

Online Cake Cutting

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Abstract

We propose an online form of the cake cutting problem. This models situations where players arrive and depart during the process of dividing a resource. We show that well known fair division procedures like cut-and-choose and the Dubins-Spanier moving knife procedure can be adapted to apply to such online problems. We propose some desirable properties that online cake cutting procedures might possess like online forms of proportionality and envy-freeness, and identify which properties are in fact possessed by the different online cake procedures.

1 Introduction

Congratulations. Today is your birthday so you take a cake into the office to share with your colleagues. At tea time, people slowly start to arrive. However, as some people have to leave early, you cannot wait for everyone to arrive before you start sharing the cake. How do you proceed fairly?

This is an example of what we call an *online* cake cutting problem. Most previous studies of cake cutting procedures have assumed that all the players are available at the time of the division. Here, players arrive and depart (either with their cake or perhaps after they have eaten their cake) as the cake is being divided. Such online problems occur in the real world as in our birthday example, but also on the internet where agents are often connecting asynchronously.

Online cake cutting poses some new challenges. On the one hand, the online aspect of such problems makes fair division more difficult than in the offline case. How can we ensure that a player does not envy another player when we may have to distribute cake to the second player before the first player is present (and we can hope to determine information about their valuation function)? On the other hand, the online aspect of such problems may make fair division easier than in the offline case. If players don't see cake that has already been distributed before they arrive, perhaps they do not envy it?

2 Online cake cutting

As is common in the literature [2], we will often assume that each player is risk averse so they maximize the minimum value of the cake that they will receive, regardless of what the other players do. A risk averse player will not choose a strategy that could yield more value if it also entails the possibility of getting less value. We will also usually assume that each player is ignorant of the value functions of the other players. We discuss relaxing these assumptions in the conclusions.

We formulate cake cutting as dividing the unit interval between the different players, where each player has a (typically additive and continuous) valuation function on the intervals that they are allocated. We do not suppose that players assigns the same value to the whole cake. Although we can normalize the valuation functions, we shall see that is not necessary as all the cake cutting procedures are *scale invariant*. Depending on the application, we may demand that players receive a continuous slice of cake or some union of slices.

In an online cake cutting problem, the players arrive in some given order. Players are allocated their cake and then depart. The order in which players depart can be fixed or can be change according to how they value the cake. For example, the player present who most values a cut slice of cake might be the next to depart. Alternatively, the player to depart might always be the player who arrived the longest time ago. We will assume that at least one player departs before the last player arrives (otherwise we can formulate this as an offline cake cutting procedure). To prevent trivial allocations, we also assume that at least one player receives some cake. However, we do not assume that all players receive cake or that all the cake is allocated. Formally an online cake cutting problem is defined by a procedure which given the valuation functions of the players who are present in the room and the number of players who will take part in total either allocates some cake to one of the present players (who then departs) or indicates that we wait until the next player arrives. This can model both a fixed arrival and departure order, as well as one in which the order depends on the valuation functions.

An important dimension of online cake cutting is what is known and by whom about the total number of players. For example, the total number of players might be known by all players. On the other hand, the players might only know a bound on the total number of players (e.g. you've invited 20 work colleagues to share your birthday cake but not all of them might turn up). However, there are several other possibilities (e.g. certain players might have complete certainty about n whilst others have complete uncertainty). In addition, an interesting generalization is when cake is being allocated before the total number of players is fixed.

3 Desirable properties

What properties do we want from an online cake cutting procedure? The literature on cake cutting studies various notions of fairness like proportionality and envy freeness, as well as various forms of strategy proofness. The generalization of cake cutting to an online setting gives rise to some natural extensions of these notions.

3.1 Proportionality

A cake cutting procedure is *proportional* iff each of the n players assigns at least $\frac{1}{n}$ of the total value to their piece(s). Unfortunately, as we shall show, online cake cutting procedures cannot always be proportional. Suppose you only like icing. The problem is that you may not be able to prevent all the cake that is iced being distributed before you enter the room. We therefore consider weaker forms of proportionality that are achievable. One more limited form of proportionality is that any player receives a fair proportion of the cake that remains when they arrive. A cake cutting procedure is *forward proportional* iff each player assigns at least $\frac{r}{n-k}$ of the total value of the cake to their pieces where r is the fraction of the total value assigned by the player to the (remaining) cake when they arrive and k is the number of players who have already left at this point.

3.2 Envy freeness

A stronger notion of fairness is envy freeness. A cake cutting procedure is *envy free* iff no player values another player's pieces more than their own. Note that envy freeness implies proportionality but not vice versa. With online cake cutting, envy freeness is also impossible to achieve in general. We therefore consider weaker forms of envy freeness that are achievable. A cake cutting procedure is *forward envy free* iff no player values the pieces of cake allocated to other players after their arrival more than their own. Players can,

however, value the cake allocated to players who have already departed more than our own cake. This models situations where, for instance, we do not envy cake we don't see being allocated, or players eat their cake before departing and we do not envy cake that has already been eaten. Note that forward envy freeness implies forward proportionality but not vice versa. Similarly, envy freeness implies forward envy freeness but not vice versa. An even weaker form of envy freeness is when a player does not envy cake that is allocated to other players whilst they are in the room. A cake cutting procedure is *immediately envy free* iff no player values the pieces of cake allocated to another player after their arrival and before their departure more than their own. Note that forward envy freeness implies immediate envy freeness but not vice versa.

3.3 Equitability

Another fairness property is equitability. A cake cutting procedure is *equitable* iff all players assign the same value to the pieces of cake to which they are allocated (and so no player envies another's valuation). For 3 or more players, equitability and envy freeness can be incompatible [2]. Equitability is a difficult property to achieve, even more so in our online setting. Unlike proportionality or envy freeness, there seems little merit in considering weaker forms of equitability. Either all players assign the same value to their allocated cake or they do not. There is no advantage to ignoring the value of the cake allocated to players who have already departed.

3.4 Efficiency

Another important notion is efficiency. Efficiency is also called Pareto optimality. A cake cutting procedure is *Pareto optimal* iff there is no other allocation to the one returned that is more valuable for one player and at least as valuable for the others. Note that Pareto optimality does not in itself ensure fairness since allocating all the cake to one player is Pareto optimal. A cake cutting procedure is *weakly Pareto optimal* iff there is no other allocation to the one returned that is more valuable for all players. A cake cutting procedure that is Pareto optimal is weakly Pareto optimal but not vice versa.

3.5 Strategy proofness

Another consideration is whether players have an incentive to act truthfully. A cake cutting procedure is *weakly truthful* iff a player will do at least as well by telling the truth whatever valuations are held by the other players [1]. A stronger notion (often called strategy proofness in social choice) is that players must not be able to profit even when they know how others value the cake. As in [3], we say that a cake cutting procedure is *truthful* iff there are no valuations where a player will do better by lying.

4 Other properties

We consider some other properties of (online) cake cutting procedures.

4.1 Surjectivity

This property has been studied in social choice but appears to have received less attention in fair division. It indicates whether the cake can be divided in every possible way. A cake cutting procedure is *surjective* iff there are valuation functions for the players such that every possible partition of the cake into n pieces is possible. Note that this definition only

considers allocations where each player receives one continuous slice of cake. However, the definition of surjectivity could be easily extended to allocations where players can receive multiple slices. Our definition of surjectivity also ignores which player receives a particular slice. If an online cake cutting procedure is surjective, then there is an arrival ordering of the players and valuation functions such that any given player can receive a particular slice.

4.2 Scale invariance

Players may have different scales for their valuation functions. Scale invariance indicates that this is unimportant. A cake cutting procedure is *scale invariant* iff the allocation of cake is unchanged when a player's valuation is uniformly multiplied by a constant factor. It turns out that scale invariance is not difficult to achieve. Indeed, all the online cake cutting procedures we shall consider here are scale invariant.

4.3 Sequentiality

In some situations we may want cake to be cut from one end. This may be the case, for instance, when the cake represents time on a shared device. An online cake cutting procedure is *sequential* iff the slice given to any player is to the left of any slice given to a player who is later to depart.

4.4 Order monotonicity

A player's allocation of cake typically depends on their arrival order. We say that a cake cutting procedure is *order monotonic* iff a player's valuation of their cake does not decrease when they are moved earlier in the arrival ordering (and all other players have the same arrival ordering). Note that as the moved player can receive cake of greater value, players who depart after them may now receive cake with less value. A positive interpretation of order monotonicity is that players are encouraged to participate as early as possible. On the other hand, players who have to arrive late may receive less value.

5 General results

The fact that some players may depart before others arrive place some fundamental limitations on the fairness of online cake cutting procedures.

Theorem 1. *No online cake cutting procedure is proportional, envy free or equitable.*

Proof: Suppose the procedure is proportional. Then every player is allocated some cake. As the cake cutting procedure is online, at least one player departs before the final player arrives. Since the valuation function of the final player to arrive is not known when the first player departs, the cake allocated to the first player to depart cannot depend on the valuation function of the final player to arrive. Similarly, the valuation function of the final player to arrive cannot change who is the first player to depart. Consider the situation in which the final player to arrive has a valuation function that only values the cake allocated to the first player to depart. Whatever cake is allocated to the final player to arrive will be of no value to them. Hence the cake cutting procedure cannot be proportional.

Suppose the procedure is envy free. We consider the case where all players have valuation functions that assign some value to every slice. Every player is allocated some cake otherwise they will envy the players who are allocated cake (and by assumption a cake cutting procedure must allocate cake to at least one player). As before, the cake allocated to the first player to depart cannot depend on the valuation function of the final player to

arrive. We now modify the valuation function of the last player to arrive so that the value of the cake remaining when the first player departs is $\frac{1}{n^2}$ of the value it was before. Even if we allocate all the remaining cake to the last player to arrive, the value of this cake cannot now equal the value they assign to the cake allocated to the first player to depart. Hence the last player to arrive will envy the first player to depart. By a similar argument, the procedure cannot be equitable. ♡

Online cake cutting procedures can, however, possess many of the other properties.

Theorem 2. *Online cake cutting procedures can be forward proportional, forward envy free, weakly Pareto optimal, truthful, scale invariant, sequential and order monotonic.*

Proof: Consider the online cake cutting procedure which allocates all the cake to the first player to arrive. ♡

Unfortunately, allocating all the cake to one player is not very fair to the other players. We therefore consider some specific online cake cutting procedures which divide the cake more equitably. It remains an important open problem to identify natural axioms that these procedures satisfy which are not satisfied by the trivial allocation of all cake to one player.

6 Online Cut-and-Choose

The cut-and-choose procedure for two players dates back to antiquity. It appears nearly three thousand years ago in Hesiod's poem *Theogony* where Prometheus divides a cow and Zeus selects the part he prefers. Cut-and-choose is also enshrined in the United Nation's 1982 Convention of the Law of the Sea where it is put forward to divide the seabed for mining. In cut-and-choose, one player cuts the cake and the other takes the "half" that they most prefer. Cut-and-choose is proportional, envy free, Pareto optimal, weakly truthful, and surjective. However, it is not equitable, nor it is truthful.

We can use cut-and-choose as the basis of an online cake cutting procedure. The first player to arrive cuts the cake and waits for the next player to arrive. Either the next player to arrive chooses this piece and departs, or the next player to arrive declines this piece and the waiting player takes this piece and departs. If more players are to arrive, the remaining player cuts the cake and we repeat the process. Otherwise, the remaining player is the last player to be allocated cake and departs with whatever is left. We assume that all players know how many players will arrive.

Running Example:. *Suppose there are three players, the first player values only $[\frac{1}{2}, 1]$, the second player values only $[\frac{1}{3}, 1]$, and the third player values only $[0, \frac{3}{4}]$. We suppose that they uniformly value slices within these intervals. If we operate the online version of cut-and-choose, the first player will arrive and cut off the slice $[0, \frac{2}{3}]$ as they assign this slice $\frac{1}{3}$ the total value of the cake. The second player then arrives. As they assign this slice with $\frac{1}{2}$ the total value of the cake and they are only expecting $\frac{1}{3}$ of the total, the second player is happy to take this slice and depart. The first player then cuts off the slice $[\frac{2}{3}, \frac{5}{6}]$ as they assign this $\frac{1}{3}$ of the total value of the cake (and $\frac{1}{2}$ of the value remaining after the second player departed with their slice). The third player then arrives. As they assign the slice $[\frac{2}{3}, \frac{5}{6}]$ with all of the total value of the remaining cake and they are only expecting $\frac{1}{2}$ of whatever remains, the third player is happy to take this slice and depart. The first player now takes what remains, the slice $[\frac{5}{6}, 1]$. It can be claimed that everyone is happy as the first player received a "fair" proportion of the cake, whilst the other two players received slices that were of even greater proportional value to them.*

This online version of the cut-and-choose procedure has many (but not all) of the desirable properties described earlier.

Theorem 3. *The online cut-and-choose procedure is forward proportional, immediately envy free, weakly truthful, surjective, scale invariant and sequential. However, it is not proportional, (forward) envy free, equitable, (weakly) Pareto optimal, truthful or order monotonic.*

Proof: Consider the player cutting the cake. As they are risk averse, and as there is a chance that they will have to take the slice of cake that they cut, they will cut a slice that is at least $\frac{1}{k}$ of the total remaining value where k is the number of players still to be allocated cake. Similarly they will not cut a slice that is more than $\frac{1}{k}$ of the total remaining value for fear that the next player to arrive will take it, leaving behind cake that if it is divided proportionally gives them a slice of small value. Hence, the procedure is forward proportional and weakly truthful. It is also immediately envy free since each slice that the cutting player sees being allocated has the same value. To demonstrate surjectivity, consider the partition that allocates the i th player with the slice $[a_i, a_{i+1}]$ where $a_1 = 0$ and $a_{n+1} = 1$. We construct a valuation for the i th player ($i < n - 1$) that assigns a value 0 to $[0, a_i]$, a value 1 to $[a_i, a_{i+1}]$, a value 0 to $[a_{i+1}, a_{i+2}]$, a value $n - i$ to $[a_{i+2}, 1]$. For the $n - 1$ th player, we construct a valuation function that assigns a value 0 to $[0, a_{n-1}]$, and values of 1 to both $[a_{n-1}, a_n]$ and $[a_n, 1]$. Finally, we construct a valuation function for the n th player that assigns a value 0 to $[0, a_n]$, and a value of 1 to $[a_n, 1]$. With these valuation functions, the i th player gets the slice $[a_i, a_{i+1}]$. Finally, it is easy to see that the procedure is scale invariant and sequential.

To show that this procedure is not proportional, (forward) envy free, equitable, (weakly) Pareto optimal truthful or order monotonic consider 4 players and a cake in which the first player places a value of 3 units on $[0, \frac{1}{4}]$, 1 unit on $[\frac{1}{4}, \frac{3}{4}]$ and 8 units on $[\frac{3}{4}, 1]$, the second player places a value of 0 units on $[0, \frac{1}{4}]$, 4 units on $[\frac{1}{4}, \frac{1}{2}]$, 8 units on $[\frac{1}{2}, \frac{5}{8}]$, and 0 units on $[\frac{5}{8}, 1]$, the third player places a value of 6 units on $[0, \frac{1}{4}]$ 0 units on $[\frac{1}{4}, \frac{1}{2}]$, 1 unit on $[\frac{1}{2}, \frac{5}{8}]$, 2 units on $[\frac{5}{8}, \frac{3}{4}]$, and 3 units on $[\frac{3}{4}, 1]$, and the fourth player places a value of 0 units on $[0, \frac{1}{4}]$, 9 units on $[\frac{1}{4}, \frac{1}{2}]$, 1 unit on $[\frac{1}{2}, \frac{3}{4}]$, and 2 units on $[\frac{3}{4}, 1]$.

If we apply the online cut-and-choose procedure, the first player will cut off and keep the slice $[0, \frac{1}{4}]$, the second player will cut off and keep $[\frac{1}{4}, \frac{1}{2}]$, The third player will now cut the cake into two pieces: $[\frac{1}{2}, \frac{3}{4}]$ and $[\frac{3}{4}, 1]$. The fourth player will take the slice $[\frac{3}{4}, 1]$, leaving the third player with the slice $[\frac{1}{2}, \frac{3}{4}]$

The procedure is not proportional as the fourth player only receives $\frac{1}{6}$ of the total value of the cake, not (forward) envy free as the first player envies the fourth player, and not equitable as players receive cake of different value. The procedure is not (weakly) Pareto optimal as allocating the first player with $[\frac{3}{4}, 1]$, the second player with $[\frac{1}{2}, \frac{3}{4}]$, the third player with $[0, \frac{1}{4}]$, and the fourth player with $[\frac{1}{4}, \frac{1}{2}]$ gives all players a slice of greater value.

The procedure is not truthful as the second player can get a larger and more valuable slice by misrepresenting their preferences and cutting the cake into the slice $[\frac{1}{4}, \frac{5}{8}]$. Finally, the procedure is not order monotonic as the value of the cake allocated to the fourth player decreases from 2 units to $\frac{3}{2}$ units when they arrive before the third player. \heartsuit

7 Online moving knife

Another class of procedure for cutting cakes uses one or more moving knives. For example, in the Dubins-Spanier procedure for n players [6], a knife is moved across the cake from left to right. When a player shouts “stop”, the cake is cut and this player takes the piece to the left of the knife. The procedure then continues with the remaining $n - 1$ players until just one player is left (who takes whatever remains). This procedure is proportional but is not envy-free. However, only the first $n - 2$ players to be allocated slices of cake can be envious.

We can use the Dubins-Spanier procedure as the basis of an online moving knife procedure. The first k players ($k \geq 2$) to arrive perform one round of a moving knife procedure

to select a slice of the cake. Whoever chooses this slice, departs. At this point, if all players have arrived, we continue the moving knife procedure with $k - 1$ players. Alternatively the next player arrives and we start again a moving knife procedure with k players. As before, we assume that all players know how many players will arrive.

Running Example: Consider again the example in which there are three players, the first player values only $[\frac{1}{2}, 1]$, the second player values only $[\frac{1}{3}, 1]$, and the third player values only $[0, \frac{3}{4}]$. If we operate the online version of the moving knife procedure, the first two players will arrive and perform one round of the moving knife procedure. The second player will be the first to call “cut” and will depart with the slice $[0, \frac{5}{9}]$ (as this has $\frac{1}{3}$ of the total value of the cake for them). The third player will now arrive and perform a round of the moving knife procedure with the first player using the remaining cake, $[\frac{5}{9}, 1]$. The third player will be the first to call “cut” and will depart with the slice $[\frac{5}{9}, \frac{47}{72}]$ (as this has $\frac{1}{2}$ the total value of the remaining value for them). The first player will then depart with what remains, the slice $[\frac{47}{72}, 1]$. It can be claimed that everyone is happy as the second and third players received a “fair” proportion of the cake that was left when they first arrived, whilst the first player received an even greater proportional value.

This online version of the moving knife procedure has the same desirable properties as the online version of the cut-and-choose procedure.

Theorem 4. *The online moving knife procedure is forward proportional, immediately envy free, weakly truthful, surjective, scale invariant and sequential. However, it is not proportional, (forward) envy free, equitable, (weakly) Pareto optimal, truthful or order monotonic.*

Proof: Suppose j players ($j > 1$) have still to be allocated cake. Consider any player who has arrived. They will call “cut” as soon as the knife reaches $\frac{1}{j}$ of the value of the cake left for fear that they will receive cake of less value at a later stage. Hence, the procedure is weakly truthful and forward proportional. The procedure is also immediately envy free as they will assign less value to any slice that is allocated after their arrival and before their departure. To demonstrate surjectivity, consider the partition that allocates the i th player with the slice $[a_i, a_{i+1}]$ where $a_1 = 0$ and $a_{n+1} = 1$. We construct a valuation for the i th player ($i < n$) that assigns a value 0 to $[0, a_i]$, a value 1 to $[a_i, a_{i+1}]$, a value $n - i$ to $[a_{i+1}, 1]$. Finally, we construct a valuation function for the n th player that assigns a value 0 to $[0, a_n]$, and a value of 1 to $[a_n, 1]$. With these valuation functions, the i th player gets the slice $[a_i, a_{i+1}]$. Finally, it is easy to see that the procedure is scale invariant and sequential.

To show that this procedure is not proportional, (forward) envy free, equitable, (weakly) Pareto optimal truthful consider again the example with 4 players used in the last proof. We suppose that $k = 2$ (i.e. at any one time, two players are watching the knife). The first player calls “cut” and departs with the slice $[0, \frac{1}{4}]$. The second player calls “cut” and departs with the slice $[\frac{1}{4}, \frac{1}{2}]$. Finally, the third player calls “cut” and departs with the slice $[\frac{1}{2}, \frac{2}{4}]$, leaving the fourth player with the slice $[\frac{3}{4}, 1]$.

The procedure is not proportional as the fourth player only receives $\frac{1}{6}$ of the total value of the cake, not (forward) envy free as the first player envies the fourth player, and not equitable as players receive cake of different value. The procedure is not (weakly) Pareto optimal as allocating the first player with $[\frac{3}{4}, 1]$, the second player with $[\frac{1}{2}, \frac{3}{4}]$, the third player with $[0, \frac{1}{4}]$ and the fourth player with $[\frac{1}{4}, \frac{1}{2}]$ gives all players a slice of greater value.

The procedure is not truthful as the second player can get a larger and more valuable slice by misrepresenting their preferences and not calling “cut” until the knife is about to reach $\frac{5}{8}$ th of the way along the cake.

Finally, to show that the procedure is not order monotonic consider 3 players and a cake in which the first player places a value of 2 units on each of $[0, \frac{1}{3}]$, $[\frac{1}{3}, \frac{2}{3}]$, and $[\frac{2}{3}, 1]$, the second player places a value of 0 units on $[0, \frac{1}{3}]$, 3 units on each of $[\frac{1}{3}, \frac{2}{3}]$ and $[\frac{2}{3}, 1]$, and the

third player places a value of 2 units on $[0, \frac{1}{6}]$, 0 units on each of $[\frac{1}{6}, \frac{1}{3}]$ and $[\frac{1}{3}, \frac{2}{3}]$, and 4 units on $[\frac{2}{3}, 1]$. As before, we suppose that $k = 2$ (i.e. at any one time, two players are watching the knife). The first player calls “cut” and departs with the slice $[0, \frac{1}{6}]$. The second player calls “cut” and departs with the slice $[\frac{1}{3}, \frac{2}{3}]$, leaving the third player with the slice $[\frac{2}{3}, 1]$. On the other hand, if the third player arrives ahead of the second player then the value of the cake allocated to them drops from 4 units to 2 units. Hence the procedure is not order monotonic. ♡

8 Online Mark-and-Choose

A possible drawback of both of the online cake cutting procedures proposed so far is that the first player to arrive can be the last player to depart. What if we want a procedure in which players can depart soon after they arrive? The next procedure has such a property. Players will depart as soon as the next player arrives (except for the last player to arrive who takes whatever cake remains). However, the new procedure is no longer sequential. It may not allocate cake from one end. In addition, the new procedure does not necessarily allocate continuous slices of cake.

In the online mark-and-choose procedure, the first player to arrive marks the cake into n pieces. The second player to arrive selects one piece to give to the first player who then departs. The second player then marks the remaining cake into $n - 1$ pieces and waits for the third player to arrive. The procedure repeats in this way until the last player arrives. The last player to arrive selects which of the two halves marked by the penultimate player should be allocated to the penultimate player. The last player then takes whatever remains.

Running Example: *Consider again the example in which there are three players, the first player values only $[\frac{1}{2}, 1]$, the second player values only $[\frac{1}{3}, 1]$, and the third player values only $[0, \frac{3}{4}]$. If we operate the online version of the mark-and-choose procedure, the first player will arrive and mark the cake into 3 equally valued pieces: $[0, \frac{2}{3}]$, $[\frac{2}{3}, \frac{5}{6}]$, and $[\frac{5}{6}, 1]$. The second player then arrives and selects the least valuable piece for the first player to take. In fact, both $[\frac{2}{3}, \frac{5}{6}]$ and $[\frac{5}{6}, 1]$ are each worth $\frac{1}{4}$ of the total value of the cake to the second player. They will therefore choose between them arbitrarily. Suppose the second player decides to give the slice $[\frac{2}{3}, \frac{5}{6}]$ to the first player. Note that the first player assigns this slice with $\frac{1}{3}$ of the total value of the cake. This leaves behind two sections of cake: $[0, \frac{2}{3}]$ and $[\frac{5}{6}, 1]$. The second player then marks what remains into two equally valuable pieces: the first is the interval $[0, \frac{7}{12}]$ and the second contains the two intervals $[\frac{7}{12}, \frac{2}{3}]$ and $[\frac{5}{6}, 1]$. The third player then arrives and selects the least valuable piece for the second player to take. The first piece is worth $\frac{7}{12}$ of the total value of the cake to the third player. As this is over half the total value, the other piece must be worth less. In fact, the second piece is worth $\frac{1}{4}$ of the total value. The third player therefore gives the second piece to the second player. This leaves the third player with the remaining slice $[0, \frac{7}{12}]$. It can again be claimed that everyone is happy as the first players received a “fair” proportion of the cake that was left when they arrived, whilst both the second and third player received an even greater proportional value.*

This procedure again has the same desirable properties as the online version of the cut-and-choose and moving knife procedures.

Theorem 5. *The online mark-and-choose procedure is forward proportional, immediately envy free, weakly truthful, surjective, and scale invariant. However, it is not proportional, (forward) envy free, equitable, (weakly) Pareto optimal, truthful, order monotonic or sequential.*

Proof: Any player marking the cake will divide it into slices of equal value (for fear that they will be allocated one of the less valuable slices). Similarly, a player selecting a slice for

another player will select the slice of least value to them (to maximize the value that they will receive next). Hence, the procedure is weakly truthful and forward proportional. The procedure is also immediately envy free as they will assign less value to the slice that they select for the departing player than the value of the slices that they mark. To demonstrate surjectivity, consider the partition that allocates the i th player with the slice $[a_i, a_{i+1}]$ where $a_1 = 0$ and $a_{n+1} = 1$. We construct a valuation for the i th player ($i < n$) that assigns a value 0 to $[0, a_i]$, a value 1 to $[a_i, a_{i+1}]$, a value $n - i$ to $[a_{i+1}, 1]$. Finally, we construct a valuation function for the n th player that assigns a value 0 to $[0, a_n]$, and a value of 1 to $[a_n, 1]$. With these valuation functions, the i th player gets the slice $[a_i, a_{i+1}]$. Finally, it is easy to see that the procedure is scale invariant.

To show that this procedure is not proportional, (forward) envy free, equitable, (weakly) Pareto optimal or truthful consider again the example with 4 players used in the last two proofs. The first player marks and is assigned the slice $[0, \frac{1}{4}]$ by the second player. The second player then marks and is assigned the slice $[\frac{1}{4}, \frac{1}{2}]$. The third player then marks and is assigned the slice $[\frac{1}{2}, \frac{3}{4}]$, leaving the fourth player with the slice $[\frac{3}{4}, 1]$.

The procedure is again not proportional as the fourth player only receives $\frac{1}{6}$ of the total value of the cake, not (forward) envy free as the first player envies the fourth player, and not equitable as players receive cake of different value. The procedure is not (weakly) Pareto optimal as allocating the first player with $[\frac{3}{4}, 1]$, the second player with $[\frac{1}{2}, \frac{3}{4}]$, the third player with $[0, \frac{1}{4}]$, and the fourth player with $[\frac{1}{4}, \frac{1}{2}]$ gives all players a slice of greater value.

The procedure is not truthful as the second player can get a larger and more valuable slice by misrepresenting their preferences and marking the cake into the slices $[\frac{1}{4}, \frac{5}{8}]$, $[\frac{5}{8}, \frac{3}{4}]$, and $[\frac{3}{4}, 1]$. In this situation, the third player will allocate the second player with the slice $[\frac{1}{4}, \frac{5}{8}]$ which is of greater value to the second player. It is also easy to see that the procedure is not sequential.

Finally, to show that the procedure is not order monotonic consider 3 players and a cake in which the first player places a value of 4 units on each of $[0, \frac{1}{3}]$, $[\frac{1}{3}, \frac{2}{3}]$ and $[\frac{2}{3}, 1]$, the second player places a value of 0 units on $[0, \frac{1}{3}]$, 6 units on $[\frac{1}{3}, \frac{2}{3}]$, and 3 units on each of $[\frac{2}{3}, \frac{5}{6}]$, and $[\frac{5}{6}, 1]$, and the third player places a value of 2 unit on $[0, \frac{1}{6}]$, 0 units on each of $[\frac{1}{6}, \frac{1}{3}]$ and $[\frac{1}{3}, \frac{2}{3}]$, and 5 units on each of $[\frac{2}{3}, \frac{5}{6}]$ and $[\frac{5}{6}, 1]$. The first player marks and is allocated the slice $[0, \frac{1}{3}]$. The second player marks and is allocated the slice $[\frac{1}{3}, \frac{2}{3}]$, leaving the third player with the slice $[\frac{2}{3}, 1]$. On the other hand, suppose the third player arrives ahead of the second player. In this case, the third player marks the cake into two slice, $[\frac{1}{3}, \frac{5}{6}]$ and $[\frac{5}{6}, 1]$. The second player allocates the third player the slice $[\frac{5}{6}, 1]$. Hence, the value of cake allocated to the third player drops from 10 units to 5 units when they go second in the arrival order. Hence the procedure is not order monotonic. \heartsuit

9 Bounded number of players

One variation of online cake cutting is when the number of players is not known but all players have the (same) upper bound, n_{max} on the number of persons to be allocated cake. We consider three cases: players know their arrival position and when the last player arrives; players do not know their arrival position but do know when the last player arrives; players do not know when the last player arrives.

9.1 Known arrival order and last player

In this case, each player knows how many players have arrived before them, and players know when no more players are to arrive. In this case, we can still operate the online cut-and-choose procedure. Given the risk averse nature of the players, each player will cut off a

slice of cake of value $\frac{1}{n_{max}-k}$ of the total where k is the number of players who have already been allocated cake.

9.2 Unknown arrival order but known last player

In this case, players do not know how many players have arrived before them, but do know when no more players are to arrive. We can again operate the online cut-and-choose procedure. The first player will cut off a slice of cake of value $\frac{1}{n_{max}-k}$ of the total where k is the number of players already allocated cake (e.g. in the first round, the first player cuts off a slice of value $\frac{1}{n_{max}}$ of the total, if this is accepted by the second player, they then cut off a slice of value $\frac{1}{n_{max}-1}$ of the total, and so on).

We can suppose that the second player to arrive will look at the cake and deduce they are the second player to arrive (since they will assign the total value of the cake to the two pieces). If they are not the last player to arrive, they will accept the offered slice if it is greater than or equal to $\frac{1}{n_{max}}$ of the total. If they are the last player to arrive, they will accept the offered slice if it is greater than or equal to $\frac{1}{2}$ of the total. Otherwise, if there are no more players are to arrive, they will take whatever cakes remain. If there are more players to arrive, they will cut off a new slice of value $\frac{1}{n_{max}-j}$ of the total where j is the number of players already allocated cake (e.g. the second player first cuts off a slice of value $\frac{1}{n_{max}-1}$ of the total, if this is accepted by the next player to arrive, the second player then cuts off a slice of value $\frac{1}{n_{max}-2}$ of the total, and so on).

We can suppose that the third (or any later) player to arrive can only deduce that they are not the first or second player to arrive. If they are not the last player to arrive, they will accept the offered slice if it is greater than or equal to $\frac{1}{n_{max}-1}$ of the total. If they are the last player to arrive, they will accept the offered slice if it is greater than or equal to $\frac{1}{2}$ of the total. Otherwise, if there are no more players are to arrive, they will take whatever cakes remain. If there are more players to arrive, they will cut off a new slice of value $\frac{1}{n_{max}-j}$ of the total where j is the number of players already allocated cake (e.g. they first cut off a slice of value $\frac{1}{n_{max}-2}$ of the total, if this is accepted by the next player to arrive, they then cut off a slice of value $\frac{1}{n_{max}-3}$ of the total, and so on).

9.3 Unknown last player

In the third case, players do know when no more players are to arrive. We now have a potential deadlock problem in operating the online cut-and-choose procedure. We need some mechanism to ensure that the last player to arrive is allocated cake. One option is to introduce a clock. If a player waits longer than a certain time, then they can take whatever cake remains. With this modification, we can again operate the online cut-and-choose procedure.

9.4 Moving knife procedures

We can also use the online moving knife procedure when there is only a bound on the number of players to be allocated cake. The results are very similar to the online cut-and-choose procedure, and depend on whether players know when the last player arrives and on whether players know how many players have been allocated cake before them.

10 Related work

There is an extensive literature on fair division and cake cutting procedures. See, for instance, [2] for an introduction. There has, however, been considerably less work on fair division problems similar to those considered here.

Thomson considers a generalization of fair division problems where the number of players may increase [7]. He explores from an axiomatic perspective whether it is possible to have a procedure in which players' allocations are monotonic (i.e. their values do not increase as the number of players increase) combined with other common properties like weak Pareto optimality.

Cloutier *et al.* consider a different generalization of the cake cutting problem in which the number of players is fixed but there are multiple cakes [5]. This can model situations where, for example, players wish to choose shifts across multiple days. Note that this problem can be reduced to multiple single cake cutting problems unless the players' valuations across cakes are linked (e.g. you prefer the same shift each day compared to different shifts).

A number of authors have studied distributed mechanisms for fair division (see, for example, [4]). In such mechanisms, players typically agree locally on deals to exchange some of the goods in their possession. The usual goal is to identify conditions under which the system converges to a fair or envy free allocation.

11 Conclusions

We have proposed an online form of the cake cutting problem. This permits us to explore the concept of fair division when players arrive and depart during the process of dividing a resource. It can be used to model situations, such as on the internet, when we need to divide resources asynchronously. There are many possible future directions for this work. One extension would be to indivisible goods. Another extension would be to undesirable goods (like chores) where we want as little of them as possible. In addition, it would be interesting to consider variants of the online cake cutting problem where players have information about the valuation functions of the other players.

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