cup X_k^m} = \mathcal{P}_X(x_1 \cup X_k^m)$ and $\mathcal{P}_{X_2^m} = \mathcal{P}_X(X_2^m)$ are priority agendas on n or fewer alternatives. Since $x_i \succsim x_m$ for all $x_i \in X$, the induction hypothesis implies that x_m is the outcome on $\mathcal{P}_X$ only if it is the Condorcet winner on X_k^m (i.e. the intersection of $x_1 \cup X_k^m$ and X_2^m).

In that case, one can prune away X_k^{m-1} from $\mathcal{P}_X$ to obtain $\mathcal{P}_{X|Y}$ where $Y \equiv X_1^{k-1} \cup x_m$. By Claim 3, $\mathcal{P}_X$ is outcome-equivalent on Y to the priority agenda $\mathcal{P}_Y$ (on n or fewer alternatives). Since $x_i \succsim x_m$ for all $x_i \in Y$, the induction hypothesis implies x_m is the outcome on $\mathcal{P}_Y$ (and hence $\mathcal{P}_{X|Y}$) only if it is the Condorcet winner on $X_1^{k-1} \cup x_m$. Combining this with the observation in the last paragraph gives the result. ■

Claim 4 *Every decision rule v implementable by priority agenda satisfies IS.*

Proof. If v is implementable by the priority agenda $\mathcal{P}_X$, then there exists a way to split X. By Claim 3, the same is true for any $X \setminus x$. By applying Claim 3 to the resulting priority agenda $\mathcal{P}_{X\setminus x}$, the same must be true for any $X \setminus \{x, y\}$. Extending this reasoning by induction, it follows that v satisfies IS. ■

Claim 5 *If $\mathcal{P}_X$ is a priority agenda s.t. the successors of the root node X are labeled B and C, then:*

$$UNE[\mathcal{P}_X, P] \in B \cap C \textit{ implies } UNE[\mathcal{P}_X(B), P] = UNE[\mathcal{P}_X(C), P].$$

Proof. The proof is by strong induction on m. The base case $m = 3$ is straightforward. For the induction step $m = n + 1$, suppose $(\succsim, \alpha)$ defines $\mathcal{P}_X$ and $X = \{x_1, ..., x_m\}$ is labeled according to $\succ^*$. By Claim 1: $B = x_1 \cup X_j^m$ for some $3 \leq j \leq m + 1$; and, $C = X_2^m$. If $j = m + 1$, then B is a singleton and there is no profile P s.t. $UNE[\mathcal{P}_X, P] \in B \cap C$. So, suppose $j \leq m$ without loss of generality. Now, fix a profile P s.t. $UNE[\mathcal{P}_X, P] = x_k$ for $j \leq k \leq m$. By way of contradiction, suppose the claim is false. Since $UNE[\mathcal{P}_X, P] = x_k$, "backward induction" leads to two possibilities for $UNE[\mathcal{P}_X(B), P] \equiv x_b$ and $UNE[\mathcal{P}_X(C), P] \equiv x_c$: (i) $b = k$ and $c \neq k$; and, (ii) $b \neq k$ and $c = k$. To establish the result, I show that each case leads to a contradiction:

(i) Consider the agenda $\mathcal{T}_{X\setminus x_1}$ (described in Claim 3) where the successors of the root are X_j^m and $C = X_2^m$. Since $UNE[\mathcal{P}_X(B), P] = x_b$, $UNE[\mathcal{P}_{X_j^m}, P] = x_b$. To see this, let B' denote the "left" successor of B in

[19] This type of argument is well-known in the literature (see e.g. Theorem 2.2 of Iglesias et al. [2014]).

$\mathcal{P}_X$. If $x_b \in B' \cap X_j^m$, then $UNE[\mathcal{P}_{X_j^m}, P] = x_b$ by the induction hypothesis. If $x_b \notin B'$, then $UNE[\mathcal{P}_{X_j^m}, P] = x_b$ as well. (Otherwise, "backward induction" gives $UNE[\mathcal{P}_X(B), P] \neq x_b$.) Since $UNE[\mathcal{P}_X, P] = x_b$, $UNE[\mathcal{P}_{X_j^m}, P] = x_b$ implies $UNE[\mathcal{T}_{X \setminus x_1}, P] = x_b$ as well. By the argument in Claim 3(ii)-(iii), $\mathcal{T}_{X \setminus x_1}$ must be outcome-equivalent to $\mathcal{P}_{X \setminus x_1} = \mathcal{P}_X(C)$. So, $x_b = UNE[\mathcal{T}_{X \setminus x_1}, P] = UNE[\mathcal{P}_X(C), P] = x_c$, which is a contradiction.

(ii) By Claim 1, X_3^m is the "right" successor of C in $\mathcal{P}_X$. Moreover, $UNE[\mathcal{P}_{X_3^m}, P] = x_c$ by the same reasoning as in (i). Now, consider the priority agenda $\mathcal{P}_{X \setminus x_2}$ on n alternatives. If $j > 3$, then the successors at the root node $X \setminus x_2$ are B and X_3^m. So, "backward induction" gives $UNE[\mathcal{P}_{X \setminus x_2}, P] = x_c$. Since $UNE[\mathcal{P}_X(B), P] = x_b$ however, this contradicts the induction hypothesis. If $j = 3$, then $\mathcal{P}_{X \setminus x_2} = \mathcal{P}_X(B)$. So, $UNE[\mathcal{P}_{X \setminus x_2}, P] = x_b$. Since the "right" successor of B in $\mathcal{P}_{X \setminus x_2}$ is X_3^m and $UNE[\mathcal{P}_{X_3^m}, P] = x_c$ however, this contradicts $UNE[\mathcal{P}_X, P] = x_c$. ∎

Proof of Proposition 2. Let $\mathcal{P}_X$ denote the priority agenda that implements v. It suffices to show preference monotonicity for X. Since $UNE[\mathcal{P}_{X|A}, P] = UNE[\mathcal{P}_X, P^A]$ by Remark 1, the result then follows for all $A \subseteq X$.

The proof is by strong induction on m. The base case $m = 3$ is straightforward. For the induction step $m = n+1$, fix a profile P s.t. $UNE[\mathcal{P}_X, P] = x$ and suppose the successors of the root X are B and C. There are two cases: (i) $x \in B \cap C$; and, (ii) $x \in B \setminus C$. **(i)** By Claim 5, $UNE[\mathcal{P}_X(B), P] = UNE[\mathcal{P}_X(C), P] = x$. Since $\mathcal{P}_X(B)$ and $\mathcal{P}_X(C)$ are priority agendas on n or fewer alternatives, $UNE[\mathcal{P}_X(B), P^x] = UNE[\mathcal{P}_X(C), P^x] = x$ by the induction hypothesis. Then, "backward induction" gives $UNE[\mathcal{P}_X, P^x] = x$. **(ii)** Since $\mathcal{P}_X(B)$ is a priority agenda on n or fewer alternatives, $UNE[\mathcal{P}_X(B), P^x] = x$ by the induction hypothesis. Since $x \notin C$, $UNE[\mathcal{P}_X(C), P^x] = UNE[\mathcal{P}_X(C), P]$. Since $UNE[\mathcal{P}_X, P] = x$, "backward induction" gives $UNE[\mathcal{P}_X, P^x] = x$. ∎

Proof of Proposition 3. Suppose $\mathcal{P}_X$ is defined by $(\succsim, \alpha)$ where $X = \{x_1, ..., x_m\}$ is labeled according to $\succ^*$. By Remark 1, it suffices to establish the result for X. The proof is by strong induction on m.

The base cases $m = 2, 3$ are straightforward. For the induction step $m = n+1$, fix a profile P s.t. $UNE[\mathcal{P}_X, P] = x_k$. (Where $x_{k-1} \sim x_k$, suppose that every alternative that amends x_{k-1} also amends x_k.) By Claim 1, the successors of the root X are: $B \equiv x_1 \cup X_j^m$ for some $3 \leq j \leq m+1$; and, $C \equiv X_2^m$. So, there are four cases to consider: (i) $j+1 \leq k \leq m$; (ii) $3 \leq k \leq j-1$; (iii) $k = j$; and, (iv) $k = 2$. To establish the result, I show that $UNE[\mathcal{P}_X^k, P] = x_k$ in each case (where $\mathcal{P}_X^k \equiv \mathcal{P}_X^{x_k}$ to simplify the notation):

(i) Since $x_k \in B \cap C$ and $UNE[\mathcal{P}_X, P] = x_k$, Claim 5 implies $UNE[\mathcal{P}_X(B), P] = UNE[\mathcal{P}_X(C), P] = x_k$. Since $\mathcal{P}_X(B)$ and $\mathcal{P}_X(C)$ are priority agendas on n or fewer alternatives and $k \neq j$, the induction hypothesis implies $UNE[\mathcal{P}_X^k(B), P] = UNE[\mathcal{P}_X^k(C), P] = x_k$. So, $UNE[\mathcal{P}_X^k, P] = x_k$.

(ii) Since $x_k \notin B$ and $UNE[\mathcal{P}_X, P] = x_k$, "backward induction" implies $UNE[\mathcal{P}_X(C), P] = x_k$. Since $\mathcal{P}_X(C)$ is a priority agenda on n or fewer alternatives, the induction hypothesis implies $UNE[\mathcal{P}_X^k(C), P] = UNE[\mathcal{P}_X(C), P] = x_k$. Since $x_k \notin B$, $UNE[\mathcal{P}_X^k, P] = x_k$ by the same kind of reasoning as Proposition 2(ii).

(iii) By the reasoning in case (i), $UNE[\mathcal{P}_X(B), P] = UNE[\mathcal{P}_X(C), P] = x_j$ and $UNE[\mathcal{P}_X^k(C), P] = x_j$. By way of contradiction, suppose $UNE[\mathcal{P}_X^k, P] \neq x_j$. Then, $UNE[\mathcal{P}_X^k, P] = x_1$ by Claim 5. Since $UNE[\mathcal{P}_X^k(C), P] = x_j$, "backward induction" implies that x_1 is majority preferred to x_j.

Since $UNE[\mathcal{P}_X^k, P] = x_1$, "backward induction" implies $UNE[\mathcal{P}_X^k(B_j), P] = x_1$ where $B_j \equiv \{x_1, x_{j-1}\} \cup X_{j+1}^m$ is the "left" successor of the root X in $\mathcal{P}_X^k$. Since the agenda at the "left" successors $B' \equiv x_1 \cup X_{j'}^m$ of B and B_j coincide and $UNE[\mathcal{P}_X^k(B_j), P] = x_1$, "backward induction" implies $UNE[\mathcal{P}_X^k(B'), P] = UNE[\mathcal{P}_X(B'), P] = x_1$. Since $UNE[\mathcal{P}_X(B), P] = x_j$ however, "backward induction" implies that x_j is majority preferred to x_1. Since this contradicts the inference drawn in the last paragraph, it follows that $UNE[\mathcal{P}_X^k, P] = x_j$.

(iv) While more involved than case (iii), the basic proof technique in this case is similar. Since $UNE[\mathcal{P}_X, P] = x_2$ and $x_2 \notin B$, "backward induction" implies $UNE[\mathcal{P}_X(C), P] = x_2$. Let $B' \equiv x_2 \cup X_{j'}^m$ and $C' \equiv X_3^m$ denote the two successors of C in $\mathcal{P}_X$. Since x_2 only appears at one terminal node below B' and $UNE[\mathcal{P}_X(C), P] = x_2$, x_2 is majority preferred to $UNE[\mathcal{P}_X(C'), P]$ and every alternative it meets on the "backward induction" path in $\mathcal{P}_X(B')$. Now, consider $\mathcal{P}_X^2$, letting $B_2 \equiv x_2 \cup X_j^m$ and $C_2 \equiv x_1 \cup X_3^m$ denote its two successors. By construction, $B_2' \equiv x_1 \cup X_{j'}^m$ and $C_2' \equiv C'$ are the two successors of C_2 in $\mathcal{P}_X^2$.

Since every alternative that amends x_1 in $\mathcal{P}_X$ also amends x_2, everything that x_2 meets on the "backward induction path" in $\mathcal{P}_X(B_2)$ is something that it meets in $\mathcal{P}_X(B')$. Since x_2 is majority preferred to all of these alternatives by the first observation in the last paragraph, $UNE[\mathcal{P}_X^2(B_2), P] = x_2$. Moreover, $UNE[\mathcal{P}_X^2(C_2'), P] = UNE[\mathcal{P}_X(C'), P]$ by the second observation in the last paragraph.

By way of contradiction, suppose $UNE[\mathcal{P}_X^2, P] \neq x_2$. Since $UNE[\mathcal{P}_X^2(B_2), P] = x_2$ and x_2 is majority preferred to $UNE[\mathcal{P}_X^2(C_2'), P] = UNE[\mathcal{P}_X(C'), P]$, "backward induction" implies $UNE[\mathcal{P}_X^2, P] = UNE[\mathcal{P}_X^2(B_2'), P]$. Since x_2 is majority preferred to everything that x_1 meets along the "backward induction path" in $\mathcal{P}_X(B_2')$, $UNE[\mathcal{P}_X^2, P] = UNE[\mathcal{P}_X^2(B_2'), P] = x_1$. Since $UNE[\mathcal{P}_X^2(B_2), P] = x_2$, x_1 is majority preferred to x_2.

Since $UNE[\mathcal{P}_X^2(B_2'), P] = x_1$, the same kind of reasoning as in the previous paragraphs establishes that $UNE[\mathcal{P}_X^2(B), P] = x_1$. Since $UNE[\mathcal{P}_X^2, P] = x_2$ however, x_2 is majority preferred to x_1. Since this contradicts the inference drawn in the last paragraph, it follows that $UNE[\mathcal{P}_X^2, P] = x_2$. ■

(c) Proof of Theorem 1 and Theorem 1*

Claim 6 *For any two profiles P, P' that coincide on A, $v(P, A) = v(P', A)$.*

Proof. The proof is by induction on $|A|$. The base case $|A| = 2$ follows from ILA. For the induction step, $v(P, A) = v(P, \{v(P, B), v(P, C)\}) = v(P, \{v(P', B), v(P', C)\}) = v(P', \{v(P', B), v(P', C)\}) = v(P', A)$ follows from IS, the induction hypothesis, and the base case. ■

Using the definition of P^A from section (a) above, one can establish an analog of Remark 1:

Claim 7 *For any decision problem (P, A), $v(P, A) = v(P^A, X)$.*

Proof. By ILA, $v(P^A, X) = ... = v(P^A, A)$. Since $v(P^A, A) = v(P, A)$ by Claim 6, $v(P^A, X) = v(P, A)$. ■

Claim 8 *Suppose (B, C) splits A for v. Then, for all $D \subseteq A$:*
- ***(i)*** *$v(P, D) = v(P, \{v(P, B \cap D), v(P, C \cap D)\})$; and,*
- ***(ii)*** *$(B \cap D, C \cap D)$ splits D if $D \neq B \cap D, C \cap D$.*

Proof. Fix some $x \in A$ and let P_x coincide with P except x is demoted to Condorcet loser on A. Then, $v(P, A \setminus x) = v(P_x, A \setminus x) = v(P_x, A) = v(P_x, \{v(P_x, B), v(P_x, C)\}) = ... = v(P, \{v(P, B \setminus x), v(P, C \setminus x)\})$ by Claim 6, ILA, and IS. Part (i) follows by repeated application of this reasoning. For part (ii), observe that $D \neq B \cap D, C \cap D$ implies $B \cap C \cap D \neq B \cap D, C \cap D$. Then, given part (i), $(B \cap D, C \cap D)$ splits D. ■

Claim 9 *For v, there is a unique way to split every issue.*

Proof. By way of contradiction, suppose (B, C) and (B', C') are distinct splittings of A. Using Claim 8(i), it can be shown that $v(P, \{v(P, B), v(P, C)\}) \neq v(P, \{v(P, B'), v(P, C')\})$ for some profile P whose Condorcet set is a cyclic triple where (B, C) and (B', C') disagree. The contradiction proves the claim. ■

Claim 10 *$\mathcal{S}_X^v$ is continuous.*

Proof. The proof is by strong induction on $m \equiv |X|$. The base cases $m = 2, 3$ follow directly from the definition of $\mathcal{S}_X^v$ and IS. For the induction step $m = n + 1$, consider the root node X of $\mathcal{S}_X^v$. Let B and C denote its two successors. By IS and the induction hypothesis, the agendas $\mathcal{S}_X^v(B)$ and $\mathcal{S}_X^v(C)$ are simple.

Let B' and C' denote the two successors of B. And, let $b \in B' \setminus C'$ ($c \in C' \setminus B'$) denote some alternative that labels one terminal node below B' (C'). To complete the proof, it suffices to show that $b \notin C$ or $c \notin C$. (The argument for C is similar.) By way of contradiction, suppose $b, c \in C$. By IS, there exists some $x \in B \setminus C$. By Claim 8, it follows that $v(P, \{x, b, c\}) = v(P, \{v(P, \{x, b, c\}), v(P, \{b, c\})\}$.

By Claim 8 and the assumption about B', the only possible splittings of $\{x, b, c\}$ are: (i) $(\{b\}, \{c, x\})$; (ii) $(\{b, x\}, \{c\})$; or, (iii) $(\{b, x\}, \{c, x\})$. By the formula in the last paragraph, each of these cases entails a contradiction: (i) $b = v(P_{xcb}, \{x, b, c\}) \neq v(P_{xcb}, \{v(P_{xcb}, \{x, b, c\}), v(P_{xcb}, \{b, c\})\} = c$; (ii) $c = v(P_{xbc}, \{x, b, c\}) \neq v(P_{xbc}, \{v(P_{xbc}, \{x, b, c\}), v(P_{xbc}, \{b, c\})\} = b$; or, (iii) both of the contradictions obtained in cases (i)-(ii). ■

Proof of Theorem 1. Using the approach described in the text, the structure of the agenda S_X^v can be determined from outcomes on Condorcet triples. By construction, S_X^v is non-repetitive. By Claim 10, S_X^v is continuous.

To show that $\mathcal{S}_X^v$ implements v, I show that $UNE[\mathcal{S}_X^v; P] = v(P, X)$ for any profile P. Since $UNE[\mathcal{S}_{X|A}^v; P] = UNE[\mathcal{S}_X^v; P^A]$ (by Remark 1) and $v(P^A, X) = v(P, A)$ (by Claim 7), $UNE[\mathcal{S}_{X|A}^v; P] = v(P, A)$ for any $A \subset X$. To see that $UNE[\mathcal{S}_X^v; P] = v(P, X)$, use "backward induction" on $\mathcal{S}_X^v$ (see the proof of Remark 1). In any terminal subgame, the UNE selects the Condorcet winner. By ILA, so does v. By deleting the Condorcet loser and continuing in this fashion, $UNE[\mathcal{S}_X^v; P] = v(P, X)$ follows immediately by IS and the construction of $\mathcal{S}_X^v$. Finally, Claim 9 ensures that $\mathcal{S}_X^v$ is the unique simple agenda implementing v. For any simple agenda $\mathcal{S}_X$ implementing v, the subgames at any node A must induce the unique splitting of A. So, $\mathcal{S}_X$ must coincide with $\mathcal{S}_X^v$. ■

Proof of Theorem 1*. (using the assumptions about v in the statement of the Theorem) ($\Rightarrow$) Since ILA is necessary for v to be implementable by agenda and v satisfies IS by assumption, the result follows from the discussion in the text. ($\Leftarrow$) Fix an issue A. Since $v(\cdot, A)$ is implementable by simple agenda, "backward induction" establishes that A can be split. Since this is true for every A, v satisfies IS. ■

(d) Proof of Theorem 2

Sub-sections (i) and (ii) establish Lemmas 1 and 2. The proof of Theorem 2 is given in sub-section (iii).

(i) Proof of Lemma 1

Claim 11 *If a^* is marginal in A for v, then it is marginal in $A \setminus x$ for all $x \in A \setminus a^*$.*

Proof. Fix any $x \in A \setminus a^*$. By way of contradiction, suppose $v(P, A \setminus x) = a^*$ for some profile P where a^* is not the Condorcet winner in $A \setminus x$. By Claim 6, $v(P_x, A \setminus x) = v(P, A \setminus x)$ for any profile P_x that coincides with P except x is demoted to Condorcet loser in A. Moreover, $v(P_x, A) = v(P_x, A \setminus x)$ by ILA. So, $v(P_x, A) = a^*$, which contradicts the assumption that a^* is marginal in A. ■

Claim 12 *For v, every issue A has at most two marginal alternatives.*

Proof. Suppose otherwise. Denote any three marginal alternatives by x, y, and z and consider the triple P_{xyz} (as defined in the text). Then, $v(P_{xyz}, \{x, y, z\}) \notin \{x, y, z\}$ by Claim 11, which is a contradiction. ■

Claim 13 *Suppose (B, C) splits A for v and a^* is marginal in A. Then:*
(i) *if $a^* \in C \setminus B$, then $(B, C) = (b, A \setminus b)$; and,*
(ii) *if $a^* \in B \cap C$ and $a^{**} \in C \setminus a^*$ is also marginal in A, then $a^{**} \in B \cap C$.*

Proof. (i) By way of contradiction, suppose $|B \setminus C| \geq 2$. Fix $b, b' \in B \setminus C$ and consider the triple $P_{a^*bb'}$. By Claim 8, $v(P_{a^*bb'}, \{a^*, b, b'\}) = v(P_{a^*bb'}, \{v(P_{a^*bb'}, \{b, b'\}), a^*\}) = a^*$. By Claim 11, this contradicts the assumption that a^* is marginal in A. **(ii)** By way of contradiction, suppose $a^{**} \notin B$. Fix some $b \in B \setminus C$ and consider the triple $P_{a^*ba^{**}}$. Then, a contradiction obtains along the same lines as (i). ■

Claim 14 *If v satisfies WM, then $\succ_v$ is asymmetric and $\succsim_v$ is complete.*

Proof. The completeness of $\succsim_v$ is a direct consequence of the asymmetry of $\succ_v$. To see that $\succ_v$ is asymmetric, suppose $y \succ_v z$ and $z \succ_v y$ for some $y, z \in X$. Let Y and Z denote the issues leading to the inferences $y \succ_v z$ and $z \succ_v y$. The proof that this amounts to a contradiction is by induction on $|Y \cup Z|$.

For $|Y \cup Z| = 4$: suppose $Y = \{a^*, y, z\}$ and $Z = \{x, y, z\}$. (Every other case is ruled out by ILA or Claim 11.) By WM and Claims 11-12, the only possible marginal alternatives in $Y \cup Z$ are: a^* and x; or, one of the two, say a^*. Now, consider the unique splitting (B, C) of $\{a^*, x, y, z\}$. There are two possibilities: (i) $a^* \in C \setminus B$; and, (ii) $a^* \in B \cap C$. **(i)** By Claim 13(i), $(B, C) = (b, \{a^*, x, y, z\} \setminus b)$ with $b \neq a^*$. By Claim 8, every possibility for b leads to a contradiction: if $b = y$, then y is not marginal in $\{a^*, y, z\}$; if $b = z$, then z is not marginal in $\{x, y, z\}$; and, if $b = x$, then y is marginal in $\{x, y, z\}$.[20] **(ii)** By Claims 8 and 13(ii), $y \in B \cap C$. So, $(B, C) = (\{a^*, y, z\}, \{a^*, x, y\})$. But, then z is not marginal in $\{x, y, z\}$.

For $|Y \cup Z| = n + 1$: Y and Z have one or two marginal alternatives (by WM and Claim 12). If both have two, then this reduces to the case $|Y \cup Z| = n$ by Claim 11. If both have one, then y or z is marginal

[20] This last case cannot occur if x is marginal in $\{a^*, x, y, z\}$.

in $Y \cup Z$ by WM and Claims 11-12. So, either y is marginal in Z or z is marginal in Y by Claim 11, both contradictions. So, suppose y and a^* are marginal in Y while z is marginal in Z. By WM and Claim 11, a^* is the only marginal alternative in $Y \cup Z$ and $a^* \notin Z$. By Claim 11, it also follows that: a^* and y are marginal in $\{a^*, y, z\}$; and, z is marginal in $\{x, y, z\}$ for any $x \in Z$.

Now, consider the splitting (B, C) of $Z^* = Z \cup a^*$. As in the base case, there are two possibilities: (i) $(B, C) = (\{b\}, Z^* \setminus b)$ with $b \neq a^*$; and, (ii) $a^* \in B \cap C$. For both, I claim that y and z must appear in the same sub-issues as a^*. **(i)** As in the base case, $b \neq y, z$. So, $\{y, z\} \subseteq C$ as claimed. **(ii)** As in the base case, $y \in B \cap C$. This, in turn, implies $z \in B \cap C$. To see why, suppose $z \in B \setminus C$ and fix some $x \in C \setminus B$. Then, as in the base case, z cannot be marginal in $\{x, y, z\}$. So, $\{y, z\} \subseteq B \cap C$ as claimed.

Continuing in the same vein on the sub-issues B and C, it follows that y and z always appear in the same sub-issues (up to the splitting of $\{a^*, y, z\}$). Now, construct the agenda $\mathcal{S}^v_{Z \cup a^*}$. By the last observation, y and z appear in exactly the same subgames of $\mathcal{S}^v_{Z \cup a^*|Z}$ (i.e. after a^* is deleted). By assumption, $v(P, Z) = y$ for some profile P where y is not the Condorcet winner in Z. Since $\mathcal{S}^v_Z$ implements v on Z by Theorem 1, $v(P, Z) = UNE[\mathcal{S}^v_Z; P] = y$. To show a contradiction, consider the related profile P_σ that permutes z and y in every voter's preference. From the symmetry of $\mathcal{S}^v_Z$, $v(P_\sigma, Z) = UNE[\mathcal{S}^v_Z; P_\sigma] = z$. But, this contradicts the assumption that z is marginal in Z and establishes that $\succ_v$ is asymmetric. ■

Claim 15 *If v satisfies WM, $\succsim_v$ is a weak order whose indifference classes contain one or two alternatives.*

Proof. Since $\succsim_v$ is complete by Claim 14, showing transitivity proves $\succsim_v$ is a weak order. Fix $x \succsim_v y \succsim_v z$. By way of contradiction, suppose $z \succsim_v x$. By WM, some alternative is marginal in $A = \{x, y, z\}$. By definition of $\succsim_v$, it then follows that A has three marginal alternatives. But, this contradicts Claim 12. This rules out the possibility that $z \succsim_v x$. Since $\succsim_v$ is complete by Claim 14, it follows that $x \succ_v z$, which shows that the indifference classes of $\succsim_v$ may contain at most two alternatives. ■

Proof of Lemma 1. Claim 15 establishes (i). To establish (ii), fix an issue A. By WM, some $x \in A$ must be marginal. Let $z \in \min_{\succsim_v} A$ and $y \equiv \min_{\succsim_v} A \setminus z$. If $x \neq y, z$, one obtains a contradiction along the lines of Claim 15. So, suppose $x = y$. Since $\succsim_v$ is complete by Claim 14, there are two possibilities. If $x \succsim_v z$, then z is marginal as well. If $z \succ_v x$, then x may be the only marginal alternative. ■

Corollary 2 *If v satisfies WM, then:*

(i) $y \succ_v z$ *if and only if there exists an* $x \in X$ *such that* $v(P_{xyz}, \{x, y, z\}) = y$*; and,*

(ii) $y \sim_v z$ *if and only if*
$$\left\{\begin{array}{c} v(P_{xyz}, \{x, y, z\}) = z \quad \text{and} \quad v(P_{xzy}, \{x, y, z\}) = y \\ \textbf{or} \\ v(P_{xyz}, \{x, y, z\}) = x \quad \text{and} \quad v(P_{xzy}, \{x, y, z\}) = x \end{array}\right\} \text{ for all } x \in X.$$

Proof. (i) ($\Leftarrow$) From the six possible splittings of $\{x, y, z\}$, $v(P_{xyz}, \{x, y, z\}) = y$ implies $v(P_{xzy}, \{x, y, z\}) \in \{x, y\}$. So, $y \succ_v z$. ($\Rightarrow$) By way of contradiction, suppose $v(P_{xyz}, \{x, y, z\}) \neq y$ for all $x \in X$. Since $y \succ_v z$, asymmetry implies $v(P_{xyz}, \{x, y, z\}) \neq z$ for all $x \in X$. So, $v(P_{xyz}, \{x, y, z\}) = x$ for all $x \in X$. By the argument in ($\Leftarrow$), $v(P_{xzy}, \{x, y, z\}) \in \{x, z\}$ for all $x \in X$. Since $y \succ_v z$, asymmetry implies $v(P_{xzy}, \{x, y, z\}) = x$ for all $x \in X$. Now, fix an issue A s.t $|A| \geq 3$ with splitting (B, C). First, observe that $y, z \in B \cap C$ or $y, z \in C \setminus B$. Otherwise, $y \in C \setminus B$ and $z \in B$ without loss of generality. If $z \in B \setminus C$, Claim 8 shows that $v(P_{xyz}, \{x, y, z\}), v(P_{xzy}, \{x, y, z\}) \neq x$ for any $x \in A$. If $z \in B \cap C$, there is a similar contradiction for $x \in B \setminus C$. Since $y, z \in B \cap C$ or $y, z \in C \setminus B$ for the splitting (B, C) of any issue A, y and z appear in the same subgames of $\mathcal{S}^v_X$. Since $y \succ_v z$, a contradiction obtains by the argument in Claim 14. **(ii)** Fix any $x \in X$. Given the six possible splittings of $\{x, y, z\}$, the result follows from (i) and the fact that $\succsim_v$ is a weak order (by Claim 14). ■

(ii) Proof of Lemma 2

Claim 16 *If v satisfies WM and (B, C) is the unique splitting of A, then $|C \setminus B| \geq 2$ implies $|B \setminus C| = 1$.*

Proof. By way of contradiction, suppose $|C \setminus B|, |B \setminus C| \geq 2$. Fix any $b, b' \in B \setminus C$ and $c, c' \in C \setminus B$. By Claim 8, $(\{b, b'\}, \{c, c'\})$ is the unique splitting of $A' = \{b, b', c, c'\}$. For all $a \in A'$, it follows that $v(P(a), A') = a$ for some profile $P(a)$ where a is not the Condorcet winner, which contradicts WM. ■

Using the definitions in section (b) above:

Claim 17 *If v satisfies WM and $A = \{a_1, ..., a_K\}$ is labeled according to $\succ_v^*$ for $K \geq 2$, then:*

$$(a_1 \cup A_j^K, A_2^K) \text{ splits } A \text{ for some } j \text{ s.t. } j \in \{3, ..., K+1\}.$$

Proof. By Claim 13(i), there are two possibilities for the unique splitting (B, C) of A: (i) either $(B, C) = (b, A \setminus b)$ with $b \neq a_K$; or, (ii) $a_K \in B \cap C$. In either case, I show that (B, C) has the form required.

(i) In this case, it suffices to show $b = a_1$. By way of contradiction, suppose $b = a_k$ for some $k \neq 1, K$. Then, $v(P_{a_K a_k a_1}, \{a_1, a_k, a_K\}) = a_k$ by Claim 8 so that $a_k \succ_v a_1$ by Lemma 2. Since $a_1 \succsim_v a_k$ by assumption, $a_k \succ_v a_1$ contradicts the fact that $\succsim_v$ is a weak order (by Claim 14). So, $b = a_1$ as required.

(ii) By Claim 16, there are two possibilities: (1) $(B, C) = (A \setminus c, A \setminus b)$; and, (2) $(B, C) = (b \cup B', A \setminus b)$ for $B' \subset A \setminus b$ and $|A \setminus B'| \geq 3$. **(1)** It suffices to show $b = a_1$ and $c = a_2$. If $a_1 \neq b, c$, then the outcomes on $\{a_1, b, c\}$ lead to the contradictions $b, c \succ_v a_1$ following the same kind of reasoning as in case (i). So, $b = a_1$ without loss of generality. If $a_2 \neq c$, then the outcomes on $\{a_1, a_2, c\}$ lead to the contradiction $c \succ_v a_2$. So, $c = a_2$. **(2)** It suffices to show: (a) $b = a_1$; (b) $a_2 \notin B'$; and, (c) $a_k \in B'$ implies $a_{k+1} \in B'$. **(a)** By the same reasoning as (1), $a_1 \notin B \cap C = B'$. If $a_1 \neq b$, then $\{a_1, b, c\}$ leads to the contradiction $b \succ_v a_1$ for $c \notin b \cup B'$. So, $b = a_1$. **(b)** If $a_2 \in B'$, then $\{a_1, a_2, c\}$ leads to the contradiction $c \succ_v a_2$ for $c \notin a_1 \cup B'$ given (a). So, $a_2 \notin B'$. **(c)** If $a_k \in B'$ and $a_{k+1} \notin B'$, then $\{a_1, a_k, a_{k+1}\}$ leads to the contradiction $a_{k+1} \succ_v a_k$ given (a). ■

Definition 9 *Given a decision rule v with revealed priority $\succsim_v$, x is **revealed to amend** $b \succ_v x$ if:*

$$v(P_{xbc}, \{b, c, x\}) = c \text{ for all } c \in X \text{ such that } b \succsim_v c \succ_v x.$$

Define α_v as follows: $b \in \alpha_v(x)$ if x is revealed to amend b and x is not revealed to amend any $a \succ_v b$.

Lemma 2 *If v satisfies WM, then α_v is an amendment rule.*

Proof. Definition 9 and Corollary 2 ensure the following: (i) $x \in \alpha_v(z) \Rightarrow x \succ_v z$; and, (iii) $[x \sim_v y \succ_v z$ and no $z' \in X$ s.t. $x \sim_v y \succ_v z' \succ_v z] \Rightarrow [x \in \alpha_v(z)$ or $w \succ_v x$ for all $w \in \alpha_v(z)]$. I show: (ii) $x \sim_v y \Rightarrow \alpha_v(x) = \alpha_v(y)$.

(ii) It suffices to show that y is revealed to amend b if x is revealed to amend b. By way of contradiction, suppose y is not revealed to amend b. By Definition 9, there exists some c s.t. $b \succsim_v c \succ_v y \sim_v x$ and, moreover, $v(P_{ybc}, \{b, c, y\}) \neq c$ for all such c. By Corollary 2, $v(P_{ybc}, \{b, c, y\}) \neq y$. Otherwise, $y \succ_v b$ which contradicts the fact that $\succsim_v$ is a weak order (by Claim 14). So, $v(P_{ybc}, \{b, c, y\}) = b$ which, by Corollary 2, implies $b \succ_v c$. Finally, Definition 9(i) implies $v(P_{xbc}, \{b, c, x\}) = c$ for all c s.t. $b \succsim_v c \succ_v x$ (since x is revealed to amend b). To summarize, $v(P_{xbc}, \{b, c, x\}) = c$ and $v(P_{ybc}, \{b, c, y\}) = b$ for some c s.t. $b \succ_v c \succ_v y \sim_v x$.

By Claim 17, the splitting of $\{b, c, x, y\}$ is $(b \cup B', \{c, x, y\})$ for $B' \subseteq \{x, y\}$. By Claim 8, $v(P_{xbc}, \{b, c, x\}) = c$ implies $x \in B'$ and $v(P_{ybc}, \{b, c, y\}) = b$ implies $y \notin B'$. So, the splitting of $\{b, c, x, y\}$ is $(\{b, x\}, \{c, x, y\})$. By Claim 8, this implies $v(P_{byx}, \{b, x, y\}) = y$ so that $y \succ_v x$ by Corollary 2, which is a contradiction. ■

(iii) Proof of Theorem 2

Proof of Theorem 2. (using the assumptions about v in the statement of the Theorem) ($\Leftarrow$) The discussion in the text following Theorem 1 shows that v satisfies ILA (i.e. any decision rule implemented by an agenda satisfies ILA). In turn, Proposition 1 and Claim 4 show that v satisfies WM and IS.

($\Rightarrow$) This follows from the fact that the simple agenda $\mathcal{S}_X^v$ from Theorem 1 coincides with the priority agenda $\mathcal{P}_X^v$ defined by $(\succsim_v, \alpha_v)$. To establish this fact, it suffices to show that the successors at the root node of $\mathcal{P}_X^v$ and $\mathcal{S}_X^v$ coincide. Extending this reasoning by induction, it follows that $\mathcal{S}_X^v$ coincides with $\mathcal{P}_X^v$.

Consider the root node $X = \{x_1, ..., x_m\}$ of $\mathcal{S}_X^v$ as labeled according to $\succ_v^*$. By Claim 17, the successors of X are $x_1 \cup X_j^m$ and X_2^m for some j s.t. $j \in \{3, ..., m+1\}$. There are two cases: **(i)** X_j^m is empty (i.e. $j = m+1$); or, **(i)** X_j^m is non-empty (i.e. $j \leq m$). The fact that the successors of X on $\mathcal{S}_X^v$ coincide with the successors on $\mathcal{P}_X^v$ follows by Claim 1 in both cases: **(i)** Claim 8 applied to (x_1, X_2^m) gives $v(P_{x_j x_1 c}, \{x_1, x_j, x\}) = x_1$ for all x_j, c s.t. $x_1 \succsim_v c \succ_v x_j$. So, no x_j s.t. $x_2 \succ_v x_j$ is revealed to amend x_1 (by Definition 9). **(ii)** Claim 8 applied to $(x_1 \cup X_j^m, X_2^m)$ gives $v(P_{x_j x_k x_1}, \{x_1, x_k, x_j\}) = x_1$ and $v(P_{x_j x_1 x_k}, \{x_1, x_k, x_j\}) = x_k$ for

$x_k \in X_2^{j-1}$. This shows that $x_k \succ_v x_j$ (by Corollary 2) and x_j is revealed to amend x_1. To see that no x_k s.t. $x_2 \succ_v x_k \succ_v x_j$ is revealed to amend x_1, it is enough to observe that $v(P_{x_k x_1 x_2}, \{x_1, x_2, x_k\}) = x_1$ for all $x_k \in X_3^{j-1}$. ■

(e) Proof of Proposition 4

Claim 18 *If every issue A s.t. $|A| \neq 1$ has two marginal alternatives, then $\mathcal{S}_X^v$ is a Euro-Latin agenda.*

Proof. Consider the splitting (B_1, X_1) of X and let a^* denote a marginal alternative in X. By Claim 13(i), there are two possibilities: (1) $a^* \in B_1 \cap X_1$ with $b_1 \in B_1 \backslash X_1$ and $x_1 \in X_1 \backslash B_1$; and, (2) $(B_1, X_1) = (b_1, X \backslash b_1)$. For (1), Claim 8 implies that $\{a^*, b_1, x_1\}$ has one marginal alternative a^*, a contradiction. So, the splitting must be (2). Continuing in the same vein on X_1 establishes that $\mathcal{S}_X^v$ is a Euro-Latin agenda. ■

Claim 19 *If every issue A s.t. $|A| \neq 2$ has a unique marginal alternative, then $\mathcal{S}_X^v$ is an Anglo-American agenda.*

Proof. Consider the splitting (B, C) of X. If $|C \backslash B| \geq 1$ (with $b \in B \backslash C$ and $c, c' \in C \backslash B$), then Claim 8 implies that $\{b, c, c'\}$ has two marginal alternatives c and c', a contradiction. This shows that $|C \setminus B| = |B \setminus C| = 1$. In other words, $(B, C) = (X \setminus c_1, X \setminus b_1)$ for some $b_1 \in B$ and $c_1 \in C$. Continuing in the same vein on $X \setminus c_1$ and $X \setminus b_1$ establishes that $\mathcal{S}_X^v$ is an Anglo-American agenda. ■

Proof of Proposition 4. (using the assumptions about v in the statement of the Theorem) Regarding the first part of the claim: the Euro-Latin procedure has two marginal alternatives for all A s.t. $|A| \neq 1$; and, the Anglo-American procedure has a unique marginal alternative for all A s.t. $|A| \neq 2$.

Regarding the second part of the claim, suppose $|X| \geq 3$. (If $|X| = 2$, the claim is trivial.) By Claim 12, there are two cases: (i) X has two marginal alternatives a_1^* and a_2^*; or, (ii) X has a unique marginal alternative.

(i) By Theorem 1 and Claim 18, it suffices to show that all A s.t. $|A| \neq 1$ have two marginal alternatives. By Claim 11, a_1^* and a_2^* are marginal in $X \setminus x$ for all $x \neq a_1^*, a_2^*$. By NP, every $X \setminus x$ has two marginal alternatives. Continuing in the same vein, the result follows by a simple inductive argument.

(ii) By Theorem 1 and Claim 19, it suffices to show that all A s.t. $|A| \neq 2$ have one marginal alternative. By way of contradiction, suppose $|X| \geq 4$ and some $X \setminus x$ has two marginal alternatives. Then, by the argument in case (i), every $A \neq X$ s.t. $|A| \neq 1$ has two marginal alternatives. To establish the contradiction, consider the splitting (B_1, X_1) of X. By the argument in Claim 18, $(B_1, X_1) = (b_1, X \setminus b_1)$. Since $\mathcal{S}_{X_1}^v$ is Euro-Latin by Claim 18, this shows that $\mathcal{S}_X^v$ is as well. It follows that X has two marginal alternatives, which is a contradiction. ■

(f) Proof of Corollary 1

For part (i), Apesteguia et al. [2014] show CP is necessary. Sufficiency follows from Claim 18 and:

Claim 20 *If v satisfies CP, then every issue A such that $|A| \neq 1$ has two marginal alternatives.*

For part (ii), Apesteguia et al. [2014] show CA is necessary. Sufficiency follows from Claim 19 and:

Claim 21 *If v satisfies CP, then every issue A such that $|A| \neq 2$ has a unique marginal alternative. What is more, this alternative coincides with the unique anti-prioritarian alternative in A when $|A| \geq 3$.*

Since the proofs of these claims are quite similar, I point out only where the differences arise:

Proof. The proof of Claim 20 (21) is by induction on $|X|$. For the base cases $|X| = 2, 3$, the claim follows from ILA and CP (CA). For the induction step, note that all $A \subset X$ satisfy the claim by the induction hypothesis. To see that X also satisfies the claim, consider the splitting (B, C) of X. There are two possibilities for the prioritarian (anti-prioritarian) alternative p in X: (i) $p \in B \cap C$; and, (ii) $p \in B \setminus C$.

For **Claim 20**: Consider $b \in B \setminus C$ and $c \in C \setminus B$. Using Claim 8, (i) leads to the contradiction that p is not prioritarian in $\{b, c, p\}$ (let alone X). So, (ii) must hold. Using the same kind of reasoning, it can be shown that $B = \{p\}$. (The idea is to consider an issue $\{b', c, p\}$ s.t. $b' \in B$, $c \in C \setminus B$. While there are several cases, a contradiction obtains for each.) By the induction hypothesis, $X \setminus p$ has two marginal alternatives. Since the splitting of X is $(\{p\}, X \setminus p)$, IS implies that these alternatives are marginal in X as well.

For **Claim 21**: Consider $b \in B$ and $c \in C \setminus B$. Using Claim 8, (ii) leads to the contradiction that p is not anti-prioritarian in $\{b, c, p\}$ (let alone X). So, (i) must hold. By the induction hypothesis, p is marginal in B and C (since it is anti-prioritarian for these issues). By IS, it then follows that p is marginal in X. Finally, by Claim 11 and the induction hypothesis, there can be no other marginal alternative in X. ■

IX. Appendix – Independence of the Axioms

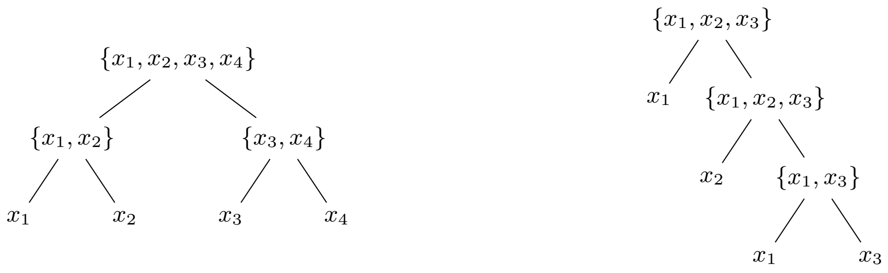

Figure 10: Agenda $\mathcal{T}_1$ (*left*) and Agenda $\mathcal{T}_2$ (*right*)

It is easy to see that the decision rule v_1 induced by $\mathcal{T}_1$ satisfies IS and ILA. To see that it violates WM, note that $v_1(P_1, \{x_1, x_2, x_3, x_4\}) = x_1$ for the profile P_1 where: x_1 is majority preferred to x_2 and x_3 but not x_4; and, x_3 is majority preferred to x_4. So, x_1 is selected without being the Condorcet winner. Since the other alternatives are symmetrically placed, there are also profiles where they are selected without being the Condorcet winner.

It is easy to see that the decision rule v_2 induced by $\mathcal{T}_2$ satisfies ILA. To see that it satisfies WM, note that $v(P, \{x_1, x_2, x_3\}) = x_3$ only if x_3 is the Condorcet winner on $\{x_1, x_2, x_3\}$. To see that it violates IS, note that $v_2(P_{123}, \{x_1, x_2, x_3\}) = x_1$ and $v_2(P_{132}, \{x_1, x_2, x_3\}) = x_2$. As such, the more preferred between x_1 and x_2 is selected for both Condorcet triples. By Table 1, this cannot be achieved with any simple agenda.

Finally, consider the decision rule v_3 on $\{x_1, x_2, x_3\}$ that selects: the majority preferred alternative between x_1 and x_2 when both are available; x_i on $\{x_i, x_3\}$; and, x_i on $\{x_i\}$. Since $(\{x_1, x_3\}, \{x_2, x_3\})$ splits $\{x_1, x_2, x_3\}$, v_3 satisfies IS. Since x_3 is trivially marginal, v_3 also satisfies WM. To see that it violates ILA, consider a profile P_3 where x_3 is the Condorcet winner on $\{x_1, x_2, x_3\}$. If v_3 satisfies ILA, then $v_3(P_3, \{x_1, x_2, x_3\}) = ... = x_3$. But, this contradicts the assumption that $v_3(P_3, \{x_1, x_2, x_3\}) = v_3(P_3, \{x_1, x_2\}) \neq x_3$.