# From Social Choice to MCDA

Thierry Marchant

Ghent University

# Outline

1. MCDA (Multicriteria Decision Analysis)
2. Some important methods
3. Conjoint measurement as a theory of MCDA
4. Social Choice as a theory of MCDA
5. Limitations
6. Open questions

# 1. MCDA (Multicriteria Decision Analysis)

# An example: choosing a car

| | Cost (€) | Max speed (km/h) | Gas (l/100km) |
|---|---|---|---|
| *a* | 18 000 | 165 | 6.2 |
| *b* | 21 000 | 185 | 7.5 |
| *c* | 17 000 | 170 | 6.0 |
| *d* | 20 500 | 185 | 9.0 |

Criteria

Alternatives

Performances

Given a performance table, which car is the best buy ?

# Second example: ranking students

Sixty students apply for a doctoral school. There are 45 places.

| | Cost (€) | Grades | Relevance |
|---|---|---|---|
| *Student 1* | 600 | 16 | High |
| *Student 2* | 450 | 14 | Low |
| … | … | … | … |
| *Student 60* | 800 | 18 | Medium |

Which students will get a grant ?

# Other examples

- Hiring a new employee
- Choosing an investment plan
- Ranking research projects
- Choosing a new railway route
- Choosing a power plant location
- …

# Notation and definitions

- $A = \{a, b, c, ...\}$ : set of alternatives or actions (e.g., cars)
- $g_1, g_2, \ldots, g_n$ : $n$ criteria or attributes, i.e. mappings from $A$ to some set (e.g., $\mathbb{R}$, {L, M, H}).
- $g_i(a)$ : performance (or evaluation) of alternative $a$ on criterion $i$.
- $\mathbf{g} = (g_1, g_2, \ldots, g_n)$ represents the performance table.

# Notation and definitions

- $\succsim(\mathbf{g})$ : a weak order representing the preferences, given **g**.

  Defined over *A* (all alternatives)

- $a \succsim(\mathbf{g})\ b$ : *a* is at least as good as *b*

- $a \succ(\mathbf{g})\ b$ : *a* is better than *b*

- $a \sim(\mathbf{g})\ b$ : *a* and *b* are indifferent

# Problem statement

- $a \gtrsim(\mathbf{g})\ b$ iff $g_1(a) \geq g_1(b)$ and $g_2(a) \geq g_2(b)$ and …
  and $g_n(a) \geq g_n(b)$
- $\gtrsim(\mathbf{g})$ is usually very incomplete.
- A less strict definition of the ranking $\gtrsim(\mathbf{g})$ is necessary.

# 2. Some important methods

# The weighted average

$a \succsim(\mathbf{g})\ b$ iff $\Sigma_i w_i g_i(a) \geq \Sigma_i w_i g_i(b)$

The weights must be elicited, according to the DM's prefs

| | Cost (€) | Max speed (km/h) | Gas (l/100km) |
|---|---|---|---|
| *a* | 15 000 | 150 | 6.7 |
| *b* | *x* | 160 | 6.7 |

Analyst: How much are you willing to pay for
an extra 10 km/h ? What is $x$ s.t. $a \sim(\mathbf{g})\ b$ ?

DM: 1000 €

Then $w_1 / w_2 = 10 / 1000 = 0.01$

# The weighted average

$a \succsim_{(\mathbf{g})} b$ iff $\Sigma_i w_i g_i(a) \geq \Sigma_i w_i g_i(b)$

*Problem*

| | Cost (€) | Max speed (km/h) | Gas (l/100km) |
|---|---|---|---|
| *a* | 15 000 | 150 | 6.7 |
| *b* | 16 000 | 160 | 6.7 |
| *c* | 17 000 | 170 | 6.7 |
| *d* | 20 000 | 240 | 6.7 |
| *e* | 21 000 | 250 | 6.7 |

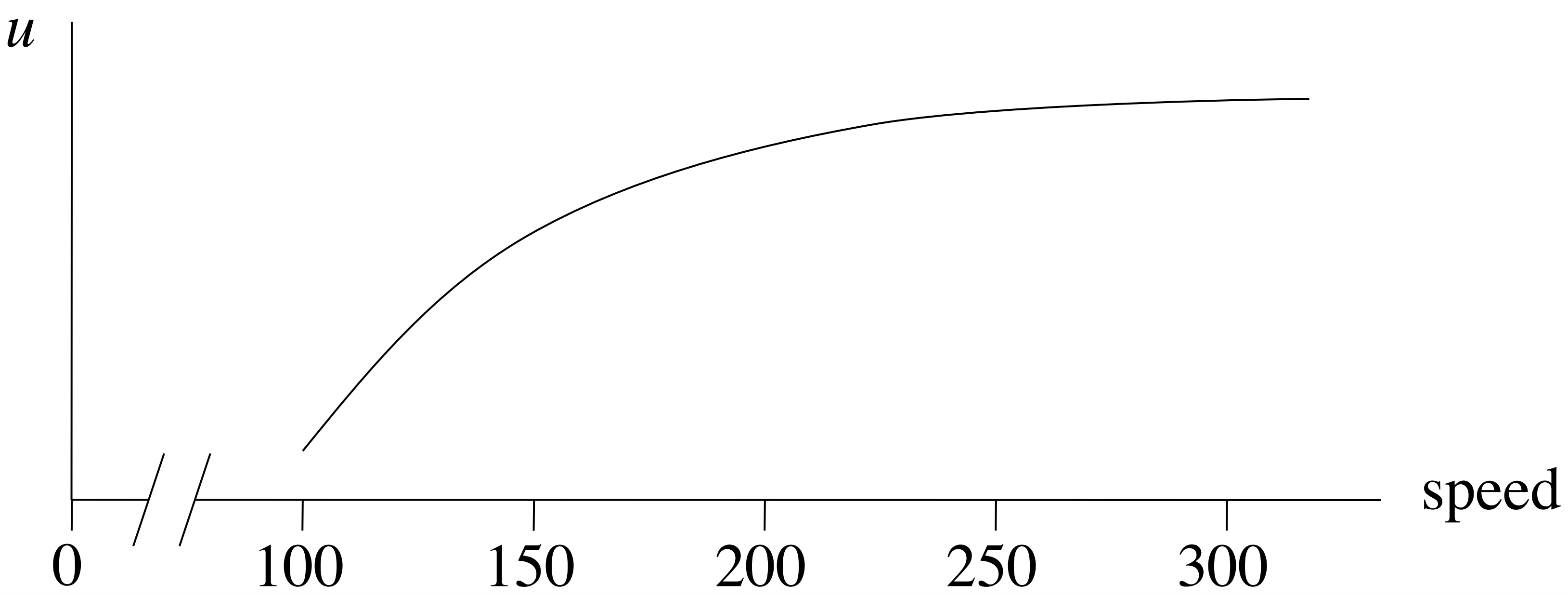
Suppose $a \sim_{(\mathbf{g})} b$. Then $b \sim_{(\mathbf{g})} c$.

And $d \sim_{(\mathbf{g})} e$.

# Additive utility (MAUT, MAVT)

$a \succsim(\mathbf{g})\ b$ iff $\Sigma_i w_i v_i(g_i(a)) \geq \Sigma_i w_i v_i(g_i(b))$

$a \succsim(\mathbf{g})\ b$ iff $\Sigma_i u_i(g_i(a)) \geq \Sigma_i u_i(g_i(b))$



# Additive utility (MAUT, MAVT)

$a \succsim(\mathbf{g})\ b$ iff $\Sigma_i\ u_i(g_i(a)) \geq \Sigma_i\ u_i(g_i(b))$

*Problems*

- Eliciting the utility functions is a tedious task
- Independence

| | Dish | Wine |
|---|---|---|
| *a* | Beef | Red |
| *b* | Beef | White |
| *c* | Fish | Red |
| *d* | Fish | White |

$a \succ(\mathbf{g})\ b \Rightarrow u_2(\text{red}) > u_2(\text{white})$

$d \succ(\mathbf{g})\ c \Rightarrow u_2(\text{white}) > u_2(\text{red})$

# Additive utility (MAUT, MAVT)

$a \succsim(\mathbf{g})\ b$ iff $\Sigma_i\ u_i(g_i(a)) \geq \Sigma_i\ u_i(g_i(b))$

*Problems*

- Eliciting the utility functions is a tedious task
- Independence

*Reaction*

- Development of many new methods since the 70's
- Avoiding one or both problems
- With new problems
- Sometimes inspired by voting methods

# Outranking methods

*Three steps*

1. Preference modelling
   - Construction of a preference relation on each criterion
2. Aggregation — Preferences aggregation
   - Aggregation of $n$ preference relations into a comprehensive relation (outranking relation)
3. Exploitation — Tournaments
   - The outranking relation is usually not directly usable (incomplete, cyclic, intransitive).

# Electre I (pref. modelling)

For every criterion $i$, define $\succsim_i$ by

$$a \succsim_i (\mathbf{g}) \; b \text{ iff } g_i(a) \geq g_i(b) - q_i \qquad (q_i \geq 0)$$

$q_i$ is an indifference threshold

Roy, B. (1971). “Problems and methods with multiple objective functions”. *Mathematical Programming*, 1:239–266.

# Electre I (aggregation)

$a \succsim(\mathbf{g})\ b$ iff $\begin{cases} \sum_{i\,:\,a \succsim_i(\mathbf{g})\, b} w_i \geq \delta & (\delta \geq 0.5) \\ \text{and} & \\ g_i(a) \geq g_i(b) - v_i \quad \forall\, i & (v_i > q_i) \end{cases}$

$a$ is at least as good as $b$

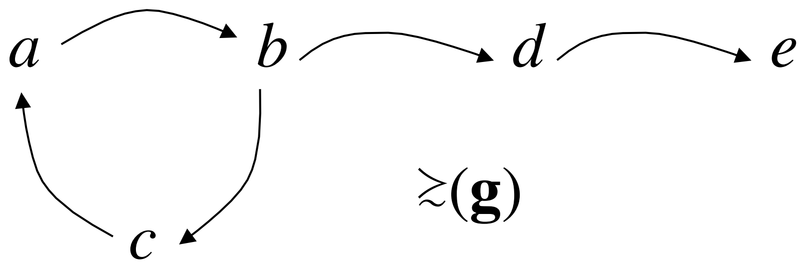
- iff the coalition of criteria s.t. $a \succsim_i (\mathbf{g})\ b$ is strong enough (weighted qualified majority) AND
- $a$ is not much worse than $b$ on each criterion (veto)

The relation $\succsim(\mathbf{g})$ is called an outranking relation.

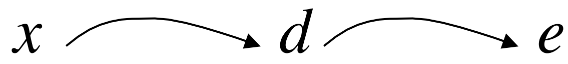
It can be incomplete, intransitive or cyclic.

# Electre I (exploitation)

1. Reduce the circuits (replace all alternatives in the circuit by a single one).




$\succsim(\mathbf{g})$




$S$

$Kernel(S) = \{x, e\}$

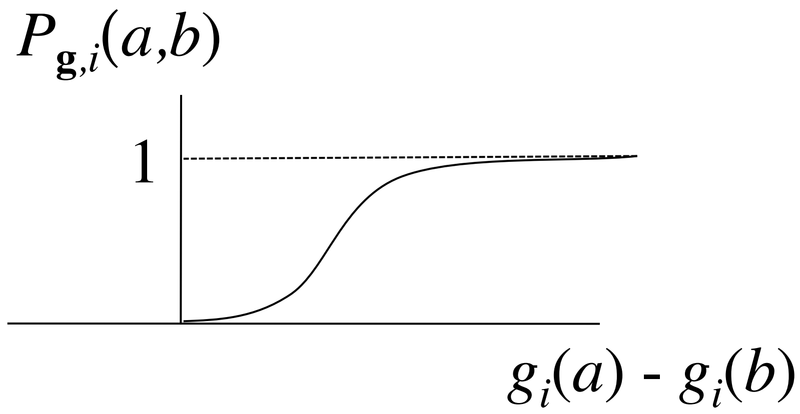
2. The <u>kernel</u> of the relation $S$ is the unique subset $B \subseteq A$ such that
   - for every $a$ not in $B$, there is $b$ in $B$ : $b\ S\ a$
   - for every $b$ in $B$, there is no $c$ in $B$ : $c\ S\ b$

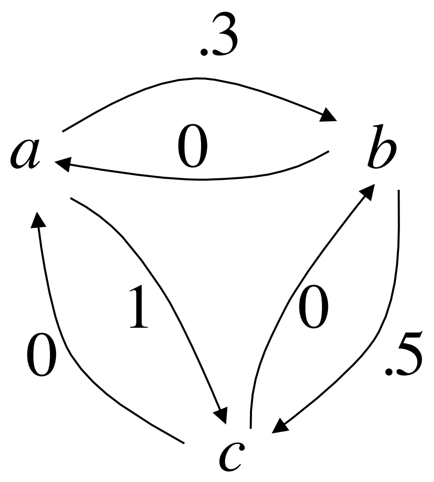
# Promethee II (Pref. modelling)

$P_{\mathbf{g},i}(a,b)$ (preference intensity) is a non decreasing function of $g_i(a)$ - $g_i(b)$

such that

- $P_{\mathbf{g},i}(a,b) \in [0,1]$ and
- $g_i(a) \leq g_i(b) \Rightarrow P_{\mathbf{g},i}(a,b) = 0$




$P_{\mathbf{g},i}$ is a valued relation.

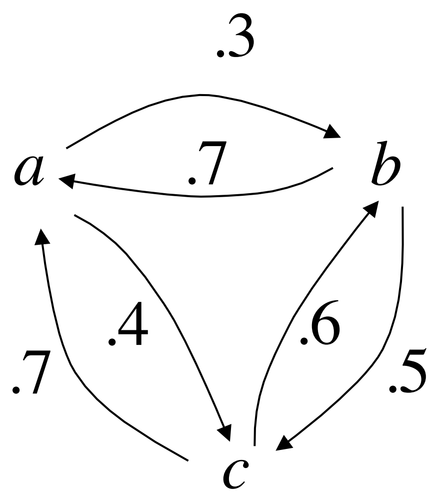
Brans, J.-P. and Vincke, Ph. (1985). A preference ranking organisation method. (The PROMETHEE method for multiple criteria decision-making). *Management Science*, 31:647–656.

# Promethee II

*Aggregation*

$\pi_{\mathbf{g}}(a,b) = \Sigma_i \; w_i P_{\mathbf{g},i}(a,b)$

$\pi_{\mathbf{g}}$ is a valued relation.




*Exploitation*

$\phi_{\mathbf{g}}(a) = \Sigma_{b \neq a} [\pi_{\mathbf{g}}(a,b) - \pi_{\mathbf{g}}(b,a)]$ ( net flow )

$a \succsim(\mathbf{g}) \; b$ iff $\phi_{\mathbf{g}}(a) \geq \phi_{\mathbf{g}}(b)$

# Many other outranking methods

QUALIFLEX
REGIME
ORESTE
ARGUS
EVAMIX
TACTIC
MELCHIOR
MAPPAC
PRAGMA
IDRA
PACMAN

# 3. Conjoint measurement as a theory of MCDA

# Conjoint measurement as a theory of MCDA

Conjoint measurement was developed in the 60's to study the numerical representation of binary relations on product sets (Debreu, Luce & Tukey, …)

It was first used in MCDA by Keeney and Raiffa (*Decisions with multiple objectives: Preferences andvalue tradeoffs*, Wiley, 1976) for analyzing MAUT. This axiomatic theory makes clear the assumptions underlying MAUT.

It was considered as not well suited for outranking methods

# 4. Social Choice as a theory of MCDA

# Social Choice as a theory of MCDA

Outranking methods are inspired from voting methods.

Why not use Social Choice Theory instead of conjoint measurement ?

Axiomatic Social Choice Theory also makes clear the assumptions underlying a method.

Since the 80's, old SCT results have been reused in MCDA or new results have been proven. Impossibility / characterization.

# Arrow's Theorem

Transitivity, Universality, Unanimity and IIA
$\Rightarrow$ Dictatorship.

*Application to the aggregation in Electre I.*

Electre I satisfies Universality, Unanimity and IIA.

It is not dictatorial.

That is why the outcome is not transitive.

# Arrow's Theorem

Transitivity, Universality, Unanimity and IIA
$\Rightarrow$ Dictatorship.

*Application to the aggregation in Promethee II.*

Promethee satisfies Universality, Unanimity and IIA.

It is not dictatorial.

That is why the outcome is not min-transitive :

$$\pi_{\mathbf{g}}(a,c) \geq \min\{\pi_{\mathbf{g}}(a,b), \pi_{\mathbf{g}}(b,c)\}$$

Fuzzy version of Arrow's Theorem :

Banerjee, A. 1994. "Fuzzy preferences and Arrow-type problems in social choice". *Social Choice and Welfare*, 11:121–130

# Arrow's Theorem

Transitivity, Universality, Unanimity and IIA
$\Rightarrow$ Dictatorship.

*Application to (aggregation-exploitation) in Promethee II.*

Promethee satisfies Transitivity, Universality and Unanimity.

It is not dictatorial.

That is why it violates IIA. (aggr. of valued relations !)

# Characterization : (aggregation-exploitation) in Promethee II

*Aggregation*

$$\pi_{\mathbf{g}}(a,b) = \Sigma_i \; w_i \, P_{\mathbf{g},i}(a,b)$$

*Exploitation*

$$\phi_{\mathbf{g}}(a) = \Sigma_{b \neq a} \, [\pi_{\mathbf{g}}(a,b) - \pi_{\mathbf{g}}(b,a)]$$

$a \succsim(\mathbf{g}) \; b$ iff $\phi_{\mathbf{g}}(a) \geq \phi_{\mathbf{g}}(b)$

$\phi_{\mathbf{g}}(a)$ can be rewritten as $\Sigma_i \; w_i \; \Sigma_{b \neq a} \, [P_{\mathbf{g},i}(a,b) - P_{\mathbf{g},i}(b,a)]$

Weighted generalized Borda rule

# Characterization : Generalized Borda rule

$B_{\mathbf{g}}(a) = \Sigma_i \, \Sigma_{b \neq a} \, [P_{\mathbf{g},i}(a,b) - P_{\mathbf{g},i}(b,a)]$

$a \succsim(\mathbf{g}) \; b$ iff $B_{\mathbf{g}}(a) \geq B_{\mathbf{g}}(b)$

*Neutrality* : $\sigma$, a permutation of $A$.

$\mathbf{g}$ and $\mathbf{h}$, two performance tables s.t.

$P_{\mathbf{h},i}(a,b) = P_{\mathbf{g},i}(\sigma(a),\sigma(b))$ for all $i$ and $a$.

Then $a \succsim(\mathbf{g}) \; b$ iff $\sigma(a) \succsim(\mathbf{h}) \; \sigma(b)$

Labels and performances do not matter.

Only preference intensities matter.

# Characterization : Generalized Borda rule

$B_{\mathbf{g}}(a) = \Sigma_i \; \Sigma_{b \neq a} \, [P_{\mathbf{g},i}(a,b) - P_{\mathbf{g},i}(b,a)]$

$a \gtrsim(\mathbf{g}) \; b$ iff $B_{\mathbf{g}}(a) \geq B_{\mathbf{g}}(b)$

*Faithfulness*: if $n = 1$ and $P_{\mathbf{g},1}$ is a weak order,
then $\gtrsim(\mathbf{g}) = P_{\mathbf{g},1}$

When possible, keep it simple

# Characterization : Generalized Borda rule

$B_{\mathbf{g}}(a) = \Sigma_i \ \Sigma_{b \neq a} [P_{\mathbf{g},i}(a,b) - P_{\mathbf{g},i}(b,a)]$

$a \succsim(\mathbf{g}) \ b$ iff $B_{\mathbf{g}}(a) \geq B_{\mathbf{g}}(b)$

*Consistency:* $\mathbf{g} = (\mathbf{g}_1, \mathbf{g}_2)$,

then $a \succsim(\mathbf{g}_1) \ b$ and $a \succsim(\mathbf{g}_2) \ b \Rightarrow a \succsim(\mathbf{g}_1, \mathbf{g}_2)$

$a \succsim(\mathbf{g}_1) \ b$ and $a \succ(\mathbf{g}_2) \ b \Rightarrow a \succ(\mathbf{g}_1, \mathbf{g}_2) \ b$

If two subsets of criteria agree, then the whole set agrees.

# Characterization : Generalized Borda rule

$$B_{\mathbf{g}}(a) = \Sigma_i \, \Sigma_{b\neq a} [P_{\mathbf{g},i}(a,b) - P_{\mathbf{g},i}(b,a)]$$

$a \succsim(\mathbf{g}) \; b$ iff $B_{\mathbf{g}}(a) \geq B_{\mathbf{g}}(b)$

*Cancellation*: $\Sigma_i \, P_{\mathbf{g},i}(a,b) = \Sigma_i \, P_{\mathbf{g},i}(b,a)$, $\forall \, a, b$

then $\succsim(\mathbf{g}) = A^2$

If the evidence in favour of *a* balances the evidence in favour of *b* for all pairs, then no winner.

# Characterization : Generalized Borda rule

$B_{\mathbf{g}}(a) = \Sigma_i \; \Sigma_{b \neq a} \, [P_{\mathbf{g},i}(a,b) - P_{\mathbf{g},i}(b,a)]$

$a \succsim(\mathbf{g}) \; b$ iff $B_{\mathbf{g}}(a) \geq B_{\mathbf{g}}(b)$

Given $P_{\bullet,i}(b,a)$, the mapping $\succsim(\bullet)$ is the Borda rule iff it satisfies Neutrality, Faithfulness, Consistency and Cancellation.

Marchant, Th. (1996). "Valued relations aggregation with the Borda method". *Journal of Multi-Criteria Decision Analysis*, 5:127–132.

# Characterization : exploitation in Promethee

Bouyssou, D. "Ranking methods based on valued preference relations: A characterization of the net flow method" *EJOR* **60,** 1992

Bouyssou, D. and Perny, P. "Ranking methods for valued preference relations: a characterization of a method based on entering and leaving flows" *EJOR*, **61**, 1992

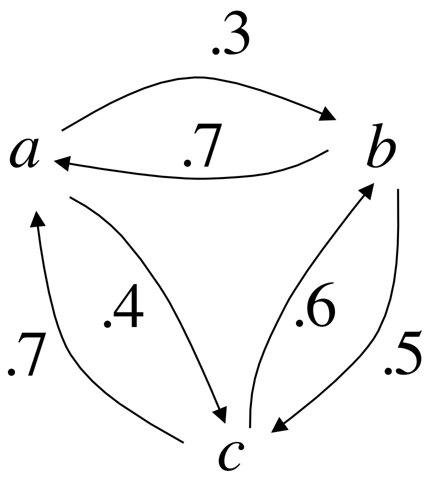
# Characterization : aggregation in Electre I

Marchant, T. "An axiomatic characterization of different majority concepts", *EJOR* **179**, 2007

# Characterization : aggregation in Tactic

Marchant, T. "An axiomatic characterization of different majority concepts", *EJOR* **179**, 2007

# Characterization : exploitation by the min




$a$ : .3
$b$ : .5
$c$ : .6

$c \succ b \succ a$

Pirlot, M. "A characterization of `min' as a procedure for exploiting valued preference relations and related results" *Journal of Multi-Criteria Decision Analysis*, **4**, 1995

# Characterization : weighted sum

Roberts, K. W. S. "Interpersonal Comparability and Social Choice Theory," *Review of Economic Studies* **47,** 1980

Bouyssou et al., *Evaluation and decision models with multiple criteria: stepping stones for the analyst*, Springer, 2006

# 4. Limitations

# Weights

Most characterizations without weights.

In most methods, doubling the weight of a criterion amounts to cloning the criterion.

| | $g_1$ | $g_2$ | $g_3$ |
|---|---|---|---|
| $a$ | 1 | 12 | 8 |
| $b$ | 3 | 19 | 6 |
| $c$ | 2 | 25 | 7 |
| $w$ | *2* | *1* | *3* |

→

| | $g_1$ | $g_1$ | $g_2$ | $g_3$ | $g_3$ | $g_3$ |
|---|---|---|---|---|---|---|
| $a$ | 1 | 1 | 12 | 8 | 8 | 8 |
| $b$ | 3 | 3 | 19 | 6 | 6 | 6 |
| $c$ | 2 | 2 | 25 | 7 | 7 | 7 |

So, existing characterizations are still valid but they leave weights unexplained.

# Alternatives set

In elections, the candidates set is often given.

In MCDA, the construction of the alternatives set is an essential step in the decision process.

Conditions about changes in $A$ should play a stronger role in MCDA

# Parameters

Voting methods are usually parameter free.

MCDA methods use plenty of parameters: weights, utility functions, indifference thresholds, veto thresholds, concordance thresholds, …

In MCDA, the value of the parameters is elicited by asking questions to the DM. For instance, with the weighted sum,
do you prefer (18 000, 165, 6.2) or (19 000, 175, 6.2) ?

If (18 000, 165, 6.2) $\succ$ (19 000, 175, 6.2) then

$w_1\,18\,000 + w_2\,165 + w_3\,6.2 > w_1\,19\,000 + w_2\,175 + w_3\,6.2$ and

$w_1 / w_2 < -10 / 1000 = -0.01$

# Parameters

In the primitives of standard social choice theory, there is no DM, no answer to questions.

# 6. Open problems

# Open problems

- Axiomatization of methods with veto
- Axiomatization of additive utility within Social Choice Theory
- Axiomatization of various methods with parameters