# The Probability of Sen's Liberal Paradox

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**Abstract**

This paper determines the probability of a conflict between acyclicity, weak Pareto, and minimal liberalism in a relatively unrestricted domain. It seems reasonable to hypothesize that the probability of a conflict between these three properties decreases as the number of individuals increases. If this were the case, Sen's Liberal Paradox would be of greater concern in small populations, such as committees, than in large populations, such as nation states. However, we conduct several numerical computations and draw the opposite conclusion. Increasing the number of individuals or the number of decisive alternative pairs increases the probability of a conflict between acyclicity, weak Pareto, and minimal liberalism, suggesting that the paradox forming preferences are not only possible in democracies, they may be probable.

## 1 Introduction

Sen's Liberal Paradox (1970) shows a fundamental conflict between liberty and democracy. Although it has been widely known that majorities can tyrannize minorities (Mill [1859]2006; Hamilton et al. [1788]1961), Sen's paradox shows that a social choice function cannot simultaneously satisfy minimal liberalism and weak Pareto over an unrestricted domain. This is akin to showing a conflict between an individual's ability to determine very limited outcomes for themselves (such as whether they sleep on their belly or back, everything else equal) and unanimous decision making – two properties that cannot come into direct conflict if liberal rights are properly assigned. The conflict emerges because the preservation of these two principles can lead to a violation of acyclicity (a necessary condition for a social choice function).

Previous studies have shown that strong restrictions on the domain of preferences (Blau 1975; Farrell 1976; Sen 1979) can eliminate the paradox. However, the frequency with which acyclicity, minimal liberalism, and weak Pareto come into conflict in a relatively unrestricted domain is an open question. If the probability of a conflict is small, then the implications of the liberal paradox are limited. The conflict between these conditions exist for some preferences, but they would not be frequent enough to cause alarm for democracy. If the probability of a conflict is large, then it would be difficult to promote specific types of liberal values and the weak Pareto criterion at the same time, as Sen suggests.

This paper attempts to determine the probability that a set of individual preferences will cause a conflict between acyclicity, minimal liberalism, and weak Pareto over a finite set of alternatives and a finite number of individuals. In this sense, it captures how likely paradox creating preferences exist within the domain of all possible preferences. This question is similar to the question asked by Niemi & Weisburg (1968), Caplin & Nalebuff (1988), and Gehrlein (2002) about the probability of voting cycles. Unlike many of these studies, this paper estimates the probability of a conflict between three properties using simulations. These simulations are based on preferences drawn from a multidimensional spatial voting model. The advantage of a multidimensional spatial voting model is that it can be used to sample a large variety of preferences where non-acyclic social rankings are expected to occur (McKelvey 1976; Schofield 1978). The advantage of simulation is that it allows for the calculation of probabilities that may be mathematically intractable.

One might hypothesize that as the number of voters, or the number of alternatives, increases, the probability of a conflict between acyclicity, weak Pareto, and minimal liberalism decreases. As such Sen's Liberal Paradox would be of greater concern in small populations, such as committees, than in large populations, such as nation states. Using numerical computations, our preliminary results suggest the opposite. The properties of Sen's Liberal Paradox are often in conflict, suggesting that his conundrum has the potential of being pervasive.

## 2 Sen's Theorem

Following the notation used by Sen (1970, 1979), let the binary relation $xPy$ indicate society's strict preference for $x$ over $y$; $xRy$ indicate that society prefers $x$ at least as much as $y$; and $xIy$ indicate that society is indifferent between the two alternatives. Similar relations can be defined for individuals using the subscript $i$.

A weak requirement in social choice theory is that social preference relations should generate a "choice set," that is, in every set of alternatives $S$, a subset of the full set of alternatives $X$, there must be a "best" alternative. A best alternative (there may be more than one) is an alternative that is at least as preferred as all other alternatives in that subset. The function that creates such a choice set is called a social decision function. Sen (1979) notes that if a preference relation is reflexive and complete,[1] then a necessary and sufficient condition for the existence of a finite choice set is acyclicity. This condition is central to the proof of his paradox.

**Definition 1** Acyclicity ($\bar{A}$)*: A social ordering is acyclical over X if and only if:*
$\forall x_1, \ldots, x_j \in X : [\{x_1 P x_2 \ \&\ x_2 P x_3 \ \&\ \ldots \ \&\ x_{j-1} P x_j\} \rightarrow x_1 R x_j]$.

[1] A preference relations is reflexive if and only if $\forall x \in S : xRx$. A preference relation is complete if and only if $\forall x, y \in S : (x \neq y) \rightarrow (xRy$ or $yRx)$.

**Definition 2** Unrestricted Domain (U)*: Every logically possible set of individual orderings is included in the domain of the social decision function.*

**Definition 3** Weak Pareto ($\bar{P}$)*: If every individual prefers alternative $x$ to alternative $y$, then society must prefer $x$ to $y$.*

**Definition 4** Minimal Liberalism ($L^*$)*: There are at least two individuals such that for each of them there is at least one pair of alternatives over which he/she is decisive, that is there is a pair $\{x, y\}$ such that if he/she prefers $x$ (respectively $y$) to $y$ (respectively $x$), then society should prefer $x$ (respectively $y$) to $y$ (respectively $x$).*[2]

The purpose of the last condition is to assure that at least two individuals are able to make one social choice, such as determining whether their walls will be pink rather than white or whether they will sleep on their belly or their back, everything else equal. With these definitions, Sen shows the following impossibility theorem.

**Theorem 1** *There is no social decision function that can simultaneously satisfy U, $\bar{P}$, and $L^*$.*

Sen completes his proof by showing that there exists a set of preferences which cause a contradiction between $\bar{A}$, $\bar{P}$, and $L^*$. This is done for two individuals with 1) non-overlapping pairs, 2) overlapping pairs with one element in common, and 3) overlapping pairs with two elements in common. The proof shows that conundrum causing preferences can occur within an unrestricted domain. Since two individuals with decisive rights over one pair of alternatives each are a subset of a larger population with more decisive rights, the theorem applies to any number of individuals and any number of alternatives.

However, the theorem does not guarantee that conundrum causing preferences will occur for all elements in the domain. As Sen writes, "The dilemma posed here may appear to be somewhat disturbing. It is, of course, not necessarily disturbing for every conceivable society, since the conflict arises with only particular configurations of individual preferences" (Sen 1970, 155). To see this, consider a society with two individuals, Muddy and Billie, who can choose over three alternatives $x$, $y$, and $z$. Muddy prefers $xP_iyP_iz$ and Billie prefers $yP_ixP_iz$. If Muddy is decisive over $\{x, y\}$ and Billie is decisive over $\{y, z\}$, then $xPy$ and $yPz$ by $L^*$. Furthermore, $xPz$ by $\bar{P}$. These social preferences maintain $\bar{A}$.

# 3 The Probability of Sen's Conundrum

After noting that paradox causing preferences occur for *only some* preferences in the domain, the natural question is how likely are such preferences? Is

[2] Sen introduces the stronger condition of Liberalism which requires that "every" individual be decisive over at least on pair of alternatives. Of course, the theorem can be shown with the stronger condition, as well.

the probability of a conflict between $\bar{A}$, $\bar{P}$, and $L^*$ affected by the size of the population and the number of alternatives available?

One reason these paradoxes might be *less* likely in larger populations than in smaller ones is that the probability of a Pareto preferred alternative decreases as the size of the population increases (Dougherty & Edward 2005). Hence, the probability of a conflict may diminish because Pareto preferred alternatives are less likely to arise. Furthermore, Blau (1975) shows that for the case of two individuals and four alternatives, only 4 of the $75^2$ possible configurations of preferences would cause a conflict. He conjectures that this probability will decrease as the number of individuals increases.

One reason these paradoxes might be *more* likely in larger populations than smaller ones is that the probability of intransitivity may increase as the size of the population increases (Niemi & Weisberg 1968).[3] Hence, we are more likely to violate acyclicity in larger populations than in smaller ones.

To determine the probability of a conflict between $\bar{A}$, $\bar{P}$, and $L^*$, we conduct a series of probability experiments using multidimensional spatial voting models and a self-written C program. In the first two sets of experiments, we assume there are $N$ individuals choosing among $A$ alternatives in a multidimensional outcome space. Each individual has an ideal point $I_i$ with Euclidean preferences. This implies that each individual prefers alternatives closer to their ideal point more than alternatives farther away. Although these assumptions do not allow for all possible combinations of preferences, they are sufficiently general to make non-acyclic social preferences likely (McKelvey 1976; Schofield 1978). Furthermore, single peaked and symmetric preferences are common in the political science literature (Poole 2005; Tsebelis 2002; Stewart 2001). Hence, they presumably model populations that researches believe occur.

## 3.1 Two Dimensions, Continuous Outcome Space

In our first probability experiment, the simulation proceeds as follows. For each trial, the program randomly draws $N$ ideal points and $A$ alternatives from a compact unit square. As our baseline study, we assume that both $N$ and $A$ are uniformly distributed. This helps to create something similar to Gehrlein's (2002) impartial culture condition, which makes all orderings equiprobable. We have also considered other distributions,[4] which will be more fully considered in future research. The program then determines individual preferences for each alternative based on which alternative is closest to the individual's ideal point and ascertains whether any alternative is Pareto preferred to another. If a Pareto preferred alterantive exists, the pair-wise preference is recorded in an $A \times A$ matrix of social preferences to indicate that the alternative numbered

[3] See Gehrlein (2002) for conclusions to the contrary.

[4] Drawing ideal points from two normally distributed clusters with means of $(.25, .25)$ and $(.75, .75)$ and a standard deviation of .10 produced smaller probabilities of a conflict than those in Table 1. However, the probability of a contradiction remained roughly 1.0 for $da = 20$ and $N \geq 41$. Presumably, the homogeneity of ideal points explains the smaller probability of a contradiction.

the same as the row index is preferred to the alternative numbered the same as the column index.

In the next phase of the trial, the program randomly assigns decisive rights, without replacement, to $N$ individuals and updates the social ranking based on those rights.[5] The number of decisive alternative pairs, da, assigned to each of the $N$ individuals is fixed as an input. If $N = 3$ and $da = 10$, then 30 pairs of alternatives are randomly assigned to the social order by one of the individual's decisive rights. In this experiment, no more than one individual is allowed to have decisive rights over the same pair of alternatives, even though Sen allowed such cases in the proof of his theorem.[6] Overlapping decisive rights are allowed. For example, Muddy and Billie cannot both be decisive over the pair $\{x, y\}$, but Muddy can be decisive over $\{x, y\}$, while Billie is decisive over $\{y, z\}$.

In the final phase of the trial, we test the strict transitivity of the social preferences and fill in preferences that can be deduced by transitivity.[7] Although testing for acyclicity directly may be more appropriate, the computational problems associated with considering triples, quadruples, quintuples, etc. for the antecedent of definition 1, makes the use of acyclicity computationally inefficient. Instead, strict transitivity is used as a rough approximation for acyclicity. This works because the probability of indifference is almost zero for a uniform distribution of alternatives in spatial models that do not allow thick indifference curves. Computations confirm this.

The program tests the strict transitivity of the social ranking and updates the social ranking as needed. If strict transitivity is violated, the trial is terminated and the contradiction is noted. If there is no contradiction, the program continues to update social preferences and tests for transitivity until their is a contradiction or it is clear that the social preference ranking does not violate transitivity. To assure that all of the deductive steps are incorporated in the test for transitivity, the evaluation is conducted once more after all transitive relationships have been updated.

Such trials are repeated to determine the relative frequency that $\bar{A}$, $\bar{P}$, and $L^*$ are in contradiction for a specific $N$ and $A$. In a large number of trials, this frequency should approximate the true probability in the population.[8].

The results of this probability experiment for $A = 50$ are presented in Table 1. As the table indicates, the probability of a contradiction between $\bar{A}$, $\bar{P}$, and $L^*$ is large, even for small populations. This suggests that Sen's paradox

[5] Condition $L^*$ requires at least two individuals to be decisive over at least one pair of alternatives. We assign $N$ individuals decisive rights over at least one pair because we believe few would find solice in the more restricted set of rights permitted by $L^*$ and because Sen considers the case of $N$ individuals having decisive rights over at least one pair of alternatives in his condition L (Sen 1970).

[6] Of course, if individuals are allowed to have decisive rights over the same pair of alternatives, then the conundrum is even more likely than the results shown here.

[7] Strict transitivity: $\forall x_1, x_2, x_3 \in X : (x_1 P x_2 \ \& \ x_2 P x_3\ ) \rightarrow x_1 P x_3]$.

[8] With 1/2 million trials, we are 95% confident that the true probability is within 0.0015 of the relative frequencies reported. This statement is based on the standard deviation of a univariate proportion, $\sqrt{\frac{\pi(1-\pi)}{T}}$, where $T$ is the number of trials.

**Table 1: Probability of a Contradiction in Two Dimensions**

| | | $da$ | |
|---|---|---|---|
| **N** | 1 | 10 | 20 |
| 3 | 0.015 | 0.676 | 0.947 |
| 41 | 0.274 | 1.000 | 1.000 |
| 60 | 0.703 | 1.000 | 1.000 |

*Note*: $A = 50$; hence, $\binom{A}{2} = 1,225$. Rounded figures indicate the probability of a contradiction between $\bar{A}$, $\bar{P}$, and $L^*$. Trials $= 500,000$.

may not be an abberation. Not only do the contradictions exist, they are fairly common in the domain of possible preferences. Furthermore, as $N$ increases, or $da$ increases, the probability of a contradiction increases as well. This implies that the paradox is more likely to occur in populations the size of a small town with a large number of alternatives and a great number of decisive rights than in committees with alternatives limited to say a short menu of exogenously formulated items.

Dougherty & Edward (2005) claim that the probability of a Pareto preferred alternative decreases as $N$ increases. Diagnostics confirm this and suggest that the decrease occurs fairly rapidly. For example, there is an average of roughly 432 Pareto preferred comparisons for $N = 3$ and $A = 50$ (Recall, for $A = 50$ there are $\binom{A}{2} = 1,225$ of possible pairs of alternatives). However, there is an average of only 3.7 Pareto preferred alternatives for $N = 41$ and $A = 50$. This means that the increased probability of a conflict between $\bar{A}$, $\bar{P}$, and $L^*$ for larger $N$ is primarily due to a direct conflict between between $\bar{A}$, and $L^*$. This explains the larger $N$ results in Table 1. For $N = 41$ and $da = 10$, 33% of the possible pairs are dictated by decisive rights ($= 410/1,225$). For $N = 60$ and $da = 20$, this shoots up to 98% of the possible pairs dictated by such rights ($= 1,200/1,225$). In such cases, a small amount of preference heterogeneity can lead to conflicts.

This highlights one of the fundamental issues in determining the probability of a conflict between $\bar{A}$, $\bar{P}$, and $L^*$. The proper ratio of decisive alternative pairs to all possible pairs appears to affect the result. However, before the reader concludes that the ratio of $Nda/\binom{A}{2}$ is the full explanation for these results, note that $A = 100$ is not sufficiently large to make the probability of a conflict between $\bar{A}$, $\bar{P}$, and $L^*$ decrease from 1. In this case, the decisive alternative pairs represent only 8% of the total number of alternative pairs possible.[9] Finding a ratio that is both desirable and feasible may be critical to determining whether Sen's Liberal Paradox should be considered pervasive.

[9] Rounding at $10^{-6}$.

In addition to more investigating the proper relationship between $N(da)$ and $A$, there are two natural extensions to these results worthy of further investigation. One extension is to try to reduce the structure of preferences imposed by a two dimensional spatial voting model. Instead, an attempt should be made to consider all possible cases in an almost completely unrestricted domain. The other is to restrict the domain in a way that more accurately models the type of preferences which might arise in an actual liberal choice situation. These ideas will be briefly addressed in the next two sections, respectively.

## 3.2 Twenty Dimensions, Continuous Outcome Space

In an attempt to reduce the structure imposed on the domain by a two dimensional spatial model, we extended the analysis to twenty dimensions. Our experiment for greater dimensions were conducted similar to the experiment described in the previous section. In each trial, the program randomly draws $N$ ideal points and $A$ alternatives from a uniform distribution on an n-dimensional hyper-cube. Individual $i's$ preference for each pair of alternatives is determined based on the shorter of the two distances between $i's$ ideal point and the two alternatives. Pareto comparisons, the assignment of decisive rights, and tests of transitivity are conducted as done before. The only difference is that the program keeps track of a greater number of dimensions.

**Table 2: Probability of a Contradiction in Twenty Dimensions**

| | | *da* | |
|---|---|---|---|
| **N** | 1 | 10 | 20 |
| 3 | 0.001 | 0.637 | 0.971 |
| 41 | 0.236 | 1.000 | 1.000 |
| 60 | x.xxx | x.xxx | 1.000 |

*Note*: $A = 50$; hence, $\binom{A}{2} = 1,225$. Rounded figures indicate the probability of a contradiction between $\bar{A}$, $\bar{P}$, and $L^*$. Trials $= 500,000$.

Results of this experiment for 20 dimensional space appear in Table 2. Notice that the probability of a contradiction between $\bar{A}$, $\bar{P}$, and $L^*$ for 20 dimensions is not greater than it is for 2 dimensions in almost every case. The one exception is $N = 3$ and $da = 20$. Part of the reasons why the figures in Table 2 are typically smaller than the figures reported in Table 1 is that the probability of a Pareto preferred alternative typically decreases as the number of dimensions increases. In the exceptional case of $N = 3$, $da = 20$, the probability increases with the number of dimensions. This conclusion is based on monitoring the average number of Pareto preferred alternatives per trial.

To confirm that the change in the number of Pareto preferred cases was explaining the differences between the $2D$ and $20D$ results, we re-configured the program so that it made no Pareto comparisons. This left the program testing the probability of a conflict between $\bar{A}$ and $L^*$. We found that with Pareto removed, the probability of a conflict between $\bar{A}$ and $L^*$ is roughly the same regardless of the number of dimensions (dimensions 3 through 6 were also explored). This suggests that the differences between comparable figures in Table 1 and Table 2 can be explained by the effects of $\bar{P}$ on the social preferences.[10]

Even though the probability of a conflict between $\bar{A}$, $\bar{P}$, and $L^*$ decreases with increasing dimensions, the probabilities are still very large for fairly small populations. Again, this suggests that Sen's paradox may be probably in medium and large populations.

## 3.3 Decisive Dimensions, Dichotomous Alternatives

As Sen and others have pointed out, sufficient restriction on the domain of alternatives can lead to an avoidance of his paradox. In an attempt to more accurately model decisive choices that Sen may have envisioned, we now consider a very different model. In this model, individual $i$ will be decisive over a pair of alternatives if and only if the difference between the two alternatives is a an attribute that individual $i$ is supposed to be decisive over. An example may illustrate the point.

Imagine that there are two individuals: Muddy and Billie, each of which want to decide whether they will sleep on their belly or their back. With two attributes and two individuals, there are $2^N$ possible states:
$s_1 = \{$Muddy belly, Billie belly$\}$
$s_2 = \{$Muddy belly, Billie back$\}$
$s_3 = \{$Muddy back, Billie belly$\}$
$s_4 = \{$Muddy back, Billie back$\}$

Muddy is decisive over $\{s_1, s_3\}$ and $\{s_2, s_4\}$. Billie is decisive over $\{s_1, s_2\}$ and $\{s_3, s_4\}$. Pareto comparisons can be made among all the alternatives. The best way to assure that each individual has decisive rights over the alternatives that differ only in terms of their decisive attributes, is to create a decisive dimension for each individual. One dimension reflects Muddy's choice to sleep

[10] At the time of submission, we have no good explanation for why the probability of a Pareto preferred alternative decreases as the number of dimensions increases. We only note that for any alternative $x$ *not* in the convex hull, the set of points Pareto preferred to $x$ is *no greater than* those circumscribed by the indifference curve centered on $I^*$, where $I^*$ is the ideal point closest to $x$. For dimensions greater than one, the set of alternatives Pareto preferred to $x$ are often smaller. Now consider $|x - I^*| = 0.1$ fixed across dimensions. For one dimension, the area within the indifference curve for $I^*$ is just less than 0.2, which is just less than 0.2 of the total interval. For two dimensions, the area within the indifference curve for $I^*$ is .031 ($= \pi(.1)^2$), which is .031 of the area of the unit square. For three dimensions, the area (or volume) within the indifference sphere for $I^*$ is .004 ($= 4/3\pi(.1)^3$) If the probability of $|x - I^*| = 0.1$ is roughly the same across dimensions, this would suggest that the potential for Pareto improvements would be smaller as the number of dimensions increases.

on his belly or back. The other dimension reflects Billie's choice to sleep on her belly or back. Each individual is decisive along this dimension *only if* the alternatives are identical on every other dimension. This set-up implies that the number of dimensions, and the number of alternatives, should be a function of the number of individuals, $N$, and the number of decisive attributes, $r$, assigned to each individual. For example if Muddy is decisive over sleeping on his belly or his back and whether to read *Lady Chatterly's Lover* or not, then $r = 2$. If Billie gets the same rights, then the number of dimensions is $Nr = 4$, and the appropriate number of alternatives is $2^{Nr} = 16$. For the case at hand, $r = 1$, the appropriate number of dimensions is 2, and $A = 4$.

Our third probability experiment tests this type of decision making over $r$ dichotomous attributes and $2^{Nr}$ alternatives. Each alternative is assigned to a corner in the n-dimensional, unit hyper-cube. For $N = 2$ and $r = 1$, this hypercube is a unit square in two dimensional space, with four corresponding alternatives: $s_1 = (0, 0)$; $s_2 = (0, 1)$; $s_3 = (1, 0)$; and $s_4 = (1, 1)$. During each trial of the experiment, the program randomly assigns preferences to each individual, presuming that each of the $A!$ possible orders are equally likely. This is done by randomly drawing one of the alternatives as the individual's most preferred alternative without replacement. The individual's second most preferred alternative is then randomly drawn from the remaining alternatives. Care is taken to assure that this is drawn with equal probability among the remaining alternatives, again without replacement. The process continues until each individual is assigned a strict order over the $A$ alternatives.

After the preferences are determined, the program assigns social preferences based on a Pareto, similar to the process described before. For example, if Muddy and Billie both prefer $s_2$ to $s_3$, then society $s_2 P s_3$.

For the minimal liberalism routine, the program compares all alternatives pairwise. It then determines whether $x_j = x_k$ on every dimension except the one representing individual $i's$ decisiveness. In such a case, the program assigns social preferences over that pair based on the preferences of individual $i$ on that pair. Such an assignment of social preferences occurs only if $x_j$ and $x_k$ are identical on every dimension except the dimension of some individual $i$. The transitivity routine then works as described previously.

Due to limited time, we ran this program only for the case of $N = 3$ and $r = 1$. In this case, $A = 8$, with 3 dimensions. Computational results suggest that the probability of a contradiction between $\bar{A}$, $\bar{P}$, and $L^*$ is 0.666. On average, 0.25 of the possible alternative pairs could be decided by Pareto.[11] Although there are a number of differences between the simulations described in this section and section 3.2, this probability can be compared to results in section 3.2 for $N = 3$, $A = 8$, $da = 4$ and 3 dimensions. In that particular case, the probability of a contradiction is 0.580, which is smaller than the probability reported here. This might loosely suggest that structuring the preferences more closely to some of the liberal examples presented by Sen may have little effect

[11] There were roughly 7 Pareto preferred alternatives per round. Hence, $7/\binom{8}{2} = .25$.

on reducing the probability of his paradox.[12]

# 4 Conclusion

A few lessons seem obvious. The Pareto principle does, of course, conflict with minimal liberalism and social decision making over certain sets of individual preferences. Sen's theorem shows this must be the case. Furthermore, the patterns which cause such conflicts appear to be quite common. Hence, what Sen introduced as a conundrum of potential conflict appears to be a conundrum of highly probable conflict. Although the probability of these conflicts can be limited by restricting the number of decisive alternative pairs, or the number of decisive individuals, we believe that few would take solace in such restrictions. The condition of minimal liberalism is quite weak and limiting liberal values to say one pair of alternatives seems to be a fairly strong limitation on liberty. If society cannot allow the preponderance of its members to be free to read what they like, sleep the way they prefer, and paint their walls their favorite color, irrespective of the preferences of others in the community, then it is not clear how society can be fully committed to liberal values and the Pareto criterion simultaneously. One of the conditions must go or the notion of a consistent social decision must be re-evaluated.

Future research will evaluate the probability of these conflicts for larger values of $da$ relative to $A$ and for different values of $N$. We will also consider various distributions of individual preferences and extend our analysis of the methods described in section 3.3, particularly to higher dimensions. Hopefully, such extensions will create a path for making social decisions that is consistent with liberalism, Pareto, and the possibility that a variety of preference patterns may occur.

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[12] The average number of Pareto preferred alternatives per round were roughly 9.6 for the case associated with section 3.2. This suggests that the structure of decisive choices, or the structure of alternatives, explains the difference between the two results.

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