Domains of social choice functions on which coalition strategy-proofness and Maskin monotonicity are equivalent[1]

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Abstract

It is known that on some social choice and economic domains, a social choice function is coalition strategy-proof if and only if it is Maskin monotonic. This equivalence provides a computational merit: Replacing coalition strategy-proofness with Maskin monotonicity significantly reduces the time required to check whether the social choice function is coalition strategy-proof, or not. This paper studies the foundation of those equivalence results. I provide a set of conditions which is sufficient for the equivalence between coalition strategy-proofness and Maskin monotonicity. Further, applying these conditions, I provide a class of domains, called "essentially strict domains," on which this equivalence holds. This yields some known and new results. An "essentially strict domain" is a domain such that each individual is endowed with a partition over the set of alternatives, and the preferences admissible to this individual are exactly the strict rankings over this partition.

Keywords— social choice function, coalition strategy-proofness, Maskin monotonicity.

0   Introduction

This paper examines logical relations between coalition strategy-proofness and Maskin monotonicity of social choice functions. Coalition strategy-proofness is a strong requirement of incentive compatibility. A social choice function is said to be **coalition strategy-proof** if no group of individuals can benefit from jointly misrepresenting their preferences, in other words, cannot manipulate the final outcome. A social choice function is said to be **Maskin monotonic** if the outcome to be chosen by the function does not vary whenever each individual switches his preference keeping or improving the relative ranking of that outcome. This property is very important in implementation theory. For example, it is well-known as a necessary condition for Nash implementation (see Maskin, 1985, 1999).

It has been observed that these two properties are strongly related to each other. The classical result by Muller and Satterthwaite (1977) asserts that on the unrestricted strict preference domain, a social choice function is strategy-proof if and only if it is Maskin monotonic.[2] Since on this domain, strategy-proofness is equivalent to coalition strategy-proofness, the theorem states the equivalence between coalition strategy-proofness and Maskin monotonicity.

In more recent studies, it has been pointed out that coalition strategy-proofness and Maskin monotonicity are equivalent even in some environments where coalition strategy-proofness is strictly stronger than strategy-proofness. Takamiya (2001, 2003) pointed out the equivalence between coalition strategy-proofness and Maskin monotonicity holds for allocation rules in a certain broad class of economies with indivisible goods, which includes some notable problems such as "marriage problems" (Gale and Shapley, 1962) and "housing markets" (Shapley and Scarf, 1974). (See also Svensson (1999) for related results.) Also in some other environments, for example the classical exchange economies, it is known that coalition strategy-proofness implies Maskin monotonicity (e.g. Barberà and Jackson, 1995).

There is an evident **computational** merit in replacing coalition strategy-proofness with Maskin monotonicity, in particular in such cases where coalition strategy-proofness is strictly stronger than strategy-proofness. To check whether the function is coalition strategy-proof by the straightforward method, one needs to check whether each **coalition** has a chance of manipulation. On the other hand, to check whether the function is Maskin monotonic or not, it only requires to check for each **individual** whether a monotonic change of his preference alters the outcome to be outputted. Evidently, the number of coalitions grows exponentially as the number of individuals grows. Thus checking coalition strategy-proofness requires exponentially larger time than checking Maskin monotonicity. For this reason, replacing coalition strategy-proofness with Maskin monotonicity significantly reduces the time required to check whether the social choice function is coalition strategy-proof, or not.

The purpose of this paper is to study the foundation of these close relationships between coalition strategy-proofness and Maskin monotonicity. I examine what conditions the domain of the social choice function should satisfy in order to have the property that coalition strategy-proofness implies Maskin monotonicity, and its converse.

The main results of this paper provide two sufficient conditions. The first condition, which is referred to as Condition A, is a sufficient condition for that coalition strategy-proofness implies Maskin monotonicity. This condition requires the domain of the social choice function to satisfy two properties. The first property says that for any individual, and any preference admissible to him, if any two alternatives are indifferent under this preference, then these alternatives are indifferent under all the preferences admissible to him. In other

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[2] "Strategy-proofness" requires that the social choice function cannot be manipulated by any single individual. That is, unlike coalition strategy-proofness, manipulations by groups are not necessarily ruled out.

words, this requires that every individual has a partition of the set of alternatives, and his admissible preferences contains only (but not necessarily all of) such preferences that any two alternatives are indifferent if, and only if, these alternatives are in the same cell of the partition. The second property that Condition A requires is that if for any preference profile in the domain, there is no pair of alternatives such that all individuals are indifferent between them. That is to say, if the alternative to be chosen shifts from one to another, then there is always someone who cares about this shift. This is equivalent to the requirement that the "join" (the coarsest common refinement) of the partitions of all the individuals that arise in the first half of this condition equals to the finest partition (i.e. the one in which each cell contains exactly one element).

The second condition, referred to as Condition B, is a sufficient condition for that Maskin monotonicity implies coalition strategy-proofness. This condition is defined as follows: Let any coalition be given. And pick up any preference profile for this coalition, which I call the first profile. Then let us fix any two alternatives, say x and y, such that y (weakly) Pareto dominates x within this coalition under the first profile. Further, pick up another arbitrary preference profile for this coalition, the second profile. Then this domain satisfies Condition B if the domain contains at least one preference profile for this coalition such that x keeps or improves its relative ranking from the first profile to this profile, and so does y from the second profile to this profile. Speaking very roughly, the third profile is a mixture of the first and the second profiles in the sense of the desirability of x and y. And Condition B requires such a mixture always exists.

Given these two sufficient conditions, we present a class of domains which satisfies both of these conditions. This class is the collection of those domains in which (i) every individual has a partition of the set of alternatives, and his admissible preferences are **exactly** such preferences that any two alternatives are indifferent if, and only if, these alternatives are in the same cell of the partition, and (ii) the join of all these partitions equals to the finest partition. Paraphrasing, such a domain is the maximal domain among those satisfying Condition A, given a list of partitions. I call these domains **essentially strict domains**.

I point out that essentially strict domains are assumed in some previous results. This observation unifies the the Muller-Satterthwaite theorem and the similar equivalence theorem by Takamiya (2003) in the context of the "generalized indivisible good allocation problems" (Sönmez, 1999), which cover various problems including well-known "housing markets" (Shapley and Scarf, 1974) and "marriage problems" (Gale and Shapley, 1962).

Furthermore, I present an application of the result regarding essentially strict domains, which, to my knowledge, has never been reported elsewhere.

1 Preliminaries

Let N denote the set of **individuals**. Assume that N is a nonempty finite set. Call any nonempty subset of N a **coalition**. Let X be the set of **alternatives** (social outcomes). X is nonempty and may be finite or infinite.

Let Q be a nonempty set. Then denote by $W(Q)$ the set of weak orderings (i.e. complete and transitive binary relations) on Q. And denote by $L(Q)$ the set of linear orderings (i.e. complete, tansitive and anti-symmetric binary relations) on Q.

For $i \in N$, call $R^i \in W(Q)$ a **preference relation** on Q of individual i. And a list $(R^i)_{i \in N}$ is called a **preference profile**. For $x, y \in Q$, xR^iy reads that to individual i, x is at least as good as y. As usual, P^i denotes the asymmetric part, and I^i denotes the symmetric part of R^i. Let $R^i \in W(Q)$ and $Q' \subset Q$. Then $\max R^i(Q)$ denotes the set of R^i-maximal elements in Q', $\{x \in Q' \mid \forall y \in Q', xR^iy\}$.

For $i \in N$, D^i denotes the set of **admissible preferences** of individual i. Assume that $D^i \subset W(X)$ for any $i \in N$. For $S \subset N$, D^S denotes the Cartesian product $\prod_{i \in S} D^i$. A **social choice function** (SCF) is a single-valued function $f : D^N \to X$. D^N is called the **domain** (of f).

Let f be a SCF. Let S be a coalition. Then we say that S manipulates f at a preference profile $R \in D^N$ if there exists some $R'^S \in D^S$ such that

$$[\forall i \in S, \ f(R^{-S}, R'^S)R^i f(R)] \ \& \ [\exists j \in S : \ f(R^{-S}, R'^S)P^j f(R)]. \tag{1}$$

Call f **coalition strategy-proof** if no coalition manipulates f at any $R \in D^N$.

For $R^i \in W(X)$, and $x \in X$, denote by $L(x, R^i)$ the set $\{y \in X \mid xR^iy\}$. That is, $L(x, R^i)$ is the lower-contour set of x relative to R^i. Call f **Maskin monotonic** if for any $i \in N$, any $R \in D^N$, and any $R^i, R'^i \in D^i$,

$$[f(R) = x \ \& \ L(x, R^i) \subset L(x, R'^i)] \Rightarrow f(R^{-i}, R'^i) = x. \tag{2}$$

We note that checking coalition strategy-proofness requires exponentially larger time than checking Maskin monotonicity: Let us bound the number of possible preferences of each individual, that is, for each i, $|D^i| \leq c$. And let the number of individuals, denoted by n, vary. Then checking coalition strategy-proofness requires to check $O(2^n c^{2n})$ times the relation (1) in the above. On the other hand, checking Maskin monotonicity requires to check the relation (2) $O(nc^{n+1})$ times. Although the both grow exponentially as n gets larger (this is because the preference domain grows larger exponentially), the former still grows exponentially faster with relative to the latter. This fact gives a **computational merit** in replacing coalition strategy-proofness with Maskin monotonicity.

2 Main results

This section presents the main results. These results provide sufficient conditions that the domain of the social choice function should satisfy to have the

property that coalition strategy-proofness implies Maskin monotonicity, and its converse. I introduce these two sufficient conditions. Let a SCF $f : D^N \to X$ be given.

Let $\mathcal{P}^i$ be a partition of X. For $x \in X$, let us denote the cell of $\mathcal{P}^i$ that contains x by $\mathcal{P}^i(x)$. Let $(\mathcal{P}^i)_{i \in N}$ be a profile of partitions of X. And let us denote by "$\bigvee$" the operation of taking the "join" (the coarsest common refinement) of the partitions.

Conditon A. There exists some profile of partitions $(\mathcal{P}^i)_{i \in N}$ such that for any $R \in D^N$, any $i \in N$ and any $x, y \in X$,

$$x \in \mathcal{P}^i(y) \Leftrightarrow xI^iy, \tag{3}$$

and

$$\bigvee_{i \in N} \mathcal{P}^i = \{\{x\} \mid x \in X\}. \tag{4}$$

In words, Condition A consists of two parts, which correspondence to the formulas (3) and (4), respectively:
(i) For any individual, and for any preference admissible to him, any two alternatives are indifferent under this preferences if, and only if, these alternatives are indifferent under all the preferences admissible to him; and
(ii) If for any preference profile, for any pair of alternatives there is at least one individual who is not indifferent between these alternatives.

Condition B. For any $S \subset N$ with $S \neq \emptyset$, any $\tilde{R}^S, \hat{R}^S \in D^S$, and any $x, y \in X$ such that $(\forall i \in S, \ y\tilde{R}^ix)$ and $(\exists i \in S : \ y\tilde{P}^ix)$, there exists $R^{\star S} \in D^S$ such that

$$\forall i \in S, \ L(x, \tilde{R}^i) \subset L(x, R^{\star i}) \ \& \ L(y, \hat{R}^i) \subset L(y, R^{\star i}). \tag{5}$$

In words, Condition B condition is defined as follows: Let any coalition be given. And pick up any preference profile for this coalition, which I call the first profile. Then let us fix any two alternatives, say x and y, such that y (weakly) Pareto dominates x within this coalition under the first profile. Further, pick up another arbitrary preference profile for this coalition, the second profile. Then the domain satisfies Condition B if the domain contains at least one preference profile for this coalition such that x keeps or improves its relative ranking from the first profile to this profile, and so does y from the second profile to this profile. Roughly speaking, the third profile is a mixture of the first and the second profiles in the sense of the desirability of x and y. And the condition requires such a mixture always exists.

Theorem 1 *Let D^N satisfy Condition A. Then if f is coalition strategy-proof, then f is Maskin monotonic.*

Proof Suppose that D^N satisfies Condition A and that f is not Maskin monotonic. Then I will show that f is not coalition strategy-proof. Since f is not Maskin monotonic, we have for some $i \in N$, some $R \in D^N$, and some $\tilde{R}^i \in D^i$,

$$L(x, R^i) \subset L(x, \tilde{R}^i) \;\&\; f(R^{-i}, \tilde{R}^i) \neq x, \tag{6}$$

where x denotes the alternative $f(R)$. Let us denote $f(R^{-i}, \tilde{R}^i)$ by y. Since D^N satisfies Condition A, there must be at least one individual j such that $xP^j y$ or $yP^j x$. Thus the set $T = \{j \in N \mid \neg x I^j y\}$ is nonempty. Then there are two cases.

(i) Assume that $i \in T$. Then either $xP^i y$ or $yP^i x$. Suppose that $xP^i y$ holds true. Then $L(x, R^i) \subset L(x, \tilde{R}^i)$ and Condition A together imply $x\tilde{P}^i y$. That is, $f(R^{-i}, R^i)\tilde{P}^i f(R^{-i}, \tilde{R}^i)$, which says i manipulate at $(R^{-i}, \tilde{R}^i)$ by reporting R^i. Thus f is not coalition strategy-proof.

In turn, suppose that $yP^i x$ holds true. Then similarly, i manipulate at R by reporting $\tilde{R}^i$, which violates coalition strategy-proofness again.

(ii) Assume that $i \notin T$. Then $xI^i y$. Let $j \in T$, which means either $xP^j y$ or $yP^j x$. Suppose that $xP^j y$. Then similarly to the case (i), $\{i, j\}$ manipulates at $(R^{-\{i,j\}}, \tilde{R}^i, R^j)$ by reporting (R^i, R^j). In turn suppose that $yP^j x$. Then $\{i, j\}$ manipulates at $(R^{-\{i,j\}}, R^i, R^j)$ by reporting $(\tilde{R}^i, R^j)$. In either way, f is not coalition strategy-proof. $\square$

Theorem 2 *Let D^N satisfy Condition B. Then if f is Maskin monotonic, then f is coalition strategy-proof.*

Proof Suppose that D^N satisfies Condition B and that f is Maskin monotonic but not coalition strategy-proof. Then there is some coalition S which manipulates at some $\tilde{R} \in D^N$ by reporting $\hat{R}^S \in D^S$. Let us denote the alternative $f(\tilde{R})$ by x, and $f(\tilde{R}^{-S}, \hat{R}^S)$ by y. Then clearly, $\forall i \in S$, $y\tilde{R}^i x$ and $\exists i \in S : y\tilde{P}^i x$. Thus Condition B implies that there is $R^{\star S} \in D^S$ such that for all $i \in S$,

$$L(x, \tilde{R}^i) \subset L(x, R^{\star i}), \tag{7}$$
$$L(y, \hat{R}^i) \subset L(y, R^{\star i}). \tag{8}$$

Since f is Maskin monotonic, (6) implies $f(\tilde{R}^{-S}, R^{\star S}) = x$. On the other hand, (7) implies $f(\tilde{R}^{-S}, R^{\star S}) = y$. These imply $x = y$, a contradiction. $\square$

3 Further results

This section provides a class of domains which satisfies both Conditions A and B. Then applying the results presented in Section 2, some previous and new results will be derived.

Essentially strict domain. Let $(\mathcal{P}^i)_{i \in N}$ be a profile of partitions of X. Then D^N is said to be the **essentially strict domain with respect to** $(\mathcal{P}^i)_{i \in N}$ if D^N is the collection of all the preference profiles R that satisfy for any $x, y \in X$,

$$x \in \mathcal{P}^i(y) \Leftrightarrow x I^i y, \tag{9}$$

and $(\mathcal{P}^i)_{i \in N}$ satisfies

$$\bigvee_{i \in N} \mathcal{P}^i = \{\{x\} \mid x \in X\}. \tag{10}$$

To paraphrase, the essentially strict domain with respect to $(\mathcal{P}^i)_{i \in N}$ is the (inclusion) maximal domain among those which satisfy Condition A given $(\mathcal{P}^i)_{i \in N}$.

It is less obvious that such a domain satisfies Condition B.

Lemma 1 *If D^N is the essentially strict domain with respect to $(\mathcal{P}^i)_{i \in N}$, then D^N satisfies Condition B.*

Proof Let $S \subset N$ with $S \neq \emptyset$, and $x, y \in X$. Let $\tilde{R}^S \in D^S$ such that $(\forall i \in S, \ y \tilde{R}^i x)$ & $(\exists i \in S, \ y \tilde{P}^i x)$, and $\hat{R}^S \in D^S$. To show that D^N satisfies Condition B, we will give $R^{\star S} \in D^S$ such that

$$\forall i \in S, \ L(x, \tilde{R}^i) \subset L(x, R^{\star i}) \ \& \ L(y, \hat{R}^i) \subset L(y, R^{\star i}). \tag{11}$$

Let $S^+ = \{i \in S \mid y \tilde{P}^i x\}$, and $S^0 = \{i \in S \mid y \tilde{I}^i x\}$. Let $R^{\star S}$ be such that for each $i \in S^+$,

$$\max R^{\star i}(X) = \mathcal{P}^i(y), \ \text{and} \tag{12}$$

$$\max R^{\star i}(X \setminus \max R^{\star i}(X)) = \mathcal{P}^i(x), \tag{13}$$

and for each $i \in S^0$,

$$\max R^{\star i}(X) = \mathcal{P}^i(y). \tag{14}$$

Note that for $i \in S^0$, $\mathcal{P}^i(x) = \mathcal{P}^i(y)$. Then evidently, $L(y, R^{\star i}) = X$ for any $i \in S$; $L(x, R^{\star i}) = X$ for any $i \in S^0$; and $L(x, R^{\star i}) = X \setminus \mathcal{P}^i(y)$ for any $i \in S^+$. Note that $L(x, \tilde{R}^i) \cap \mathcal{P}^i(y) = \emptyset$. Thus $R^{\star S}$ satisfies (11). $\square$

Now we obtain the following result applying Theorems 1 and 2.

Theorem 3 *Let D^N be an essentially strict domain. Then f is coalition strategy-proof if, and only if, f is Maskin monotonic.*

In the following, we will derive two known results and one new result as special cases of Theorem 3. First, let the partition profile $(\mathcal{P}^i)_{i \in N}$ be such that for each i, $\mathcal{P}^i$ is $\{\{x\} \mid x \in X\}$. Then the essentially strict domain with respect to $(\mathcal{P}^i)_{i \in N}$ coincides with $L(X)^N$. This yields the well-known Muller-Satterthwaite theorem.

Corollary 1 (Muller and Satterthwaite, 1977) *Let $D^N = L(X)^N$. Then f is coalition strategy-proof if, and only if, it is Maskin monotonic.*

The second application is for "generalized indivisible good allocation problems," as defined in Sönemz (1999). This class of allocation problems contains well-known "marriage problems" (Gale and Shapley, 1962) and "housing markets" (Shapley and Scarf, 1974) as special cases. A **generalized indivisible good allocation problem** (an **allocation problem**, henceforth) is a list $(N, \Omega, \mathcal{A}, R)$.[3] Here N is the set of individuals, as we have defined in Section 1. Ω is the set of goods, which is assumed to be a nonempty finite set. An "allocation" is a set-valued function $x : N \rightarrow\rightarrow \Omega$ such that $\{x(i) \mid i \in N\}$ is a partition of Ω. $\mathcal{A}$ is the set of feasible allocations. And R is a preference profile belonging to the domain D^N defined as follows:

$$D^N := \{R \mid \forall i \in N, \ R^i \in W(\mathcal{A}) \ \& \ (\forall x, y \in \mathcal{A}, \ x I^i y \Leftrightarrow x(i) = y(i))\}. \quad (15)$$

That is, every individual has preferences that exhibit no consumption externalities, and are strict over their own assignments.

Obviously, in this case, D^N is an essentially strict domain with respect to $(\mathcal{P}^i)_{i \in N}$, where for each $i \in N$, $\mathcal{P}^i$ is the partition such that for any $x, y \in \mathcal{A}$, $x \in \mathcal{P}(y) \Leftrightarrow x(i) = y(i)$.

In this setting, we consider the set of allocation problems $\{(N, \Omega, \mathcal{A}, R) \mid R \in D^N\}$, and SCFs $f : D^N \rightarrow \mathcal{A}$. Then we have the following known result as a corollary to Theorem 3.

Corollary 2 (Takamiya, 2003) *Let f be a SCF in a setting of allocation problems. Then f is coalition strategy-proof if, and only if, it is Maskin monotonic.*

The third application is a class of settings which are interpreted as environments of "distributive work." Consider a domain D^N defined as follows: Fix an individual $d \in N$. And assume $D^d = L(X)$. This yields the partition $\mathcal{P}^d = \{\{x\} \mid x \in X\}$ which satisfies (9). For each $i \in N \setminus \{d\}$, let $\mathcal{P}^i$ be an arbitrary partition of X. Then let D^N be the essentially strict domain with respect to $(\mathcal{P}^i)_{i \in N}$. In words, D^N is an essentially strict domain in which at least one individual has exactly all the linear orderings over X as the admissible preferences.

This domain has an interpretation like the following: Imagine a team of people who are to complete one complex work. There is one distinct individual called a "director" who takes care of the whole picture. Each of the other members of the team is assigned his part of the work for which he takes the responsibility, and he only cares about the components of outcomes relevant to the part assigned to him. Thus he is indifferent over any two outcomes between which the components relevant to his part are the same. But the director takes care of all components thus his admissible preferences are strict over all outcomes.

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[3] The complete definition of the problem includes initial endowments, which are superfluous for the present purpose thus dropped here.

Corollary 3 *Let f be a SCF in a setting of the distributive work environment. Then f is coalition strategy-proof if, and only if, it is Maskin monotonic.*

To my knowledge, this result has never been reported elsewhere. This shows that the introduction of essentially strict domains not only synthesizes known results but produces some new insights.

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