Three Hierarchies of Simple Games Parameterized by "Resource" Parameters

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Abstract

This paper contributes to the program of numerical characterization and classification of simple games outlined in the classic monograph of von Neumann and Morgenstern (1944). We suggest three possible ways to classify simple games beyond the classes of weighted and roughly weighted games. To this end we introduce three hierarchies of games and prove some relations between their classes. We prove that our hierarchies are true (i.e., infinite) hierarchies. In particular, they are strict in the sense that more of the key "resource" (which may, for example, be the size or structure of the "tie-breaking" region where the weights of the different coalitions are considered so close that we are allowed to specify either winningness or nonwinningness of the coalition), yields the flexibility to capture strictly more games.

1 Introduction

A simple game is a mathematical object that is used in economics and political science to describe the distribution of power among coalitions of players [10, 11]. Recently simple games have been studied as access structures of secret sharing schemes [2]. They have also appeared in a variety of mathematical and computer science contexts under various names, e.g., monotone boolean [5] or switching functions and threshold functions [6]. Simple games are closely related to hypergraphs, coherent structures, Sperner systems, clutters, and abstract simplicial complexes. The term "simple" was introduced by von Neumann and Morgenstern (1944) because in this type of games players strive not for monetary rewards but for power, and each coalition is either all-powerful or completely ineffectual. However these games are far from being simple.

An important class of simple games—well studied in economics—is the weighted majority games [10, 11]. In such a game every player is assigned a real number, his weight. The winning coalitions are the sets of players whose weights total at least q, a certain threshold. However, it is well known that not every simple game has a representation as a weighted majority game [10]. The first step in attempting to characterize nonweighted games was the introduction of the class of roughly weighted games [9]. Formally, a simple game G on the player set $P = [n] = \{1, 2, \ldots, n\}$ is *roughly weighted* if there exist nonnegative real numbers $w_1, \ldots, w_n$ and a real number q, called the *quota*, not all equal to zero, such that for $X \in 2^P$ the condition $\sum_{i \in X} w_i > q$ implies X is winning, and $\sum_{i \in X} w_i < q$ implies X is losing. This concept realizes a very common idea in social choice that sometimes a rule needs an additional "tie-breaking" procedure that helps to decide the outcome if the result falls on a certain "threshold." Taylor and Zwicker [9] demonstrated the usefulness of this concept. Rough weightedness was studied by Gvozdeva and Slinko [4], where it was characterized in terms of trading transforms, similar to the characterization of weightedness by Elgot [3] and Taylor and Zwicker [8].

It might seem that nonweighted games and even games without rough weights are weird. However, an important observation of von Neumann and Morgenstern [10, Section 53.2.6] states that they "correspond to a different organizational principle that deserves closer study." In some of these games, as they noted, all the minimal winning coalitions are minorities and at the same time "no player has any advantage over any other" (e.g., the

Fano game introduced later). This is an attractive feature for secret sharing as in the case of large number of users it is advantageous to keep minimal authorized coalitions relatively small. This is may be why weighted threshold secret sharing schemes were largely ignored and were characterized only recently [1].

The parameter of the first hierarchy reflects the balance of power between small and large coalitions; the larger this parameter the more powerful some of the small coalitions are. Gvozdeva and Slinko [4] proved that for a game G that is not roughly weighted there exists a certificate of nonweightedness (see the definition in Section 2) of the form

$$\mathcal{T} = (X_1, \ldots, X_j, P; Y_1, \ldots, Y_j, \emptyset), \tag{1}$$

where $X_1, \ldots, X_j$ are winning coalitions of G, P is the grand coalition, and $Y_1, \ldots, Y_j$ are losing coalitions. However, sometimes it is possible to have more than one grand coalition in the certificate. This may occur when coalitions $X_1, \ldots, X_j$ are small but nonetheless winning.

A certificate of nonweightedness of the form

$$\mathcal{T} = (X_1, \ldots, X_j, P^\ell; Y_1, \ldots, Y_j, \emptyset^\ell) \tag{2}$$

will be called ℓ-potent of length $j + \ell$. Each game that possesses such a certificate will be said to belong to the class of games $\mathcal{A}_q$, where $q = \ell/(j + \ell)$. The parameter q can take values in the open interval $(0, \frac{1}{2})$. We will show that $\mathcal{A}_p \supseteq \mathcal{A}_q$ for any p and q such that $0 < p \le q < \frac{1}{2}$ and that the inclusion $\mathcal{A}_p \supseteq \mathcal{A}_q$ is strict as soon as $p < q$.

Another hierarchy emerges when we allow several thresholds instead of just one in the case of roughly weighted games. We say that a simple game G belongs to the class $\mathcal{B}_k, k \in \{1, 2, 3, \ldots\}$, if there are k thresholds $0 < q_1 \le q_2 \le \cdots \le q_k$ and any coalition with total weight of players smaller than q_1 is losing, any coalition with total weight greater than q_k is winning . We also impose an additional condition that, if a coalition X has total weight $w(X)$ which satisfies $q_1 \le w(X) \le q_k$, then $w(X) = q_i$ for some i. All games of the class $\mathcal{B}_1$ are roughly weighted. In fact, as we'll prove in Section 4 almost all roughly weighted games to this class: $\mathcal{B}_1$ is exactly the class of roughly weighted games with nonzero quota. We will show that the Fano game [4] belongs to $\mathcal{B}_2$ but does not belong to $\mathcal{B}_1$. We prove that $\mathcal{B}$-hierarchy is strict, that is,

$$\mathcal{B}_1 \subsetneq \mathcal{B}_2 \subsetneq \cdots \subsetneq \mathcal{B}_\ell \subsetneq \cdots,$$

with the union of these classes being the class of all simple games.

Yet another way to capture more games is by making the threshold "thicker." We here will not use a point but rather an interval $[a, b]$ for the threshold, $a \le b$. That is, all coalitions with total weight less than a will be losing and all coalitions whose total weight is greater than b winning. This time—in contrast with the k limit of $\mathcal{B}_k$—we do not care how many different values weights of coalitions falling in $[a, b]$ may take on. A good example of this situation would be a faculty vote, where if neither side controls a 2/3 majority (calculated in faculty members or their grant dollars), then the Dean would decide the outcome as he wished. We can keep weights normalized so that the lower end of the interval is fixed at 1. Then the right end of the interval α becomes a "resource" parameter. Formally, a simple game G belongs to class $\mathcal{C}_\alpha$ if all coalitions in G with total weight less than 1 are losing and every coalition whose total weight is greater than α is winning. We show that the class of all simple games is split into a hierarchy of classes of games $\{\mathcal{C}_\alpha\}_{\alpha \in [1,\infty)}$ defined by this parameter. We show that as α increases we get strictly greater descriptive power, i.e., strictly more games can be described, that is, if $\alpha < \beta$, then $\mathcal{C}_\alpha \subsetneq \mathcal{C}_\beta$. In this sense the hierarchy is strict. This strict hierarchy result, and our strict hierarchy results for hierarchies $\mathcal{A}$ and $\mathcal{B}$, have very much the general flavor of hierarchy results found in computer science:

more resources yield more power (whether computational power to accept languages as in a deterministic or nondeterministic time hierarchy theorem, or as is the case here, description flexibility to capture more games).

The strictness of the latter hierarchy was achieved because we allowed games with arbitrary (but finite) numbers of players. The situation will be different if we keep the number of players n fixed. Then there is an interval $[1, s(n)]$ such that all games with n players belong to $\mathcal{C}_{s(n)}$ and $s(n)$ is minimal with this property. There will be also finitely many numbers $q \in [1, s(n)]$ such that the interval $[1, q]$ represents more n-player games than any interval $[1, q']$ with $q' < q$. We call the set of such numbers the nth *spectrum* and denote it $\mathrm{Spec}(n)$. We also call a game with n players *critical* if it belongs to $\mathcal{C}_\alpha$ with $\alpha \in \mathrm{Spec}(n)$ but does not belong to any $\mathcal{C}_\beta$ with $\beta < \alpha$. We calculate the spectrum for $n < 7$ and also produce a set of critical games, one for each element of the spectrum. We also try to give a reasonably tight upper bound for $s(n)$.

All three of our hierarchies provide measures of how close a given game is to being a simple weighted voting game. That is, they each quantify the nearness to being a simple weighted voting game (e.g., hierarchies $\mathcal{B}$ and $\mathcal{C}$ quantify based on the extent and structure of a "flexible tie-breaking" region). And the main theme and contribution of this paper is that we prove for each of the three hierarchies that allowing more quantitative distance from simple weighted voting games yields *strictly more* games, i.e., the hierarchies are proper hierarchies.

2 Preliminaries

Definition 1. *A simple game is a pair $G = (P, W)$, where W is a subset of the power set 2^P satisfying the monotonicity condition:*

if $X \in W$ and $X \subsetneq Y \subseteq P$, then $Y \in W$,

and $W \notin \{\emptyset, 2^P\}$ (nontriviality assumption).

Elements of the set W are called *winning coalitions*. We also define the set $L = 2^P \setminus W$ and call elements of this set *losing coalitions*. A winning coalition is said to be *minimal* if every its proper subset is a losing coalition. Due to monotonicity, every simple game is fully determined by the set of its minimal winning coalitions. A player which does not belong to any minimal winning coalitions is called *dummy*.

For $X \subseteq P$ we will denote its complement $P - X$ as X^c.

Definition 2. *A simple game is called proper if $X \in W$ implies that $X^c \in L$ and is called strong if $X \in L$ implies that $X^c \in W$. A simple game that is proper and strong is called a constant-sum game.*

The following definition is given as it has appeared in [4].

Definition 3. *A simple game $G = (P, W)$ is called roughly weighted if there exist nonnegative real numbers $w_1, \ldots, w_n$ and a nonnegative real number q, not all equal to zero, such that for $X \in 2^P$ the condition $\sum_{i \in X} w_i < q$ implies $X \in L$ and $\sum_{i \in X} w_i > q$ implies $X \in W$. We say that $[q; w_1, \ldots, w_n]$ is a rough voting representation for G; the number q is called the* quota.

Example 1 (The Fano game)**.** *This important example first appeared in [10, Section 53.2.6]. Let $P = [7]$ be identified with the set of seven points of the projective plane of order two, called the Fano plane. Let us take the seven lines of this projective plane as minimal winning coalitions:*

$$\{1,2,3\},\ \{3,4,5\},\ \{1,5,6\},\ \{1,4,7\},\ \{2,5,7\},\ \{3,6,7\},\ \{2,4,6\}. \tag{3}$$

We will denote them by $X_1, \ldots, X_7$, respectively. This, as is easy to check, defines a constant-sum game the Fano. As we will see slightly later, it has no rough voting representation. As we can see from the list of minimal winning coalitions they are all minorities, yet symmetry makes all players equal in this example.

We remind the reader that a sequence of coalitions

$$\mathcal{T} = (X_1, \ldots, X_j; Y_1, \ldots, Y_j) \tag{4}$$

is a trading transform [9] if the coalitions $X_1, \ldots, X_j$ can be converted into the coalitions $Y_1, \ldots, Y_j$ by rearranging players. This can also be expressed as

$$|\{i : a \in X_i\}| = |\{i : a \in Y_i\}| \qquad \text{for all } a \in P.$$

We say that the length of $\mathcal{T}$ is j.

Definition 4. *A trading transform $(X_1, \ldots, X_j; Y_1, \ldots, Y_j)$ with all coalitions $X_1, \ldots, X_j$ winning and all coalitions $Y_1, \ldots, Y_j$ losing is called a certificate of nonweightedness. This certificate is said to be potent if the grand coalition P is among $X_1, \ldots, X_j$ and the empty coalition is among $Y_1, \ldots, Y_j$.*

Elgot proved (using a different terminology) that the existence of a certificate of nonweightedness implies that the game is not weighted and that every nonweighted game has one. Taylor and Zwicker [9] showed that for a nonweighted game with n player this certificate can be found of length at most 2^{2^n}; Gvozdeva and Slinko [4] lowered this bound to $(n+1)2^{\frac{1}{2}n\log_2 n}$.

Theorem 1 (Criterion of rough weightedness [4])**.** *A simple game G with n players is roughly weighted iff for no positive integer $j \le (n+1)2^{\frac{1}{2}n\log_2 n}$ does there exist a potent certificate of nonweightedness of length j.*

In Example 1 the following eight winning coalitions $X_1, \ldots, X_7, P$ of the Fano game can be transformed into the following eight losing coalitions: $X_1^c, \ldots, X_7^c, \emptyset$. So the sequence

$$(X_1, \ldots, X_7, P; X_1^c, \ldots, X_7^c, \emptyset) \tag{5}$$

is a potent certificate of nonweightedness for this game. So the game is not roughly weighted, thanks to Theorem 1.

Theorem 2 ([4])**.** *The following games are roughly weighted:*

- *every game with 4 or less players,*
- *every strong or proper game with 5 or less players, and*
- *every constant sum game with 6 or less players.*

Definition 5 ([9], p. 6)**.** *We say that a player p in a game is a* dictator *if p belongs to every winning coalition and to no losing coalition. If all coalitions containing player p are winning, this player is called a* passer*. A player p is called a* vetoer *if p is contained in the intersection of all winning coalitions.*

Proposition 1 ([4])**.** *Suppose G is a simple game with n players. Then G is roughly weighted if any one of the following three conditions holds:*

(a) G has a passer.

(b) G has a vetoer.

(c) G has a losing coalition that consists of $n-1$ players.

Due to Proposition 1(a) there is one trivial way to make any game roughly weighted. This can be done by adding an additional player and making her a passer. Then we can introduce rough weights by assigning weight 1 to the passer and weight 0 to any other player and setting the quota equal to 0. Note, that if the original game is not roughly weighted, then such rough representation is unique. In our view, adding a passer trivializes the game but does not make it closer to a weighted majority game; this is why in definitions of our hierarchies $\mathcal{B}$ and $\mathcal{C}$ we disallow thresholds to be equal 0.

As in [4] we would like to represent trading transforms algebraically. Let $T = \{-1, 0, 1\}$ and let $T^n = T \times T \times \cdots T$ (n times). With any pair (X, Y) of subsets $X, Y \in [n]$ we define

$$\mathbf{v}_{X,Y} = \chi(X) - \chi(Y) \in T^n,$$

where $\chi(X)$ and $\chi(Y)$ are the characteristic vectors of subsets X and Y, respectively.

Let now $G = (P, W)$ be a simple game. We will associate an algebraic object with G. For any pair (X, Y), where X is winning and Y is losing, we put the pair in correspondence with the vector $\mathbf{v}_{X,Y}$. The set of all such vectors we will denote $I(G)$ and will call the ideal of the game. Saying that $(X_1, \ldots, X_j; Y_1, \ldots, Y_j)$ is a certificate of nonweightedness is equivalent to saying that the following vector sum of the ideal is $\mathbf{0}$: $\mathbf{v}_{X_1,Y_1} + \mathbf{v}_{X_2,Y_2} + \cdots + \mathbf{v}_{X_j,Y_j} = \mathbf{0}$. An ℓ-potent certificate $(X_1, \ldots, X_j, P^\ell; Y_1, \ldots, Y_j, \emptyset^\ell)$ will be represented as

$$\mathbf{v}_{X_1,Y_1} + \mathbf{v}_{X_2,Y_2} + \cdots + \mathbf{v}_{X_j,Y_j} + \ell \cdot \mathbf{1} = \mathbf{0},$$

where $\mathbf{1}$ is a vector whose entries are each 1.

3 The $\mathcal{A}$-Hierarchy

This hierarchy of classes $\mathcal{A}_\alpha$ tries to capture the richness of the class of games that do not have rough weights, and does so by introducing a parameter $\alpha \in (0, \frac{1}{2})$. As we already discussed, the larger this parameter the more power is given to some relatively small coalitions. Our method of classification is based on the existence of potent certificates of nonweightedness for such games [4].

Definition 6. *Let q be a rational number. A game G belongs to the class $\mathcal{A}_q$ of $\mathcal{A}$-hierarchy if G possesses a potent certificate of nonweightedness*

$$(X_1, \ldots, X_m; Y_1, \ldots, Y_m), \tag{6}$$

with ℓ grand coalitions among $X_1, \ldots, X_m$ and ℓ empty coalitions among $Y_1, \ldots, Y_m$, such that $q = \ell/m$. If α is irrational,we set $\mathcal{A}_\alpha = \bigcap_{q<\alpha} \mathcal{A}_q$.

It is easy to see that, if $q \geq \frac{1}{2}$, then $\mathcal{A}_q$ is empty. Indeed, suppose $q \geq \frac{1}{2}$ and $\mathcal{A}_q$ is not empty. Then there is a game G with a certificate of nonweightedness

$$\mathcal{T} = (X_1, \ldots, X_k, P^m; Y_1, \ldots, Y_k, \emptyset^m) \tag{7}$$

with $m \geq k$. This is not possible since m copies of P contain more elements than are contained in the sets $Y_1, \ldots, Y_k$ taken together and so (7) is not a trading transform. So our hierarchy consists of a family of classes $\{\mathcal{A}_\alpha\}_{\alpha \in (0, \frac{1}{2})}$. We would like to show that this hierarchy is strict, that is, a smaller parameter captures more games.

Proposition 2. *If $0 < \alpha \leq \beta < \frac{1}{2}$, then $\mathcal{A}_\alpha \supseteq \mathcal{A}_\beta$.*

Proof. It is sufficient to prove this statement when α and β are rational. Suppose that we have a game G in $\mathcal{A}_\beta$ that possesses a certificate of length n_1 with k_1 grand coalitions and $\beta = k_1/n_1$. Let $\alpha = k_2/n_2$. We can then represent these numbers as $\beta = m_1/n$ and $\alpha = m_2/n$, where $n = \operatorname{lcm}(n_1, n_2)$. Since $\alpha \leq \beta$, we have $m_2 \leq m_1$. Since $n = n_1 h$ and $m_1 = k_1 h$ for some integer h, we can now combine h certificates for G to obtain one with length n and m_1 grand coalitions. We reclassify the $m_1 - m_2$ grand coalitions into ordinary winning coalitions, and we will get a certificate for G of length n with m_2 grand coalitions. So $G \in \mathcal{A}_\alpha$. $\square$

We say that a game G is *critical* for $\mathcal{A}_\alpha$ if it belongs to $\mathcal{A}_\alpha$ but does not belong to any $\mathcal{A}_\beta$ with $\beta > \alpha$.

Theorem 3. *If $0 < \alpha < \beta < \frac{1}{2}$, then $\mathcal{A}_\alpha \supsetneq \mathcal{A}_\beta$.*

Proof. First, we will construct a two-parameter family of simple games. For any integers $a \geq 2$ and $b \geq 2$ let $G = ([a^2 + a + b + 1], W)$ be a simple game for which a coalition X is winning, exactly if $|X| > a^2 + 1$ or X contains a subset whose characteristic vector is a cyclic permutation of $(\underbrace{1, \ldots, 1}_{a+1}, \underbrace{0, \ldots, 0}_{a^2+b})$.

Let $X_1, \ldots, X_{a^2+a+b+1}$ be winning coalitions, whose characteristic vectors are cyclic permutations of $(\underbrace{1, \ldots, 1}_{a+1}, \underbrace{0, \ldots, 0}_{a^2+b})$. Also let $Y_1, \ldots, Y_{a^2+a+b+1}$ be losing coalitions, whose characteristic vectors are cyclic permutations of

$$(\underbrace{1, \ldots, 1}_{a}, 0, \underbrace{1, \ldots, 1}_{a}, 0, \underbrace{1, \ldots, 1}_{a}, 0, \ldots, \underbrace{1, \ldots, 1}_{a}, 0, 0, 1, \underbrace{0, \ldots, 0}_{b-1}),$$

where there are a groups of symbols $\underbrace{1, \ldots, 1}_{a}, 0$.

This game possesses the following potent certificate of nonweightedness

$$\mathcal{T} = (X_1, \ldots, X_{a^2+a+b+1}, P^{a^2-a}; Y_1, \ldots, Y_{a^2+a+b+1}, \emptyset^{a^2-a}). \tag{8}$$

So $G \in \mathcal{A}_{\frac{a^2-a}{2a^2+b+1}}$. Let us prove that G is critical for this class, that is, it does not belong to any $\mathcal{A}_{q'}$ for $q' > q$. Note that the vectors $\mathbf{v}_i = \mathbf{v}_{X_i, Y_i}$ belong to the ideal of this game. Note also that the sum of all coefficients of $\mathbf{v}_i$ is $\mathbf{v}_i \cdot \mathbf{1} = a - a^2$ and that for any other vector $\mathbf{v} \in I(G)$ from the ideal of this game we have $\mathbf{v} \cdot \mathbf{1} \geq a - a^2$.

Suppose G also has a potent certificate of nonweightedness

$$(A_1, \ldots, A_s, P^t; B_1, \ldots, B_s, \emptyset^t). \tag{9}$$

with $q' = \frac{t}{t+s} > \frac{a^2-a}{2a^2+b+1} = q$. The latter is equivalent to $\frac{a^2+a+b+1}{a^2-a} > \frac{s}{t}$. Let $\mathbf{u}_i = \mathbf{v}_{A_i, B_i} \in I(G)$, then (9) can be written as

$$\mathbf{u}_1 + \mathbf{u}_2 + \cdots + \mathbf{u}_s + t \cdot \mathbf{1} = \mathbf{0}.$$

As $\mathbf{u}_i \cdot \mathbf{1} \geq a - a^2$, taking the dot product of both sides with $\mathbf{1}$ we get $t(a^2+a+b+1) \leq s(a^2-a)$, which is equivalent to $\frac{a^2+a+b+1}{a^2-a} \leq \frac{s}{t}$, so we have reached a contradiction.

We will now show that any rational number between 0 and $\frac{1}{2}$ is representable as $\frac{a^2-a}{2a^2+1+b}$ for some positive integers $a \geq 2$ and $b \geq 2$. Let $\frac{p}{q} \in (0, \frac{1}{2})$. Then $q - 2p > 0$ and it is possible to choose a positive integer k such that $k^2 p(q - 2p) - kq - 3 > 0$. Take $a = kp$ and $b = k^2 p(q - 2p) - kq - 1$. Substituting these values we get $\frac{a^2-a}{2a^2+1+b} = \frac{p}{q}$. $\square$

4 $\mathcal{B}$-Hierarchy

The $\mathcal{B}$-hierarchy generalizes the idea behind rough weightedness to allow more "points of flexibility."

Definition 7. *A simple game $G = (P, W)$ belongs to $\mathcal{B}_k$ if there exist real numbers $0 < q_1 \le q_2 \le \cdots \le q_k$, called thresholds, and a weight function $w\colon P \to \mathbb{R}^{\ge 0}$ such that*

(a) if $\sum_{i \in X} w(i) > q_k$, then X is winning,

(b) if $\sum_{i \in X} w(i) < q_1$, then X is losing,

(c) if $q_1 \le \sum_{i \in X} w(i) \le q_k$, then $w(X) = \sum_{i \in X} w(i) \in \{q_1, \dots, q_k\}$.

Games from $\mathcal{B}_k$ will be sometimes called k-rough.

The condition $0 < q_1$ in Definition 7 is essential. If we allow the first threshold q_1 be zero, then every simple game can be represented as a 2-rough game. To do this we assign weight 1 to the first player and 0 to everyone else. It is also worthwhile to note that adding a passer does not change the class of the game, that is, a game G belongs to $\mathcal{B}_k$ iff the game G' obtained from G by adding a passer belongs to $\mathcal{B}_k$. This is because a passer can be assigned a very large weight. Thus $\mathcal{B}_1$ consists of the roughly weighted simple games with nonzero quota.

Example 2. *We know that the Fano game is not roughly weighted. Let us assign weight 1 to every player of this game and select two thresholds $q_1 = 3$ and $q_2 = 4$. Then each coalition whose weight falls below the first threshold is in L, and each coalition whose total weight exceeds the second threshold is in W. If a coalition has total weight of three or four, i.e., its weight is equal to one of the thresholds, it can be either winning or losing. Thus the Fano is a 2-rough game.*

Theorem 4. *For every natural number $k \in \mathbb{N}^+$, there exists a game in $\mathcal{B}_{k+1} \setminus \mathcal{B}_k$.*

Proof. We will construct a simple game that is a $(k+1)$-rough but not k-rough. Let $G_{k+1,n} = ([n], W)$ be a simple game with $n = 2k+4$ players. We have $k+2$ types of players with the ith type consisting of two elements $2i-1$ and $2i$. The set of minimal winning coalitions of this game is $W^m = \{\{2i-1, 2i\} \mid i = 1, 2, \dots, k+2\}$.

If we assign weight 1 to every player, then $G_{k+1,n}$ is $(k+1)$-rough game with thresholds $q_1 = 2$, $q_2 = 3$, $\dots$, $q_{k+1} = k+2$. Let us assume that this game is j-rough for some $j < k+1$, and let w be the new weight function and let $r_1, \dots, r_j$ be the new thresholds. By $\max\{a, b\}$ let us denote the element of the set $\{a, b\}$, that has the bigger weight (relative to w). We know that $w(\max\{2i-1, 2i\}) \ge r_1/2 > 0$ for each type i. Consider losing coalition $\{\max\{1,2\}, \max\{3,4\}\}$ with one player from the first type and one from the second type. It has weight

$$w(\{\max\{1,2\}, \max\{3,4\}\}) = w(\max\{1,2\}) + w(\max\{3,4\}) \ge \frac{r_1}{2} + \frac{r_1}{2} = r_1.$$

Assume the worst-case scenario, i.e., that $w(\{\max\{1,2\}, \max\{3,4\}\})$ is equal to r_1. Let us then create a new losing coalition $\{\max\{1,2\}, \max\{3,4\}, \max\{5,6\}\}$ by adding a new player from the third type. It is easy to see that

$$r_1 = w(\{\max\{1,2\}, \max\{3,4\}\}) < w(\{\max\{1,2\}, \max\{3,4\}, \max\{5,6\}\}).$$

So the weight of the new coalition is at least r_2. Assume the worst-case scenario again, and make the weight of $\{\max\{1,2\}, \max\{3,4\}, \max\{5,6\}\}$ be equal to r_2. Proceed by adding a player from the next type to the losing coalition in this manner. At the jth step we will have

$$\begin{aligned} r_j = & w(\{\max\{1,2\},\ldots,\max\{2j+1,2j+2\}\}) < \\ & w(\{\max\{1,2\},\ldots,\max\{2j+1,2j+2\},\max\{2j+3,2j+4\}\}). \end{aligned}$$

The coalition that was constructed last is losing since it does not contain two players from the same type. So it cannot have weight greater then r_j, which it does. This is a contradiction. Thus $G_{k+1,n}$ is not j-rough for any $j < k+1$. □

In all examples above the number of thresholds of a simple game is equal to the cardinality of the largest losing coalition minus the cardinality of the smallest minimal winning coalition plus one. This is not always the case.

Example 3. *Let $G = ([7], W)$ be a simple game with minimal winning coalitions $\{1,2\}, \{6,7\}, \{3,4,5\}$ and all coalitions of four players except $\{2,3,4,6\}$. This game is not roughly weighted, because we have the following potent certificate of nonweightedness*

$$\begin{aligned} \mathcal{T} = & \{\{1,2\}^7, \{3,4,5\}^9, P; \{2,3,5\}^3, \{2,3,4\}^3, \\ & \{2,3,6\}, \{2,3,7\}, \{1,3,4\}, \{1,3,5\}, \{1,4,5\}^6, \emptyset\}. \end{aligned}$$

Let us assign weight 0 to the third player and $\frac{1}{2}$ to everyone else. Then the following hold:

- $w(\{1,2\}) = w(\{6,7\}) = w(\{3,4,5\}) = 1$ *and* $w(\{2,3,4,6\}) = \frac{3}{2}$.
- *If X is winning coalition with four or more players, then $w(X) \geq \frac{3}{2}$.*
- *If X is losing coalition with three players, then $w(X) \in \{1, \frac{3}{2}\}$.*
- *If X is losing coalition with fewer than three players, then $w(X) \leq 1$.*

Thus G is a 2-rough game with thresholds 1 and $\frac{3}{2}$. Note that the third player has weight zero but he is not a dummy.

5 $\mathcal{C}$-hierarchy

Let us consider another extension of the idea of rough weightedness. This time we will use a threshold interval instead of a single threshold or (as in $\mathcal{B}$-hierarchy) a collection of threshold points. It is convenient to "normalize" the weights so that the left end of our threshold interval is 1. We do not lose any generality by doing this.

Definition 8. *We say that a simple game $G = (P, W)$ is in the class $\mathcal{C}_\alpha$, $\alpha \in \mathbb{R}^{\geq 1}$, if there exists a weight function $w\colon P \to \mathbb{R}^{\geq 0}$ such that for $X \in 2^P$ the condition $w(X) > \alpha$ implies that X is winning, and $w(X) < 1$ implies X is losing. Games from $\mathcal{C}_\alpha$ will be sometimes called rough$_\alpha$.*

The roughly weighted games with nonzero quota form the class $\mathcal{C}_1$. From Example 2 we can conclude that the Fano game is in $\mathcal{C}_{4/3}$ (by giving each player weight $1/3$). We also note that adding or deleting a passer does not change the class of the game.

Definition 9. *We say that a game G is* critical *for $\mathcal{C}_\alpha$ if it belongs to $\mathcal{C}_\alpha$ but does not belong to any $\mathcal{C}_\beta$ with $\beta < \alpha$.*

It is clear that if $\alpha \leq \beta$, then $\mathcal{C}_\alpha \subseteq \mathcal{C}_\beta$. However, we can show more.

Proposition 3. *Let c and d be natural numbers with $1 < d < c$. Then there is a simple game G that is rough$_{c/d}$, but that for each $\alpha < c/d$ is not rough$_\alpha$.*

Proof. Define a game $G = (P, W)$, where $P = [cd]$. Similarly to the proof of Theorem 4 we have c types of players with d players in each type and the different types do not intersect. Winning coalitions are sets with more than $c+1$ players and also sets having at least d players from the same type. By i_j we will denote the ith player of jth type.

If we assign weight $1/d$ to each player, then the lightest winning coalition (d players from the same type) has weight 1 and the heaviest losing coalition has weight c/d. Thus G belongs to $\mathcal{C}_{c/d}$.

Let us show that G is not rough$_\alpha$ for any $\alpha < c/d$. Suppose G is rough$_\alpha$ relative to a weight function w. Let $\max\{1_j, \ldots, d_j\}$ be the element of the set $\{1_j, \ldots, d_j\}$ that has the biggest weight relative to w.

For any type j we know that $w(\max\{1_j, 2_j, \ldots, d_j\}) \geq \frac{1}{d}$. The coalition

$$Y = \{\max\{1_1, \ldots, d_1\}, \ldots, \max\{1_c, \ldots, d_c\}\}$$

is losing by definition. Moreover, it has weight $w(Y) \geq c/d$. So c/d is the smallest number that can be taken as α so that G is rough$_\alpha$. □

Theorem 5. *For each $1 \leq \alpha < \beta$, it holds that $\mathcal{C}_\alpha \subsetneq \mathcal{C}_\beta$.*

Proof. We know that $\mathcal{C}_\alpha \subseteq \mathcal{C}_\beta$. If β is a rational number, then by Proposition 3 there exists a game G that is rough$_\beta$ but is not rough$_\alpha$. If β is an irrational number, then choose a rational number r, such that $\alpha < r < \beta$. By Proposition 3 there exists a game G that is rough$_r$ but is not rough$_\alpha$. So $\mathcal{C}_\alpha \subsetneq \mathcal{C}_r$. All that remains to notice is that $\mathcal{C}_r \subseteq \mathcal{C}_\beta$. □

Theorem 6. *Let G be a simple game that is not roughly weighted and is critical for $\mathcal{C}_a$. Suppose G also belongs to $\mathcal{A}_q$ for some $0 < q < \frac{1}{2}$. Then*

$$a \geq \frac{1-q}{1-2q}.$$

Proof. Obviously we can assume that q is rational. Since G is in $\mathcal{A}_q$, it possesses a certificate of nonweightedness $\mathcal{T}$ of the kind

$$\mathcal{T} = (X_1, \ldots, X_t, P^s; Y_1, \ldots, Y_t, \emptyset^s).$$

Suppose we have a weight function $w\colon P \to \mathbb{R}^{\geq 0}$ instantiating $G \in \mathcal{C}_\alpha$. Then since $w(X_i) \geq 1$ and $w(P) \geq a$, we have

$$w(X_1) + \cdots + w(X_t) + sw(P) \geq t + sa. \tag{10}$$

On the other hand, $w(Y_i) \leq a$ and

$$w(Y_1) + \cdots + w(Y_t) \leq ta. \tag{11}$$

From these two inequalities we get $t + sa \leq ta$ or $a \geq \frac{t}{t-s}$. Since $q = \frac{s}{t+s}$ we obtain $a \geq \frac{1-q}{1-2q}$, which proves the theorem. □

6 Degrees of Roughness of Games with Small Number of Players

First, we will derive bounds on the largest number $s(n)$ of the spectre Spec(n).

Theorem 7. $\frac{1}{2}\left\lfloor\frac{n}{2}\right\rfloor \le s(n) \le \frac{n-2}{2}$.

Proof. Let G be a game with n players. Without loss of generality we can assume that G doesn't contain passers. Moreover the maximal value of $s(n)$ is achieved on games that are not roughly weighted. By Proposition 1 the biggest losing coalition contains at most $n-2$ players and the smallest winning coalition has at least two players. If we assign weight $\frac{1}{2}$ to every player, then G is in $\mathcal{C}_{(n-2)/2}$.

We can use a game similar to the one from Theorem 4 to prove the lower bound. Suppose our game has n players. If n is odd, then one player will be a dummy. The remaining $2\left\lfloor\frac{n}{2}\right\rfloor$ players will be divided into $\left\lfloor\frac{n}{2}\right\rfloor$ pairs: $\{1,2\}$, $\{2,3\}$, $\ldots$, $\{m-1,m\}$, where $m = \left\lfloor\frac{n}{2}\right\rfloor$. These pairs are declared minimal winning coalitions. Given any weight function w we have $w(\max\{2i-1,2i\}) \ge \frac{1}{2}$ for each i. Then

$$w(\{\max\{1,2\},\ldots,\max\{m-1,m\}\}) \ge \frac{m}{2},$$

while this coalition is losing. So $s(n) \ge m/2$ which proves the lower bound. $\square$

Now let us calculate the spectra for $n \le 6$. By Theorem 2 all games with four players are roughly weighted. Since we may assume that the game does not have passers we may assume that the quota is nonzero. Hence we have $\mathrm{Spec}(4) = \{1\}$.

Let $G = ([n], W)$ be a simple game. The problem of finding the smallest α such that $G \in \mathcal{C}_\alpha$ is a linear programming question. Indeed, let $W^{\min}$ and $L^{\max}$ be the set of minimal winning coalitions and the set of maximal losing coalitions, respectively. We need to find the minimum α such that the following system of linear inequalities is consistent:

$$\begin{cases} w(X) \ge 1 & \text{for } X \in W^{\min}, \\ w(Y) \le \alpha & \text{for } Y \in L^{\max}. \end{cases}$$

This is equivalent to the following optimization problem:

Minimize: w_{n+1}.
Subject to: $\sum_{i\in X} w_i \ge 1$ and $\sum_{i\in Y} w_i - w_{n+1} \le 0$; $X \in W^{min}, Y \in L^{max}$.

Theorem 8. *Spec*$(5) = \left\{1, \frac{6}{5}, \frac{7}{6}, \frac{8}{7}, \frac{9}{8}\right\}$.

Proof. Let G be a critical game with five players. If G has a passer, then as was noted, the passer can be deleted without changing the class of G, hence $G \in \mathcal{C}_1$. If G has no passers and does not belong to $\mathcal{C}_1$, then it is not roughly weighted. By Theorem 2 each game that is not roughly weighted is not strong (recall Definition 2) and is not proper. Thus we have a winning coalition X such that X^c is also winning and a losing coalition Y such that Y^c is also losing.

By Proposition 1 we may assume that the cardinalities of both X and Y are 2. Without loss of generality we assume that $X = \{1,2\}$ and $X^c = \{3,4,5\}$. Note that Y cannot be contained in X^c as otherwise Y^c contains X and is not losing. So without loss of generality we assume that $Y = \{1,5\}, Y^c = \{2,3,4\}$.

We have two levels of as yet unclassified coalitions, which can be set either losing or winning:

$$\begin{aligned} &\text{level 1}: \{1,3,4\},\{1,3,5\},\{1,4,5\},\{2,3,5\},\{2,4,5\}, \\ &\text{level 2}: \{1,3\},\{1,4\},\{2,5\},\{3,5\},\{4,5\}. \end{aligned}$$

We wrote Maple code using the "LPSolve" command. First we choose losing coalitions on level 1 and delete all subsets of them from level 2. We add every unclassified coalition

α	Minimal winning coalitions and maximal losing coalitions	Weight representation
$\frac{9}{8}$	$W^{min} = \{\{1,2\},\{1,3,5\},\{1,4,5\},\{3,4,5\}\}$, $L^{max} = \{\{1,5\},\{1,3,4\},\{2,3,4\},\{2,3,5\},\{2,4,5\}\}$	$w_1 = \frac{5}{8}, w_2 = \frac{3}{8}, w_5 = \frac{4}{8}$, $w_3 = w_4 = \frac{2}{8}$
$\frac{8}{7}$	$W^{min} = \{\{1,2\},\{2,5\},\{1,3,4\},\{3,4,5\}\}$, $L^{max} = \{\{1,3,5\},\{1,4,5\},\{2,3,4\}\}$	$w_1 = w_5 = \frac{3}{7}, w_2 = \frac{4}{7}$, $w_3 = w_4 = \frac{2}{7}$
$\frac{7}{6}$	$W^{min} = \{\{1,2\},\{1,4,5\},\{3,4,5\}\}$, $L^{max} = \{\{1,3,4\},\{1,3,5\},\{2,3,4\},\{2,3,5\},\{2,4,5\}\}$	$w_1 = w_2 = \frac{3}{6}$, $w_3 = w_4 = w_5 = \frac{2}{6}$
$\frac{6}{5}$	$W^{min} = \{\{1,2\},\{1,3\},\{1,4\},\{2,5\},\{3,5\},\{4,5\}\}$, $L^{max} = \{\{1,5\},\{2,3,4\}\}$	$w_1 = w_5 = \frac{3}{5}$, $w_2 = w_3 = w_4 = \frac{2}{5}$

Table 1: Examples of critical simple games for every number of 5th spectrum

from level 1 to winning coalitions. After that we choose losing coalitions on level 2. We run through all possible combinations of losing coalitions on both levels and solve the respective linear programming problems.

The results of these calculations are displayed in Table 1. □

Theorem 9. *The 6th spectrum Spec(6) is the set*

$$\left\{1, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \frac{7}{6}, \frac{8}{7}, \frac{9}{7}, \frac{9}{8}, \frac{10}{9}, \frac{11}{9}, \frac{11}{10}, \frac{12}{11}, \frac{13}{10}, \frac{13}{11}, \frac{13}{12}, \frac{14}{11}, \frac{14}{13}, \frac{15}{13}, \frac{15}{14}, \frac{16}{13}, \frac{16}{15}, \frac{17}{13}, \frac{17}{14}, \frac{17}{15}, \frac{17}{16}, \frac{18}{17}\right\}.$$

Proof. Is omitted due to lack of space. The code and the list of critical games are available from the authors. □

7 Conclusion and Further Research

Economics has studied extensively weighted majority games. This class was previously extended to the class of roughly weighted games [9, 4]. However, many games are not even roughly weighted and some of these games are important both for theory and applications. In this paper we introduce three hierarchies, each of which partitions the class of games without rough weights according to some parameter that can be viewed as capturing some resource - either a measure of our flexibility on the size and structure of the tie-breaking region or allowing certain types of certificates of nonweightedness. It is important to look for further connections between the classes of the three hierarchies, and we commend that direction to the interested reader.

In this paper we studied only the $\mathcal{C}$-spectrum here. Some interesting questons about this spectrum still remain, especially the bounds for $s(n)$ are of considerable interest. It is interesting to study both the $\mathcal{A}$-spectrum and $\mathcal{B}$-spectrum as well.

Acknowledgments

During a talk on roughly weighted games at the March 2010 Dagstuhl Seminar on Computational Foundations of Social Choice, Bill Zwicker asked whether those games were merely the start of some hierarchy. That question motivated this paper, and we are deeply grateful to Bill both for that question and for encouragement and helpful conversations. We are also very grateful to Edith Elkind for helpful conversations.

Hemaspaandra was partly supported by NSF grants CCF-0426761 and CCF-0915792 and a Friedrich Wilhelm Bessel Research Award from the Alexander von Humboldt Foundation. Gvozdeva and Slinko were supported by the University of Auckland Faculty of Science Research Development Fund 3624495/9844.

References

[1] A. Beimel, T. Tassa, and E. Weinreb. Characterizing Ideal Weighted Threshold Secret Sharing. In: *Theory of Cryptography.* Lecture Notes in Computer Science. Springer Berlin / Heidelberg, Volume 3378/2005: 600–619, 2005

[2] G.R. Blakley. Safeguarding cryptographic keys. *Proceedings of the National Computer Conference* 48: 313–317, 1979.

[3] C.C. Elgot. Truth Functions Realizable by Single Threshold Organs, *Proc. of the Second Annual Symposium on Switching Circuit Theory and Logical Design*, 225–245, 1961.

[4] T. Gvozdeva and A. Slinko. Weighted and Roughly Weighted Simple Games, Mathematical Social Sciences. to appear.

[5] A.D. Korshunov. Monotone Boolean functions. *Russian Math. Surveys* 58(5): 929–1001, 2003.

[6] S. Muroga. *Threshold logic and Its Applications.* Wiley Interscience, NY, 1971.

[7] M. Richardson. On finite projective games. *Proceedings of the American Mathematical Society* 7: 458–465, 1956.

[8] A.D. Taylor and W.S. Zwicker. A characterization of weighted voting. *Proceedings of the American Mathematical Society* 115: 1089–1094, 1992.

[9] A.D. Taylor and W.S. Zwicker. *Simple games.* Princeton University Press. Princeton. NJ, 1999.

[10] J. von Neumann and O. Morgenstern. *Theory of games and economic behavior. Third edition.* Princeton University Press. Princeton, NJ, 1953.

[11] L.S. Shapley. Simple Games: An Outline of the Descriptive Theory. *Behavioral Science* 7: 59–66, 1962.

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