Cost-sharing of continuous knapsacks

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Abstract

This paper provides a first insight into cost sharing rules for the continuous knapsack problem. Assuming a set of divisible items with weights from which a knapsack with a certain weight constraint is to be filled, different such (classes of) rules are discussed. Those - based on individual approvals of the items - optimally fill the knapsack and share the cost of the knapsack among the individuals. Using various reasonable properties of continuous knapsack cost sharing rules, we provide three characterization results.

1 Introduction

Cost allocation in combinatorial optimization problems has been intensively discussed in recent years (see [14] for a summary). The major focus has been on the minimum cost spanning tree problem, the earliest and most widely investigated cost sharing problem in this area (e.g. [3], [4], [10]). There the interest lies mainly in the fair division of the cost of creating a network in which each agent is connected directly or indirectly to a source. A second emphasis has been on scheduling and queuing problems, i.e., on the problem of optimally processing jobs of different lengths or weights on a single server (e.g. [8], [12], [13]).

The above problem of finding minimum cost spanning trees has a major advantage among combinatorial optimization problems. Its optimal solution can be found in polynomial time. Only then, i.e., in the case of finding such an optimal solution "quickly", does it seem to make sense to talk about fairly sharing the costs, because otherwise any changes to the setting could make it impossible to find the new cost allocation in reasonable time. The focus could only be on fixed solutions.

Among the combinatorial optimization problems, the knapsack problem is concerned with efficiently filling a weight-restricted knapsack with items from a set of items with possibly different weights and profits. Efficiency in that respect means maximizing some profit function based on the items' profits. In case of indivisible items, this problem is typically NP-hard. One exception is the continuous knapsack problem in which the items are divisible and therefore the solution could contain a certain fraction of one item.

In usual cost sharing problems such as the bankruptcy problem ([1], [16]) or the minimum cost spanning tree problem, "objective" preferences such as costs or claims play a major role in determining a fair cost allocation. This will be different in our framework, where we focus on the approval or disapproval of certain items by individuals ([5]). The social welfare of a set of items is simply defined by the total number of approvals for the single items in the set ([6]). This could be seen as a first step towards using (binary) preference information in determining a fair cost allocation.

The setting used in this paper can be summarized as follows: we start with a certain knapsack (a capacity, time interval, etc.) and a set of items over which individuals have

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binary preferences. Each of the items has a (possibly different) weight. First, the goal is to fill the knapsack such that social welfare, (i.e., the sum of approvals) is maximized. Then the attempt is to fairly divide the cost of the knapsack (or maintaining the capacity, or using the time) among the individuals.

As an example consider a multi-national research project that has some pre-determined cost. Space and/or time constraints might limit the number of researchers (out of a pool of potential candidates) that can participate. In addition, the possible candidates might be forced to use the provided resource for their specific research for different amounts of time. The potential financing countries of the research project might approve and disapprove of different researchers. The question now is how to select the set of researchers and how to distribute the cost among the participating countries.[2]

In principle we are concerned with sharing the cost of a selected set of non-rival items that provides different utilities or payoffs to the individuals. Cost allocation aspects in such a binary knapsack problem have been considered before by Dror [9] and certain rules such as the Shapley value or the equal charge method have been suggested. In this paper we want to introduce and characterize (a family of) possibly interesting continuous knapsack cost sharing rules.

The following section establishes the formal framework, defines the continuous knapsack problem, and introduces reasonable properties of continuous knapsack cost sharing rules. Section 3 first introduces a whole family of such rules and then focuses on two rules of which characterization results are provided. Section 4 concludes the paper.

2 Preliminaries

Let $\mathcal{N} = \{1, \ldots, n\}$ denote a set of individuals, and $I = \{1, \ldots, m\}$ a set of items. With each item $j \in I$, we associate a positive weight $w_j \in \mathbb{R}_+$. The weights are summarized by the vector $\omega \in \mathbb{R}^m_+$, where the j-th entry ω_j corresponds to w_j.
Each individual $i \in \mathcal{N}$ partitions the set I into a set A_i of items she approves of and a set of items she disapproves of. For $i \in \mathcal{N}$, the vector representation $a_i \in \{0,1\}^m$ turns out to be useful, where the j-th entry $a_{i,j} = 1$ if individual i approves of item j, and $a_{i,j} = 0$ if i disapproves of j. These vectors are captured by means of an $n \times m$ matrix A, whose rows correspond to the vectors a_i; i.e., $A = (a_{i,j})_{i \in \mathcal{N}, j \in I}$.
$A \ominus a_i$ denotes the matrix resulting from A by deleting the row corresponding to a_i. Let B be a $k \times m$ matrix for some $k \in \mathbb{N}$. For some $b \in \{0,1\}^m$, $B \oplus b$ is the $(k+1) \times m$ matrix created by concatenating to B a $(k+1)$-st row β and setting $\beta = b$.
For $j \in I$, let $\mathcal{N}_j$ be the set of individuals of $\mathcal{N}$ who approve of j, i.e., $\mathcal{N}_j = \{i \in \mathcal{N} : j \in A_i\}$. The value p_j of item $j \in I$ is defined as the number of individuals that approve of j. Formally, $p_j := |\{i \in \mathcal{N} : j \in A_i\}| = |\mathcal{N}_j|$.

Given a capacity constraint (or weight bound) W, we can represent a knapsack cost sharing problem as the quadruple $(\mathcal{N}, A, \omega, W)$. A solution to this problem assigns to each individual a cost share. However, one of the major problems in this combinatorial optimization exercise is its computational complexity, i.e., finding an optimal knapsack is NP-hard. Hence, we need to restrict ourselves to a special setting of the knapsack problem. Therefore we assume the items to be divisible, i.e., a solution may contain fractions of (at most) one item. This is called the continuous knapsack problem introduced in the following subsection.

[2] A "fraction" of a researcher could be seen as a part-time worker.

2.1 The continuous knapsack

The following definition introduces a well-known optimization problem:

Definition 2.1 *(Continuous Knapsack Problem)*

Given a set $I = \{1, \dots, m\}$ of items, and, for each $j \in I$, positive real numbers p_j and w_j, the continuous knapsack problem is the following problem:[3]

$$\begin{array}{rl} & \max \sum_{j \in I} p_j x_j \\ s.t. & \sum_{j \in I} w_j x_j \leq W \\ & x_j \in [0,1] \end{array}$$

It is known that the continuous knapsack problem can be solved in polynomial time (see [11]). In what follows, we assume that the items are sorted in a way such that

$$\frac{p_1}{w_1} > \frac{p_2}{w_2} > \ldots > \frac{p_m}{w_m} \tag{1}$$

Note that in practice, the strict inequalities in (1) are not a limitation, since these may always be reached by arbitrarily small "perturbations" of the weights or by modifying the accuracy of measurement. In theory (compare [11]), inequality (1) ensures that the unique solution the entity chooses is determined by

$$x_j := \begin{cases} 1 & \text{for } j = 1, \dots, s-1 \\ \frac{1}{w_s}(W - \sum_{i=1}^{s-1} w_i) & \text{for } j = s \\ 0 & \text{for } j > s \end{cases} \tag{2}$$

where s is defined by

$$\sum_{j=1}^{s-1} w_j < W \quad \text{and} \quad \sum_{j=1}^{s} w_s \geq W$$

The corresponding objective function value z is given by $z = \sum_{j \in I} p_j x_j = \sum_{j=1}^{s-1} p_j + \frac{p_s}{w_s}(W - \sum_{i=1}^{s-1} w_i)$.
Item s is called *split item.*[4] For an optimal solution $X = (x_1, x_2, \dots, x_m)$, we abbreviate $X_+ = \{j \in I : x_j > 0\} = \{1, \dots, s\}$. In what follows, and in order to simplify notation, x_j is identified with its value in the optimal solution of the considered continuous knapsack problem.

2.2 Dividing a continuous knapsack

Let the quadruple $(\mathcal{N}, A, \omega, W)$ be given. From the previous section we know that a solution can be calculated in polynomial time. Now, the goal is to divide the cost of the optimally packed knapsack among the individuals in a fair manner. In that respect, we first have to determine the cost of the knapsack. In this paper, we assume that every unit of weight imposes a cost of one, and therefore the total cost of the knapsack is equal to the weight

[3] In our approach we will focus on maximizing a sort of utilitarian social welfare given by the sum of approvals. This might, however, not be the only way to implement a fair solution. More egalitarian approaches could also be considered at that stage.

[4] Note that possibly $x_s = 1$ holds in the optimal solution. That is, the split item s is not necessarily "split", i.e., $0 < x_s < 1$ need not hold.

constraint W. However, dividing then the weight w_j by W for each $j \in I$ and setting $W = 1$ does not change the structure of the problem (and, in particular, the optimal solutions of the corresponding continuous knapsack problems are identical). Thus, in the major part of the paper it is assumed that $W = 1$. In that case, the continuous knapsack cost sharing problem is denoted by the triple $(\mathcal{N}, A, \omega)$, and we refer to the corresponding continuous knapsack problem as the pair (A, ω).

In general, a continuous knapsack cost sharing rule is a function $\phi : (\mathcal{N}, A, \omega, W) \to \mathbb{R}^n_+$. The i-th entry ϕ_i of ϕ is interpreted as the share of the cost that individual i has to carry.

In the following we define some desirable properties for a continuous knapsack cost sharing rule, trying to capture certain aspects of fairness.

Properties of cost sharing rules.

The first requirement – frequently used in the literature in various contexts – is that the total cost of the knapsack should be allocated exactly.

Efficiency: A cost allocation rule ϕ is *efficient*, if $\sum_{i=1}^{n} \phi_i(\mathcal{N}, A, \omega, W) = W$.

For the sake of readability, the remaining properties (except additivity) are defined for the case $W = 1$. However, the definitions coincide with the ones for the general case.
The second property, widely used e.g. in scheduling problems ([13]), represents the idea that voters should not benefit from "splitting" into several voters with disjoint sets of approved items (or, the other way round, in case their approved items are disjoint, "merging" into a single voter). At the same time, the remaining voters should not be disadvantaged if certain voters "split up" (or "merge"). In principle this should prevent the creation of fake identities, i.e., the individual possibility to manipulate the fair division process.[5]

To illustrate the idea of splitting, let voter i approve of items $1, 2, 3$. Replacing voter i by voters i_j approving of item j only, $1 \le j \le 3$, should have the result that the sum of the cost shares of the three voters i_j has to be equal to the cost share of voter i in the original problem. In the following definition, given a set of individuals $\mathcal{N}'$, $A'_{i'}$ refers to the set of approved items of $i' \in \mathcal{N}'$ (and $a'_{i'}$ denotes the corresponding vector of approvals).

Split-proofness: Let $i \in \mathcal{N}$. Let $\mathcal{N}' = (\mathcal{N} \setminus \{i\}) \cup \{i_1, \dots, i_r\}$, such that sets A'_{i_ℓ} form a partition of A_i, i.e., $\biguplus_{\ell=1}^{r} A'_{i_\ell} = A_i$. Let $A' = A \oplus (a'_{i_1} \oplus \ldots \oplus a'_{i_r}) \ominus a_i$.
A cost allocation rule ϕ is called *split-proof*, if

- $\phi_i(\mathcal{N}, A, \omega) = \sum_{j=1}^{|A_i|} \phi_{i_j}(\mathcal{N}', A', \omega)$ and
- $\phi_h(\mathcal{N}', A', \omega) \le \phi_h(\mathcal{N}, A, \omega)$ for all $h \in \mathcal{N} \setminus \{i\}$

Remark. Note that for a split-proof rule ϕ, the first of the above conditions implies that $\sum_{h \in \mathcal{N} \setminus \{i\}} \phi_h(\mathcal{N}, A, \omega) = \sum_{h \in \mathcal{N} \setminus \{i\}} \phi_h(\mathcal{N}', A', \omega)$. Thus, the mild second condition implies that $\phi_h(\mathcal{N}', A', \omega) = \phi_h(\mathcal{N}, A, \omega)$ holds for all $h \in \mathcal{N} \setminus \{i\}$. To see this, assume that the share of an individual h becomes strictly smaller in problem $(\mathcal{N}', A', \omega)$. Then, for at least one $h' \in \mathcal{N} \setminus \{j\}$ we must have $\phi_{h'}(\mathcal{N}', A', \omega) > \phi_{h'}(\mathcal{N}, A, \omega)$, in contradiction to the above definition.

Since each of the following two properties refers to an instance $(\mathcal{N}, A, \omega)$, for the sake of brevity we write ϕ_i instead of $\phi_i(\mathcal{N}, A, \omega)$ for $i \in \mathcal{N}$.

[5] It has to be added though, that the property is probably less compelling in this setting compared to scheduling problems, as fake identities are not allowed to overlap with their (sets of) approvals.

The first property reflects the compelling idea, well-known in the literature, that the cost allocation should not depend on the label of the individual.

Anonymity: Let $i, i' \in \mathcal{N}$. A cost allocation rule ϕ is called *anonymous*, if $(A_i = A_{i'} \Rightarrow \phi_i = \phi_{i'})$.

The second requirement is similar to the usual dummy-property. It states that an individual who only approves of items not in the optimal solution, should not be charged. A "totally unhappy" individual should not be forced to carry the knapsack or contribute to its costs.

Dummy: If $x_j = 0$ for all $j \in A_i$, then $\phi_i = 0$.

The following property applies non-manipulability arguments to situations in which pairs of individuals, that only approve of one single item, try to improve their situation by switching their approvals. It requires their cost shares to be exactly the same, i.e., providing absolutely no incentive to get involved into such switches.

Switch-proofness: Given $(\mathcal{N}, A, \omega)$, let $A_i = \{j\}$, $A_{i'} = \{j'\}$ with $x_j = x_{j'} = 1$. Let $(\mathcal{N}, \tilde{A}, \omega)$ with $\tilde{a}_h = a_h$ for all $h \in \mathcal{N} \setminus \{i, i'\}$ and $\tilde{a}_f = a_g$ for $f, g \in \{i, i'\}$, $f \neq g$. Then $\phi_k(\mathcal{N}, A, \omega) = \phi_k(\mathcal{N}, \tilde{A}, \omega)$ for all $k \in \mathcal{N}$.

A further reasonable property requires the division process to be independent of a possible sequential structure, i.e., if the knapsack is divided into two different and smaller knapsacks that together have exactly the same weight constraint as before, then applying the sharing rule to each of the smaller knapsacks separately should lead to the same total cost share as applying the rule to the original knapsack. This property will be called additivity and has been used, e.g., by [7] w.r.t. rights problems.

Additivity: Let $W^{(1)}, W^{(2)} \in \mathbb{R}_+$ with $W^{(1)} + W^{(2)} = 1$. Let $\phi^{(1)} = \phi(\mathcal{N}, A, \omega, W^{(1)})$, and let $X^{(1)}$ be the optimal solution of $(A, \omega, W^{(1)})$. Let $\tilde{A} = (\tilde{a}_{ij})_{i \in \mathcal{N}, j \in I}$ such that, for $i \in \mathcal{N}$, $\tilde{a}_{ij} = 0$ if $x_j^{(1)} = 1$ and $\tilde{a}_{ij} = a_{ij}$ otherwise.
In addition, let $\tilde{\omega} \in \mathbb{R}_+^m$ such that $\tilde{\omega}_j = (1 - x_j^{(1)})\omega_j$ for $j \in X_+^{(1)}$ with $0 < x_j^{(1)} < 1$, and $\tilde{\omega}_j = \omega_j$ otherwise. Let $\phi^{(2)} = \phi(\mathcal{N}, \tilde{A}, \tilde{\omega}, W^{(2)})$. Then, ϕ is *additive*, if $\phi = \phi^{(1)} + \phi^{(2)}$.

The final property is concerned with the changes in the cost shares given a minimal weight-change of a non-split item contained in the optimal solution of the continuous knapsack problem, keeping the remaining weights unchanged. It is exclusively concerned with situations in which everyone approves of exactly one item. A minimal weight change in that respect is one in which the optimal solution does not change, i.e., the set of items in the optimal solution before and after the weight change is identical.

Definition 2.2 *Given $(\mathcal{N}, A, \omega)$, let X be an optimal solution of the continuous knapsack problem (A, ω) with $X_+ = \{1, \ldots, s\}$ and $x_s < 1$. For some $j < s$, let $\tilde{w}_j < w_j$ and $\tilde{\omega} = (w_1, \ldots, w_{j-1}, \tilde{w}_j, w_{j+1}, \ldots, w_m)$.*
We call $\tilde{w}_j$ insignificantly smaller than w_j, if for the optimal solution $\tilde{X}$ of $(A, \tilde{\omega})$, we have $\tilde{X}_+ = X_+$.

Now, let the weight of j insignificantly decrease in the sense of the above definition, and let each individual approve of exactly one item. Then, weight-monotonicity states that all those that approve of the item that became insignificantly smaller should face a decrease in their cost share relative to the change in the value of the objective function. The formal definition of this condition is as follows:

Weight-monotonicity: Let $\tilde{w}_j$ be insignificantly smaller than w_j. Then, for all $i \in \mathcal{N}$ with $A_i = \{j\}$, $\frac{\phi_i(\mathcal{N}, A, \tilde{\omega})}{\phi_i(\mathcal{N}, A, \omega)} = \frac{z}{\tilde{z}}$, where $\tilde{z}$ denotes the objective function value of the optimal solution of $(A, \tilde{\omega})$.

3 Characterizations

In what follows, we consider a continuous knapsack cost sharing problem $(\mathcal{N}, A, \omega)$ where (as previously) X with $X_+ = \{1, \ldots, s\}$ corresponds to the optimal solution of the continuous knapsack problem (A, ω).

We now want to investigate, whether certain combinations of the previous properties can be used to determine specific reasonable cost sharing rules. Our first result establishes a full description of the family of efficient rules, that satisfies the dummy property, split-proofness and switch-proofness. As a second result, we present the characterization of a special representative of this family by adding weight-monotonicity. Finally, a characterization of another reasonable cost sharing rule is given.

Theorem 3.1 *The efficient rules that satisfy the dummy property, split-proofness and switch-proofness are exactly the functions* ϕ^c *with* $0 \leq c \leq \frac{1}{\sum_{i<s} p_i}$*, defined by* $(\forall i \in \mathcal{N})$

$$\phi_i^c(\mathcal{N}, A, \omega) = c \cdot \sum_{j \in A_i} x_j + \mathbb{1}_{A_i}(s) \cdot \frac{1 - cz}{p_s}$$

Proof. First, we show that $\phi_i^c \geq 0$ holds for all $i \in \mathcal{N}$, i.e., ϕ^c is indeed a cost sharing rule. Since $c \geq 0$ holds, we obviously have $\phi_i^c \geq 0$ for i with $s \notin A_i$. If $s \in A_i$, then

$$\begin{aligned} \phi_i^c &= \textstyle\sum_{j \in A_i \setminus \{s\}} x_j c + x_s c + \frac{1-cz}{p_s} = \sum_{j \in A_i \setminus \{s\}} x_j c + x_s c + \left(\frac{1-c\sum_{i=1}^{s-1} p_i - c p_s x_s}{p_s}\right) \\ &= \textstyle\sum_{j \in A_i \setminus \{s\}} x_j c + \left(\frac{1-c\sum_{i=1}^{s-1} p_i}{p_s}\right) \end{aligned}$$

Due to $c \geq 0$, we have $\sum_{j \in A_i \setminus \{s\}} x_j c \geq 0$; in addition, $1 - c\sum_{i=1}^{s-1} p_i \geq 0$ holds because of $c \leq \frac{1}{\sum_{i=1}^{s-1} p_i}$. Thus, $\phi_i^c \geq 0$ holds in the case $s \in A_i$ as well.

Now, it is shown that each of the axioms is satisfied by the proposed rule.
The dummy property is obviously satisfied. Now, consider $\sum_{i \in \mathcal{N}} \phi_i^c = \sum_{i \in \mathcal{N}} c \sum_{j \in A_i} x_j + \sum_{i \in \mathcal{N}} \mathbb{1}_{A_i}(s) \frac{1}{p_s}(1 - cz) = c \sum_{i \in \mathcal{N}} \sum_{j \in A_i} x_j + \frac{1}{p_s}(1 - cz) \sum_{i \in \mathcal{N}} \mathbb{1}_{A_i}(s)$. Since item j is approved by exactly p_j individuals of $\mathcal{N}$, it holds that $\sum_{i \in \mathcal{N}} \sum_{j \in A_i} x_j = \sum_{j \in I} p_j x_j = z$, and $\sum_{i \in \mathcal{N}} \mathbb{1}_{A_i}(s) = p_s$. Hence, $\sum_{i=1}^{n} \phi_i^c = cz + \frac{1}{p_s}(1 - cz)p_s = 1$, which proves efficiency.
For a fixed $i \in \mathcal{N}$, let $(\mathcal{N}', A', \omega)$ be as described in the definition of split-proofness. Note that the optimal solution X' of (A', ω) is also the optimal solution of (A, ω), and the respective objective function values z' and z coincide. Thus,

$$\sum_{\ell=1}^{r} \phi_{i_\ell}^c(\mathcal{N}', A', \omega) = \sum_{\ell=1}^{r} (c \sum_{j \in A'_{i_\ell}} x_j + \mathbb{1}_{A'_{i_\ell}}(s) \frac{1}{p_s}(1 - cz)) = c \sum_{\ell=1}^{r} \sum_{j \in A'_{i_\ell}} x_j + \frac{1}{p_s}(1 - cz) \sum_{\ell=1}^{r} \mathbb{1}_{A'_{i_\ell}}(s)$$

By construction, $\sum_{\ell=1}^{r} \sum_{j \in A'_{i_\ell}} x_j = \sum_{j \in A_i} x_j$, and $\sum_{\ell=1}^{r} \mathbb{1}_{A'_{i_\ell}}(s) = \mathbb{1}_{A_i}(s)$. Hence, $\sum_{\ell=1}^{r} \phi_{i_\ell}^c(\mathcal{N}', A', \omega) = c \sum_{j \in A_i} x_j + \mathbb{1}_{A_i}(s) \frac{1}{p_s}(1 - cz) = \phi_i^c(\mathcal{N}, A, \omega)$. I.e., ϕ^c is split-proof.
For switch-proofness, let $A_i = \{j\}$ and $A_{i'} = \{j'\}$ such that $x_j = x_{j'} = 1$. Let $\tilde{A}$ be built from A because i and i' "switch" their items (as in the definition of switch-proofness). Then, $\phi_k(\mathcal{N}, A, \omega) = c = \phi_k(\mathcal{N}, \tilde{A}, \omega)$ for $k \in \{i, i'\}$, since the optimal solutions of (A, ω) and $(\tilde{A}, \omega)$ coincide. The latter fact obviously implies $\phi_k(\mathcal{N}, A, \omega) = \phi_k(\mathcal{N}, \tilde{A}, \omega)$ for all $k \in \mathcal{N} \setminus \{i, i'\}$ as well.

On the other hand, assume there is a rule ψ that satisfies the stated conditions. Now in order to create the new instance $(\mathcal{N}', A', \omega)$ from $(\mathcal{N}, A, \omega)$, replace each voter i with the voters $i_1, \ldots, i_{|A_i|}$ such that $|A'_{i_\ell}| = 1$ for each $1 \leq \ell \leq |A_i|$ and $\bigcup_{\ell=1}^{|A_i|} A'_{i_\ell} = A_i$. Because of split-proofness, we know that

$$\sum_{\ell=1}^{|A_i|} \psi_{i_\ell}(\mathcal{N}', A', \omega) = \psi_i(\mathcal{N}, A, \omega) \tag{3}$$

holds for each $i \in \mathcal{N}$.
Obviously, the optimal solutions of (A, ω) and (A', ω) coincide; let X be such an optimal solution, with $X_+ = \{1, \ldots, s\}$. Note that the objective function value is given by

$$z = p_1 x_1 + \ldots p_s x_s = p_1 + \ldots p_{s-1} + p_s x_s$$

First, we show that ψ is anonymous. Let $i, j \in \mathcal{N}$ with $A_i = A_j$. Starting with instance $(\mathcal{N}', A', \omega)$, create instance $(\mathcal{N}', \tilde{A}', \omega)$ by applying a "switch" between the individuals i_k and j_k, $k \in \{1, \ldots, |A_i|\}$, i.e., $\tilde{A}'_g = A'_h$ holds for $g, h \in \{i_k, j_k\}$. Now, switch-proofness and the fact that ψ is a function imply $\psi_{i_k}(\mathcal{N}, A', \omega) = \psi_{i_k}(\mathcal{N}, \tilde{A}', \omega) = \psi_{j_k}(\mathcal{N}, A', \omega)$ for all $k \in \{1, \ldots, |A_i|\}$. Thus, $\psi_i(\mathcal{N}, A, \omega) = \sum_{\ell=1}^{|A_i|} \psi_{i_\ell}(\mathcal{N}', A', \omega) = \sum_{\ell=1}^{|A_i|} \psi_{j_\ell}(\mathcal{N}', A', \omega) = \psi_j(\mathcal{N}, A, \omega)$ is satisfied; i.e., ψ is anonymous.
Let $i, i' \in \mathcal{N}'$ with $A'_i = \{j\}$, $A'_{i'} = \{j'\}$ and $j, j' < s$. Then, perform a switch between i and i' and call the new instance $(\mathcal{N}', A^*, \omega)$. Because of split-proofness, we can assume that the last two rows of each A and A^* correspond to a'_i and $a'_{i'}$ (in the same order). Note that in A^*, the row a'_i displays ${A^*}_{i'}$ and the row $a'_{i'}$ displays ${A^*}_i$ respectively. Thus, since ψ is a function, we must have $\psi_i(\mathcal{N}', A', \omega) = \psi_{i'}(\mathcal{N}', A^*, \omega)$. However, switch proofness yields that $\psi_{i'}(\mathcal{N}', A', \omega) = \psi_{i'}(\mathcal{N}', A^*, \omega)$. Hence, we must have $\psi_i(\mathcal{N}', A', \omega) = \psi_{i'}(\mathcal{N}', A', \omega)$. Therefore, for some $c \geq 0$, $\psi_{g'}(\mathcal{N}', A', \omega) = c$ must hold for all $g' \in \mathcal{N}'$ with $A'_{g'} = \{h'\}$ and $x_{h'} = 1$.
Anonymity together with the dummy property implies that, for some $c_s, c \in \mathbb{R}_+ \cup \{0\}$,

$$\psi_{i'}(\mathcal{N}', A', \omega) = \begin{cases} c_s & \text{if } A_{i'} = \{s\} \\ c & \text{if } A_{i'} = \{j' : j' < s\} \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Efficiency yields

$$1 = \sum_{i' \in \mathcal{N}'} \psi_{i'}(\mathcal{N}', A', \omega) = \sum_{i' \in \mathcal{N}'_s} \psi_{i'} + \sum_{j<s} \sum_{i' \in \mathcal{N}'_j} \psi_{i'} \tag{5}$$

Note that, by construction, for each $j \in I$, $|\mathcal{N}'_j| = p_j$. Equation (5) can hence be rewritten as

$$1 = p_s c_s + c \cdot (p_1 + p_2 + \ldots + p_{s-1}) \tag{6}$$

Recall that $z = p_1 + p_2 + \ldots + p_{s-1} + x_s p_s$, or, equivalently, $\sum_{i=1}^{s-1} p_i = z - x_s p_s$. Substituting the last equality in (6), we get

$$\begin{array}{rcl} 1 - p_s c_s & = & c(z - x_s p_s) \\ \Leftrightarrow \quad c_s & = & \frac{1-cz}{p_s} + x_s c \end{array} \tag{7}$$

With (3) and (4), we get $\psi_i(\mathcal{N}, A, \omega) = \sum_{\ell=1}^{|A_i|} \psi_{i_\ell}(\mathcal{N}', A', \omega) = \sum_{j \in A_i \setminus \{s\}} x_j c + c_s \cdot \mathbb{1}_{A_i}(s)$. With (7), this yields

$$\psi_i(\mathcal{N}, A, \omega) = \begin{cases} \sum_{j \in A_i} x_j c & \text{if } s \notin A_i \\ \sum_{j \in A_i} x_j c + \frac{1-cz}{p_s} & \text{if } s \in A_i \end{cases}$$

Analogously to the beginning of the proof, it follows that $0 \leq c \leq \frac{1}{\sum_{i<s} p_i}$ must hold for ψ to be a cost sharing rule. Therewith, $\psi = \phi^c$. $\square$

A representative of the above family of rules is derived from the idea, that a voter's cost share should exclusively depend on the total number of the items in the optimal knapsack she approves of, relative to the total number of approvals for the entire knapsack (in each case taking fractional values into account[6]). In particular, if someone likes twice as many items (included as a whole) from the knapsack than another individual, then she should also be given a cost share twice as high. Obviously this cost sharing rule is not concerned with weights of items or number of approvals for one specific item. Formally, this rule can be defined as follows:

Definition 3.1 *Given a problem $(\mathcal{N}, A, \omega)$, the simple proportional continuous knapsack cost sharing rule is defined as $(\forall i \in N)$*

$$\phi_i^{sol}(\mathcal{N}, A, \omega) = \frac{\sum_{j \in A_i} x_j}{z}$$

The rule ϕ^{sol} can be characterized as follows.

Theorem 3.2 $\phi_i^{sol}(\mathcal{N}, A, \omega)$ *is the only efficient and split-proof rule that satisfies dummy, switch-proofness, and weight-monotonicity.*

Proof. ϕ^{sol} belongs to the family ϕ^c (setting $c = \frac{1}{z}$. Hence, due to Theorem 3.1, it is sufficient to show that ϕ^{sol} is the only among the rules ϕ^c that satisfies weight-monotonicity. It is easy to verify that ϕ^{sol} satisfies weight-monotonicity. To proof the other direction, we follow the argumentation of the above proof. Consider instance $(\mathcal{N}', A', \omega)$ (of the above proof) and assume $x_s < 1$. Decrease the weight of item j from w_j insignificantly to $\tilde{w}_j$ for some $j < s$ such that $x'_s = 1$ in the optimal solution $\tilde{X}$ (with objective function value $\tilde{z}$ of $(A', \tilde{\omega})$, where $(\mathcal{N}', A', \tilde{\omega})$ denotes this new instance). Call the new shares (according to (4)) c'_s and c'; note that due to $x'_s = 1$, with analogous arguments as in the proof of Theorem 3.1, from switch-proofness we get $c'_s = c'$.
From efficiency, we thus get $1 = p_s c'_s + c' \cdot (p_1 + p_2 + \ldots + p_{s-1}) = c'(p_1 + p_2 + \ldots + p_s) = c' \cdot \tilde{z}$. Therewith, $c' = \frac{1}{\tilde{z}}$. Weight-monotonicity, however, implies $\frac{\psi_i(\mathcal{N}', A', \omega)}{\psi_i(\mathcal{N}', A', \tilde{\omega})} = \frac{c'}{c} = \frac{z}{\tilde{z}}$ for $i \in \mathcal{N}'$ with $A_i = \{j\}$. Hence, $c = \frac{1}{z}$ follows. Thus, ψ corresponds to ϕ^c with $c = \frac{1}{z}$, i.e., $\psi = \phi^{sol}$. $\square$

The above rule puts its focus purely on the proportion of individual approvals to total approvals. This might seem unreasonable or inefficient in certain situations for two reasons: First, where extensive weight differences between the single items can be observed, a rule being sensitive to weights and weight changes might be preferable. Second, the more individuals approve of a certain item in the knapsack, the lower should probably be their cost share, if one assumes a non-rival good whose cost it imposes on the knapsack does not depend on the number of approvals. Hence, if we replace switch-proofness and weight-monotonicity with additivity, we characterize a rule, that takes into account the "inefficiency" $\frac{w_j}{p_j}$ of item $j \in I$ directly. The cost sharing rule is defined as follows:

Definition 3.2 *Given a problem $(\mathcal{N}, A, \omega)$, the weight-and-approval-based proportional continuous knapsack cost sharing rule is defined as $(\forall i \in N)$*

$$\phi_i^e(\mathcal{N}, A, \omega) = \sum_{j \in A_i} \frac{w_j}{p_j} x_j$$

[6] I.e, if a fraction of an item is included in the knapsack, then only the respective fraction of the approval is taken into account.

The rule ϕ^e can be characterized as follows:

Theorem 3.3 *$\phi_i^e(\mathcal{N}, A, \omega)$ is the only efficient and split-proof rule that satisfies dummy, anonymity, as well as additivity.*

Proof. For readability, we write ϕ instead of ϕ^e within this proof. We first show that all these axioms are satisfied by ϕ.

$$\sum_{i \in \mathcal{N}} \phi_i = \sum_{i \in \mathcal{N}} \sum_{j \in A_i} \frac{w_j}{p_j} x_j = \sum_{j \in I} \sum_{i \in \mathcal{N}_j} \frac{w_j}{p_j} x_j = \sum_{j \in I} p_j \frac{w_j}{p_j} x_j = \sum_{j \in I} w_j x_j$$

However, the last sum in the above expression corresponds to 1 because X is an optimal solution of (A, ω); thus, ϕ is efficient.

Split-proofness, dummy and anonymity are obviously satisfied.

For additivity, let $W^{(1)}, W^{(2)} \in \mathbb{R}_+$ with $W^{(1)} + W^{(2)} = 1$. Note that $x_j = 0$ implies $x_j^{(1)} = 0$ and $x_j^{(2)} = 0$. Thus, it is sufficient to consider the items $\{1, \ldots, s\}$. By construction, $X_+^{(1)} = \{1, ..., \ell\}$ for some $\ell \leq s$.

Case 1: $x_\ell^{(1)} = 1$. By construction, this means that there is no voter that approves of any of the items $\{1, ..., \ell\}$ in instance $(\mathcal{N}, \tilde{A}, \tilde{\omega}, W^{(2)})$. Thus, $x_j^{(2)} = 0$ for all $1 \leq j \leq \ell$. Vice versa, we have $x_j^{(1)} = 0$ and $x_j^{(2)} = x_j$ for all $j \in \{\ell+1, \ldots, s\}$. In addition, $\tilde{w}_j = w_j$ holds for $j \in \{\ell+1, \ldots, s\}$. Hence, $\phi_i^{(1)} + \phi_i^{(2)} = \sum_{j \in A_i} \frac{w_j}{p_j} x_j^{(1)} + \sum_{j \in A_i} \frac{\tilde{w}_j}{p_j} x_j^{(2)} = \sum_{j \in A_i} \frac{w_j}{p_j} x_j = \phi_i$.

Case 2: $0 < x_\ell^{(1)} < 1$. Then, in instance $(\mathcal{N}, \tilde{A}, \tilde{\omega}, W^{(2)})$, each of the items $\{1, \ldots, \ell - 1\}$ has zero approvals. Thus the ranking analogous to (1) (restricted to the remaining items) is

$$\frac{p_\ell}{\tilde{w}_\ell} > \frac{p_{\ell+1}}{\tilde{w}_{\ell+1}} > \ldots > \frac{p_s}{\tilde{w}_s} > \ldots > \frac{p_m}{\tilde{w}_m}$$

because the number of approvals of these items remains unchanged, and only the weight of item ℓ has decreased (compared to the original instance).

Case 2a: $\ell \neq s$. By the choice of x_s and $\tilde{w}_\ell$, $W^{(2)} = \sum_{k=\ell}^{s-1} \tilde{w}_\ell + x_s w_s$ must hold. Thus, $X_+^{(2)} = \{\ell, \ell+1, \ldots, s\}$, and $x_\ell^{(2)} = \ldots = x_{s-1}^{(2)} = 1$ and $x_s^{(2)} = x_s$. As in the above case, by construction for all $\ell + 1 \leq j \leq s$ we have $\tilde{w}_j = w_j$. Note that for $j \neq \ell$, $x_j^{(1)} + x_j^{(2)} = x_j$. Thus, if $\ell \notin A_i$, we get

$$\phi_i^{(1)} + \phi_i^{(2)} = \sum_{j \in A_i} \frac{w_j}{p_j} x_j^{(1)} + \sum_{j \in A_i} \frac{\tilde{w}_j}{p_j} x_j^{(2)} = \sum_{j \in A_i} \frac{w_j}{p_j} (x_j^{(1)} + x_j^{(2)}) = \sum_{j \in A_i} \frac{w_j}{p_j} x_j = \phi_i \tag{8}$$

Let $\ell \in A_i$. By construction, $\tilde{w}_\ell = (1 - x_\ell^{(1)}) w_\ell$. With $x_\ell^{(2)} = 1$, analogously to equation (8) we get

$$\begin{aligned} \phi_i^{(1)} + \phi_i^{(2)} &= \sum_{j \in A_i \setminus \{\ell\}} \tfrac{w_j}{p_j} (x_j^{(1)} + x_j^{(2)}) + \tfrac{w_\ell}{p_\ell} x_\ell^{(1)} + \tfrac{\tilde{w}_\ell}{p_\ell} x_\ell^{(2)} \\ &= \sum_{j \in A_i \setminus \{\ell\}} \tfrac{w_j}{p_j} x_j + \tfrac{w_\ell}{p_\ell} x_\ell^{(1)} + \tfrac{w_\ell}{p_\ell} (1 - x_\ell^{(1)}) \\ &= \sum_{j \in A_i} \tfrac{w_j}{p_j} x_j \\ &= \phi_i \end{aligned}$$

Case 2b: $\ell = s$. For $1 \leq j \leq s - 1$, we thus have $x_j^{(1)} = x_j = 1$ and $x_j^{(2)} = 0$.

By construction, $\tilde{w}_s = (1 - x_s^{(1)}) w_s$ and $W^{(2)} = (x_s - x_s^{(1)}) w_s$. Hence, $x_s^{(2)} = \frac{1}{\tilde{w}_s} W^{(2)} = \frac{1}{\tilde{w}_s} (x_s - x_s^{(1)}) w_s = \frac{x_s - x_s^{(1)}}{1 - x_s^{(1)}}$. As a consequence,

$$\frac{w_s}{p_s} x_s^{(1)} + \frac{\tilde{w}_s}{p_s} x_s^{(2)} = \frac{w_s}{p_s} \left(x_s^{(1)} + (1 - x_s^{(1)}) \frac{x_s - x_s^{(1)}}{1 - x_s^{(1)}} \right) = \frac{w_s}{p_s} x_s$$

Therewith, $\phi_i^{(1)} + \phi_i^{(2)} = \sum_{j \in A_i} \frac{w_j}{p_j} x_j = \phi_i$ holds in this case as well. I.e., ϕ is additive.

Assume there is a rule ψ that satisfies efficiency, split-proofness, dummy, anonymity, as well as additivity. As in the above proofs, create a new problem $(\mathcal{N}', A', \omega)$ from $(\mathcal{N}, A, \omega)$ by replacing each voter i with the voters $i_1, \ldots, i_{|A_i|}$ such that $|A'_{i_\ell}| = 1$ for each $1 \leq \ell \leq |A_i|$ and $\bigcup_{\ell=1}^{|A_i|} A'_{i_\ell} = A_i$. Since ψ is split-proof, we get

$$\sum_{\ell=1}^{|A_i|} \psi_{i_\ell}(\mathcal{N}', A', \omega) = \psi_i(\mathcal{N}, A, \omega) \quad \text{for all } i \in \mathcal{N} \tag{9}$$

Since ψ is efficient and split-proof,

$$1 = \sum_{i \in \mathcal{N}} \psi_i(\mathcal{N}, A, \omega) = \sum_{i \in \mathcal{N}} \sum_{\ell=1}^{|A_i|} \psi_{i_\ell}(\mathcal{N}', A', \omega) = \sum_{k \in I} \sum_{i \in \mathcal{N}_k} \psi_i(\mathcal{N}', A', \omega) \tag{10}$$

Because ψ is anonymous, it holds that for each $j \in I$, $\psi_i(\mathcal{N}', A', \omega) = \psi_{i'}(\mathcal{N}', A', \omega) =: \delta_j$ for $i, i' \in \mathcal{N}_j$. Due to the dummy property, we have

$$\delta_j = 0 \ \forall j > s \tag{11}$$

Thus, (10) is equivalent to

$$1 = \sum_{j=1}^{s} p_j \cdot \delta_j \tag{12}$$

In what follows, we make use of additivity. In the first step, let $W^{(1)} = w_1$ and $W^{(2)} = \sum_{j=2}^{s-1} w_j + w_s x_s$. Then, the optimal solution of $(A', \omega, W^{(1)})$ is given by packing item 1 in the knapsack, i.e., $x_1^{(1)} = 1$ and $x_j^{(1)} = 0$ for $j > 1$. Anonymity implies that, for $j \in I$, there are $\delta_j^{(1)}, \delta_j^{(2)} \in \mathbb{R}_+$ such that $\delta_j^{(1)} = \psi_i(\mathcal{N}', A', \omega, W^{(1)})$ and $\delta_j^{(2)} = \psi_i(\mathcal{N}', \tilde{A}', \tilde{\omega}, W^{(2)})$ for $i \in \mathcal{N}_j$. Clearly, $\delta_j^{(1)} = 0$ if $j \geq 2$ because of the dummy property. Hence, anonymity and efficiency imply $p_1 \delta_1^{(1)} = w_1$, and thus $\delta_1^{(1)} = \frac{w_1}{p_1}$. By construction, in instance $(\mathcal{N}, A', \tilde{\omega}, W^{(2)})$, there is no voter who approves of item 1. By the dummy property, this means $\delta_1^{(2)} = 0$. Because of additivity, we have

$$\delta_1 = \delta_1^{(1)} + \delta_1^{(2)} = \frac{w_1}{p_1} \tag{13}$$

In the second step, let $W^{(1)} = w_1 + w_2$ and $W^{(2)} = \sum_{j=3}^{s-1} w_j + w_s x_s$. The optimal solution of $(A', \omega, W^{(1)})$ is $x_1^{(1)} = x_2^{(1)} = 1$ and $x_j^{(1)} = 0$ for $j > 2$. The dummy property yields $\delta_j^{(1)} = 0$ for $j > 2$. This fact and efficiency imply

$$w_1 + w_2 = p_1 \delta_1^{(1)} + p_2 \delta_2^{(1)} \tag{14}$$

Note that there is no voter who approves of one of the items $\{1, 2\}$ in instance $(\mathcal{N}, A', \tilde{\omega}, W^{(2)})$. Thus, $\delta_j^{(2)} = 0$ for $j \in \{1, 2\}$; because of additivity, this means $\delta_j = \delta_j^{(1)}$ for $j \in \{1, 2\}$. In particular, with $\delta_1 = \frac{w_1}{p_1}$ (see (13)), this turns equation (14) into

$$\begin{array}{rrcl} & w_1 + w_2 & = & p_1 \frac{w_1}{p_1} + p_2 \delta_2 \\ \Leftrightarrow & \delta_2 & = & \frac{w_2}{p_2} \end{array}$$

Repeating this argumentation, after a total of $s-1$ steps we have $\delta_k = \frac{w_k}{p_k}$ for all $1 \leq k \leq s-1$. Considering the instance $(\mathcal{N}, A', \omega)$, from (12) we know that $1 = \sum_{k=1}^{s} p_k \delta_k$ holds (due to efficiency). Thus, we have $1 = \sum_{k=1}^{s-1} w_k + p_s \delta_s$. On the other hand, $1 = \sum_{k=1}^{s-1} w_k + w_s x_s$ holds because of the choice of x_s (see 2). Combining the two last equalities yields $p_s \delta_s = w_s x_s$, and thus $\delta_s = \frac{w_s x_s}{p_s}$. With (11), we have

$$\delta_j = \begin{cases} \frac{w_j}{p_j} & \text{for } j < s \\ \frac{w_s}{p_s} x_s & \text{for } j = s \\ 0 & \text{for } j > s \end{cases}$$

Hence, equation (9) and the definition of δ_j imply $\psi_i(\mathcal{N}, A, \omega) = \sum_{j \in A_i} \frac{w_j}{p_j} x_j$. I.e., ψ and ϕ^e coincide. $\square$

4 Conclusion

In this paper we have investigated cost sharing w.r.t. the continuous knapsack problem. Instead of costs or claims, we used the number of approvals to determine the optimal solution. To share the costs of the knapsack, we first introduced a whole family of cost sharing rules, and then provided explicit characterizations of two particular rules. The first rule assumed each item in the knapsack to impose the same cost, and made the individuals pay purely relative to their number of approved items. An interesting question in that respect would be to analyse the incentives to state one's true preferences. The second rule, however, was aware of both, the weight of the items in the knapsack and the number of individuals that approve of each item. It seems absolutely reasonable that those individuals who almost exclusively approve of items in the knapsack and/or approve of heavier items in the knapsack should carry a larger share of the cost. Based on various reasonable properties for continuous knapsack cost sharing rules, we provided characterization results for the two solution methods. Of course, the rules discussed in this paper are perhaps of an obvious kind, not taking too much care of the step of finding the optimal solution. However, many extensions seem possible and of interest for future research. On the one hand, further different - and probably less obvious - sharing rules could be introduced and analysed. On the other hand, more preference information, such as complete individual rankings, and - in addition - different types of objective functions could be used in the process of finding the optimal solution.

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