Approximate Judgement Aggregation

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Abstract

We analyze judgement aggregation problems in which a group of agents independently votes on a set of complex propositions that has some interdependency constraint between them (e.g., transitivity when describing preferences). We generalize the current results by studying approximate judgment aggregation. That is, we relax the main two constraints assumed in the current literature. We relax the consistency constraint by measuring the fraction of inputs for which an aggregation mechanism returns an inconsistent result and we relax the independence constraint by defining a measure for the dependance of the aggregation for an issue on the votes on other issues. We define the problem of measuring the impact of such small relaxation on the class of satisfying aggregation mechanisms and raise the question of whether there exists an agenda for which the expansion of this class is non-trivial. We show that the recent works for preference aggregation of Kalai and Mossel fit into this framework. We prove that, as in the case of preference aggregation, in the case of a subclass of premise-conclusion agendas, the set of satisfying aggregation mechanisms does not extend non-trivially when relaxing the constraints.
A corollary from our result for the xor premise-conclusion agenda is a generalization of the classic result for local property testing of linearity of boolean functions.

Keywords: approximate aggregation, discursive dilemma, premise-conclusion agenda, inconsistency index, dependency index

1 Introduction

Assume a committee of three referees needs to review a paper for a conference. Each of the referees judges the paper individually for originality and for quality (assumed to be pass/fail questions) and approves the paper only if it passes both criteria. The three referees cast their votes simultaneously and we assume no strategic behavior on their behalf. Now assume that both the first and second referee think that the paper is original enough and both the second and third referee think it stands in the quality standards of the conference. Then we have that although a minority of the committee (one out of three) thinks the paper should pass, for each issue separately there is a supporting majority (two out of three). This discrepancy between the majority vote on premises (quality and originality) and the majority vote on the conclusion (pass) was presented by Kornhauser and Sager in 1986[13] and was later named 'The Doctrinal Paradox'. Such discrepancy phenomena can happen when the 'accepted opinions' is restricted to be other sets as well (e.g., Condorcet Paradox for preference aggregation) and is the subject of a growing body of works in economics, political science, philosophy, law, and other related disciplines. (A survey of this field can be found in [14])

Abstract aggregation can be formalized in the following way. There is a committee of n individuals (also called voters) that needs to decide on m boolean issues (that is, each question has exactly two possible answers `True` and `False`[1]). Each individual holds an **opinion** which is an answer for each of the issues. We denote the answer of the $i^{\underline{th}}$ voter

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[1] There is some literature also on aggregating non-boolean issues, e.g., [20] and [7], but this is outside the scope of this paper.

for the j$^{\underline{\text{th}}}$ issue by X_i^j and the vector of all opinions in the committee (called **profile**) by $X \in (\{0,1\}^m)^n$ (For the ease of presentation we will identify `True` with 1 and `False` with 0). Like in the example above, not all opinions are acceptable (one cannot accept a non-original paper). We assume a non-empty set $\mathbb{X}$ of $\{0,1\}^m$ called the **agenda** is given. The opinions in $\mathbb{X}$ are called the **consistent** opinions and only these opinions are held by voters[2]. For instance the **conjunction agenda**, which is the agenda described in the example, is defined to be the set $\{000, 010, 100, 111\}$[3]. Another example is the **preference agenda**. In this agenda the consistent opinions represent the linear orders over a set of candidates $\{c_1, c_2, \ldots, c_s\}$ and the issues are the $\binom{s}{2}$ pair-wise comparisons between candidates[4,5].

An aggregation mechanism is a function that defines for any profile the **aggregated opinion** ($F : (\{0,1\}^m)^n \to \{0,1\}^m$). There are two desired properties for aggregation mechanism, independence and consistency. **Independence** states that the aggregated opinion on the j$^{\underline{\text{th}}}$ issue, $F^j(X)$ depends solely on the opinions on that issue X^j. **Consistency** of the aggregation mechanism states that whenever all the members of the committee hold consistent opinions, i.e., $X \in \mathbb{X}^n$, F returns a consistent opinion as well, i.e., $F(X) \in \mathbb{X}$.

For instance, issue-wise majority satisfies independence but also, as can be seen in the accept-paper example, might lead to an inconsistent result for the conjunction agenda and hence does not satisfy consistency. Similarly, the Condorcet Paradox[15] shows that, for the preference agenda, issue-wise majority might lead to an inconsistent result. The natural question is whether one can find other aggregation mechanisms that satisfy independence and consistency. Answering this question, Arrow's theorem[1] shows that (under mild and natural constraint[6]) the only aggregation mechanisms that satisfy independence and consistency are the dictatorships. For other agendas one can find similar theorems that characterize the class of consistent and independent aggregation mechanism to be a very small and unnatural class. For instance, for the conjunction agenda (under the same mild and natural constraint[6]) the only aggregation mechanisms that satisfy independence and consistency are the oligarchies (The oligarchy of a coalition S returns for each issue `True` if all voters in S voted `True` for that issue). In a recent work Dokow and Holzman ([5],[6]) proved a generalization of these results characterizing the set of consistent and independent aggregation mechanism for several large families of agendas.

Lately there is a series of works coping with impossibility results in Social Choice using approximations (e.g., [11] and [10]). The version of approximation we define in this work is studying independence aggregation mechanisms that are almost consistent in the sense that they return a consistent aggregated opinion for the vast majority of the inputs [7]. We quantify being almost consistent by defining the **inconsistency index**.

Definition 1.1 (Inconsistency Index).
For an agenda $\mathbb{X}$ and an aggregation mechanism F for that agenda, the inconsistency index

[2]For instance those might be the legal opinions, logic consistent opinions, or rational according to other criteria so one can assume that any 'reasonable' individual should hold only consistent opinions.

[3]I.e., the third bit ia a conjunction of the first two.

[4]For instance, for $s = 3$ the issues are '$c_1 \overset{?}{\geqslant} c_2$', '$c_2 \overset{?}{\geqslant} c_3$', and '$c_3 \overset{?}{\geqslant} c_1$' and the consistent opinions are $\{001, 010, 100, 110, 101, 011\}$.

[5]A related model that can be found in the literature is 'Judgement Aggregation'. In this model the issues are logical propositions over a set of variables and a consistent opinion is an assignment to these variables (so not every combination of truth values for the proposition is achievable). From our perspective the model we describe is more general since we allow any agenda. Dokow and Holzman[5] proved that the two models are equivalent in the sense that each set of consistent opinions can be described using a proposition set (although not uniquely).

[6]Pareto - Whenever all the voters hold the same opinion, this is the aggregated opinion.

[7]In most of this work we leave the independence constraint intact and relax the consistency constraint. However, as we show in section 6, one can relax the independence constraint as well and get similar results.

is defined to be the probability to get an inconsistent result.

$$IC^{\mathbb{X}}(F) = \Pr\left[F(X) \notin \mathbb{X} \mid X \in \mathbb{X}^n\right]$$

assuming uniform probability over the inputs.

This definition assumes a uniform distribution over the opinions for each voter and that voters draw their opinions independently (**Impartial Culture Assumption**). This assumption, while certainly unrealistic, is the natural choice in this kind of work and is discussed further in section 2.

In addition we use the usual Hamming distance between two aggregation mechanisms ($d^{\mathbb{X}}(F, G) = \Pr\left[F(X) \neq G(X) \mid X \in \mathbb{X}^n\right]$) and derive from it a distance between an aggregation mechanism and a collection of aggregation mechanisms ($d^{\mathbb{X}}(F, \mathcal{G}) = \min_{G \in \mathcal{G}} d^{\mathbb{X}}(F, G)$).

It is easy to see that when F is close to G and G is consistent, F is close to being consistent, i.e., $IC(F)$ is small. Our main question is whether there are other aggregation mechanisms that are close to being consistent (Formally, $IC^{\mathbb{X}}(F) \leqslant d^{\mathbb{X}}(F, G)$)).

For the preference agenda, recent works of Kalai[12] and Mossel[17] prove such bounds

Theorem ([12])**.** *There exists an absolute constant K such that the following holds: For any $\epsilon > 0$ and any aggregation mechanism F for the preference agenda over 3 candidates that satisfies: F is balanced[8], F is independent, and $IC(F) < K\epsilon$, there exists an aggregation mechanism G that satisfies consistency and independence such that $d(F, G) < \epsilon$.*

In this paper we prove similar theorems for a family of agendas: premise-conclusion agendas in which every issue is either a premise or a conclusion of at most two premises. In a premise-conclusion agendas the issues are divided into two types: premises and conclusions. Each conclusion j is characterized by a boolean function Φ_j over the premises and an opinion is consistent if the answers to the conclusion issues are attained by applying the function Φ_j on the answers to the premise issues.

$$\mathbb{X} = \left\{x \in \{0,1\}^m \mid x^j = \Phi^j(\text{premises}) \quad \text{for every conclusion issue } j\right\}$$

For instance the conjunction agenda is a premise-conclusion agenda with two premises and one conclusion and we mark this by notating the agenda as $\langle A, B, A \wedge B\rangle$. In some cases the division to premises and conclusion might be non-unique. For instance for the xor agenda $\mathbb{X} = \{001, 010, 100, 111\}$ one can define it as a premise-conclusion agenda both as $\langle A, B, A \oplus B\rangle$ and as $\langle A, A \oplus C, C\rangle$.

The main result of this paper is:

Theorem (Theorem 4.1)**.** *For any $\epsilon > 0$ and $n \geqslant 1$, there exists $\delta = poly\left(\frac{1}{n}, \epsilon\right)$, such that for every premise-conclusion agenda in which each issue is either a premise, or a conclusion of at most two premises, if F is an aggregation mechanism for $\mathbb{X}$ over n voters satisfying independence and $IC(F) < \delta$, then there exists an aggregation mechanism G that satisfies consistency and independence such that $d(F, G) < \epsilon$.*
Moreover, one may take $\delta = Cn^{-2}\epsilon^5$ for some absolute constant C.

From the theorem it follows that, whenever the inconsistency index of is small enough ($O(n^{-7})$), the distance to the class of independent consistent mechanisms is small too (poly(n)-small. I.e., bounded from above by one over a polynomial of n) and hence proves that for these agendas the class of satisfying aggregation mechanisms does not expand much when relaxing the consistency constraint.

[8]For every pair of candidates, a and b, it holds that the probability that F ranks a above b is exactly 1/2.

The general statement follows easily from the analysis of three basic cases: The conjunction agenda $\langle A, B, A \wedge B \rangle$, the xor agenda $\langle A, B, A \oplus B \rangle$, and the id agenda $\langle A, A \rangle$

We use two different techniques in the proofs. For the conjunction agenda we study influence measures[9] of voters on the issue-aggregating functions and for the xor agenda we use Fourier analysis of the issue-aggregating functions.[10]

Notice the question of approximate aggregation has a close relation to the field of local property testing. In this field we query a function at a small number of (random) points testing for a global property (In our case the property is being a consistent independent aggregation mechanism). And indeed one can see our characterization for the xor agenda as a generalization of the result of Blum, Luby, and Rubinfeld ([3], [2]) that shows that a function f that passes the linearity test with high probability[11] is close to linear.

An open question is whether one can find such bounds for any agenda or whether there exists an agenda for which the class of aggregation mechanisms that satisfy consistency and independence expands non trivially when we relax the consistency and independence constraints.

We proceed to describe the structure of the current paper. In Section 2 we describe the formal model of aggregation mechanisms. In section 3 we give the two main examples we deal with, preference aggregation and premise-conclusion aggregation. In section 4 we state the motivation to deal with approximate aggregation, we describe the known results for preference approximate aggregation by Kalai and Mossel and state our main result for approximate aggregation for premise-conclusion agendas. In sections 5 we outline the proof of the main theorem. In section 6 we define a measure that relaxes the independence constraint and show that any result for approximate aggregation for independent aggregation mechanisms (which is the case in our main theorem) can be translated to the more general definition relaxing both constraints. Section 7 concludes.

2 The model

We define the model similarly to [5] (which is Rubinstein and Fishburn's model [20] for the boolean case)

We consider a **committee** of n individuals that needs to decide on m issues. An **opinion** is a vector $x = (x_1, x_2, \ldots, x_m) \in \{0,1\}^m$ denoting an answer to each of the issues. An opinion **profile** is a matrix $X \in \left(\{0,1\}^m\right)^n$ denoting the opinions of the committee members so an entry X_i^j denotes the vote of the $\text{i}^{\underline{\text{th}}}$ voter for the $\text{j}^{\underline{\text{th}}}$ issue, the $\text{i}^{\underline{\text{th}}}$ row of it X_i states the votes of the $\text{i}^{\underline{\text{th}}}$ individual on all issues, and the $\text{j}^{\underline{\text{th}}}$ column of it X^j states the votes of each of the individuals on the $\text{j}^{\underline{\text{th}}}$ issue. In addition we assume that an **agenda** $\mathbb{X} \in \{0,1\}^m$ of the **consistent** opinions is given.

The basic notion in this field is an **aggregation mechanism** which is a function that returns an **aggregated opinion** (not necessarily consistent) for every profile $\left(F : \left(\{0,1\}^m\right)^n \to \{0,1\}^m\right)$[12].

An aggregation mechanism satisfies **Independence** (and we say that the mechanism is **independent**) if for any two consistent profiles X and Y and an issue j, if $X^j = Y^j$ (all individuals voted the same on the $\text{j}^{\underline{\text{th}}}$ issue in both profiles) then $(F(X))^j = (F(Y))^j$ (the aggregated opinion for the $\text{j}^{\underline{\text{th}}}$ issue is the same for both profiles). This means that F

[9] Both the known influence (Banzhaf power index) and a new measure we define:The ignorability of a voter.

[10] The proof for the id case is trivial.

[11] which is equivalent to that the aggregation mechanism for $\langle A, B, A \oplus B \rangle$ that uses f for each of the issues has small inconsistency index.

[12] We define the function for all profiles for simplicity but we are not interested in the aggregated opinion in cases one of the voters voted an inconsistent opinion.

satisfies independence if one can find m boolean functions $f^1, f^2, \ldots, f^m : \{0,1\}^n \to \{0,1\}$ s.t. $F(X) \equiv \left(f^1(X^1), f^2(X^2), \ldots, f^m(X^m)\right)$. Notice this property is a generalization of the IIA property for social welfare functions (aggregation mechanism for the preference agenda) so a social welfare function satisfies IIA iff it satisfies independence as defined here (when the issues are the pair-wise comparisons). An independent aggregation mechanism satisfies **systematicity** if $F(X) = \left\langle f(X^1), \ldots, f(X^m)\right\rangle$ for some issue aggregating function, i.e., all issues are aggregated using the same function. We will use the notation $\left\langle f^1, f^2, \ldots, f^m\right\rangle$ for the independent aggregation mechanism that aggregates the $j^{\underline{th}}$ issue using f^j.

The main measure we study in this paper is the **inconsistency index** $IC^{\mathbb{X}}(F)$ of a given aggregation mechanism F and a given agenda $\mathbb{X}$ (as defined in the introduction). This measure is a relaxation of the **consistency** criterion that is usually assumed in current works[13]. We defining this measure by

$$IC^{\mathbb{X}}(F) = \Pr\left[F(X) \notin \mathbb{X} \mid X \in \mathbb{X}^n\right]$$

assuming uniform distribution of the profiles. In cases the context is clear we omit the agenda and notate it by $IC(F)$.

This definition includes two major assumptions on the opinion profile distribution. First, we assume the voters pick their opinions independently and from the same distribution. Second, we assume a uniform distribution over the (consistent) opinions for each voter(**Impartial Culture Assumption**). The uniform distribution assumption, while certainly unrealistic, is the natural choice for proving 'lower bounds' on $IC(F)$. That means, proving results of the format 'Every aggregation mechanism of a given class has inconsistency index of at least ...'. In particular, the lower bound, up to a factor δ, applies also to any distribution that gives each preference profile at least a δ fraction of the probability given by the uniform distribution. Note that we cannot hope to get a reasonable bound result for every distribution. For instance, since for every aggregation mechanism we can take a distribution on profiles for which it returns a consistent opinion.

2.1 Boolean Functions

Since this work deals with binary functions (for aggregating issues), we need to define several notions for this framework as well. To ease the presentation, throughout this paper we will identify `True` with 1 and `False` with 0 and use logical operators on bits and bit vectors (using entry-wise semantics).

Let $f : \{0,1\}^n \to \{0,1\}$ be a boolean function. f is the **oligarchy** of a coalition S if it is of the form: $f(x) = \prod_{i \in S} x_i$. This means that f returns 1 if all the members of S voted 1. We denote by **`Olig`** the class of all 2^n oligarchies. Two special cases of oligarchies are the constant 1 function which is the oligarchy of the empty coalition and the dictatorships which are oligarchies of a single voter.

f is a **linear** function if is it of the form $f(x) = \bigoplus_{i \in S} x_i$ for some coalition S[14]. This means that f returns 1 if an even number of the members of S voted 1. We denote by **`Lin`** the class of all 2^n linear functions. Two special cases of linear functions are the constant 1 function which is the xor function over the empty coalition and the dictatorships which are xor of a single voter.

We say that f satisfies the **Pareto** criterion is $f(\bar{0}) = 0$ and $f(\bar{1}) = 1$[15]. I.e., when all

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[13] F satisfies consistency if $IC(F) = 0$.

[14] An equivalent definition is: $\forall x, y :\ f(x) + f(y) = f(x + y)$ when the addition is in $\mathbb{Z}_2$ and $\mathbb{Z}_2^n$, respectively.

[15] In the literature this criterion is sometimes referred to as Unanimity, e.g., in [14]. We choose to follow [6] and refer to it as Pareto to distinguish between it and the unanimity function which is the oligarchy of $\{1, 2, \ldots, n\}$.

the individuals voted unanimously 0 then f should return 0 and similarly for the case of 1.

We define two different measures for the influence of an individual on a function $f : \{0,1\}^n \to \{0,1\}$. Both definitions use the uniform distribution over $\{0,1\}^n$ (which is consistent with the assumption we have on the profile distribution).

- The **influence**[16] of a voter i on f is defined to be the probability that he can flip the result by changing his vote.

$$I_i(f) = \Pr[f(x) \neq f(x \oplus e_i)]$$

($x \oplus e_i$: e_i = the $\text{i}^{\underline{\text{th}}}$ elementary vector. It is equivalent to flipping the $\text{i}^{\underline{\text{th}}}$ bit $0 \leftrightarrow 1$)

- The (zero-)**ignorability** of a voter i on f is defined to be the probability that f returns 1 when i voted 0.

$$P_i(f) = \Pr[f(x) = 1 \mid x_i = 0)]$$

(We did not find a similar index defined in the voting literature or in the cooperative games literature).

In addition we define a distance function over the boolean functions. The distance between two functions $f, g : \{0,1\}^n \to \{0,1\}$ is defined to be the probability of getting a different result (normalized Hamming distance). $d(f,g) = \Pr[f(x) \neq g(x)]$. From this measure we will derive a distance from a function to a set of functions by $d(f,\mathcal{G}) = \min_{g \in \mathcal{G}} d(f,g)$

One more notation we are using in this paper is x_J for a binary vector $x \in \{0,1\}^n$ and a coalition $J \subseteq \{1,2,\ldots n\}$ for notating the entries of x that correspond to J.

3 Agenda Examples

A lot of natural problems can be formulated in the framework of aggregation mechanisms. In this paper we concentrate on two examples: (strict) preference aggregation and the class of premise-conclusion agendas. Among other interesting natural agendas in this framework that were studied one can find the equivalence agenda[9] and the membership agenda [21][16].

3.1 Preference Aggregation

Aggregation of preferences is one of the oldest aggregation frameworks studied. In this framework there are s candidates and each individual holds a full strict order over them. We are interested in Social Welfare Functions which are functions that aggregate n such orders to an aggregated order. As seen in [18] and [4], this problem can be stated naturally in our framework by defining $\binom{s}{2}$ issues[17].

3.2 Premise-conclusion agendas

In a premise-conclusion agendas the issues are divided into two types: k premises and $(m-k)$ conclusions. The conclusion issues are boolean functions over the k premises, $\Phi : \{0,1\}^k \to \{0,1\}^{m-k}$. An opinion is consistent if the answers to the conclusion issues are attained by applying the function Φ on the premise issues.

$$\mathbb{X} = \left\{x \in \{0,1\}^m \mid x^j = \Phi^j(x_1,\ldots,x_k) \quad j = k+1,\ldots,m\right\}$$

In this paper we prove results to the following two specific premise-conclusion agendas. We later derive results to a general family of premise-conclusion agendas.

[16] In the simple cooperative games regime, this is also called the Banzhaf power index of player i in the game f.

[17] The issue $\langle i,j \rangle$ (for $i < j$) represents whether an individual prefers c_i over c_j.

3.2.1 Conjunction Agenda (Doctrinal Paradox Agenda)

In the (2-premises) conjunction agenda $\langle A, B, A \wedge B\rangle$ there are three issues to decide on and the consistency criterion is defined to be that the third issue is a conjunction of the first two. A common description of the problem is of a group of judges or jurors that should decide whether a defendant is liable under a charge of breach of contract. Each of them should decide on three issues: whether the contract was valid (p), whether there was a breach (q) and whether the defendant is liable (r). In their decision making they are constrained by the legal doctrine that the defendant is only liable if the contract was valid and if there was indeed a breach ($r \iff (p \wedge q)$).

3.2.2 Xor Agenda

Similarly, in the (2-premises) xor agenda $\langle A, B, A \oplus B\rangle$ there are three issues to decide on and the consistency criterion is defined to be that the third issue is `True` if the first two answers are equal. An equivalent way to define this agenda is constraining the number of `True` answers to be odd.

4 Approximate Aggregation Results

In this paper we are interested in studying whether relaxing the consistency constraint, i.e., taking $IC(F) = \Pr\left[F(X) \notin \mathbb{X} \mid X \in \mathbb{X}^n\right]$ to be small (while restricting ourselves to independent aggregation mechanisms), extends non-trivially the set of satisfying aggregation mechanisms, i.e. entails that $d(F, \mathcal{C}(\mathbb{X})) = \min_{G\in\mathcal{C}(\mathbb{X})} \Pr\left[F(X) \neq G(X) \mid X \in \mathbb{X}^n\right]$ is small (taking $\mathcal{C}(\mathbb{X})$ to be the class of aggregation mechanisms that satisfies consistency and independence). More specifically we are interested in theorems of the following form (For a given agenda $\mathbb{X}$):

Theorem. *For any $\epsilon > 0$ and $n \geqslant 1$, there exists $\delta = \delta\left(\frac{1}{n}, \epsilon\right)$, such that if F is an aggregation mechanism for $\mathbb{X}$ over n voters satisfying independence and $IC(F) < \delta$, then there exists an aggregation mechanism G that satisfies consistency and independence such that $d(F, G) < \epsilon$.*

Notice that such a theorem can be trivially satisfied by $\delta(\epsilon, n) = 0$. We seek better bounds. Particulary, we are interested that whenever ϵ is small (e.g., $\frac{1}{\text{poly}(n)}$), then so is δ. E.g., taking δ to be *poly* $\left(\frac{1}{n}, \epsilon\right)$.

We find the motivation for dealing with the field of approximate aggregation in three different disciplines.

- The consistent characterization are often regarded as 'impossibility results' in the sense that they 'permit' a very restrictive set of aggregation mechanisms. (e.g., Arrow's theorem tells us that there is no 'reasonable' way to aggregate preferences). Extending this theorems to approximate aggregation characterizations sheds light on these impossibility results by relaxing the constraints.
- The questions of Aggregation Theory have often roots in Philosophy, Law, or Political Science. Results on approximate aggregations support the discussion that started in the works of Arrow[1] and Kornhauser and Sager[13] and searches for ways to deal with scenarios in which it is needed to aggregate such opinions.
- The CS field of Local Property Checking of Boolean Functions deals with the problem of deciding whether a given function has a given property (e.g., linearity) or whether it

is 'far' from any object having the property. The works in the field consider randomized algorithms that query the function at points of their choice, and seek algorithms which query the function at relatively few points (For a survey of this field see [8]). The question of checking locally for a global property is very close to the framework of approximate aggregation (whether there exists an aggregation mechanism that is far from the set of independent and consistent aggregation mechanisms but still does not fail for most profiles). And indeed, the analysis of such randomized algorithm deals with very similar expressions to the inconsistency index and hence results from the field of approximate aggregation can be easily translated to the field of property testing for the property 'belongs to the class of consistent aggregation mechanism'. Special interest should be in results that restrict the aggregation mechanisms to systematic aggregation mechanisms (For instance Blum, Luby, and Rubinfeld's result ([3],[2]) can be seen as a result for approximate aggregation using systematic aggregation mechanisms for the xor agenda.).

The first work studying approximate aggregation was done for the preference agenda over three candidate by Kalai[12] (although without stating the general framework of approximate aggregation). In this paper he proved the following bound for approximate aggregation mechanisms.

Theorem ([12])**.** *There exists an absolute constant K such that the following holds: For any $\epsilon > 0$ and any aggregation mechanism F for the preference agenda over 3 candidates that satisfies: F is balanced[18], F is independent, and $IC(F) < K\epsilon$, there exists an aggregation mechanism G that satisfies consistency and independence such that $d(F,G) < \epsilon$.*

This theorem was extended by Mossel[17] for preference agendas over any number of candidates and non-balanced aggregation mechanisms but with worse dependence of $IC(F)$ in ϵ (instead of linear as above).

Our main theorem gives bounds for every premise-conclusion agenda in which every conclusion is a function of at most two of the premises.

Theorem 4.1 (Main theorem)**.**
For any $\epsilon > 0$ and $n \geqslant 1$, there exists $\delta = poly\left(\frac{1}{n}, \epsilon\right)$, such that for every premise-conclusion agenda in which each issue is a premise, a conclusion of one premise, or a conclusion of two premises, if F is an aggregation mechanism for $\mathbb{X}$ over n voters satisfying independence and $IC(F) < \delta$, then there exists an aggregation mechanism G that satisfies consistency and independence such that $d(F,G) < \epsilon$.
Moreover, one may take $\delta = Cn^{-2}\epsilon^5$ for some absolute constant C.

5 Proof Sketch

We prove this theorem by proving it explicitly for three specific agendas: the id agenda $\langle A, A\rangle$, the xor agenda $\langle A, B, A \oplus B\rangle$, and the conjunction agenda $\langle A, B, A \wedge B\rangle$. Since every boolean function on two bits can be reduced to one of the cases $f(x,y) = x$, $f(x,y) = y$, $f(x,y) = x \wedge y$, and $f(x,y) = x \oplus y$ by negating the inputs and output (which is renaming of opinions in our framework) we get theorem 4.1 using induction on the number of conclusions.

Below we sketch the proof idea for the xor agenda and conjunction agenda. The proofs of the more technical lemmas can be found in the full version.

[18] For every pair of candidates, a and b, it holds that the probability that F ranks a above b is exactly 1/2.

5.1 Proof for the xor agenda

For the agenda $\langle A, B, A \oplus B\rangle$ we prove:

Theorem 5.1. *For any $\epsilon < \frac{1}{6}$ and any independent aggregation mechanism F: If $IC(F) \leqslant \epsilon$, then there exists an aggregation mechanism G that satisfies consistency and independence such that $d(F, G) \leqslant 3\epsilon$.*

Proof sketch.
Technique[19]: The proof uses the Fourier representation of boolean functions. That means representing the functions as linear combinations of the linear boolean functions.

Given an independent aggregation mechanism $F = \langle f, g, h\rangle$ we analyze the expression $\mathbb{E}[f(x)g(y)h(xy)]$ when x and y are sampled uniformly and independently. On one hand we show that $\mathbb{E}[f(x)g(y)h(xy)] = 1 - 2IC(F)$. On the other hand we show that $\mathbb{E}[f(x)g(y)h(xy)] = \sum_{\chi \in \mathtt{Lin}} \widehat{f}(\chi)\widehat{g}(\chi)\widehat{h}(\chi)$ when $\left|\widehat{f}(\chi)\right|$ equals $1 - 2\min\left(d\left(f, \chi\right), d\left(f, -\chi\right)\right)$. Hence, when $IC(F)$ is small then this sum is close to one and hence there exists a linear function such that f,g, and h are close to it (up to negation). Noticing that for any linear function χ, $\langle \chi, \chi, \chi\rangle$ and the permutations of $\langle -\chi, -\chi, \chi\rangle$ are consistent independent aggregation mechanism for this agenda gives us the result.

5.2 Proof for the conjunction agenda

For the agenda $\langle A, B, A \wedge B\rangle$ we prove:

Theorem 5.2. *For any $\epsilon > 0$ and any independent aggregation mechanism F: If $IC(F) \leqslant \epsilon$, then there exists an aggregation mechanism G that satisfies consistency and independence such that $d(F, G) < 5\sqrt[5]{n^2\epsilon}$.*

Proof sketch.
Technique: The main insight in the proof is that we can bound the product of the influence of a voter on f and the ignorability of the same voter for g (and vice versa) using the inconsistency index of F by $P_i(f) \cdot I_i(g) \leqslant 4IC(F)$.

Let $F = \langle f, g, h\rangle$ be an aggregation mechanism that satisfies $IC(F) \leqslant \epsilon$. In case that f (or g) is close enough to the constant zero function, F is close to the consistent aggregation mechanism $\langle 0, g, 0\rangle$.

Otherwise, we define for a given function $f : \{0,1\}^n \to \{0,1\}$ and a coalition J (the junta), the junta function $f^J : \{0,1\}^n \to \{0,1\}$. It is derived from f in the following way:

$$f^J(x) = \text{majority}\left\{f(y) \mid y_J = x_J\right\}.$$

I.e., for a given input, f^J reads only the votes of the junta members, iterates over all the possible votes for the members outside the junta, and returns the more frequent result (assuming uniform distribution over the votes of the voters outside J).
We define f^J and g^J with regard to the junta of all the voters with small ignorability for either f or g. We prove that f^J and g^J are close to f and g, respectively and that there exists an issue aggregation function $h^\star$ such that $\left\langle f^J, g^J, h^\star\right\rangle$ is a consistent aggregation mechanism that is close to F.

There is a known characterization of the consistent independent aggregation mechanism for the conjunction agenda. (This characterization is a direct corollary from a series of works in the more general framework of aggregation, E.g., [19], [5]. We include a proof of it in the full version)

[19] The proof is similar to the analysis of the BLR (Blum-Luby-Rubinfeld) linearity test done in [2].

Lemma 5.3.
Let $f, g, h : \{0,1\}^n \to \{0,1\}$ be three voting functions satisfying $IC(\langle f, g, h\rangle) = 0$. Then either $f = h \equiv 0$, or $g = h \equiv 0$, or $f = g = h \in$ `Olig`.

A corollary from this theorem and theorem 5.2 is a characterization of the approximate aggregation mechanisms for this agenda. Actually ,in the proof of theorem 5.2 we get a tighter characterization that distinguishes between the two cases of consistent independent aggregation mechanism.

6 General Definition of Approximate Aggregation

In this paper we defined approximate aggregation by leaving the independence constraint intact and relaxing the consistency constraint. In this section we show that under a more general definition of approximate aggregation that relaxes both constraints we get similar results for any agenda and hence we do not lose much by restricting ourselves to the narrower definition.

Let $\mathbb{X}$ be an agenda and let F be an aggregation mechanism for that agenda. We define the **dependency index** as a measure for 'not satisfying independence'.

Definition 6.1 (dependency index)**.**
For an agenda $\mathbb{X}$ and an aggregation mechanism F for that agenda, the dependency index $DI^{\mathbb{X}}(F)$ is defined by

$$DI^{\mathbb{X}}(F) = \max_{j=1,\ldots,m} DI^{j,\mathbb{X}}(F) \quad \text{when} \quad DI^{j,\mathbb{X}}(F) = \mathop{\mathbb{E}}_{X \in \mathbb{X}^n}\left[\Pr_{Y \in \mathbb{X}^n}\left[F(X) \neq F(Y) | X^j = Y^j\right]\right]$$

That is, $DI^{j,\mathbb{X}}(F)$ is the probability that the following test for dependence of aggregating issue j on other issues fails (returns `False`):

Choose a profile X uniformly at random.
Choose a profile Y that agrees with X on issue j uniformly at random.
Return whether $F(X) \neq F(Y)$

We are interested in theorems of the form (for a given agenda $\mathbb{X}$):

Theorem. *For any $\epsilon > 0$ and $n \geqslant 1$, there exist $\delta_{IC}, \delta_{DI} > 0$[20], such that if F is an aggregation mechanism for $\mathbb{X}$ over n voters satisfying $IC(F) \leqslant \delta_{IC}$ and $DI(F) \leqslant \delta_{DI}$, then there exists an aggregation mechanism G that satisfies consistency and independence such that $d(F, G) < \epsilon$.*

It is easy to see that theorems of this form are generalizations of theorems of the form we proved in this paper and one can easily derive approximate aggregation results for independent aggregation mechanisms ($DI(F) = 0$) from theorems of the above general form.

It turns out that one can derive theorems the other way too using the following proposition.[21]

Proposition 6.1. *Let G be an aggregation mechanism for an agenda over m issues that satisfies $DI(G) \leqslant \delta_{DI}$. Then there exists an independent aggregation mechanism F that satisfies $d(F, G) \leqslant 2m\delta_{DI}$*

Given a result in the following format (which is the format we proved for in this paper):

Let $\delta : [0,1] \to [0,1]$ be a function s.t. for any $\epsilon > 0$: If F is an aggregation mechanism satisfying independence and $IC(F) \leqslant \delta(\epsilon)$, then there exists an aggregation mechanism G that satisfies consistency and independence such that $d(F, G) < \epsilon$.

[20] We would like δ_{IC}, δ_{DI} to not be too small. For instance we would like them to be *poly* $\left(\frac{1}{n}, \epsilon\right)$.
[21] Due to space limitations we omit the proof. It can be found in the full version of this paper.

We will define $\delta_{IC} = \frac{1}{2}\delta\left(\frac{\epsilon}{2}\right)$, $\delta_{DI} = \frac{1}{4m}\min\left(\delta\left(\frac{\epsilon}{2}\right), \epsilon\right)$. Now, let G be an aggregation mechanism that satisfies $IC(G) \leqslant \delta_{IC}$ and $DI(G) \leqslant \delta_{DI}$. Then based on proposition 6.1 there is an independent aggregation mechanism F such that $d(F,G) \leqslant 2m\delta_{DI}$. It is easy to see that $IC(F) \leqslant IC(G) + d(F,G)$ and for any aggregation mechanism H, $d(G,H) \leqslant d(F,H) + d(F,G)$ and hence there exists an aggregation mechanism H that satisfies consistency and independence such that $d(G,H) < \epsilon$.[22]

Notice that the dependency of δ_{IC} and δ_{DI} in ϵ and n (for instance, being polynomial in these parameters) is 'inherited' from the dependency of δ in ϵ and n. Therefore, such result will be similar in quality to the result for approximate aggregation mechanism that satisfies independence and we do not lose much by restricting ourselves to studying approximate aggregation by mechanisms that satisfying independence when analyzing a given agenda.

7 Summary and Future Work

In this paper we defined the issue of approximate aggregation which is a generalization of the study of aggregation mechanisms that satisfy consistency and independence. We defined measures for the relaxation of the consistency constraint (inconsistency index IC) and for the relaxation of the independence constraint (dependency index DI) .

We proved that relaxing these constraints does not extend the set of satisfying aggregation mechanisms in a non-trivial way for any premise-conclusion agenda in which every conclusion can be stated as a function of at most two of the premises. Particulary we calculated the dependency between the extension of this class (ϵ) and the inconsistency index ($\delta(\epsilon)$) (although maybe not strictly) for any premise-conclusion agenda of three issues. The relation we proved includes dependency on the number of voters (n). In both the works that preceded us for preference agendas (Kalai[12] and Mossel[17]) the relation did not include such a dependency. An interesting question is whether such a dependency is inherent for premise-conclusion agendas or whether it is possible to prove a relation that does not depend on n.

A major assumption in this paper is the uniform distribution over the inputs which is equivalent to assuming i.i.d uniform distribution over the premises. We think that our results can be extended for other distributions (still assuming voters' opinions are distributed i.i.d) over the space over premises' opinions which seem more realistic.

Immediate extensions for this work can be to extend our result to more complex premise-conclusion agendas and generalize our results for three issues premise-conclusion agenda and Kalai and Mossel's works for the preference agenda to get a unified bound for any three issues agenda.

A major open question is whether one can find an agenda for which relaxing the constraints of independence and consistency extends the class of satisfying aggregation mechanisms in a non-trivial way.

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[22] For specific agendas, one might be able to strengthen some of these inequalities using the structure of $\mathbb{X}$ to get a stronger bounds on δ_{IC} and δ_{DI}.

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