From Arrow's Impossibility to Schwartz's Tournament Equilibrium Set

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PREFERENCE AGGREGATION IN MULTIAGENT SYSTEMS

Overview

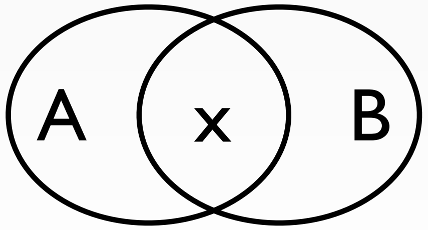
- Strictly axiomatic approach to social choice
  - Search for reasonable SCFs with a solid axiomatic foundation
  - Choice consistency and rationalizability
  - Variable agendas and variable electorates
- From the impossible to the possible
  - Arrovian impossibilities
  - Three escape routes
  - Scoring rules
  - Top cycle and uncovered set
  - Minimal covering set and tournament equilibrium set
- The tractable and the intractable
  - Polynomial-time algorithms and hardness results

Choice Theory

- Let U be a universe of alternatives.
- Alternatives are chosen from feasible subsets.
  - Throughout this talk, the set of feasible sets $\mathcal{F}(U)$ contains all finite and non-empty subsets of U.
- A choice function is a function $S : \mathcal{F}(U) \rightarrow \mathcal{F}(U)$ such that $S(A) \subseteq A$.
- Rationality and consistency conditions impose restrictions on choices in variable feasible sets.
  - Example: A choice function S with $S(\{a,b\}) = \{a\}$ and $S(\{a,b,c\}) = \{b\}$ seems unreasonable.

Rational Choice

- Def. (Richter, 1966): S is rationalizable if there exists a relation R on U such that for each feasible set A,
$$S(A) = \{a \in A : x \; P \; a \text{ for no } x \in A\},$$
where P is the strict part of R.
    - Acyclicity of P is necessary and sufficient.
- Typical rationalizing relations (Samuelson, 1938; Herzberger, 1973)
    - Base relation: $a \; \overline{R}_S \; b$ iff $a \in S(\{a, b\})$
    - Revealed preference relation: $a \; R_S \; b$ iff $a \in S(X)$ for some X with $b \in X$
- Typical consistency conditions (Sen, 1971)
Let A, B be feasible sets and $x \in A \cap B$.
    - Contraction (α): if $x \in S(A \cup B)$ then $x \in S(A) \cap S(B)$
    - Expansion (γ): if $x \in S(A) \cap S(B)$ then $x \in S(A \cup B)$



Rationality and Consistency

- Theorem (Sen, 1971): S is rationalizable iff it satisfies α and γ, i.e., for all feasible sets A, B and $x \in A \cap B$,
  $x \in S(A \cup B)$ iff $x \in S(A) \cap S(B)$.
  - R_S and $\overline{R}_S$ are identical and rationalize S.

- Stronger forms of rationality and consistency
  - Rationalizability via a transitive and complete relation
  - Weak axiom of revealed preference (WARP) (Samuelson, 1938):
    if $B \subseteq A$ and $S(A) \cap B \neq \emptyset$ then $S(A) \cap B = S(B)$
  - Theorem (Arrow, 1959): A choice function is transitively rationalizable iff it satisfies WARP.
    - WARP ⇔ α & β+ (Bordes, 1976)
  - Rationalizability via a quasi-transitive relation (only the strict part needs to be transitive)

From Choice to Social Choice

- Let N be a finite set of voters and $\mathcal{R}(U)$ the set of all transitive and complete relations over U.
- A social choice function (SCF) is a function $f : \mathcal{R}(U)^N \times \mathcal{F}(U) \to \mathcal{F}(U)$ such that $f(R, A) \subseteq A$.
  - For a given preference profile, every SCF induces a choice function and all rationality and consistency conditions can be readily applied.
  - Arrow's (1951) impossibility theorem, as formulated for SCFs, uses transitive rationalizability.

> *[...] the arbitrariness of power of which Arrow's case of dictatorship is an extreme example, lingers in one form or another even when transitivity is dropped, so long as* some *regularity is demanded (such as the absence of cycles).*
>
> Amartya Sen (1995)

Independence & Monotonicity

- Desirable independence conditions
  - IIA (Independence of Irrelevant Alternatives): Choice only depends on preferences over alternatives in the feasible set
  - Neutrality: IIA & choice is independent of the names of alternatives

- Desirable conditions on choice from {a,b}
  - Non-imposition (NI): There are preference profiles such that a and b are the only choice, respectively.
  - Pareto-optimality: If a is unanimously strictly preferred to b, then b is not chosen
  - Monotonicity: If a is chosen, then it is also chosen when it is reinforced (& NI)
  - Positive responsiveness: If a is chosen, then it is chosen uniquely when it is reinforced (& NI)

Fairness Conditions

- A coalition is decisive if it can single-handedly decide choice from any two-element set {a,b} (e.g., a majority of voters).
  - Formally, if all voters in a decisive coalition strictly prefer a to b, then a is chosen uniquely.
  - Pareto-optimality precisely says that the grand coalition is decisive.
- Undesirable coalitions of voters
  - Dictator: Decisive coalition with only one element
  - Vetoer: Voter who can force an alternative into the choice set (he can veto the exclusion)
    - Oligarchy: Decisive coalition of vetoers
  - Collegium: Non-empty intersection of all decisive coalitions
- Anonymity: Choice is independent of the names of the voters

Arrovian Impossibility Results

Anonymity	Neutrality	Positive Responsiveness	Rationalizability	*Condorcet/May (1785,1952)*
No Dictator	IIA	Pareto-Optimality	Transitive Rationalizability	*Arrow (1951)*
No Vetoer/Oligarchy	IIA	Pareto-Optimality	Quasi-Transitive Rationalizability	*Gibbard (1969)*
No Vetoer	IIA	Positive Responsiveness	Rationalizability	*Mas-Colell & Sonnenschein (1972)*
No Vetoer	Neutrality	Monotonicity	Rationalizability	*Blau & Deb (1977)*
No Collegium	---	Pareto-Optimality	Rationalizability	*Brown/Banks (1975,1995)*

No Collegium	Neutrality	Positive Responsiveness	Transitive Rationalizability	(WARP)
No Vetoer	IIA	Monotonicity	Quasi-Transitive Rationalizability	
No Dictator	---	Pareto-Optimality	Rationalizability	(α & γ)

weaker

What now?

- Three ways to escape from these results:
  - ignore consistency (and impose other restrictions instead)
    - Smith (1973), Young (1975)
  - only require expansion consistency
    - Bordes (1976), Moulin (1986)
  - weaken consistency conditions
    - B. & Harrenstein (2009)
- From now on, we assume for convenience that individual preferences are linear (anti-symmetric) and there is an odd number of voters.
  - some results hold without this restriction, some have been generalized by using additional axioms, some have not been generalized in a satisfactory way

Escape Route #1

Ignore consistency
(and impose other restrictions instead)

Interlude: Borda vs. Condorcet

- Jean-Charles Chevalier de Borda (1733 – 1799)
    - mathematician, physicist, and sailor
    - participated in the construction of the standard-meter (1/10.000.000 of the distance between the north pole and the equator)

- Marie Jean Antoine Nicolas Caritat, Marquis de Condorcet (1743 – 1794)
    - philosopher and mathematician
    - early advocate of equal rights and opponent of the death penalty

Family of Scoring Rules

- For a fixed number of alternatives m, a score vector is a vector $s=(s_1, \ldots, s_m)$ such that $s_1 \geq \ldots \geq s_m$ and $s_1 > s_m$.
  - The (cumulative) score of an alternative is the sum of scores s_i it receives for being ranked ith.
- A scoring rule is an SCF that chooses those alternatives from feasible set A that have the highest score according to some scoring vector of size |A|.
  - Examples
    - Borda's rule: $s=(|A|-1, |A|-2, \ldots, 0)$
      - The score assigned by a single voter corresponds to the number of alternatives he ranks lower.
      - Borda proposed this method to the French Academy of Sciences in 1770. It was then used for 20 years until it was abolished by Napoleon Bonaparte.
    - plurality rule: $s=(1, 0, \ldots, 0)$
    - anti-plurality: $s=(1, \ldots, 1, 0)$

Family of Condorcet Extensions

- Alternative a is a Condorcet winner if, for every other alternative b, there is a majority of voters who prefer a to b.
- A Condorcet extension is an SCF that uniquely chooses a Condorcet winner whenever one exists.
  - Example
    - Copeland's rule: Choose those alternatives that win most pairwise comparisons according to majority rule.

Scoring Rules and Condorcet Extensions

- For two alternatives, majority rule is the only scoring rule and the only Condorcet extension.
- Proposition (Condorcet, 1785): Borda's rule is no Condorcet extension when there are more than two alternatives.
- Theorem (Fishburn, 1973): No scoring rule is a Condorcet extension when there are more than two alternatives.
  - Proof:

	6	**3**	**4**	**4**
s_1	a	c	b	b
s_2	b	a	a	c
s_3	c	b	c	a

Properties of Borda's Rule

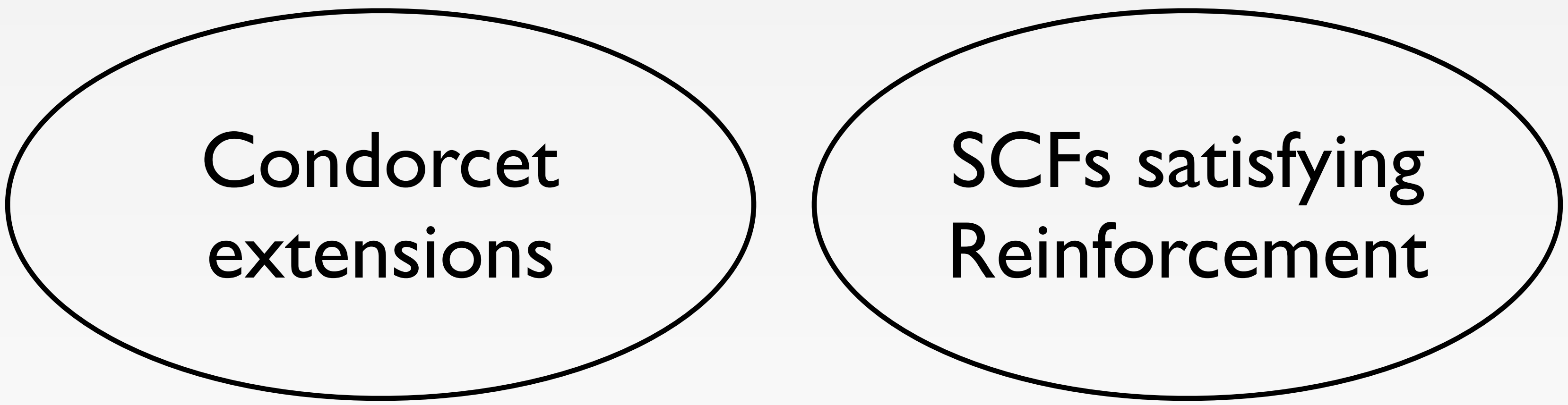
- Borda's rule has a special role within the class of scoring rules.
  - Borda's rule chooses the alternatives with the highest average rank in individual rankings.
  - Theorem (Smith, 1973): A Condorcet winner is never the alternative with the lowest Borda score. Borda's rule is the only scoring rule for which this is the case.
  - Theorem (Gehrlein et al., 1978): Borda's rule maximizes the probability over all scoring rules that a Condorcet winner is chosen whenever it exists.
  - There are a number of appealing axiomatic characterizations of Borda's rule (e.g., Young, 1978).
- Is there an SCF that combines the appeal of Borda's rule and Condorcet's principle?

Variable Electorates

- One of the most remarkable results in social choice theory characterizes scoring rules in terms of a variable electorate.
- An SCF satisfies reinforcement when all alternatives that are chosen simultaneously by two disjoint sets of voters are precisely the alternatives chosen by the union of both sets.
  - This is precisely the equivalent of α & γ for a variable electorate!
- Loosely speaking, an SCF satisfies continuity if negligible fractions of voters have no influence on the choice set.
- Theorem (Young, 1975): An SCF is a scoring rule iff it is neutral, anonymous, reinforcement, and satisfies continuity.

The Dilemma of Social Choice

- Theorem (Young et al., 1978): No Condorcet extension satisfies reinforcement when there are more than two alternatives.
- Two centuries after Borda and Condorcet, it turns out that the rationales between both ideas are incompatible.



- For social welfare functions, the intersection of these two sets contains exactly one neutral function: Kemeny's rule! (Young et al., 1978)

Escape Route #2

Only require expansion consistency

Contraction vs. Expansion

- Recap
  - rationalizability $\Leftrightarrow$ α & γ
  - transitive rationalizability $\Leftrightarrow$ α & β^+
  - numerous impossibility results involving various forms of rationalizability
- Sen (1977) showed that most of the Arrovian impossibilities remain intact when substituting rationalizability with contraction consistency (α and even substantially weakened versions of α).
- However, expansion consistency (on its own) is unproblematic.

Majoritarian SCFs

- A majoritarian SCF is an SCF that satisfies anonymity, neutrality, positive responsiveness, and binariness (i.e., the choice set only depends on the pairwise comparisons within the feasible set).
  - Choice only depends on the base relation, which is furthermore fixed to be the majority relation.
- The majority relation is asymmetric and complete, i.e., it can be represented by a tournament graph.
  - If a is preferred to b by a majority of voters we will say that a dominates b (a>b)
  - An undominated alternative is a Condorcet winner.
  - Vertices with maximal degree are Copeland winners.
  - Notation: dominion $D(a)=\{b \mid a>b\}$ and dominators $\overline{D}(a)=\{b \mid b>a\}$

The Top Cycle

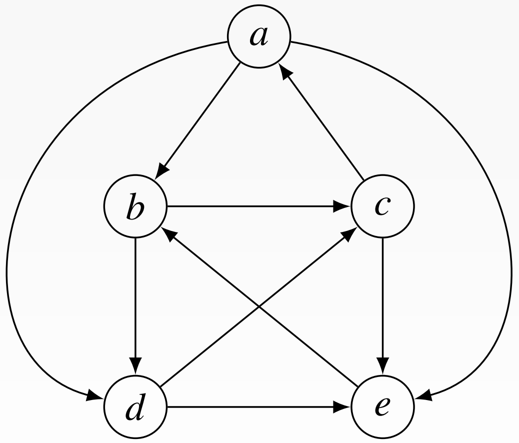
(Good, 1971; Smith, 1973)


John I. Good

- A dominating set is a set of alternatives such that every alternative in the set dominates every outside alternative.
  - The set of all dominating sets is totally ordered by set inclusion.
- The minimal dominating set is called the top cycle (TC).
  - also known as GETCHA or Smith set
- Theorem (Bordes, 1976): The top cycle is the smallest majoritarian SCF satisfying β^+.
- How can we efficiently compute the top cycle?

TC (linear-time algorithm)

- Algorithm for computing TC_a, the minimal dominating set containing a given alternative a
  - Initialize working set B with {a} and then iteratively add all alternatives that dominate an alternative in B until no more such alternatives can be found.
  - Computing TC_a for every alternative a and then choosing the smallest set yields an $O(n^3)$ algorithm.
- Alternatives with maximal degree (the Copeland winners) are always contained in TC (and linear-time computable).
  - **procedure** $TC(A, \succ)$
    $B \leftarrow C \leftarrow CO(A, \succ)$
    loop
    $C \leftarrow \bigcup_{a \in C} \overline{D}_{A \setminus B}(a)$
    if $C = \emptyset$ **then return** B **end if**
    $B \leftarrow B \cup C$
    end loop



More on the Top Cycle

- Theorem (Deb, 1977): The top cycle consists precisely of the maximal elements of the asymmetric part of the transitive closure of the dominance relation.
  - Alternative linear-time algorithm using Kosaraju's or Tarjan's algorithm for finding strongly connected components
- There is a first-order expression for membership in TC (B., Fischer, & Harrenstein; 2009):

  $$TC(x) \leftrightarrow \forall y \forall z (\forall v (z \geq^3 v \rightarrow z \geq^2 v) \wedge z \geq^2 x \rightarrow z \geq^2 y)$$

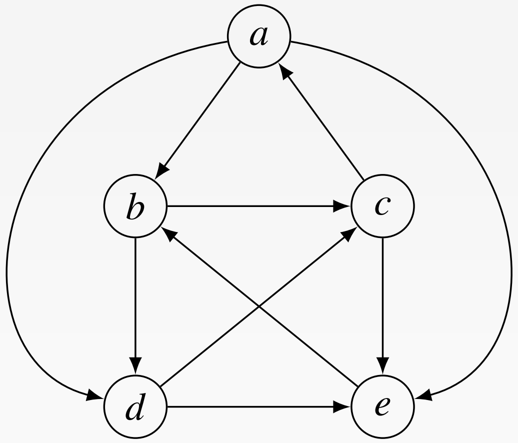
  - Computing TC is in AC^0
- The top cycle is very large.
  - In fact, it is so large that it may contain Pareto-dominated alternatives (when there are more than three alternatives).

The Uncovered Set


Nicholas Miller

(Fishburn, 1977; Miller, 1980)

- Covering relation: a covers b if D(b)⊂D(a).
  - The covering relation is a transitive subrelation of the dominance relation.
- The uncovered set (UC) consists of all uncovered alternatives.
  - UC contains the Condorcet winners of inclusion-maximal subsets that admit a Condorcet winner.
- Example
  - UC = {a,b,c,d}



- Theorem (Moulin, 1986): The uncovered set is the smallest majoritarian Condorcet extension satisfying γ.

UC (algorithm)

- Straightforward n^3 algorithm
- Equivalent characterization of UC
  - UC consists precisely of those alternatives that reach every other alternative on a domination path of length at most two. (Shepsle & Weingast, 1984).
- Algorithm via matrix multiplication
  - Fastest known matrix multiplication algorithm (Coppersmith & Winograd, 1990): $O(n^{2.38})$
  - Matrix multiplication is believed to be feasible in linear time ($O(n^2)$).

procedure $UC(A, \succ)$
  for all $i, j \in A$ **do**
    if $i \succ j \vee i = j$ **then** $m_{ij} \leftarrow 1$
    else $m_{ij} \leftarrow 0$ **end if**
  end for
  $M \leftarrow (m_{ij})_{i,j \in A}$
  $U \leftarrow (u_{ij})_{i,j \in A} \leftarrow M^2 + M$
  $B \leftarrow \{i \in A \mid \forall j \in A : u_{ij} \neq 0\}$
  return B

Escape Route #3

Weaken consistency conditions

From Alternatives to Sets

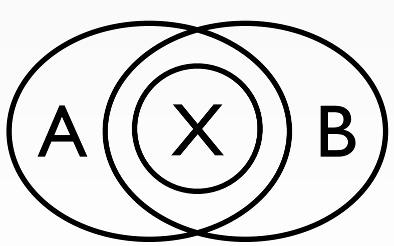
- Choice functions yield sets of alternatives, yet rationality and consistency conditions are defined in terms of alternatives.
  - Rationalizing relations are defined on alternatives
  - α & γ: An alternative in the intersection of two feasible sets is chosen in both sets iff it is also chosen in the union of both sets
- Redefining these conditions by making reference to the entire set of chosen alternatives, rather than its individual elements allows us to circumvent Arrovian impossibilities.

Set-Rationalizable Choice

- Def.: S is set-rationalizable if there is a relation $R \subseteq \mathcal{F}(U) \times \mathcal{F}(U)$ such that for each $A \in \mathcal{F}(U)$ there is no $X \in \mathcal{F}(A)$ with $X \; P \; S(A)$ where P is the strict part of R.

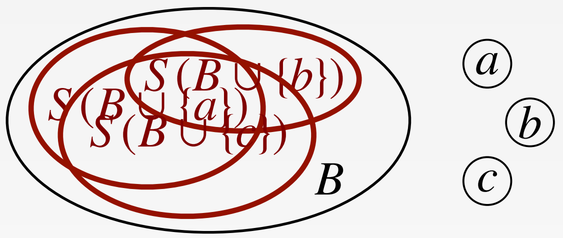
- Rationalizing relations
  - Base relation: $A \; \overline{R}_S \; B$ iff $A = S(A \cup B)$
  - Revealed preference relation: $A \; \widehat{R}_S \; B$ iff $A = S(X)$ for some X with $B \subseteq X$

- Consistency conditions
  Let A, B be feasible sets and $X \subseteq A \cap B$.
  - Contraction ($\widehat{\alpha}$): if $X = S(A \cup B)$ then $X = S(A)$ and $X = S(B)$
  - Expansion ($\widehat{\gamma}$): if $X = S(A)$ and $X = S(B)$ then $X = S(A \cup B)$



Stability

- A notion of stability for choice sets inspired by von Neumann & Morgenstern (1944)
- Let S be an arbitrary choice function.
- A set of alternatives B is S-stable in a feasible set A if
  B = {a∈A | a∈S(B∪{a})}.
  - Equivalently, B is S-stable iff
    S(B)=B (internal stability), and
    a∉S(B∪{a}) for all a∈A\B (external stability).



  - If every feasible set admits a unique inclusion-minimal S-stable set, we define $\widehat{S}$ as the choice function that returns this set.
  - Proving that a choice function $\widehat{S}$ is well-defined is often very tricky.
- A choice function S is self-stable if $\widehat{S}$ is well-defined and $S = \widehat{S}$.

Various Characterizations

- Proposition (B. et al., 2009): A choice function is quasi-transitively rationalizable iff it satisfies α, $\hat{\alpha}$, and $\hat{\gamma}$.
  - As a consequence, WARP $\Rightarrow$ $\hat{\alpha}$ & $\hat{\gamma}$.
- Theorem (B. et al., 2009): A choice function is set-rationalizable iff it satisfies $\hat{\alpha}$.
- Theorem (B. et al., 2009): A choice function is self-stable iff it satisfies $\hat{\alpha}$ and $\hat{\gamma}$.
- Are there reasonable set-rationalizable (or even self-stable) SCFs?

Set-Rationalizable SCFs

- Proposition (B. et al., 2009): No scoring rule satisfies $\hat{\alpha}$.
  - Proof:

3	2	1
a	b	c
c	a	b
b	c	a

$$S(R, \{a, b, c\}) = \{a\}$$
$$S(R, \{a, b\}) = \{a, b\}$$

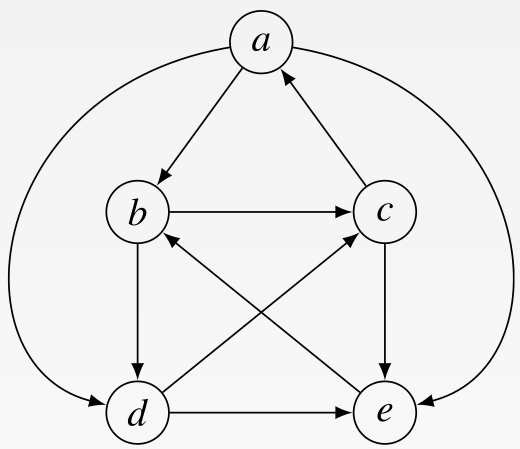
  - The same holds for all weak Condorcet extensions and runoff scoring rules.
    - Plurality, Borda, Kemeny, Dodgson, Maximin, Nanson, Bucklin, Hare (STV), etc. are not set-rationalizable.
- BUT: A handful of Condorcet extensions are set-rationalizable and even self-stable!

The Minimal Covering Set

(Dutta; 1988)

Bhaskar Dutta

- The minimal covering set is the smallest UC-stable set: $MC = \widehat{UC}$
  - A covering set is a set of alternatives B such that $a \notin UC(B \cup \{a\})$ for all alternatives $a \notin B$.
  - Theorem (Dutta, 1988): The set of all covering sets is closed under intersection.
    - A unique minimal covering set is guaranteed to exist.



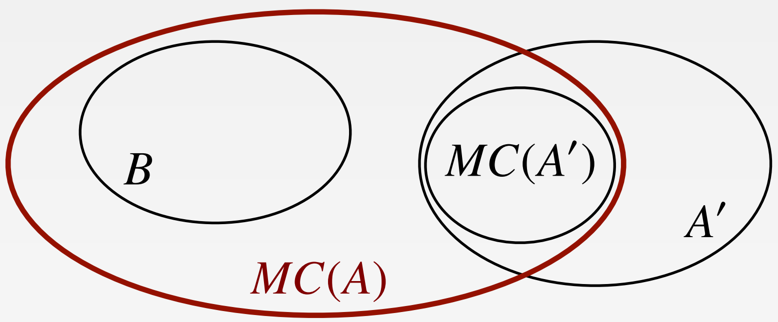
- Example
  - Covering sets: {a,b,c,d,e}, {a,b,c,d}, and {a,b,c}
  - MC = {a,b,c}
- Theorem (Dutta, 1988): The minimal covering set is the smallest majoritarian Condorcet extension satisfying $\hat{\alpha}$ and γ^*.

MC (complexity)

- No obvious reason why computing MC should be in NP
  - Verifying whether a set is a covering set is easy, verifying minimality is not.
  - Checking whether a set is MC and checking whether an alternative is contained in MC is in coNP.
    - A covering set is *not* minimal if there exists a proper covering subset.
- Straightforward iterative algorithms do not work
  - start with entire set and remove alternatives
    - there may be no covering sets in between entire set and MC
  - start with singleton and add alternatives
    - unclear which of the alternatives that are not covered by the current working set should be included

MC (algorithm)

- Three insights needed for polynomial-time algorithm
  - Lemma: If $B \subseteq MC(A)$ and $A' = \bigcup_{a \in A \setminus B} (UC(B \cup \{a\}) \cap \{a\})$ then $MC(A') \subseteq MC(A)$.



    - For every proper subset of MC, the lemma tells us how to find another disjoint and non-empty subset of MC.
  - Lemma (Laffond, Laslier, & Le Breton; 1993): Every tournament game contains a unique Nash equilibrium, the support of which (the so-called bipartisan set BP) is contained in MC.
  - The bipartisan set can be computed via linear programming.

MC (algorithm, ctd.)

- Theorem (B. and Fischer, 2008): The minimal covering set can be computed in polynomial time.

procedure $MC(A, \succ)$
  $B \leftarrow BP(A, \succ)$
  loop
    $A' \leftarrow \bigcup_{a \in A \setminus B} (UC(B \cup \{a\}) \cap \{a\})$
    if $A' = \emptyset$ **then return** B **end if**
    $B \leftarrow B \cup BP(A', \succ)$
  end loop

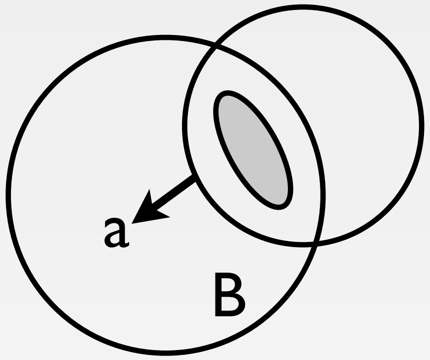
procedure $BP(A, \succ)$
  for all $i, j \in A$ **do**
    if $i \succ j$ **then** $m_{ij} \leftarrow 1$
    else if $j \succ i$ **then** $m_{ij} \leftarrow -1$
    else $m_{ij} \leftarrow 0$ **end if**
  end for
  $$s \in \{s \in \mathbb{R}^n \mid \begin{array}{ll} \sum_{j \in A} s_j \cdot m_{ij} \leq 0 & \forall i \in A \\ \sum_{j \in A} s_j = 1 & \\ s_j \geq 0 & \forall j \in A\} \end{array}$$
  $B \leftarrow \{a \in A \mid s_a > 0\}$
  return B

Tournament Equilibrium Set

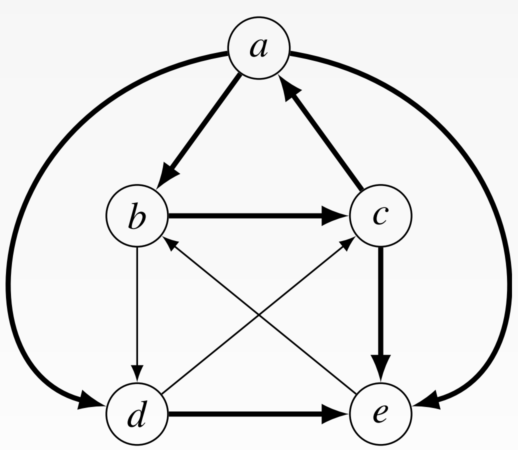
(Schwartz, 1990)

Thomas Schwartz

- Let S be an arbitrary SCF.
  - A non-empty set of alternatives B is S-retentive, if $S(\overline{D}(a)) \subseteq B$ for all $a \in B$.
  - Idea: No alternative in the set should be "properly" dominated by an outside alternative.



- $\mathring{S}$ is a new SCF that yields the union of all minimal S-retentive sets.
  - $TC = \mathring{TRIV}$
  - $\boxed{TEQ = \mathring{TEQ}}$
    - recursive definition
    - unique fixed point of ring-operator
    - Example: TEQ = {a,b,c}



The Mystery of TEQ

- Theorem (Laffond et al., 1993; B., 2009): The following statements are equivalent:
  - Every tournament contains a unique minimal TEQ-retentive set.
  - TEQ satisfies $\widehat{\alpha}$ and $\widehat{\gamma}$ (and thus is set-rationalizable and self-stable).
  - TEQ satisfies monotonicity for more than two alternatives.
- Furthermore, these statements imply that TEQ is contained in MC.
- All or nothing: Either TEQ is a most appealing SCF or it is severely flawed.
- Theorem (B., Fischer, Harrenstein, Mair; 2010): Deciding whether an alternative is contained in TEQ is NP-hard.
  - The best known upper bound is PSPACE!

Summary & Conclusion

- Standard rationality and consistency conditions lead to devastating impossibility results (among which Arrow's is the most prominent).
- Three ways to escape from these results:
    - ignore consistency (and impose other restrictions instead)
        - Scoring rules can be characterized by a consistency condition with respect to a variable electorate.
        - All Condorcet extensions fail to satisfy this condition.
    - only require expansion consistency
        - TC and UC can be characterized using β^+ and γ, respectively.
    - weaken consistency conditions
        - There is a small, but appealing, class of set-rationalizable SCFs, which contains TC, MC, and (maybe) TEQ.
        - MC can be computed in polynomial time while TEQ is NP-hard.

Recommended Books

- K. Arrow, A. Sen, & K. Suzumura: Handbook of Social Choice and Welfare (2002)
- D. Austen-Smith & J. Banks: Positive Political Theory I (1999)
- W. Gärtner: A Primer in Social Choice Theory (2009)
- J.-F. Laslier: Tournament Solutions and Majority Voting (1997)
- H. Moulin: Axioms of Cooperative Decision Making (1988)
- A. Sen: Social Choice Theory, in the Handbook of Mathematical Economics (1986)
- A. Taylor: Social Choice and the Mathematics of Manipulation (2005)