On the Axiomatic Characterization of Runoff Voting Rules

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Abstract

Runoff voting rules such as single transferable vote (STV) and Baldwin's rule are of particular interest in computational social choice due to their recursive nature and hardness of manipulation, as well as in (human) practice because they are relatively easy to understand. However, they are not known for their compliance with desirable axiomatic properties, which we attempt to rectify here. We characterize runoff rules that are based on scoring rules using two axioms: a weakening of local independence of irrelevant alternatives and a variant of population-consistency. We then show, as our main technical result, that STV is the only runoff scoring rule satisfying an independence-of-clones property. Furthermore, we provide axiomatizations of Baldwin's rule and Coombs' rule.

1 Introduction

In the general theory of voting, voters each rank a set of alternatives, and based on these input rankings a voting rule determines an aggregate ranking of the alternatives (or merely a winner). (Due to the possibility of ties, often multiple aggregate rankings or winners are allowed to be returned.) The framework is extremely general and so finds applications in many different settings. The voters can be, e.g., people or software agents; the alternatives can be, e.g., political representatives or joint plans. Given this generality, it is perhaps no surprise that no one voting rule has emerged as the one-size-fits-all best option. When comparing the relative merits of different voting rules in a specific context, various criteria can play a role. In social choice theory, many *axioms* have been defined, which are desirable abstract properties that a voting rule should satisfy. Examples of axioms are *anonymity*, stating (informally) that all voters should be treated equally, and *monotonicity*, stating (informally) that increased support should not harm an alternative. Famous impossibility theorems, like the ones by Arrow [1] and Gibbard and Satterthwaite [13, 19], have demonstrated that every voting rule suffers from some flaws, as certain combinations of desirable properties are incompatible. On the other hand, there are nice axiomatic characterizations that state that, for certain combinations of axioms, one and only one voting rule satisfies all of them simultaneously (e.g., see [25]). Besides the compliance with choice-theoretic axioms, other desiderata have been considered. In recent years, there has been an extensive focus on *computational* aspects of voting rules. From this perspective, a good voting rule is one for which determining the outcome is computationally easy, whereas undesirable behavior such as strategic manipulation is computationally intractable [4, 6, 11, 12].

Practical concerns give rise to another criterion that is more difficult to formalize: in order to be adopted for making important collective decisions, a voting rule should be transparent and easy to understand for all participants. This is certainly the case for *(positional) scoring rules* such as *plurality* and *Borda's rule*. Unfortunately, scoring rules are not a panacea. They are not Condorcet-consistent: an alternative that outperforms every other alternative in pairwise comparisons may yet fail to win. They can be easily manipulated, both in an intuitive sense and in some[1] (but not all[2]) precise computational senses. And,

[1] See Bartholdi, III et al. [2].

[2] See Conitzer et al. [7], Davies et al. [8], and Betzler et al. [3].

they do not satisfy *independence of clones*: the introduction of a new alternative that is an (almost) exact copy of another alternative, so that these two alternatives are ranked adjacent to each other in all votes, can change the winner.[3] Independence of clones may be less important in, for example, an election for a political representative, where a charismatic, high-visibility potential candidate who would be both willing and able to copy another candidate's platform may not be available. On the other hand, independence of clones seems particularly important when voting over plans in AI.[4] For a specific example, consider a planning problem where a robot needs to cross a shallow stream. The robot's two options are to put on rain boots and cross the stream (which could involve putting on the left boot first or the right boot first), or to walk some distance to cross a bridge. The voters here could be different algorithms that rank the plans with respect to certain criteria, for example the time each option will take or the likelihood that the robot will get wet. If the first option is considered as two separate plans, then one would want to use a voting rule that is independent of clones.

Relatedly, in highly anonymous (say, Internet) contexts, independence of clones seems important even when voting over representatives, as it is easy to create an additional online identity. Indeed, cloning has already received some attention in the computational social choice community. For instance, Elkind et al. [10] have analyzed the structure of clone sets and Elkind et al. [9] have studied the computational complexity of manipulating an election by cloning. We note that voting rules that satisfy independence of clones cannot be manipulated by cloning.

As it turns out, some of the drawbacks of scoring rules can be remedied by moving to *runoff* rules, in which weak alternatives are repeatedly eliminated, until only a single alternative, the winner, remains. Examples are *single transferable vote* (STV), in which the alternative with the weakest plurality score is repeatedly eliminated, and *Baldwin's rule*, in which Borda's rule is used rather than plurality. (There is also *Nanson's rule*, in which all alternatives with a below average Borda score are eliminated.) Unlike Borda's rule from which it derives, Baldwin's rule is Condorcet-consistent (as is Nanson's). Similarly, STV is known to possess desirable properties that plurality does not. Notably, it satisfies independence of clones, and this will be the focus of much of this paper.

Our contribution. The contribution of this paper is threefold.

- We characterize runoff scoring rules using two axioms. The first of these axioms is a weakening of *local independence of irrelevant alternatives* (LIIA), which was used by Young [24] to characterize Kemeny's rule. The second is a variant of *consistency*, which features prominently in an important axiomatic characterization of scoring rules [21, 23].[5]
- We show that STV is the only runoff scoring rule that is independent of clones. Thus, STV is characterized by the combination of three axioms. (We are aware of one other axiomatic characterization that uses an independence-of-clones criterion, namely by Laslier [14], who uses it to characterize an SCF known as the *essential set*.)
- We demonstrate the versatility of our approach by deriving a number of further axiomatic characterizations of specific runoff scoring rules, on the backs of earlier characterizations of specific scoring rules.

[3] Tideman [22] has shown that plurality and Borda's rule are not independent of clones. In Appendix A, we generalize these results and show that no non-trivial scoring rule is independent of clones.

[4] Schulze [20] makes a similar point about voting over plans.

[5] Indeed, Smith [21] was already looking ahead to runoff scoring rules, stating that "It would be interesting, but perhaps very difficult, to characterize [runoff scoring rules]".

2 Preliminaries

Let A be a finite set of *alternatives*. For a subset $B \subseteq A$, a (strict) *ranking* of B is a permutation of B. The set of all rankings of B is denoted by $\mathcal{L}(B)$. We will often focus on the alternative that is ranked last in a ranking. For a given ranking $r = (a_1, \ldots, a_m)$, the last-ranked alternative a_m is called the *bottom element* of r, denoted $bottom(r)$. For a set of rankings $\{r_1, \ldots, r_\ell\}$, $bottom(\{r_1, \ldots, r_\ell\}) = \bigcup_{i=1}^{\ell} bottom(r_i)$ is the set of all alternatives that are the bottom element of some ranking in the set.

Let N be a finite set of voters. The preferences of voter $i \in N$ are represented by a ranking $R_i \in \mathcal{L}(A)$. A *preference profile* is a list $R \in \mathcal{L}(A)^N$ containing a ranking R_i for each voter $i \in N$. For $B \subseteq A$, let $R|_B$ denote the preference profile in which all elements in $A \setminus B$ have been removed from all rankings in R. Furthermore, the *rank distribution* of an alternative a under R is defined as the vector $d(a, R) \in \mathbb{N}^{|A|}$ whose j-th entry is given by the number of voters that rank a in position j.

We distinguish between two types of aggregation functions. A *social choice function* (SCF) f associates with every preference profile R a non-empty set $f(R) \subseteq A$ of alternatives. A *social preference function* (SPF) f associates with every preference profile R a non-empty set $f(R) \subseteq \mathcal{L}(A)$ of rankings of A. Note that ties are allowed due to the fact that the output set can have size greater than 1.

An SCF or SPF is *neutral* if permuting the alternatives in the individual rankings also permutes the set of chosen alternatives, or the set of chosen rankings, in the exact same way. An SCF or SPF is *anonymous* if the set of chosen alternatives, or the set of chosen rankings, does not change when the voters are permuted. An SCF or SPF is called *symmetric* if it is both neutral and anonymous. Throughout this paper, we only consider symmetric SCFs and SPFs.

For social preference functions, we will also consider the following basic properties that we consider quite mild.

Definition 1. *An SPF f satisfies*

- weak unanimity *if $R_i = r$ for all $i \in N$ implies $r \in f(R)$;*
- weak decisiveness *if there exists a preference profile R on at least two alternatives with $|f(R)| = 1$; and*
- continuity at the bottom *if for any two preference profiles R and R' with $bottom(f(R)) = \{a\}$, there exists a natural number k such that $bottom(f(kR \cup R')) = \{a\}$.*[6]

A *scoring rule* is an SCF that is defined by a sequence $s = (s^n)_{n \geq 1}$, where for each $n \in \mathbb{N}$, $s^n = (s^n_1, \ldots, s^n_n) \in \mathbb{Q}^n$[7] is a *score vector*[8] of length n. For a preference profile R on k alternatives, the score vector s^k is used to allocate points to alternatives: the score $s^k(a)$ of a under R is given by $s^k(a) = s^k \cdot d(a, R)^\intercal$, where $d(a, R)^\intercal$ is the transpose of the rank distribution of a under R. (Note that if two alternatives have the same rank distribution, they have the same score for every score vector.) The outcome of the scoring rule is the alternative (or set of alternatives in the case of a tie) with maximal score. Examples of scoring rules are *plurality* ($s^n = (1, 0, \ldots, 0)$), *veto* ($s^n = (0, \ldots, 0, -1)$), and *Borda's rule* ($s^n = (n-1, n-2, \ldots, 0)$). We consider two scoring rules s and t identical if for each $n \in \mathbb{N}$,

[6] The preference profile $kR \cup R'$ consists of k copies of R and one copy of R'.

[7] The requirement that scores are rational is required for the proof of Lemma 5. All other results hold even when we allow scores to be real numbers.

[8] Note that we do *not* impose the condition that $s^n_1 \geq \ldots \geq s^n_n$. This will result in us having to deal with some unintuitive rules in Section 4, but we obtain a more complete result this way, which will also be useful later on.

the score vector t^n is an affine transformation of s^n. For instance, the scoring rule given by $s^n = (-1, -2, \ldots, -n)$ is identical to Borda's rule.

Every scoring rule s gives rise to a *(one-at-a-time) runoff scoring rule* as follows. As long as the number of alternatives is greater than or equal to two, iteratively eliminate one alternative with the *lowest* score (according to s) from all rankings. The runoff scoring rule corresponding to s is the SPF that outputs the ranking in which alternatives appear in reverse elimination order (the alternative that is eliminated first is ranked last, and so on).

An important issue is how ties are handled. In this paper, we use *parallel-universes tie-breaking (PUT)*, i.e., the outcome of a runoff scoring rule is the union of all rankings that result from the iterative procedure described above for *some* way of breaking the ties. The following three SPFs will be of particular interest in this paper: *Single transferable vote (STV)* is the runoff scoring rule that is based on plurality; *Coombs' rule* is the runoff scoring rule that is based on veto; and *Baldwin's rule* is the runoff scoring rule that is based on Borda's rule.[9]

Example 1. *Consider the following preference profile with four voters and three alternatives.*

$$2 \times (a, b, c)$$
$$1 \times (b, c, a)$$
$$1 \times (c, a, b)$$

Breaking ties using PUT, it can be easily checked that STV selects the three following rankings: (a, b, c), (a, c, b), *and* (c, a, b). *Coombs' rule and Baldwin's rule do not encounter any ties for this profile and uniquely select* (a, b, c).

3 Main Axioms

In this section, we introduce three axioms that we will use to characterize STV. The first axiom states that collective rankings can be constructed recursively by excluding last-ranked alternatives. For a ranking $r \in \mathcal{L}(B)$ and an alternative $a \notin B$, let (r, a) denote the ranking of $B \cup \{a\}$ that agrees with r on B and has $bottom((r, a)) = a$.

Definition 2. *An SPF f satisfies* independence of bottom alternatives *if for all preference profiles R,*

$$f(R) = \bigcup_{a \in bottom(f(R))} \{(r, a) : r \in f(R|_{A \setminus \{a\}})\}.$$

Independence of bottom alternatives is implied by local independence of irrelevant alternatives as introduced by Young [24].

The second axiom is a variant of consistency [21, 23] that focuses on last-ranked (instead of first-ranked) alternatives: If two groups of voters, N and N', collectively rank sets of alternatives C and D last, respectively, then the set of alternatives ranked last by $N \cup N'$ is precisely $C \cap D$, if this is non-empty.

Definition 3. *An SPF f satisfies* consistency at the bottom *if for all $R \in \mathcal{L}(A)^N$ and $R' \in \mathcal{L}(A)^{N'}$ with $N \cap N' = \emptyset$,*

$$bottom(f(R \cup R')) = bottom(f(R)) \cap bottom(f(R'))$$

whenever the set on the RHS is non-empty.

[9]Nanson's rule is not a runoff scoring rule in our sense, as it generally eliminates more than one alternative at a time. See Niou [17] for the difference between Nanson's rule and Baldwin's rule.

Finally, we consider *independence of clones*. This property was introduced by Tideman [22] and Zavist and Tideman [26] for SCFs, and we adapt it to SPFs. We say an SPF is independent of clones if, after ignoring all clones that are not ranked highest, cloning operations do not affect the set of collective rankings. More precisely, a *cloning operation* $\mathcal{C}$ transforms a preference profile R into another preference profile $R^{\mathcal{C}}$ in which one of the alternatives, say a, has been replaced by a set of *clones*. In every ranking R_i, the clones keep the relative position of a with respect to all other alternatives; the ranking among the clones themselves is arbitrary. For a ranking r of the set of all alternatives, including any clones that have been introduced by cloning operation $\mathcal{C}$, we let $\lceil r \rceil_{\mathcal{C}}$ denote the ranking of the set of all original alternatives that is obtained from r by deleting all clones and inserting a at the position where the highest-ranked clone of a was ranked in r. Then:

Definition 4. *An SPF f satisfies* independence of clones *if for all preferences profiles R and all cloning operations $\mathcal{C}$,*

$$\lceil f(R^{\mathcal{C}}) \rceil_{\mathcal{C}} = f(R).$$

Example 2. *In the profile given in Example 1, replace alternative a by two clones a and a' such that the profile becomes*

$$1 \times (a, a', b, c)$$
$$1 \times (a', a, b, c)$$
$$1 \times (b, c, a, a')$$
$$1 \times (c, a, a', b)$$

It is easily verified that Baldwin's rule selects (c, a, a', b), among other rankings. Since $\lceil (c, a, a', b) \rceil_{\mathcal{C}} = (c, a, b)$ was not selected by Baldwin's rule in the original profile, this example shows that Baldwin's rule is not independent of clones.

4 Axiomatic Characterization of STV

Our main result is a characterization of STV in terms of the axioms that were introduced in the previous sections. We will show that STV is the only symmetric SPF that satisfies the three axioms introduced in Section 3 and the basic properties in Definition 1.

We start by checking that STV indeed satisfies the conditions. It is easy to check that STV is symmetric and satisfies the basic properties in Definition 1. Being a runoff scoring rule, STV also satisfies independence of bottom alternatives and consistency at the bottom (see Lemma 1). It is left to show that STV is independent of clones. Tideman [22] has proved that STV is independent of clones when interpreted as an SCF. We recap the argument and extend it to also show clone independence in the SPF setting.

Recall that STV is based on the plurality scoring rule. Since only first-ranked alternatives receive points under plurality, in the first round of STV, the only effect of a cloning operation is that the points of the cloned alternative are distributed among its clones, while the number of points remains the same for all other alternatives. This leaves only two possibilities for the first alternative to be eliminated: either it is the alternative that would have been eliminated in the original profile, or some clone is eliminated. In the former case, the elimination order is not affected. In the latter case, the points of the eliminated clone transfer exclusively to other clones. Applying this reasoning iteratively, it follows that (1) as long as there is more than one clone left, the elimination order does not change (when ignoring the elimination of clones), and (2) the last remaining clone accumulates all the points of the other clones, ending up with exactly the number of points the cloned alternative received in the original profile. Therefore, from the point on where only one clone is left, the elimination procedure proceeds exactly the same as in the original profile. This shows that the SPF version of STV satisfies independence of clones.

The remainder of this section is devoted to showing that STV is indeed the *only* symmetric SPF satisfying the aforementioned set of axioms. We start by characterizing runoff scoring rules.

Lemma 1. *Let f be a symmetric SPF satisfying continuity at the bottom. Then f satisfies independence of bottom alternatives and consistency at the bottom if and only if it is a runoff scoring rule.*

Proof. We first show that every runoff scoring rule satisfies independence of bottom alternatives and consistency at the bottom. Let s be the sequence of score vectors on which f is based. Independence of bottom alternatives follows immediately from the definition of a runoff scoring rule (and from our assumption that PUT is used in case of ties). As for consistency at the bottom, observe that the set of last-ranked alternatives $bottom(f(R))$ coincides with the set of *winners* for the scoring rule $\bar{s}$, which is given by $\bar{s}^k_j = -s^k_j$ for all $1 \leq j \leq k$. All scoring rules are consistent [21, 23], and consistency of $\bar{s}$ implies consistency at the bottom of f.

For the other direction, let f be a symmetric SPF satisfying continuity at the bottom, independence of bottom alternatives, and consistency at the bottom. Independence of bottom alternatives implies that f is uniquely determined by the SCF, call it g, that selects $bottom(f(R))$. Symmetry of f implies that g is symmetric as well. Furthermore, continuity at the bottom of f implies continuity of g and consistency at the bottom of f implies consistency of g. We can therefore apply the result by Smith [21] and Young [23], which states that a symmetric SCF is continuous and consistent if and only if it is a scoring rule. It follows that the bottom elements of $f(R)$ are selected by a scoring rule. Therefore, f must be the corresponding runoff scoring rule. □

Since Smith [21] has shown that no runoff scoring rule is monotonic, Lemma 1 yields the following impossibility result.

Corollary 1. *There does not exist a symmetric SPF that satisfies independence of bottom alternatives, consistency at the bottom, continuity at the bottom, and monotonicity.*

From now on, we will identify a runoff scoring rule f with the sequence $s = (s^k)_{k \in \mathbb{N}}$ that defines the scoring rule on which f is based. Call a score vector s^k *trivial* if $k \geq 2$ and $s^k_i = s^k_j$ for all $1 \leq i, j \leq k$. The following lemma shows that a weakly decisive runoff scoring rule can only be independent of clones if all score vectors are non-trivial.

Lemma 2. *Let s be a weakly decisive runoff scoring rule. If s is independent of clones, then s^n is non-trivial for all $n \geq 2$.*

Proof. Assume that there exists $n \geq 2$ such that s^n is trivial. We will show that s is not independent of clones. Let $k \geq 2$ be the smallest integer such that s^k is trivial. Weak decisiveness implies that $k \geq 3$ (if s^2 were trivial, we would always have a tie in the last round and therefore multiple output rankings). Consider the profile consisting of one voter with preferences

$$1 \times (c_1, c_2, \ldots, c_{k-1})$$

Since s^{k-1} is non-trivial, at least one of the alternatives, say c_i, is *not* ranked last in any universe. Therefore, none of the rankings output by s has c_i at the bottom. Now, clone one of the alternatives that is ranked last in the first round in some universe (i.e., an alternative in a position with the lowest score). After cloning we are in the case of k alternatives, and since s^k is trivial, all alternatives score equally. In particular, there is some universe where c_i is eliminated first. Thus, s is not independent of clones. □

We go on to show that, under mild conditions, it is possible to construct a preference profile, including one pair of clones, such that two alternatives (disjoint from the clone set) obtain an arbitrarily lower score than all other alternatives. This construction will later be useful because it enables us to push any two alternatives to the bottom of the ranking, no matter how many points these alternatives received in the original profile.

Lemma 3. *Let s^n be a non-trivial score vector with $n \geq 5$ and $s^n \neq (1,0,1,0,1)$. For every $M \in \mathbb{N}$, there exists a preference profile R on a set A of n alternatives with the following properties:*

- *there are two alternatives a and a' that appear in consecutive positions in every ranking and that have the same rank distribution,*
- *there are two alternatives c and d with $\{c,d\} \cap \{a,a'\} = \emptyset$ that have the same rank distribution, and*
- *$s^n(c) + M = s^n(d) + M < \min\{s^n(b) : b \in A \setminus \{c,d\}\}$.*

Proof. Let s^n be a score vector with $n \geq 5$ and $s^n \neq (1,0,1,0,1)$. To avoid cluttered notation, we omit the superscript and write $s = (s_1, \ldots, s_n)$ instead of $s^n = (s_1^n, \ldots, s_n^n)$.

We begin by showing that there exist distinct $i, j, k \in \{1, \ldots, n\}$ such that

$$i, j \neq k+1 \text{ and } s_k + s_{k+1} > s_i + s_j. \tag{1}$$

For a contradiction, assume that it is not possible to find distinct i, j, k satisfying (1). Let $x = s_1$ and $y = s_2$. Then it must be the case that every pair of consecutive scores in $s_3, \ldots, s_n$ sums to $x + y$, specifically $s_{n-1} + s_n = x + y$, or else we could choose $k = n-1$. But this tells us that $s_2 + s_3 = x + y$, and so $s_3 = x$. Continuing in this way, we see that the score vector s must have the form $(x, y, x, \ldots)$. Non-triviality of s implies $x \neq y$. If $n \geq 6$, then we could choose $k = 1$ and i, j to be the two lowest scoring remaining positions. This choice would satisfy (1) because $x + y > 2\min\{x, y\}$. Thus, $n = 5$ and there are only two possibilities for s: either $s = (1,0,1,0,1)$ or $s = (0,1,0,1,0)$. The former case is excluded by assumption. In the latter case we can choose $k = 4$, $i = 1$, and $j = 3$ to satisfy (1).

We therefore know that it is always possible to find distinct i, j, k satisfying (1). We now distinguish two cases.

Case 1: $|\{s_1, \ldots, s_{k-1}, s_{k+2}, \ldots, s_n\}| > 1$.

In this case, we construct a profile R' with $(n-4)$ voters as follows. Each voter places a and a' in positions k and $k+1$, respectively, and places c and d in the lowest and second lowest scoring remaining positions, respectively. We know by assumption that $s(a) + s(a') > s(c) + s(d)$. Now place the remaining alternatives so that each is placed an equal numbers of times in all other positions. This is easily achieved by fixing an arbitrary permutation of $A \setminus a, a', c, d$ and "rotating" these $n-4$ alternatives through the $n-4$ remaining positions in such a way that all these alternatives have the same rank distribution under R'. Now for each voter in R', create three extra voters. This bigger profile, call it R'', thus has $4(n-4)$ voters. Half of these voters rank a ahead of a' and half rank c ahead of d in such a way that $d(a, R'') = d(a', R'')$ and $d(c, R'') = d(d, R'')$. By the placement of c and d, these two alternatives have a strictly lower score than every other alternative. Defining $R = pR''$ for a sufficiently large natural number p yields a profile in which c and d lose by an arbitrarily large points margin.

Case 2: $s_1 = \ldots = s_{k-1} = s_{k+2} = \ldots = s_n$.

We will show that the construction described in Case 1 still works, with a few minor modifications. Without loss of generality, we can assume that $s_1 = \ldots = s_{k-1} = s_{k+2} = \ldots = s_n = 0$. First suppose that s_k and s_{k+1} are both non-zero. Then $s_k + s_{k+1} > s_i + s_j = 0$

implies that s_k and s_{k+1} cannot both be negative. If one number is positive and the other is negative, we can choose i to be the position with $s_i < 0$, change $k, k+1$ to be any adjacent positions, and j to be the lowest scoring remaining position, which will be a zero position: $n \geq 5$ guarantees that at least one zero position still remains. Next suppose that s_k and s_{k+1} are both positive. Then we simply need to replace k by either $k+1$ or $k-1$, and choose i and j to be any two of the remaining $(n-3)$ unoccupied positions which score zero.

This leaves only the case where all entries in s^n are zero, except for one. Suppose first that the non-zero entry, say s_i, is negative. Then we can have half the voters place c and the other half place d in position i, and thus we get that all alternatives score zero except for c and d, whose score n is negative. Lastly, if s_i is positive (w.l.o.g. $s_i = 1$), then choose three consecutive entries which are either $(0,0,1)$ or $(1,0,0)$ (the assumption $n \geq 5$ guarantees that such positions can be found). Construct a profile P' with $(n-2)$ voters. All place c and d outside of the chosen consecutive positions. One places a in the scoring position and a' adjacent to a, and another places a' in the scoring position an a adjacent to it. The remaining $(n-4)$ voters each place a different one of the remaining $(n-4)$ alternatives in the scoring position; each of these voters can place a and a' in the consecutive zero positions. As before, we now extend this profile to a new profile P'' having $4(n-2)$ voters, such that $d(a, P'') = d(a', P'')$ and $d(c, P'') = d(d, P'')$. In the profile P'' all alternatives score a positive number of points except c and d, so we can multiply it by a sufficiently large number to yield a profile P where c and d score arbitrarily less points than any other alternative. □

We will use the construction in Lemma 3 to show that the score vectors of a clone-independent runoff scoring rule must have a particularly simple form.

Definition 5. *A score vector s^n of length n is a* plurality/veto combination *if $s_i^n = s_j^n$ for all $2 \leq i, j \leq n-1$.*

By definition, all score vectors of length three or smaller are plurality/veto combinations. When dealing with plurality/veto combinations, we will always assume that the score vector is normalized so that $s_2^n = \ldots = s_{n-1}^n = 0$.

The following is our key lemma.

Lemma 4. *Let s be a weakly decisive runoff scoring rule. If s is independent of clones, then s^n is a plurality/veto combination for all $n \in \mathbb{N}$.*

Proof. Let s be a weakly decisive runoff scoring rule that is independent of clones. Trivially, s^n is a plurality/veto combination for all $n \leq 3$. We first consider the case $n \geq 5$, and deal with the four-alternative case later.

Let $n \geq 5$ and suppose that s^n is not a plurality/veto combination. Then there exists $2 \leq k \leq n-2$ such that $s_k^n \neq s_{k+1}^n$. We will show that s is not independent of clones.

Assume for now that $s^n \neq (1,0,1,0,1)$. We construct a two-voter profile such that two alternatives, say c and d, have the same rank distribution in the case of $n-1$ alternatives but obtain different scores in the n alternative profile that results from cloning one alternative, say a, into clones a and a'.

In the case $k < n-2$, consider the $n-1$ alternative profile

$$1 \times (a, \ldots, c, \ldots, b, d)$$
$$1 \times (b, \ldots, d, \ldots, a, c)$$

where c and d are each once ranked k-th and once n-th. After cloning a, the profile becomes

$$1 \times (a, a', \ldots, c, \ldots, b, d)$$
$$1 \times (b, \ldots, d, \ldots, a, a', c)$$

Here, c obtains one $(k+1)$-th ranking and one n-th ranking, and d obtains one k-th ranking and one n-th ranking.

In the case $k = n-2$, consider the $n-1$ alternative profile

$$1 \times (c, b, \ldots, d, a)$$
$$1 \times (d, a, \ldots, c, b)$$

and clone a to obtain

$$1 \times (c, b, \ldots, d, a, a')$$
$$1 \times (d, a, a', \ldots, c, b)$$

so that d receives a first and a k-th ranking and c receives a first and a $k+1$-th ranking. In both cases, $s_k \neq s_{k+1}$ implies that c and d have different scores after cloning.

We now apply Lemma 3 to append a profile of votes such that c and d have the same rank distribution but score the lowest of all alternatives in the cloned profile. Hence we ensure that the same alternative (either c or d) is ranked uniquely last in every universe after cloning, whereas before cloning both were ranked last in at least one universe or neither was ranked last in any universe. This contradicts independence of clones.

Next we deal with the case $s^5 = (1, 0, 1, 0, 1)$ (which does not allow us to apply Lemma 3). By the above reasoning we may assume that s^n is a plurality/veto combination for all $n > 5$. Write $s^6 = (w, 0, 0, 0, 0, z)$, $w, z \in \mathbb{Q}$. Consider the ten voter, five alternative profile

$$2 \times (a, e, c, d, b)$$
$$2 \times (a, d, b, c, e)$$
$$2 \times (a, b, d, c, e)$$
$$2 \times (c, a, d, b, e)$$
$$2 \times (c, a, d, e, b)$$

Note that $s^5(\cdot) = 6$ for all five alternatives. Therefore, every alternative is eliminated first in some parallel universe. We will now show that s is not independent of clones. The cloning operation will depend on the values of w and z.

If $w > 0$, we clone a in such way that a and a' have the same rank distribution. In the cloned profile, $s^6(a) = s^6(a') > 0 = s^6(d)$. If $w < 0$ then we similarly clone a, and now $s^6(a) = s^6(a') = 3w > 4w = s^6(c)$. In both cases, a is no longer the first alternative eliminated in any universe, contradicting independence of clones. Lastly suppose $w = 0$. Then Lemma 2 implies $z \neq 0$, as otherwise s^6 would be trivial. We can now clone e and apply reasoning that is completely analogous to before, in both the $z < 0$ and $z > 0$ cases.

Finally, we consider the case $n = 4$. By the above, we may assume s^n is a plurality/veto combination for all $n > 4$. Assume for the sake of contradiction that s^4 is not a plurality/veto combination, so $s_2^4 \neq s_3^4$. Consider one voter with preferences (a, b, c, d) over four alternatives, so that $s^4(b) \neq s^4(c)$. Now we clone a so that the voter's preferences become (a, a', b, c, d). Since s^5 is a plurality/veto combination, we know $s^5(b) = s^5(c)$. We can apply Lemma 3 to append a profile of votes such that b and c score lower than all other alternatives, and hence both b and c are first eliminated in at least one universe. Before cloning, however, one of b and c was *not* eliminated first in any universe. This contradicts independence of clones. □

The previous lemma shows that independence of clones requires score vectors of the form $s^n = (x_n, 0, \ldots, 0, y_n)$. A closer analysis allows us to narrow down the possibilities even further: the following lemma implies that both x_n and y_n must be non-negative, and in the case where there exist $n', n'' \geq 3$ such that both $x_{n'}$ and $y_{n''}$ are positive, the fraction $\frac{x_n}{y_n}$ has to be constant for all $n \geq 3$ (since they can never both be zero by weak decisiveness).

Lemma 5. *Let s be a weakly decisive runoff scoring rule that satisfies independence of clones. Then, for all $n \geq 3$, $s^n = (x_n, 0, \ldots, 0, y_n)$ with $x_n \geq 0$, $y_n \geq 0$, and $x_n y_{n+1} = x_{n+1} y_n$.*

Proof. Let s be a weakly decisive runoff scoring rule that satisfies independence of clones. Lemma 4 implies that $s^n = (x_n, 0, \ldots, 0, y_n)$ for all $n \geq 3$. It is left to show that $x_n \geq 0$, $y_n \geq 0$, and $x_n y_{n+1} = x_{n+1} y_n$ for all $n \geq 3$. We prove these statements in two steps.

- Step I: $x_n \geq 0$ and $y_n \geq 0$ for all $n \geq 3$.
- Step II: $x_n y_{n+1} = x_{n+1} y_n$ for all $n \geq 3$.

Proof of Step I. Suppose there exists $n \geq 3$ such that $x_n, y_n < 0$. Consider a cyclic profile on alternatives $c_1, c_2, \ldots, c_{n-1}$:

$$2 \times (c_1, c_2, \ldots, c_{n-1})$$
$$2 \times (c_{n-1}, c_1, c_2, \ldots)$$
$$\vdots$$
$$2 \times (c_2, \ldots, c_{n-1}, c_1)$$

In this profile, all alternatives have the same rank distribution and, thus, the same score. It follows that every alternative is eliminated first in some universe. Now clone c_1 in such a way that from every pair of voters with the same preferences (in the $(n-1)$ alternative profile), exactly one voter ranks c_1 ahead of c_1', we see that each of the clones of c_1 is placed in the negative scoring positions only half as often as the other alternatives. Thus neither of the clones is eliminated first in any universe.

Now suppose $x_n \geq 0$ and $y_n < 0$ for some $n \geq 4$. Consider the following profile on $n-1$ alternatives:

$$2 \times (\ldots, a, b)$$
$$2 \times (\ldots, b, a)$$

Clearly, $s^n(a) = s^n(b)$. Now clone a, such that a and a' score the same number of points. Alternative b now has twice as many last place votes as a or a', and no other alternative has any, so b is now the loser in every universe. An analogous argument applies to the case where $x_n < 0$ and $y_n \geq 0$ for some $n \geq 4$.

We finally deal with the cases where $x_3 < 0$ or $y_3 < 0$. Suppose that $y_3 < 0$ but $x_3 \geq 0$. Then we can take two voters with preferences (a, b, c), and c is the unique alternative that is eliminated first. But if we clone a in such a way that a and a' each get one first place vote, then we are now in the situation where b is the first alternative to be eliminated in some universe, because $x_4 \geq 0$ and $y_4 \geq 0$. Again, an analogous argument discards the case where $x_3 < 0$ and $y_3 \geq 0$. This concludes the proof of Step I.

Proof of Step II. Without loss of generality, we can assume that $x_n, y_n \in \mathbb{N}$ for all $n \geq 3$. Assume for contradiction that there exists $k \geq 3$ such that $x_k y_{k+1} \neq x_{k+1} y_k$. We will show that s is not independent of clones by constructing a profile such that a cloning operation changes the order of elimination of alternatives.

We first consider the case $k \geq 4$. Construct a profile as follows. Take a very large number of voters. Have exactly y_k voters rank b in first position and exactly x_k voters rank c in last position - these groups may or may not overlap. Among the rest of the voters, distribute first and last place votes equally between the remaining alternatives. Thus $s^k(b) = s^k(c) = x_k y_k$

and $s^k(x) = p(x_k + y_k)$ for all $x \in A \setminus \{b, c\}$, where p depends on the total number of voters and is chosen sufficiently large such that $s^k(x) \gg s^k(b)$. This is always possible since at least one of x_k and y_k is positive. Now clone one of the other alternatives, say a, and have half of the voters rank a ahead of a'. By assumption, $s^{k+1}(b) = y_k x_{k+1} \neq x_k y_{k+1} = s^{k+1}(c)$. Of the remaining alternatives, a scores the lowest with $s^{k+1}(a) = s^{k+1}(a') = \frac{1}{2}p(x_{k+1} + y_{k+1})$. As long as p is large enough that $s^{k+1}(a) > s^{k+1}(b)$ and $s^{k+1}(a) > s^{k+1}(c)$ (and we can choose p large enough so that this is true), then b and c still score the lowest of all alternatives in the new cloned profile. Thus one of b or c is now the loser in every parallel universe.

We now consider the case $k = 3$. The profile

$$
\begin{aligned}
2y_3 &\times (a, b, c) \\
2x_3 &\times (c, a, b)
\end{aligned}
$$

yields the following scores: $s^3(a) = 2x_3y_3$, $s^3(b) = 2x_3y_3$, and $s^3(c) = 2(x_3^2 + y_3^2) > 2x_3y_3$. Thus, a and b each lose in some parallel universe.

Now we clone c and obtain the profile

$$
\begin{aligned}
y_3 &\times (a, b, c, c') \\
y_3 &\times (a, b, c', c) \\
x_3 &\times (c, c', a, b) \\
x_3 &\times (c', c, a, b)
\end{aligned}
$$

with $s^4(a) = 2y_3x_4$, $s^4(b) = 2x_3y_4$, and $s^4(c) = s^4(c') = y_3y_4 + x_3x_4$. Our assumption that $x_3y_4 \neq y_3x_4$ implies $s^4(a) \neq s^4(b)$. So if we can show that $s^4(c) > s^4(a)$ or $s^4(c) > s^4(b)$, then the clone manipulation will be complete.

We need to distinguish several cases. First suppose that all of x_3, x_4, y_3, y_4 are non-zero and that furthermore $y_3y_4 + x_3x_4 \leq 2x_3y_4$ and $y_3y_4 + x_3x_4 \leq 2y_3x_4$. Rearranging, we get

$$
\frac{y_3}{x_3} + \frac{x_4}{y_4} \leq 2 \quad \text{and} \quad \frac{y_4}{x_4} + \frac{x_3}{y_3} \leq 2.
$$

Letting $\frac{y_3}{x_3} = u$ and $\frac{x_4}{y_4} = v$ yields

$$
u + v \leq 2 \quad \text{and} \quad \frac{1}{u} + \frac{1}{v} \leq 2,
$$

where by assumption, $u \neq \frac{1}{v}$. In particular, u and v are not both 1. Since $\frac{1}{u} + \frac{1}{v}$ is decreasing in u and v, we can assume that $u+v = 2$ for the purpose of satisfying this pair of inequalities. The second equation becomes

$$
\frac{1}{u} + \frac{1}{2-u} \leq 2,
$$

which yields

$$
u^2 - 2u + 1 = (u-1)^2 \leq 0.
$$

It is easily shown that the only solution is when $x_3 = y_3$ and $x_4 = y_4$, which violates the assumption that $x_3y_4 \neq x_4y_3$.

Thus, neither of the c clones score the lowest and either a is first eliminated in every universe or b is, so we have exhibited a clone manipulation.

Next, suppose that at least one of x_3, x_4, y_3, y_4 is zero, but $s^4(c) > 0$. Then $s^4(c) > \min\{s^4(a), s^4(b)\} = 0$, so we have exhibited a clone manipulation.

Lastly, consider the case where $s^4(c) = 0$. Since we have assumed that $x_3y_4 \neq x_4y_3$, the only way this is possible is for $x_3 = y_4 = 0$ or $x_4 = y_3 = 0$. These two cases are symmetric, so we consider $x_3 = y_4 = 0$. Then we have $s^3 = (0, 0, 1)$ and $s^4 = (1, 0, 0, 0)$. We can exhibit a clone manipulation involving two voters with preferences (a, b, c). By cloning a and having exactly one voter rank a ahead of a', neither of the a clones is eliminated first in any universe, whereas before cloning a was eliminated first in some universe. □

We finally arrive at a full characterization of the class of scoring rules that give rise to clone-independent runoff scoring rules.

Lemma 6. *A weakly decisive runoff scoring rule s is independent of clones if and only if one of the following four cases holds:*

- $s^2 = (1, 0)$ *and* $s^n = (x, 0, \ldots, 0, 1)$ *for all* $n \geq 3$ *and some* $x > \frac{1}{2}$*,*
- $s^2 = (0, 1)$ *and* $s^n = (1, 0, \ldots, 0, x)$ *for all* $n \geq 3$ *and some* $x > \frac{1}{2}$*,*
- $s^n = (1, 0, \ldots, 0)$ *for all* $n \geq 2$*, or*
- $s^n = (0, \ldots, 0, 1)$ *for all* $n \geq 2$*.*

Proof. The rule where $s^n = (1, 0, \ldots, 0)$ for all $n \geq 2$ is STV, and we have argued at the beginning of Section 4 that STV is independent of clones. The (peculiar) rule where $s^n = (0, \ldots, 0, 1)$ for all $n \geq 2$ is independent of clones for the exact same reason.

If $s^n = (x, 0, \ldots, 0, y)$ for all $n \geq 3$, with constant $x, y \geq 0$, then it is impossible to exhibit a clone manipulation for any profile *on three or more alternatives.* The argument is identical to the argument that STV is independent of clones, with the key points being that (1) cloning an alternative can not affect the score of any other alternative, and (2) when a clone is eliminated, its score transfers exclusively to the remaining clones.

Now suppose that $s^2 = (1, 0)$. (The case $s^2 = (0, 1)$ follows by symmetry, and it is straightforward to check that any remaining rule not stated in the theorem is ruled out by Lemma 5—recalling, of course, that we consider an affine transformation of a score vector to be the same score vector.) If s is independent of clones then we know from Lemma 5 that either $s^3 = (1, 0, 0)$ or $s^3 = (x, 0, 1)$ for some $x \geq 0$. The former case would dictate that $s^n = (1, 0, \ldots, 0)$ for all $n \geq 2$, i.e., s is STV. Analogously, the latter case yields $s^n = (x, 0, \ldots, 0, 1)$ for all $n \geq 3$. It remains to be shown that, in the latter case, s is independent of clones if and only if $x > \frac{1}{2}$.

Suppose first that $x \leq \frac{1}{2}$ and consider the profile

$$2 \times (a, b)$$

so that s selects only the ranking (a, b). Now clone b into b and b' to obtain the profile

$$1 \times (a, b, b')$$
$$1 \times (a, b', b)$$

In the cloned profile, $s^3(b) = s^3(b') = 1 \geq 2x = s^3(a)$. So there is some universe in which a is the first alternative eliminated, and we have exhibited a clone manipulation.

Next let $x > \frac{1}{2}$ and consider a two-alternative profile

$$k_a \times (a, b)$$
$$k_b \times (b, a)$$

with at least one voter, so $k_a + k_b \geq 1$. As a first step, we note that we can restrict attention to cloning operations in which only one of the alternatives is cloned. To see this, suppose that we clone both alternatives. Then at some step, there will be some alternative with only one clone not yet eliminated (while more than one clone of the other alternative remains). For the purposes of a clone manipulation, we are only interested in the step at which this final clone is eliminated, and thus the order of elimination prior to this step is irrelevant. So we could have achieved an identical outcome by cloning only one alternative at the outset.

We now distinguish two cases.

Case 1: $k_a = k_b$. In this case, s selects both (a, b) and (b, a). Suppose that a successful cloning operation exists. Without loss of generality, assume that b is cloned. A successful cloning operation is one for which in every parallel universe, there is some clone of b which is eliminated *after* a. (It is not possible to have a as the top alternative in all universes, since by the time that only one clone of b remains, the profile has returned to the original profile in which a and b are tied.)

Suppose we clone b into j clones $b^{(1)}, \dots, b^{(j)}$, for some $j \geq 2$ (note that setting $j = 1$ does not change the profile, or the set of output rankings). If it is the case that one of the clones is eliminated first in some universe, then the cloned profile reduces to a case with only $(j-1)$ clones and we need only have considered this $(j-1)$ case to start with. Therefore, we may assume that a is uniquely the first alternative to be eliminated in the cloned profile. That is,

$$s^{j+1}(a) < \min\{s^{j+1}(b^{(i)}) : 1 \leq i \leq j\} \leq \frac{1}{j} \sum_i s^{j+1}(b^{(i)}),$$

where the second inequality uses the fact that the lowest scoring clone of b must score less than the average score of all the clones.

Recalling that $s^{j+1} = (x, 0, \dots, 0, 1)$, we have that

$$\sum_i s^{j+1}(b^{(i)}) = k_a + k_b x \qquad \text{and} \qquad s^{j+1}(a) = k_a x + k_b.$$

We use this, and the assumption that $k_a = k_b$, to rewrite the inequality

$$s^{j+1}(a) = k_a x + k_b < \frac{1}{j}(k_a + k_b x) = \frac{1}{j}(k_b + k_a x) = \frac{s^{j+1}(a)}{j},$$

which is not possible for any $j \geq 2$. Thus, there does not exist a successful cloning operation.

Case 2: $k_a \neq k_b$. Without loss of generality, suppose that $k_a > k_b$, so s uniquely selects (a, b). In order for a cloning operation to be successful, we require s to select b (or a clone of b) as the winner in some universe. For this purpose, cloning a will not be useful. (If we were to do so, we would require that s eliminates all clones of a in succession while not eliminating b. But at the final elimination step, one clone of a would remain and the profile would be identical to the uncloned profile, implying that b would be eliminated in all universes.) Therefore, the only hope of a successful cloning operation is to clone b into j clones $b^{(1)}, \dots, b^{(j)}$.

For a successful cloning operation, we need there to exist some universe in which a is the first alternative eliminated, i.e.,

$$s^{j+1}(a) = k_a x + k_b \leq \min\{s^{j+1}(b^{(i)}) : 1 \leq i \leq j\} \leq \frac{1}{j} \sum_i s^{j+1}(b^{(i)}) = \frac{1}{j}(k_a + k_b x).$$

Rearranging, and using the assumption that $x > \frac{1}{2}$, we get

$$\frac{1}{2} < x \leq \frac{k_a - j k_b}{j k_a - k_b},$$

which gives us

$$\frac{k_b}{k_a} < \frac{j-2}{1-2j}.$$

By the requirement that $k_a > k_b \geq 0$, the LHS of this inequality is necessarily non-negative. And by the requirement that $j \geq 2$, the RHS is necessarily non-positive. Therefore, the inequality is impossible to satisfy. As a consequence, there does not exist a successful cloning operation. □

We are now ready to prove our main result.

Theorem 1. *STV is the unique symmetric SPF satisfying independence of bottom alternatives, consistency at the bottom, independence of clones, and the properties in Definition 1.*

Proof. We have already discussed that STV satisfies all the properties. As to whether any other SPF could satisfy all the properties, Lemma 1 implies that such an SPF must be a runoff scoring rule. Then, Lemma 6 characterizes runoff scoring rules that are independent of clones. We finally argue that among these, only STV satisfies weak unanimity, as follows. We claim that if a runoff scoring rule s satisfies weak unanimity, then $s^n_{n-1} \geq s^n_n$ for all $n \geq 2$. Suppose this is not true for some $n \in \mathbb{N}$. Then we can consider a profile of one voter with preferences $(a_1, \ldots, a_n)$ over the n alternatives. Then, a_n is not the first alternative eliminated in any parallel universe, since $s^n_{n-1} < s^n_n$. This violates weak unanimity. Hence all rules listed in Lemma 6 other than STV violate weak unanimity. □

5 Other Characterization Results

We can use Lemma 1 to obtain characterizations of two additional runoff scoring rules, leveraging existing characterizations of scoring rules.[10]

Definition 6. *An SPF satisfies the* bottom-majority criterion *if $bottom(f(R)) = bottom(r)$ whenever more than half of the voters agree on a ranking r.*

Definition 7. *An SPF f satisfies the* ranking-majority criterion *if $f(R) = \{r\}$ whenever more than half of the voters agree on a ranking r.*

Theorem 2. *Coombs' rule is the unique symmetric SPF satisfying independence of bottom alternatives, consistency at the bottom, the ranking-majority criterion, and continuity at the bottom.*

Proof. Lemma 1 requires that the rule is a runoff scoring rule. Lepelley [15] showed that plurality is the only scoring rule that satisfies the majority criterion. Equivalently, veto is the only scoring rule such that $bottom(f(R)) = \{x\}$ whenever a majority of voters rank x last. We now show that veto is also the only scoring rule which satisfies the (weaker) bottom-majority criterion. Clearly veto does satisfy this. Now consider some other scoring rule g. Take a profile R in which a majority of voters rank a in last place but $bottom(g(R)) \neq a$. Such a profile exists because g is not veto. Define $b = bottom(g(R))$ and modify R in the following way. For all those voters who ranked a last, move b to a position which scores the lowest out of all the positions not occupied by a. This does not increase the score of b—in particular, b still scores less than a. Now place the other alternatives such that all voters who rank a last now vote identically, so that more than half the voters now agree on their ranking of the alternatives. The loser may have changed but it still can not be a since the score of b is strictly smaller than the score of a. Therefore, g does not satisfy the bottom-majority criterion.

[10] For an overview of such characterizations, see Merlin [16].

Hence, among all runoff scoring rules, Coombs' rule uniquely satisfies the ranking-majority criterion. because it is the only runoff scoring rule guaranteed to eliminate the alternative ranked in the bottom position by a majority (should one exist) at every step. □

In a similar fashion, we can characterize Baldwin's rule.

Definition 8. *An SPF f is* Condorcet-consistent *if a Condorcet winner is ranked first in all rankings in $f(R)$.*

Theorem 3. *Baldwin's rule is the unique symmetric SPF satisfying independence of bottom alternatives, consistency at the bottom, continuity at the bottom, and Condorcet-consistency.*

Proof. Lemma 1 requires that the rule is a runoff scoring rule. Smith [21] showed that Borda's rule is the only scoring rule that never ranks a Condorcet loser first. Equivalently, it is the only scoring rule that guarantees that a Condorcet winner cannot be ranked last. Thus, Baldwin's rule is the only runoff scoring rule that is guaranteed not to eliminate a Condorcet winner in any round.[11] □

Note that Nanson's rule is also Condorcet-consistent. However, it is not a runoff scoring rule according to our definition.

6 Non-Scoring Runoff Rules

So far, we have only considered runoff rules where a scoring rule is used to determine which alternative is ranked last. If we drop the criterion of consistency at the bottom, we can also choose the last alternative using another rule. One particularly remarkable way of doing so is to use *the inverted STV rule* to choose which alternative is ranked last. The inverted STV rule is the runoff scoring rule given by $s^n = (0, \ldots, 0, 1)$. (Equivalently, it is STV applied to the inverted votes.) Eliminate that alternative, and repeat. We call this *nested runoff voting*, as it involves a runoff rule within a runoff rule.

Example 3. *For the preference profile given in Example 1, nested runoff voting selects (a, b, c) and (b, c, a).*

As it turns out, this rule is independent of clones.

Proposition 1. *Nested runoff voting satisfies independence of bottom alternatives and independence of clones (but not consistency at the bottom).*

Proof. It fails consistency at the bottom because inverted STV, which is used to determine the bottom alternative, is not a scoring rule. It satisfies independence of bottom alternatives because it is a runoff rule. By Lemma 6, inverted STV is independent of clones, so cloning does not affect which alternative is ranked last at any point (with the possible exception of a clone being ranked last instead of the original alternative that it cloned). As a result, nested runoff voting also satisfies independence of clones. □

Therefore, nested runoff voting serves as an example of an SPF that satisfies independence of bottom alternatives and independence of clones, but not consistency at the bottom. Furthermore, any runoff scoring rule other than those characterized by Lemma 6 (such as Baldwin's or Coombs' rule) satisfies independence of bottom alternatives and consistency at the bottom, but not independence of clones. SPFs that satisfy consistency at the bottom and independence of clones but not independence of bottom alternatives appear harder to find.

[11] The fact that Baldwin's rule is the unique runoff scoring rule that satisfies the Condorcet criterion has been previously observed by Smith [21].

7 Conclusion

In this paper, we focused on the axiomatic characterization of runoff scoring rules. While we gave axiomatizations of Baldwin and Coombs' rules as well, our main contribution was an axiomatic characterization of STV based on the independence of clones criterion.

As far as we are aware, STV is the only known example of a neutral voting rule that is both independent of clones and NP-hard to manipulate by a single manipulator.[12] This is so because the only neutral variant of ranked pairs is the variant using PUT [5], and this variant is not independent of clones (by Example 1 in [26]); Schulze's rule is easy to manipulate by a single manipulator [18]; and we are not aware of any other non-trivial rules that are independent of clones. In our view, independence of clones is a very important criterion, and we do not yet have a good understanding of what properties make a rule independent of clones. Hence, characterizing rules that satisfy independence of clones is an important direction for future research.

If we were to view STV as a social *choice* function rather than an SPF, and relax our definition of independence of clones accordingly, we do not know whether other non-trivial runoff scoring rules would satisfy the property. Indeed, our proofs relied heavily on being able to alter *some* position in the ranking. However, we are not aware of any runoff scoring rules that are independent of clones in the SCF context, other than STV.

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A Scoring Rules are not Independent of Clones

We show that no scoring rule is independent of clones, even when viewed in the SCF setting. That is, for any scoring rule, it is possible to change the set of winners by cloning one or more alternatives.

Let $\mathcal{C}$ be a cloning operation that transforms profile R into profile $R^{\mathcal{C}}$ by substituting alternative a with a set $C(a)$ of clones. For a subset B of alternatives (possibly including clones), let $B|_{\mathcal{C}}$ be the "decloned vesion" of B, i.e.,

$$B|_{\mathcal{C}} = \begin{cases} (B \setminus C(a)) \cup \{a\} & \text{if } B \cap C(a) \neq \emptyset \\ B & \text{otherwise.} \end{cases}$$

Definition 9. *An SCF f satisfies* independence of clones *if for all preference profiles R and all cloning operations $\mathcal{C}$,*

$$f(R^{\mathcal{C}})|_{\mathcal{C}} = f(R).$$

Say that a scoring rule $s = (s^n)_{n\geq 1}$ is *non-trivial* if there is at least one $n \geq 2$ such that s^n is non-trivial.

Theorem 4. *No non-trivial scoring rule is independent of clones.*

Proof. Let $s = (s^n)_{n\geq 1}$ be a non-trivial scoring rule. Note that if we can exhibit a clone manipulation for some profile on $n \geq 3$ alternatives, then s is not independent of clones. Thus, we can restrict attention to profiles with at least three alternatives.

We first show that if s^n is trivial for some $n \geq 3$, then we can exhibit a clone manipulation. Suppose that s^{n-1} is non-trivial. Then we can consider one voter with preferences $(c_1, \ldots, c_{n-1})$, and the set of winners excludes at least one alternative by non-triviality. But if we clone any alternative c_i to produce c_i and c_i', then the set of winners now includes everyone. Similarly, if s^n is trivial but s^{n+1} non-trivial, then we can consider the one voter profile $(c_1, \ldots, c_n)$, in which all alternatives are in the winning set. Since s^{n+1} is non-trivial, there is some $k \leq n$ with $s_k^{n+1} \neq s_{k+1}^{n+1}$. We can choose an alternative c_i to clone into c_i

and c_i' such that in the cloned profile, positions k and $k+1$ will be held by alternatives other than the clones. Thus at least one of these alternatives is no longer in the winning set.

Define the inverse score vector $\bar{s}^n = (-s_1^n, \ldots, -s_n^n)$. Applying Lemma 3 to $\bar{s}^n$, we see that it is possible to create a profile where two alternatives get an arbitrarily *large* score according to s^n, not an arbitrarily small one.

We now show that in order to be independent of clones, it is necessary (but not sufficient) for all score vectors to be plurality/veto combinations. From the proof of Lemma 4, we see that if there exists $n \geq 5$ for which the score vector is not a plurality/veto combination (except for the case of $s^5 = (1, 0, 1, 0, 1)$), then we can change the set of winners by cloning from the $(n-1)$ alternative case. To rule out the possibility that $s^5 = (1, 0, 1, 0, 1)$, we can use the same construction as in the proof of Lemma 4, but reasoning about the winning alternative(s) rather than the losing one(s). And to see that s^4 must be a plurality/veto combination, we can again use the same construction as in the proof of Lemma 4.

Thus, for every $n \geq 3$, the score vector s^n must be of the form $s^n = (x, 0, \ldots, 0, y)$ for some x, y with $x \neq 0$ or $y \neq 0$. Suppose without loss of generality that $x \neq 0$. We distinguish two cases.

Case 1: $x \neq -y$. Consider the following cyclic profile on $n-1$ alternatives.

$$\begin{gathered} 2 \times (c_1, c_2, \ldots, c_{n-1}) \\ 2 \times (c_{n-1}, c_1, c_2, \ldots) \\ \vdots \\ 2 \times (c_2, \ldots, c_{n-1}, c_1) \end{gathered}$$

Since every alternative has the same rank distribution, the winning set consists of all alternatives. Now clone one alternative, say c_1, into c_1 and c_1'. It is possible to rank the clones in such a way that both c_1 and c_1' are ranked first exactly once and ranked last exactly once. The other alternatives are still ranked in each of these positions twice. Since $x \neq -y$, the score of the clones is distinct from the score of the other alternatives, and the set of winners has changed.

Case 2: $x = -y$. Suppose that $x > 0$. The case $y > 0$ can be treated in the same way. Consider the profile

$$\begin{gathered} 2 \times (a, b, \ldots) \\ 2 \times (b, a, \ldots) \end{gathered}$$

and note that a and b are both chosen as winners. Now clone a to obtain the profile

$$\begin{gathered} 1 \times (a, a', b, \ldots) \\ 1 \times (a', a, b, \ldots) \\ 1 \times (b, a', a, \ldots) \\ 1 \times (b, a, a', \ldots) \end{gathered}$$

It is now clear that b has the highest score of all alternatives, so b is the unique winner. □