From Election Fraud to Finding the Dream Team: A Study of the Computational Complexity in Voting Problems and Stability in Hedonic Games

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Ich versichere an Eides Statt, dass die vorliegende Dissertation von mir selbständig und ohne unzulässige fremde Hilfe unter Beachtung der „Grundsätze zur Sicherung guter wissenschaftlicher Praxis an der Heinrich-Heine-Universität Düsseldorf“ erstellt worden ist.

Des Weiteren erkläre ich, dass ich eine Dissertation in der vorliegenden oder in ähnlicher Form noch bei keiner anderen Institution eingereicht habe.

Teile dieser Arbeit wurden bereits in den folgenden Schriften veröffentlicht bzw. zur Publikation angenommen: [EPR11], [EFR+15a], [RS12a], [EFR+15b], [RS13], [BRR+12], [RRS+14], [RRS+15], [RRS14], [FRR+14], [FRR+15], [LRR+15].

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Zusammenfassung

Die vorliegende Dissertation untersucht komplexitätstheoretische Eigenschaften verschiedener Wahlprobleme und kooperativer Spiele mit hedonischen Präferenzen.

Für die Wahlsysteme Bucklin Voting und Fallback Voting werden verschiedene Arten der Einflussnahme im Hinblick auf ihre Berechnungskomplexität untersucht. Wir präsentieren eine breite Analyse der beiden Wahlsysteme bezüglich der klassischen Berechnungskomplexität von Manipulation, Bestechung und des Swap-Bribery-Problems. Darüber hinaus untersuchen wir ebenfalls eine für das Wahlsystem Fallback Voting angepasste Variante von Extension Bribery. Hinsichtlich der Komplexität von Wahlkontrolle erweitern wir bereits bekannte Analysen, indem wir die parametrisierte Komplexität von Kontrolle durch Hinzufügen/Entfernen von Kandidaten oder Wählern betrachten, wobei der Parameter jeweils die Anzahl der hinzugefügten/entfernten Kandidaten bzw. Wähler ist. Ergänzend dazu präsentieren wir die erste experimentelle Untersuchung von Kontrollproblemen, in der wir die Berechnungskomplexität von Kontrolle in zufällig erzeugten Wahlen empirisch untersuchen. Neben Bucklin und Fallback Voting untersuchen wir in diesem Zusammenhang ebenfalls die Pluralitätsregel.

Des Weiteren analysieren wir die Komplexität des Margin-of-Victory-Problems und untersuchen dessen Verhältnis zu destruktiver ungewichteter Bestechung und zeigen NP-Vollständigkeit dieser beiden Probleme für das Cup-Protokoll. Für die exakte Variante des Margin-of-Victory-Problems, die wir erstmals definieren und analysieren, zeigen wir DP-Vollständigkeit für das Schulze-System, das Cup-Protokoll und die Familie von Copeland$^{\alpha}$-Systemen. Darüber hinaus definieren wir eine weitere Variante – den Swap Margin of Victory – und zeigen dessen enge Verbindung zu der destruktiven Variante des Swap-Bribery-Problems mit Einheitspreisen. Für das Cup-Protokoll ist dieses Problem NP-vollständig, während wir für k-Approval und bestimmte Scoring-Protokolle Polynomialzeitalgorithmen angeben.

Darüber hinaus führen wir eine neue Variante des Possible-Winner-Problems für gewichtete Wahlen ein, in denen die Präferenzen der Wähler vollständig gegeben, die Gewichte der Wähler jedoch unbestimmt sind. Für den Fall, dass die Gewichte aus den nicht-negativen rationalen Zahlen gewählt werden, zeigen wir, dass dieses Problem sowohl für Scoring-Protokolle als auch für Bucklin Voting und Fallback Voting in deterministischer Polynomialzeit lösbar ist.

Für hedonische Spiele mit feind-basierten Präferenzen widmen wir uns der Komplexitätsanalyse der Probleme, für ein gegebenes solches Spiel zu entscheiden, ob eine wundervoll stabile bzw. strikt kernstabile Koalitionsstruktur existiert. Wir verbessern bekannte untere Schranken für diese Probleme und zeigen deren DP-Härte. Darüber hinaus ist es uns gelungen zu zeigen, dass ein coDP-Härte-Beweis gleichzeitig Härte für die Klasse Θ_2^p (parallelen Zugriff auf NP) impliziert. Damit wäre die exakte Komplexität des Existenzproblems bezüglich wundervoller Stabilität vollständig beschrieben, da dieses Problem bekanntermaßen in dieser Komplexitätsklasse enthalten ist.

Außerdem führen wir eine neue Klasse hedonischer Spiele ein, in denen jeder Spieler seine Mitspieler in Freunde, Feinde und neutrale Spieler aufteilt und für die Menge der Freunde und der Feinde jeweils eine schwache Präferenzordnung angibt. Um diese Präferenzen über Spielern zu Präferenzen über Koalitionen zu erweitern, verwenden wir eine verallgemeinerte Bossong-Schweigert-Erweiterung. Da es in diesen FEN-hedonischen Spielen (FEN steht für „Friend/Enemy/Neutral“) jedoch unvollständige Präferenzen geben kann, also Paare von Koalitionen existieren können, die anhand dieser Präferenzen nicht vergleichbar sind, definieren wir sogenannte Vergleichbarkeitsfunktionen mit kardinalen Werten basierend auf Borda-ähnlichen Scoring-Vektoren. Für diese echte Unterklasse von additiv separablen hedonischen Spielen analysieren wir die Komplexität von Verifikations- und Existenzproblemen bezüglich vieler bekannter Stabilitätskonzepte wie der Nash-Stabilität oder der (strikten) Kernstabilität.

Abstract

In this thesis we study computational aspects of different voting problems and cooperative games with hedonic preferences.

For two well-studied voting systems, namely Bucklin and fallback voting, we present a detailed study of the computational complexity of common manipulative attacks on elections. We fully describe the classical worst-case complexity of manipulation, bribery, and swap bribery in both voting systems and furthermore study extension bribery tailored to fallback elections. We extend existing studies of the complexity of electoral control in Bucklin and fallback elections by investigating control by adding/deleting candidates or voters parameterized by the number of added/deleted candidates or voters, respectively. We complement these results with the first experimental evaluation of electoral control based on randomly generated elections, which also provides results for plurality elections.

Furthermore, we study the complexity of the margin of victory problem and its relation to destructive unweighted bribery. We show that for the cup rule, both destructive unweighted bribery and the margin of victory problem are NP-complete. Beyond that, we introduce and study two new variants of this problem: exact margin of victory and swap margin of victory. The exact variant can be shown to be DP-complete for the Schulze and the cup rule, as well as for the family of Copeland$^{\alpha}$ systems. The swap margin of victory problem, on the other hand, can be shown to be solvable in deterministic polynomial time for k-approval and certain positional scoring rules, while for the cup rule this problem is NP-complete. We furthermore show the close connection of the swap margin of victory problem and destructive swap bribery with unit costs.

Moreover, we define a new notion of the possible winner problem for weighted elections in which the uncertainty lies in the voters' weights and the complete preferences of the voters are given. We study this problem in detail for nonnegative rational weights and show that for positional scoring rules and Bucklin and fallback voting this problem can be solved in deterministic polynomial time.

In the context of hedonic games we study the computational complexity of wonderful stability existence and strict core stability existence in enemy-based hedonic games. We improve the best known lower bounds for these problems by establishing DP-hardness results. We furthermore prove that coDP-hardness of these problems directly implies hardness for Θ_2^p (parallel access to NP), which in turn would resolve the question of the exact complexity of the former problem, as it is known to be contained in this complexity class.

Beyond that, we introduce a new class of hedonic games in which each player divides her co-players into friends, enemies, and other (neutral) players and furthermore provides a weak ranking of her friends and a weak ranking of her enemies. These preferences over players are extended to preferences over coalitions using a generalized Bossong-Schweigert extension principle. The thus defined FEN-hedonic games may have incomplete preferences, meaning that there can be pairs of coalitions that are incomparable with respect to this preference extension. We suggest to break these incomparabilities by defining cardinal comparability functions based on Borda-like scoring vectors leading to additively separable preferences. We show that this class of Borda-induced FEN-hedonic games is a strict subclass of additively separable hedonic games and provide a detailed study on the computational complexity of the existence and verification problems of commonly studied stability concepts such as Nash stability and (strict) core stability.

Contents

1 Introduction

Processes of collective decision making encounter us in our every-day life on a permanent basis. We take part in political elections and delegate our sovereign power to representatives who themselves participate in parliamentary decision processes. In courts composed of several judges, the individual judgings have to be aggregated to a collective judgment. Rather basic tasks such as planning a family vacation or finding a suitable restaurant or movie for a diverse group of people are settings of preference aggregation. A less obvious (but anyhow just as ubiquitous) application of preference aggregation in a wider sense is the design of web search engines as these algorithms have to provide a specifically ordered output that presumably lists the "best" search results first.

Searching further for typical examples of preference aggregation procedures shows that many of these processes are conducted in an electronic environment, for example online auctions and the implementation of recommender systems; in some countries even political elections are held electronically by installing voting machines at the polling places. Thus, the analysis of such processes from a technical perspective is of high interest.

1.1 The Theory of Computational Social Choice

Computational social choice is an interdisciplinary research area at the intersection of *computer science* and *social choice theory*, which has a two-dimensional structure.

On the one hand, methods of collective decision making are applied to fields in computer science such as the development of web applications [GMH+99, DKN+01, TZ16] and *multiagent systems* in *artificial intelligence* [SL09, ER97] in a wider sense. Here, existing algorithms or approaches for handling collective decision making (tailored to the specific given setting) are analyzed and may be even improved based on social-choice-theoretic properties these mechanisms fulfill or fail to fulfill.

On the other hand, aggregation rules from social choice are looked at through the lenses of computer science: Established aggregation rules are analyzed with respect to their algorithmic properties, for example, the computational complexity of their running time depending on the input size or their computational resistance against manipulative attacks [FHH10]. Computational properties that might be desirable for aggregation mechanisms used in certain settings can be motivation for defining new variants of such rules. Another interesting line of research is that of computer-assisted theorem proving, that, based on the approach taken by Tang and Lin [TL09], has been further developed in the work of Geist and Endriss [GE11], Brandl et al. [BBG+15], and Brandt and Geist [BG14].

This exemplary list is far from being exhaustive and only provides a superficial description without capturing the full variety of research directions that have been taken in the context of computational social choice. The main areas, which are surveyed in the textbooks edited by Rothe [Rot15] and Brandt et al. [BCE+16], and the work of Chevaleyre et al. [CEL+07], Brandt et al. [BCE13], and Endriss [End11, End14] include *judgment aggregation*, *multiagent resource allocation*, *voting theory*, and computational aspects of *cooperative game theory*.[1]

Voting as a mechanism for collective decision making has been used ever since the Athenian democracy. More than 2000 years later, in the 18th century anno Domini, the seminal work by Condorcet [Con85] constituted the theory of social choice theory – a rapidly growing and evolving science focusing on the design and analysis of voting systems (and other mechanisms for preference aggregation) with respect to axiomatic properties, see [Mou88, BF02, Tid06]. Two of the presumably most famous and discussed results are *Arrow's Impossibility Theorem* [Arr63] and the *Gibbard-Satterthwaite Theorem* [Gib73, Sat75] (and its extensions by Gärdenfors [Gär76] and Duggan and Schwartz [DS00]). The latter essentially states that for elections with at least three candidates, any reasonable voting system is manipulable in the sense that voters can benefit from reporting insincere preferences. This result triggered a rich line of research regarding the computational complexity of manipulative attacks on elections, see [BTT89, CSL07, BTT92, FHH09], which we continue in this thesis.

Chapters 3–6 of this thesis study problems from voting theory and computational aspects in certain cooperative games, namely hedonic games. For the remaining fields of computational social choice, judgment aggregation and multiagent resource allocation, we provide short overviews of important references including surveys and recently published research.

Judgment Aggregation The model of *judgment aggregation* formalizes situations in which a group of judges, who have individual positions regarding a given set of logically connected issues, have to aggregate their positions to a collective judgment. This model was formally introduced by List and Pettit [LP02] and has been intensely studied. A comprehensive overview of these studies can be found in the surveys by List and Puppe [LP09] and Endriss et al. [EGP12], and the book chapters by Baumeister et al. [BER15] and Endriss [End16].

Recent papers study the complexity of computing a collective judgment for given aggregation methods, see the work of Lang and Slavkovic [LS14], Endriss et al. [EHS15], and Endriss and de Haan [EH15], while the complexity of bribery, manipulation, and control is addressed by Baumeister et al. [BEE+15, BRS15, BEE+13, BEE+12].

Multiagent Resource Allocation In the context of *multiagent resource allocation (MARA)*, methods of dividing a given resource among a set of agents are defined and studied, see the work of Chevaleyre et al. [CDE+06] for an overview of such settings and their applications. Two subareas can be distinguished depending on whether the given resource is divisible or not.

If the former is the case, we are in the research field of *cake cutting*, where the cake is a synonym for the given heterogeneous divisible resource and the agents, who may have differ-

[1] Note that this is a rather coarse division into subareas since there are many overlappings leading to a fruitful field of study.

ent preferences over different parts of this cake, aim at dividing it in a fair and efficient way. Formally introduced in the textbooks by Brams and Taylor [BT96] and Robertson and Webb [RW98], the development and analysis of cake cutting protocols has found much attention. The book chapters by Lindner and Rothe [LR15b] and Procaccia [Pro16] provide a detailed overview of the state of the art. Very recently, Aziz and Mackenzie [AM15] answered a glaring open question in this field by presenting the first discrete and bounded cake cutting algorithm that guarantees an envy-free allocation of the cake among four agents.

MARA settings with indivisible resources model situations in which the resource is a set of indivisible goods which has to be allocated to the participating agents depending on their preferences over (bundles) of these goods. Much of the progress that has been made regarding the study of such allocation mechanisms is covered in the book chapters by Lang and Rothe [LR15a] and Bouveret et al. [BCM16]. Recent research has, amongst others, focused on MARA settings in which the agents provide ordinal preferences, see for example the work of Baumeister et al. [BBL+14], Nguyen et al. [NBR15], and Aziz et al. [AGM+14], while other recent results investigate the complexity of social welfare optimization and the optimization of allocation procedures by bilateral swaps of the resources between agents, see the work of Nguyen et al. [NNR+14] and Damamme et al. [DBC+15], respectively.

1.2 Organization of this Thesis

The first two sections in Chapter 2 give elementary definitions from graph and complexity theory building a basis for further definitions and notions that will be defined in the course of this thesis. In Section 2.3 we define the voting systems that are relevant for the presented studies. The chapter concludes in Section 2.4 with an informal introduction to the voting problems that will be analyzed in Chapters 3, 4, and 5. This section furthermore gives a detailed overview of connections between these problems with respect to their complexity. The formal definitions of these problems together with real-life examples and further motivation can be found in the corresponding chapters in which the results are presented.

Chapter 3 is devoted to the analysis of Bucklin and fallback voting: Each section of this chapter starts with a preliminary part in which the central voting problem and its studied variants are formally defined, pointers to related work are given, and a selection of known results for other voting systems is presented. Following that, the results central to this thesis are given. Section 3.1 comprises the study of the complexity of *manipulation* in Bucklin and fallback voting while Section 3.2 focuses on *electoral control*. The latter provides a detailed two-part study of the complexity of control problems starting with a worst-case analysis which is complemented by the, to the best of our knowledge, first experimental study on control complexity, which also provides results for plurality voting. Section 3.3 focuses on the complexity of *bribery* including the standard scenario as well as *campaign management* problems such as swap bribery and extension bribery.

Chapter 4 provides an analysis of the complexity of different variants of the *margin of victory* problem for several voting systems and gives an overview of the relation to destructive bribery scenarios. We introduce the new variants *exact margin of victory* and *swap margin of* *victory* and present results for Schulze and cup elections for the exact variant and results for elections held under positional scoring rules and the cup rule for the swap margin of victory.

In Chapter 5 we define a new variant of the *possible winner* problem in which the uncertainty lies in the distribution of the voters' weights. This problem is studied for the case of nonnegative rational weights for positional scoring rules and Bucklin and fallback voting.

Chapter 6 starts with an introduction to the field of game theory and defines the concept of hedonic game. In detail, different representations of the preferences in hedonic games are discussed and stability concepts that are commonly studied are presented. In Section 6.2 the complexity of the existence of wonderfully stable and strictly core-stable coalition structures in *enemy-based hedonic games* is analyzed. In Section 6.3, we introduce so-called *FEN-hedonic games*, in which each player separates the other participating players into friends, enemies, and neutral players and she is further allowed to ordinally rank the players in her friend set and her enemy set. After providing motivation for and a detailed discussion of this new variant, we finally present results regarding the complexity of existence and verification problems of common stability concepts when incomparabilities are broken with *Borda-like comparability functions*.

We conclude the thesis with a summary of the presented results and pointers to promising future work in Chapter 7.

2 Preliminaries

In this chapter we give basic definitions that will be used throughout this thesis. To ensure a compact presentation of the basics, we will only define those notions here that are repeatedly used in different chapters. Specific notions, on the other hand, that are only relevant for particular chapters or sections will be defined therein.

We denote with $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ the set of natural numbers, integer numbers, rational numbers, and real numbers, respectively. The set of natural numbers does not contain the number 0. We denote $\mathbb{N} \cup \{0\}$ with $\mathbb{N}_0$. For $\mathbb{F} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$, we write $\mathbb{F}_{\geq y} = \{x \in \mathbb{F} \mid x \geq y\}$. The set $\mathbb{F}_{\leq y}$ is defined analogously. We call x a string over a finite, nonempty alphabet Σ, if x is a finite sequence of letters from Σ and we denote with $|x|$ its length. Σ^* denotes the set of all strings over Σ. The cardinality of a set A and a list B is denoted by $\|A\|$ and $\|B\|$, respectively, and for two sets $C, D \subseteq \Sigma^*$ we write $C - D = \{x \in \Sigma^* \mid x \in C \text{ and } x \notin D\}$. The complement of a set $A \subseteq \Sigma^*$ is defined by $\overline{A} = \{x \in \Sigma^* \mid x \notin A\}$.

2.1 Graph Theory

We start with some useful terms from graph theory based on the textbooks by Diestel [Die05] and Gurski et al. [GRR+10] .

Definition 2.1 (Undirected Graph) *An* undirected graph *G is a pair $G = (V, E)$ consisting of a vertex set V and a set of edges E connecting (not necessarily all of) the vertices. Formally, an edge $e \in E$ is a pair $e = \{v, v'\}$ of two vertices $v \neq v' \in V$.*

We call two vertices $v, v' \in V$ in a graph $G = (V, E)$ *adjacent* if and only if there is an edge $e = \{v, v'\} \in E$ connecting the two. If for each pair of vertices $v, v' \in V$ there is an edge connecting them, we call the graph G *complete*. Let $V' \subseteq V$ be a subset of the vertex set and $E' \subseteq E$ a subset of the edge set. We say that the graph $G_{V'} = (V', E')$ is an *induced subgraph of G* when for E' it holds that for each $v, v' \in V'$ there is an edge $e \in E'$ if and only if $e \in E$ holds. A *path between two vertices $v, v' \in V$* in a graph $G = (V, E)$, is (if existent) a sequence of edges $(\{v, v_1\}, \{v_1, v_2\}, \ldots, \{v_{k-1}, v_k\}, \{v_k, v'\})$ leading from v to v', where all vertices on the path are pairwise distinct. A *cycle* is a path starting from a vertex $v \in V$ that ends in the same vertex, that is, a path of the form $(\{v, v_1\}, \{v_1, v_2\}, \ldots, \{v_{k-1}, v_k\}, \{v_k, v\})$. Vertices that are connected by an edge are also called *neighbors*. Based on that we call all the vertices a certain vertex v is adjacent to the *neighborhood of the vertex v*, and denote this set by $\mathcal{N}(v) = \{v' \in V - \{v\} \mid \exists \{v, v'\} \in E\}$. The set $\mathcal{N}[v] = \mathcal{N}(v) \cup \{v\}$ denotes the *closed neighborhood of the vertex v*. The *degree* of a vertex $v \in V$ is $deg(v) = \|\mathcal{N}(v)\|$.

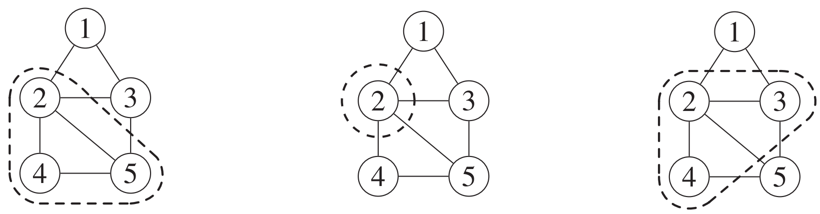
The term of neighborhood can also be defined for sets of vertices in a graph. Let $V' \subseteq V$ be such a subset of vertices in a graph $G = (V, E)$, then we define the *neighborhood of* V' to be $\mathcal{N}(V') = \bigcup_{v \in V'} \mathcal{N}(v)$. The *closed neighborhood of* V' is then $\mathcal{N}[V'] = \mathcal{N}(V') \cup V'$. With the following notions we can define interesting parameters of a graph.

Definition 2.2 (*k*-Clique, Dominating Set, Vertex Cover) *Let $G = (V, E)$ be an undirected graph and $V' \subseteq V$ be a subset of vertices. We call V'*

- *a* k-clique *if and only if the induced subgraph $G_{V'}$ is complete and $\|V'\| = k$.*
- *a* dominating set *if and only if for each $v \in V - V'$ there is a vertex $v' \in V'$ with $\{v, v'\} \in E$.*
- *a* vertex cover *if and only if for each edge $e \in E$ it holds that $e \cap V' \neq \emptyset$.*

Note that for $k \geq 2$, every k-clique contains k'-cliques for $1 \leq k' < k$. The size of a biggest clique in a graph is called the *clique number* $\omega(G)$ while the *domination number* $\gamma(G)$ is the size of a smallest dominating set in G. With $\tau(G)$ we denote the so-called *vertex cover number* giving the size of a smallest vertex cover in G. For a given vertex $v \in V$ we say that the *clique number of* v is the size of a biggest clique v is part of and we denote this number by $\omega_G(v)$.

Example 2.3 *Let $G = (V, E)$ be an undirected graph with five vertices $V = \{1, 2, 3, 4, 5\}$ and the edges $E = \{\{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,5\}, \{4,5\}\}$. Figure 2.1 shows the graphical representation of G.*



(a) 3-clique in G **(b)** Dominating set in G **(c)** Vertex cover in G

Figure 2.1: Example of an undirected graph G

The biggest complete induced subgraph of G contains three vertices, thus $\omega(G) = 3$. One of the three 3-cliques in G, namely $\{2, 4, 5\}$ is displayed in Figure 2.1a. The other two are $\{2, 3, 5\}$ and $\{1, 2, 3\}$. The smallest dominating set in G consists of the single vertex 2, see Figure 2.1b. The vertex 2 is adjacent to every other vertex in G and thus $\{2\}$ fulfills the conditions for a dominating set. Since $1 \leq \gamma(G) \leq \|V\|$ trivially holds, $\{2\}$ is the smallest possible dominating set in G. The graph G has several vertex covers and Figure 2.1c shows a minimal vertex cover. The vertices in $\{1, 2, 5\}$ form another vertex cover, but no smaller cover can be found, thus it holds that $\tau(G) = 3$.

For two disjoint sets of vertices $V_1, V_2 \subseteq V$ we say that V_1 *is independent of* V_2 if the vertices from the two subsets are not connected, that is if $\{\{v_i, v_j\} \mid v_i \in V_1 \text{ and } v_j \in V_2\} \cap E = \emptyset$. If a graph consists of two or more independent subsets of vertices, the graph is called *disconnected* and we call the subsets *independent components* of the graph. Now we turn to a special family of graphs, so-called *trees*.

Definition 2.4 (Tree, Rooted Tree) *An undirected graph $G = (V, E)$ is called a* tree *if it does not contain any cycles and is connected. Nodes with degree* 1 *are called* leaves. *A* rooted tree *is a tree in which a fixed distinct vertex $r \in V$ is the so-called* root.

Note that in a rooted tree, the root r is never called a leaf, even if $deg(r) = 1$. Furthermore, all nodes except for the leaves are called *inner nodes*. The neighbors of the root r are the *child-nodes* of r, having r as their so-called *parent-node*. The remaining inner nodes are both a child-node of their parent-node and the parent-node of their remaining neighbors. Nodes with the same parent node are called *siblings* and the *height of a rooted tree* is the length of the longest path from the root to a leaf. A *complete binary tree* is a special rooted tree where the root and each inner node have exactly two children and there are 2^k leaves if the height of the tree is k.

Example 2.5 *Figure 2.2 shows a disconnected graph with fourteen vertices consisting of two independent components that are trees.*[1]

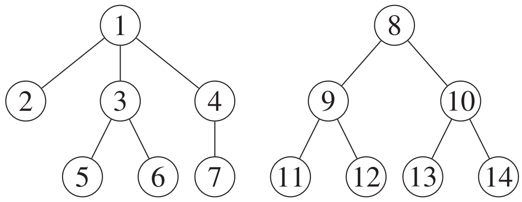


Figure 2.2: Example of a disconnected graph consisting of two trees

The right component of the graph is a complete binary tree of height 2 *with root* 8*, inner nodes* 9 *and* 10*, and the leaves* 11*,* 12*,* 13*, and* 14*. Vertices* 9 *and* 10 *are siblings having the same parent node* 8*, and are thus the child-nodes of* 8*. The left component is not a binary tree as the root* 1 *has three child-nodes.*

The notion of graphs can be extended by assigning a direction to the edges.

Definition 2.6 (Directed Graph, Complete Directed Graph) *A pair $G = (V, E)$ is a* directed graph*, where V is a set of vertices and E is a set of directed edges. An edge from vertex v to vertex v' is denoted by (v, v') for $v \neq v'$. We call a directed graph* complete *if for every pair of vertices $v, v' \in V$ both directed edges (v, v') and (v', v) are in E.*

A *path from vertex v to vertex v'* in a directed graph is, if existent, a sequence of edges $((v, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k), (v_k, v'))$ leading from v to v' and all vertices on the path are pairwise distinct. In *weighted (directed) graphs* we assign to each edge an integer weight.[2]

Definition 2.7 (Strongest Path) *Let $G = (V, E)$ be a directed weighted graph and $v, v' \in V$ be two vertices in G. The* strength of a path *is defined to be the smallest weight any of the edges on the path has. The* strongest path *from a vertex v to a vertex v' is then a path with maximum strength among all existent paths from v to v' in the graph.*

[1]Such graphs are also called forests.

[2]Clearly, also rational or real weights can be allowed.

Example 2.8 *Let $G = (V, E)$ be a directed weighted graph with five vertices in $V = \{1, 2, 3, 4, 5\}$, the edge set $E = \{(1,2),(1,4),(2,3),(2,5),(4,1),(4,5),(5,2),(5,3)\}$, and the weights displayed in Figure 2.3.*

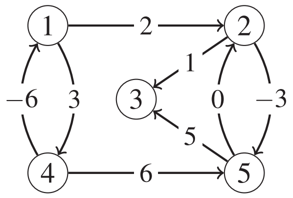


Figure 2.3: Example of a directed weighted graph G

Possible paths from 1 *to* 3 *are* $((1,2),(2,3))$, $((1,2),(2,5),(5,3))$, $((1,4),(4,5),(5,3))$, *and* $((1,4),(4,5),(5,2),(2,3))$ *and Table 2.1 shows the weights on the different paths. The minimal weight in the third column is the weight of the path, and thus we see that the weight of the strongest path from* 1 *to* 3 *is* 3.

path	weights on the path	minimal weight
$((1,2),(2,3))$	$\{2,1\}$	1
$((1,2),(2,5),(5,3))$	$\{2,-3,5\}$	−3
$((1,4),(4,5),(5,3))$	$\{3,6,5\}$	**3**
$((1,4),(4,5),(5,2),(2,3))$	$\{3,6,0,1\}$	0

Table 2.1: Weights of the different paths from 1 to 3 in graph G, where the weight of the strongest path is displayed in boldface

2.2 Complexity Theory

In the field of complexity theory the computational complexity of problems is studied with respect to different measures such as time and space. The time complexity of a problem is, intuitively speaking, the number of steps an algorithm needs to solve the problem depending on the size of the input. As for any problem there clearly might be trivial cases for which a solution is easy to find, we are interested in the worst-case complexity of problems. The theoretical analysis of the problems studied within this thesis focuses on their worst-case time complexity. In this section we introduce those complexity classes and notions needed for this analysis. If not stated otherwise, the definitions can be found in the textbooks by Rothe [Rot05, Rot08], Papadimitriou [Pap94], Garey and Johnson [GJ79], and Downey and Fellows [DF99].

The $\mathcal{O}$-notation, which is very useful to estimate an algorithm's running time, describes the intuition that a function f does not grow faster than another function g, where a finite number of exceptions is allowed.

Definition 2.9 ($\mathcal{O}$-Notation) *Let f and g be two functions $f, g : \mathbb{N} \to \mathbb{N}$. It holds that*

$$f \in \mathcal{O}(g) \iff (\exists c, n_0 \in \mathbb{N})(\forall n \geq n_0)[f(n) \leq c \cdot g(n)].$$

2.2.1 Classical Complexity

As common in the literature, we define a classical decision problem $\mathscr{D}$ to be a language over a finite alphabet Σ, that is $\mathscr{D} \subseteq \Sigma^*$, containing the yes-instances of the problem. For an instance d of the problem $\mathscr{D}$ we write $d \in \mathscr{D}$ if and only if d is a yes-instance.

Turing [Tur36] introduced the concept of Turing machine as a computational model, on which the remaining definitions in this section base. We start with defining the two presumably best known complexity classes containing decision problems decidable in deterministic and nondeterministic polynomial time, respectively.

Definition 2.10 (P and NP) *The complexity class* P *contains all problems that are accepted by a deterministic Turing machine in polynomial time. The class* NP *contains those problems that are accepted by a nondeterministic Turing machine in polynomial time.*

Since every deterministic Turing machine is, by definition, a nondeterministic one, the inclusion $\mathrm{P} \subseteq \mathrm{NP}$ holds. The question whether this inclusion is strict is one of the famous *Millennium Problems*, which is yet unresolved.[3]

When a problem can be shown to be contained in a specific complexity class, this establishes an upper bound for its complexity. Lower bounds, however, can be defined with notions of *hardness* for a given complexity class, say $\mathfrak{C}$, which allow us to determine whether a problem is at least as hard to solve as the hardest problems contained $\mathfrak{C}$. We denote with FP the set of functions $f : \Sigma^* \to \Sigma^*$ that are computable in deterministic polynomial time.

Definition 2.11 (Polynomial-Time Many-One Reduction) *Let $\mathfrak{C}$ be a complexity class and let $\mathscr{D}_1$ and $\mathscr{D}_2$ be two decision problems.*

1. *We say that $\mathscr{D}_1$ is* polynomial-time many-one reducible *to $\mathscr{D}_2$, denoted by $\mathscr{D}_1 \leq_{\mathrm{m}}^{\mathrm{p}} \mathscr{D}_2$, if and only if*
$$(\exists f \in \mathrm{FP})(\forall x \in \Sigma^*)[x \in \mathscr{D}_1 \iff f(x) \in \mathscr{D}_2].$$
2. *We say that $\mathscr{D}_1$ is* hard for a complexity class $\mathfrak{C}$ *if and only if $\mathscr{D}_2 \leq_{\mathrm{m}}^{\mathrm{p}} \mathscr{D}_1$ for all $\mathscr{D}_2 \in \mathfrak{C}$.*
3. *A problem $\mathscr{D}$ is* complete for a complexity class $\mathfrak{C}$ *if it is hard for $\mathfrak{C}$ and contained in $\mathfrak{C}$.*

Clearly, the polynomial-time many-one reduction is transitive, thus, it directly follows that for two decision problems with $\mathscr{D}_1 \leq_{\mathrm{m}}^{\mathrm{p}} \mathscr{D}_2$, it holds that hardness for a complexity class $\mathfrak{C}$ of the problem $\mathscr{D}_1$ directly implies hardness of $\mathscr{D}_2$ for this class, as well.

Assuming that P is a proper subset of NP, NP-hard problems can be considered to be intractable while the complexity class P represents efficiently solvable problems. For an interesting discussion on this dogma, we refer the reader to the textbooks by Rothe [Rot05, Rot08] and to Section 2.2.2.

Cook [Coo71] and Levin [Lev73] independently established the first NP-completeness proof by showing the problem SATISFIABILITY to be NP-complete.

[3] For further information, see the website of the *Clay Mathematics Institute* (CMI) `http://www.claymath.org/millennium-problems/p-vs-np-problem`.

Satisfiability (SAT)	
Given:	A boolean formula φ in conjunctive normal form.
Question:	Is φ satisfiable, i.e., is there a truth assignment for which φ evaluates to true?

A boolean formula φ over the variables $x_1, x_2, \ldots, x_n$ is in *conjunctive normal form* if it is of the form $\varphi(x_1, x_2, \ldots, x_n) = \bigwedge_{i=1}^{m} C_i$, where $C_i = \bigvee_{j=1}^{k_i} \ell_{i,j}$ are the so-called *clauses* of φ consisting of the disjunction of the *literals* $\ell_{i,j}$ over the variables $x_1, x_2, \ldots, x_n$. We say that a formula φ is in *disjunctive normal form* if and only if it is of the form $\varphi(x_1, x_2, \ldots, x_n) = \bigvee_{i=1}^{m}(\bigwedge_{j=1}^{k_i} \ell_{i,j})$. The problem 3-SAT denotes a variant of SATISFIABILITY that is restricted on boolean formulas in 3-CNF, meaning that each clause has at most 3 literals.

Following the result by Cook [Coo71] and Levin [Lev73], a variety of decision problems from different disciplines such as graph theory, network design, or number theory were shown to be NP-complete, see Garey and Johnson [GJ79] for a comprehensive collection. Here, we formally state those decision problems that will be used repeatedly in the course of this thesis.

Exact Cover by Three-Sets (X3C)	
Given:	A set $B = \{b_1, b_2, \ldots, b_{3m}\}$, $m > 1$, and a collection $\mathscr{S} = \{S_1, S_2, \ldots, S_n\}$ of subsets $S_i \subseteq B$ with $\|S_i\| = 3$ for each i, $1 \leq i \leq n$.
Question:	Is there a subcollection $\mathscr{S}' \subseteq \mathscr{S}$ such that each element of B occurs in exactly one set in $\mathscr{S}'$?

Note that X3C is trivial to solve for $m = 1$. Another well-studied and commonly used decision problem is the PARTITION problem.

Partition	
Given:	A set $A = \{1, \ldots, k\}$ and a list $(a_1, \ldots, a_k)$ of nonnegative integers with $\sum_{i=1}^{k} a_i = 2K$, where K is some positive integer.
Question:	Is there a set $A' \subseteq A$ such that $\sum_{i \in A'} a_i = \sum_{i \notin A'} a_i = K$?

Recall Section 2.1 for the definition of the following graph-theoretic problem.

Clique	
Given:	An undirected graph $G = (V, E)$ and a positive integer k.
Question:	Is there a clique of size at least k in G, i.e., is $\omega(G) \geq k$?

The complexity class coNP contains those problems $\mathscr{D}$ for which $\overline{\mathscr{D}}$ is in NP. Note that for an NP-hard problem $\mathscr{D}$ each of the reductions $\overline{\mathscr{D}} \leq_{\mathrm{m}}^{\mathrm{p}} \mathscr{D}'$ and $\mathscr{D} \leq_{\mathrm{m}}^{\mathrm{p}} \overline{\mathscr{D}'}$ can be used to establish coNP-hardness of $\mathscr{D}'$.

Papadimitriou and Yannakakis [PY84] introduced the class DP containing those decision problems that can be written as the difference of two NP-sets: $\mathrm{DP} = \{\mathscr{D}_1 - \mathscr{D}_2 \mid \mathscr{D}_1, \mathscr{D}_2 \in \mathrm{NP}\}$. Besides other problems, complete problems for this class are the so-called *exact variants* of NP-hard problems, such as EXACT VERTEX COVER.

EXACT VERTEX COVER (XVC)	
Given:	An undirected graph $G = (V, E)$ and a positive integer k.
Question:	Is $\tau(G) = k$, i.e., is the size of a smallest vertex cover in G exactly k?

By changing the question to whether $\tau(G) \leq k$ holds, the NP-complete problem VERTEX COVER problem is defined. One way of proving a given decision problem to be DP-hard is to find a reduction from another problem that is already known to be DP-hard. An example of a natural DP-complete problem mentioned by Papadimitriou and Yannakakis [PY84] is the following.

SAT-UNSAT	
Given:	Two boolean formulas φ_1 and φ_2.
Question:	Does $(\varphi_1 \in \text{SAT}) \wedge (\varphi_2 \notin \text{SAT})$ hold?

Another way to show DP-hardness is to apply the following tool by Wagner [Wag87].

Lemma 2.12 (Wagner [Wag87]) *Let $\mathscr{D}_1$ be some* NP*-hard problem, and let $\mathscr{D}_2$ be any set. If there exists a polynomial-time computable function f such that, for any two instances x_1 and x_2 of $\mathscr{D}_1$ for which $x_2 \in \mathscr{D}_1$ implies that $x_1 \in \mathscr{D}_1$, we have*

$$(x_1 \in \mathscr{D}_1) \wedge (x_2 \notin \mathscr{D}_1) \iff f(x_1, x_2) \in \mathscr{D}_2, \tag{2.1}$$

then $\mathscr{D}_2$ is DP*-hard.*

Cai et al. [CGH+88, CGH+89] introduce a generalization of DP to a hierarchy of complexity classes called the *boolean hierarchy over* NP, denoted by $\text{BH(NP)} = \bigcup_{k \geq 0} \text{BH}_k(\text{NP})$, where $\text{BH}_k(\text{NP}) = \{\mathscr{D}_1 - (\mathscr{D}_2 - (\cdots - (\mathscr{D}_{k-1} - \mathscr{D}_k)\cdots)) \mid \mathscr{D}_i \in \text{NP} \text{ and } \mathscr{D}_k \subseteq \mathscr{D}_{k-1} \subseteq \cdots \subseteq \mathscr{D}_1\}$ are the levels of the hierarchy. It holds that $\text{P} = \text{BH}_0(\text{NP})$, $\text{NP} = \text{BH}_1(\text{NP})$, and $\text{DP} = \text{BH}_2(\text{NP})$.

The concept of Turing machine can be extended by giving a machine access to a so-called *oracle set $\mathscr{D}$* which itself is a decision problem. During a computation such an *oracle Turing machine* can ask queries of the form $x \in \Sigma^*$ to the oracle set and gets the answer "yes" or "no" depending on whether $x \in \mathscr{D}$ or $x \notin \mathscr{D}$. We denote deterministic and nondeterministic oracle Turing machines with polynomial running time by *DPOTM* and *NPOTM*, respectively.

Definition 2.13 (Polynomial-Time Turing Reduction) *Let $\mathscr{D}_1$ and $\mathscr{D}_2$ be two decision problems. We say that $\mathscr{D}_1$*

1. polynomial-time Turing reduces *to $\mathscr{D}_2$, denoted by $\mathscr{D}_1 \leq_{\text{T}}^{\text{p}} \mathscr{D}_2$, if and only if there is a DPOTM accepting $\mathscr{D}_1$ with oracle $\mathscr{D}_2$.*
2. nondeterministic polynomial-time Turing reduces *to $\mathscr{D}_2$, denoted by $\mathscr{D}_1 \leq_{\text{T}}^{\text{NP}} \mathscr{D}_2$, if and only if there is a NPOTM accepting $\mathscr{D}_1$ with oracle $\mathscr{D}_2$.*

For a complexity class $\mathfrak{C}$ we define two notions of closure: $\mathrm{P}^{\mathfrak{C}} = \{\mathscr{D} \mid (\exists \mathscr{D}' \in \mathfrak{C})[\mathscr{D} \leq_{\mathrm{T}}^{\mathrm{p}} \mathscr{D}']\}$ and $\mathrm{NP}^{\mathfrak{C}} = \{\mathscr{D} \mid (\exists \mathscr{D}' \in \mathfrak{C})[\mathscr{D} \leq_{\mathrm{T}}^{\mathrm{NP}} \mathscr{D}']\}$. Papadimitriou and Zachos [PZ83] introduced the complexity class $\Theta_2^p = \mathrm{P}^{\mathrm{NP}[\log]}$ containing those decision problems that can be decided by a DPOTM which asks $\mathscr{O}(\log(n))$ sequential Turing queries to an NP oracle. Hemachandra [Hem89] and Köbler et al. [KSW87] independently showed that DPOTMs asking a set of precomputed queries in parallel to an NP oracle can solve the same set of problems; this complexity class is known as $\mathrm{P}_{||}^{\mathrm{NP}}$ and it thus holds that $\Theta_2^p = \mathrm{P}^{\mathrm{NP}[\log]} = \mathrm{P}_{||}^{\mathrm{NP}}$. Decision problems can be shown to be hard for this class with the following tool which has also been introduced by Wagner [Wag87].

Lemma 2.14 (Wagner [Wag87]) *Let $\mathscr{D}_1$ be some* NP*-hard problem, and let $\mathscr{D}_2$ be any set. If there exists a polynomial-time computable function f such that, for all $k \geq 1$ and any $2k$ instances $x_1, \dots, x_{2k}$ of $\mathscr{D}_1$ for which $x_j \in \mathscr{D}_1$ implies that $x_i \in \mathscr{D}_1$ for $i < j$, we have*

$$\|\{i \,|\, x_i \in \mathscr{D}_1\}\| \text{ is odd} \iff f(x_1, x_2, \dots, x_{2k}) \in \mathscr{D}_2, \tag{2.2}$$

then $\mathscr{D}_2$ is Θ_2^p-hard.

Chang and Kadin [CK95] introduce structural properties of decision problems that are particularly interesting with respect to completeness for, amongst others, the levels of the boolean hierarchy and Θ_2^p.

Definition 2.15 (Chang and Kadin [CK95]) *We say that a decision problem $\mathfrak{D}$* has AND_ω functions *if for all $n \in \mathbb{N}$ it holds $\{< D_1, D_2, \dots, D_n > \mid D_1 \in \mathfrak{D} \wedge D_2 \in \mathfrak{D} \wedge \dots \wedge D_n \in \mathfrak{D}\} \leq_{\mathrm{m}}^{\mathrm{p}} \mathfrak{D}$.*

Their findings relevant to this thesis are summarized in the lemma below.

Lemma 2.16 (Chang and Kadin [CK95]) *Let $\mathfrak{D}$ be a decision problem.*

1. *If $\mathfrak{D}$ is* NP*-complete, it has* AND_ω *functions.*
2. *If $\mathfrak{D}$ is* DP*-complete, it has* AND_ω *functions.*
3. *If $\mathfrak{D}$ is complete for any class of the boolean hierarchy higher than the second level, it cannot have* AND_ω *functions, unless the boolean hierarchy collapses to the second level.*
4. *If $\mathfrak{D}$ is Θ_2^p-complete, it has* AND_ω *functions.*

The class Θ_2^p is contained in another hierarchy of complexity classes above NP: the *polynomial hierarchy*, which goes back to the work of Meyer and Stockmeyer [MS72] and Stockmeyer [Sto76]. Inductively, the levels of this hierarchy are defined by

$$\Sigma_0^p = \Delta_0^p = \Pi_0^p = \mathrm{P}, \Sigma_{i+1}^p = \mathrm{NP}^{\Sigma_i^p}, \Delta_{i+1}^p = \mathrm{P}^{\Sigma_i^p}, \Pi_{i+1}^p = \mathrm{co}\Sigma_{i+1}^p, \text{ for } i \geq 0.$$

This thesis focuses on the first levels of this hierarchy: $\Sigma_0^p = \mathrm{P}$, $\Sigma_1^p = \mathrm{NP}^{\mathrm{P}} = \mathrm{NP}$, $\Pi_1^p = \mathrm{coNP}$, and for $i = 2$ particularly on Σ_2^p. Meyer and Stockmeyer [MS72] propose the following representation of the class Σ_2^p by alternating quantifiers, which is useful to show Σ_2^p membership for a given decision problem $\mathscr{D}$.

Lemma 2.17 (Meyer and Stockmeyer [MS72]) *A decision problem $\mathscr{D}_1$ is contained in Σ_2^p if and only if there exists a set $\mathscr{D}_2 \in \mathrm{P}$ and a polynomial p such that for each $x \in \Sigma^*$ it holds that*

$$x \in \mathscr{D}_1 \iff (\exists y \in \Sigma^*)(\forall z \in \Sigma^*)[|y| \le p(|x|) \wedge |z| \le p(|x|) \Longrightarrow (x,y,z) \in \mathscr{D}_2].$$

One natural Σ_2^p-complete problem that will be used in a hardness proof in Chapter 6 is the following (see the survey by Schaefer and Umans [SU02a, SU02b] for other natural Σ_2^p-complete problems).

2-QUANTIFIED 3-DNF-SAT	
Given:	Two sets $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ of boolean variables and a boolean formula $\varphi(X,Y)$ over $X \cup Y$ in disjunctive normal form where each of the conjunctive clauses consists of exactly three distinct literals.
Question:	Is there a truth assignment τ_X for the variables in X such that for every truth assignment τ_Y for the variables in Y the formula $\varphi(X,Y)$ evaluates to true under τ_X, τ_Y?

We conclude this part by presenting the relation of the just defined complexity classes where inclusions are indicated by lines from left to right. Note that none of these inclusions is known to be strict:

$$\mathrm{P} \begin{smallmatrix} \diagup \mathrm{NP} \diagdown \\ \diagdown \mathrm{coNP} \diagup \end{smallmatrix} \mathrm{DP} — \Theta_2^p — \Sigma_2^p$$

2.2.2 Parameterized Complexity

Downey and Fellows [DF99] introduced the theory of parameterized complexity theory which extends the theory of classical complexity theory in the following sense: Instead of measuring the worst-case complexity of a problem in the instance size n only, the structural properties of a problem are taken into account by including a secondary measurement.

A *parameterized decision problem* is a language $\mathscr{L} \subseteq \Sigma^* \times \mathbb{N}$ with elements $(x,k) \in \mathscr{L}$, where $x \in \Sigma^*$ is the problem instance in the classical sense and $k \in \mathbb{N}$ is the *parameter*.

Definition 2.18 (Fixed-parameter Tractability) *A parameterized problem $\mathscr{L}$ is called* fixed-parameter tractable *if there exists some computable function f such that for each input (x,k) of size $n = |(x,k)|$, it can be determined in time $\mathscr{O}(f(k) \cdot n^c)$ whether or not $(x,k) \in \mathscr{L}$, where c is a constant.*

FPT denotes the parameterized complexity class containing all fixed-parameter tractable problems and it can be seen as the parameterized analogon to P. The VERTEX COVER problem parameterized by the size of a solution is contained in FPT as it can be solved in $O(2^k n)$, where here n is the number of vertices in the instance graph, and k is the solution size. This shows that an NP-hard problem might be efficiently solvable in practice when it is fixed-parameter tractable and the parameter is small enough in typical instances.

By defining the concept of parameterized reduction, hardness for parameterized complexity classes can be defined.

Definition 2.19 (Parameterized Reduction) *Let $\mathfrak{C}$ be a parameterized complexity class with $\mathscr{L}, \mathscr{L}' \in \mathfrak{C}$. We say that*

1. $\mathscr{L}$ parameterizedly reduces to $\mathscr{L}'$ *if there is a function $f : \Sigma^* \times \mathbb{N} \to \Sigma^* \times \mathbb{N}$ such that for each (x,k),*
   - *$f(x,k) = (x',k')$ can be computed in time $\mathscr{O}(g(k) \cdot p(|x|))$ for some function g and some polynomial p, and*
   - *$(x,k) \in \mathscr{L}$ if and only if $(x',k') \in \mathscr{L}'$, where $k' \leq g(k)$ and k' depends only on k;*
2. $\mathscr{L}$ *is* hard for $\mathfrak{C}$ *if every problem in $\mathfrak{C}$ parameterizedly reduces to $\mathscr{L}$; and*
3. $\mathscr{L}$ *is* complete for $\mathfrak{C}$ *if it both belongs to $\mathfrak{C}$ and is hard for $\mathfrak{C}$.*

The class XP contains those parameterized decision problems solvable in time $O(n^{g(k)})$ for some function g and it contains the main hierarchy of parameterized complexity classes, called the W-*hierarchy*.

$$\mathrm{FPT} = \mathrm{W}[0] \subseteq \mathrm{W}[1] \subseteq \mathrm{W}[2] \subseteq \cdots \subseteq \mathrm{W}[t] \subseteq \cdots \subseteq \mathrm{XP}.$$

Problems that are complete for classes $\mathrm{W}[t]$ for $t \geq 1$ are considered to be intractable with respect to the given parameter and $\mathrm{W}[1]$ can be seen as a strong parameterized analogon of NP. A natural $\mathrm{W}[1]$-complete problem is the CLIQUE problem parameterized with the solution size. The following parameterized variant of DOMINATING SET is complete for $\mathrm{W}[2]$.

k-DOMINATING SET (k-DS)	
Given:	A graph $G = (V,E)$ and a positive integer $k \leq \|V\|$.
Parameter:	k.
Question:	Is there a dominating set of size at most k in G?

Note that showing $\mathrm{W}[t]$-hardness of a problem for $t > 1$ directly establishes $\mathrm{W}[1]$-hardness of the problem, as well. For further background on parameterized complexity theory, see the monographs by Niedermeier [Nie06] and Flum and Grohe [FG06].

2.3 Elections and Voting Systems

Given a selection of different choices, a common task for a group of individuals is to determine the best of the given alternatives. There are various ways of achieving this goal reaching from randomly or dictatorially choosing the winner, to holding an election where the alternatives serve as candidates and the individuals are the voters. To successfully find a winner or a set of winners in a given election, two issues have to be specified: firstly, how the voters can or have to express their opinion in form of their vote and secondly, how the winner is determined from these votes.

Definition 2.20 (Voting System) *For a given* voting system $\mathscr{E}$ *we define an* $\mathscr{E}$ election *to be a tuple* (C,V)*, where C is a finite set of candidates and V is a finite list of votes.* $\mathscr{E}$ *defines the representation of the votes in V and furthermore provides a procedure for the winner determination.* $\mathscr{E}(C,V)$ *denotes the set of* $\mathscr{E}$ winners.

In the literature, voting systems that output a set of winners are also called *social choice correspondences*, whereas when the winner determination has to provide a unique winner, the system is a so-called *social choice function*. Note that V does not have to be given as a list containing each single ballot, but can rather be *represented succinctly* by listing all occurring distinct ballots and storing for each ballot the number of voters that have cast this exact vote.

In the social choice literature a variety of different voting systems has been designed and each of these systems has its assets and drawbacks depending on the purpose the election is held for. To categorize these voting systems, a comprehensive collection of properties has been established. Two of these properties that capture the essence of democratic processes are *non-dictatorship* and *citizen's sovereignty*: A voting system $\mathscr{E}$ is called *non-dictatorial* if for $\|V\| \geq 1$ there is no single voter $v \in V$ such that the outcome of an election solely depends on v's preference. From the candidate's perspective, the property *citizen's sovereignty* which is fulfilled if for each $c \in C$ there exists a set of voters making c an $\mathscr{E}$ winner, guarantees that every participating candidate, at least in theory, has a chance of being an $\mathscr{E}$-winner. It is safe to assume that these two properties should be fulfilled by any reasonable voting system. Two other properties that are highly relevant are *anonymity* and *neutrality*. For the former to hold, a voting system has to ensure that renaming the voters does not change the outcome of an election. The latter is an analogon regarding the set of candidates: A voting rule is called *neutral* if the outcome does not depend on the candidates' naming, i.e., if any two candidates are swapped in each vote, the outcome changes accordingly.

A comprehensive, but not exhaustive selection of voting systems and analyses regarding their social choice properties can be found in the book chapters by Baumeister and Rothe [BR15] and Zwicker [Zwi16], and the work of Brams and Fishburn [BF02], Moulin [Mou88], Tideman [Tid06], and Rothe et al. [RBL$^+$11].

Depending on the voting system, the winner determination procedure can be very involved. Thus, for practical reasons (and also from a theoretical point of view, as we will see in the Chapters 3, 4, and 5) the computational complexity of a voting system's winner determination is of high interest. For the *Young*, *Dodgson*, and *Kemeny* rule, for instance, Hemaspaandra et al. [HHR97, HSV05] and Rothe et al. [RSV03] showed $\mathrm{P}_{||}^{\mathrm{NP}}$-completeness of the winner problem, see also the interesting recent work by Betzler et al. [BBN14].

For voting systems proceeding in several rounds, such as *single transferable vote (STV)* or *ranked pairs*, the issue of how and especially at what point of the winner determination process ties are broken is essential for their complexity: Breaking ties by applying the so-called *parallel universes tie-breaking* (instead of breaking ties whenever they occur during the procedure by a beforehand fixed tie-breaking order) can increase the complexity of the winner problem from tractability to NP-hardness, see the intriguing work of Conitzer et al. [CRX09] and Brill and Fischer [BF12]. For the voting systems that we define in the following, the set of winners can be determined in deterministic polynomial time.

Most of the voting systems are originally defined for elections in which all voters are equal with respect to their influence on the election's outcome. To allow modeling scenarios in which this assumption is not reasonable, the concept of *weighted election* can be defined. In a weighted election, each vote $v_i \in V$ is associated with a nonnegative integer weight w_i which implies that the voter v_i is counted as if w_i voters with weight 1 would have cast the same ballot. Unweighted elections can be viewed as special weighted elections in which all voters have unit weight.

With only one exception, all voting systems that we define in the following subsections expect the voters to provide a preference in form of a *linear order* which implies that it has to be complete, transitive, and irreflexive. Thus, the voters have to rank all candidates in C, furthermore if a is better ranked than b and b is in turn better ranked than c the ranking has to rank a better than c, and finally the ranking has to be strict, meaning that no ties between candidates are allowed. For a candidate set $C = \{a,b,c,d\}$ a preference of a voter preferring c to b to d to a is denoted by $c > b > d > a$. Example 2.21 gives us an election where the votes are linear preferences and demonstrates how the votes of an election are adapted when candidates are deleted. We extend this example in the following subsections to illustrate the voting systems that will be defined.

Example 2.21 *Let (C,V) be an election with five candidates $C = \{a,b,c,d,e\}$ and six voters in $V = \{v_1,v_2,v_3,v_4,v_5,v_6\}$ with the following preferences.*

$\mathbf{v_1}:$	$b > c > a > d > e$	$\mathbf{v_3},\mathbf{v_4}:$	$e > b > a > d > c$
$\mathbf{v_2}:$	$a > d > c > b > e$	$\mathbf{v_5},\mathbf{v_6}:$	$c > a > e > b > d$

If candidates in C are deleted, they are omitted in each vote leading to a new list of voters. In the subelection (C',V) with $C' = \{a,c,d\}$, for example, the voters have the following preferences:

$\mathbf{v_1}:$	$c > a > d$	$\mathbf{v_3},\mathbf{v_4}:$	$a > d > c$
$\mathbf{v_2}:$	$a > d > c$	$\mathbf{v_5},\mathbf{v_6}:$	$c > a > d$

The concept of transitive preferences captures the model of *rational voters*. There are several voting systems also allowing *irrational*, that is, intransitive votes, but these are not studied in the scope of this thesis. For further information and an interesting discussion, we refer the interested reader to the work of Faliszewski et al. [FHH+09b].

2.3.1 Positional Scoring Rules

We start with *positional scoring rules*, a well-known and intensely studied family of voting systems (see for example [BF02, HH07]). For an election (C,V) with $\mathfrak{m}$ candidates and $\mathfrak{n}$ voters, positional scoring rules are defined by a so-called *scoring vector* $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_{\mathfrak{m}})$ with $\alpha_i \geq \alpha_{i+1}$ for $1 \leq i \leq \mathfrak{m}-1$, that defines the points each candidate gets from each vote depending on her position in the vote: Candidate $c \in C$ gets α_i points from a vote $v \in V$ in which c is positioned on position i. The $\vec{\alpha}$*-score* of a candidate is the sum of all points she gains from all votes in V. The candidates with the highest $\vec{\alpha}$-scores are the $\vec{\alpha}$*-winners*. This definition gives a generalized characterization of various voting systems. Betzler and Dorn [BD10] define a subclass of positional scoring rules, so-called *pure scoring rules* having the property that whenever there are $\mathfrak{m} \geq 2$ candidates, the scoring vector $\vec{\alpha}$ for an $\mathfrak{m}$-candidate election can be obtained from a scoring vector $\vec{\alpha}'$ for an $(\mathfrak{m}-1)$-candidate election by adding an additional value α_k such that the above inequalities for α_i, $i \in \{1,\ldots,\mathfrak{m}\}$ hold. We will present some of the most prominent positional scoring rules, that all are pure scoring rules, and define them by specifying the scoring vector: The *Borda rule* (or *Borda voting*, *Borda count*), see [Bor81], is defined by the scoring vector $\vec{\alpha} = (\mathfrak{m}-1, \mathfrak{m}-2, \ldots, 2, 1, 0)$. Other known systems are *plurality voting* with $\vec{\alpha} = (1, 0, \ldots, 0)$, the *veto rule* (or *veto voting*), where $\vec{\alpha} = (1, 1, \ldots, 1, 0)$, and *k-approval (k-AV)*, where for a given k with $1 \leq k \leq \mathfrak{m}$, $\alpha_i = 1$ for $1 \leq i \leq k$ and $\alpha_i = 0$ otherwise. Note that plurality can also be defined as 1-approval and likewise the veto rule is equivalent to $(\mathfrak{m}-1)$-approval.

Example 2.22 continues Example 2.21 and shows the winner determination in positional scoring rules for a selection of different scoring vectors.

Example 2.22 *Let (C,V) be the election defined in Example 2.21. Table 2.2 shows the scores and the winners in (C,V) for the Borda rule, veto voting, plurality voting,* 2*-approval, and* 3*-approval.*

Voting System	**a**	**b**	**c**	**d**	**e**	**winners**
Borda	**16**	13	13	3	12	a
plurality	1	1	**2**	0	**2**	$\{c,e\}$
veto	**6**	**6**	4	4	4	$\{a,b\}$
2-AV	**3**	**3**	**3**	1	2	$\{a,b,c\}$
3-AV	**6**	3	4	1	4	a

Table 2.2: Scores and winners in the election from Example 2.21 for different positional scoring rules

Columns 2 *through* 6 *show the scores the candidates obtain for the scoring vectors of the different voting systems. The highest overall points are given in boldface.*

2.3.2 Bucklin and Fallback Voting

In Chapter 3 we focus on the voting systems Bucklin and fallback voting. *Bucklin voting* is a well-known voting system that was used already in the first decade of 1900, see [HH26]. It is also known under the name *Grand Junction voting* as James W. Bucklin (the eponym for the more common name "Bucklin voting") promoted it to be used in Grand Junction, Colorado in 1909. After that election it was also used in several other U.S. American cities until 1917.[4] For an analysis of the system's social choice properties we refer the reader to the work of Rothe et al. [RBL+11] and Tideman [Tid06].

[4] See the interesting website `http://www.electology.org/bucklin` for a list of cities using this system.

In Bucklin voting the winner determination procedure proceeds as follows: For a Bucklin election (C,V) with $\mathfrak{m}$ candidates we define the so-called *level i score of a candidate* $c \in C$ to be the number of votes in V positioning c among the first i, $1 \leq i \leq \mathfrak{m}$, positions and we denote it by $score_V^i(c)$. If V is clear from the context, we will omit the subscript in the notation and simply use $score^i(c)$ instead. Furthermore we define $maj(V) = \lfloor \|V\|/2 \rfloor + 1$ to be the majority threshold. The *Bucklin score of c* is defined as the smallest level i on which c reaches the majority threshold, that is $score^i(c) \geq maj(V)$. Finally, those candidates with the smallest Bucklin score, say k, and the biggest level k score are the *level k Bucklin winners* of the election.

The variant of Bucklin in which all candidates with smallest Bucklin score are winners, is called *simplified Bucklin*, but this variant will not be further analyzed in this thesis.

Fallback voting is a hybrid voting system that was introduced by Brams and Sanver [BS09] that combines Bucklin voting and *approval voting*. Approval voting, also defined by Brams and Fishburn [BF78, BF83] (see also the textbook edited by Laslier and Sanver [LS10]), is not a preference-based voting system as the voters are not asked to provide a ranking of the candidates, but to indicate their approval or disapproval of each candidate. For a compact representation, the ballots are given as so-called approval vectors in $\{0,1\}^{\|C\|}$ in which each position represents a fixed candidate and the entry indicates whether the voter approves of the candidate (1) or disapproves (0). All candidates with a maximal number of approvals are the *approval winners*.

Fallback voting combines the two voting systems in the following way: In a first step each voter $v \in V$ is asked to provide her so-called approval strategy S_v specifying the candidates that v approves of. In a second step, each voter has to rank only the candidates in S_v and we denote this ranking by $\overrightarrow{S_v}$. The ballots have the form

$$v: \overrightarrow{S_v} \mid C - S_v.$$

The winner determination makes use of Bucklin voting in the following sense: For a fallback election (C,V) all level k Bucklin winners are so-called *level k fallback winners*, if they exist. If there are no Bucklin winners due to the disapprovals, all approval winners are *fallback winners*.

With the above definitions in mind it is obvious that Bucklin elections are special fallback elections in which all voters approve of all candidates. This fact is interesting when analyzing the complexity of certain control and campaign management problems (see Sections 3.2 and 3.3) where this relation allows to transfer, on the one hand, lower bounds found for Bucklin voting to fallback voting and, on the other hand, upper bounds found for fallback elections to Bucklin elections. But note that it highly depends on the voting problem at hand whether or not known results can be transferred.

Example 2.23 *Recall election (C,V) from Example 2.21. The Bucklin winner can be directly determined as the ballots have the right representation. There are six voters in V, thus the majority threshold $maj(V)$ is reached with four points. The left part of Table 2.3 shows the different scores on the different levels in (C,V).*

We see that the three candidates a,c, and e all reach the majority threshold on the third level (marked by the highlighted row), thus the Bucklin score in the election is 3*. Since a has the highest level* 3 *score, a is the unique level* 3 *Bucklin winner in* (C,V).

	(C,V)					(C,V')				
	a	*b*	*c*	*d*	*e*	*a*	*b*	*c*	*d*	*e*
score^1	1	1	2	0	2	1	1	1	0	2
score^2	3	3	3	1	2	2	2	2	1	2
score^3	**6**	3	**4**	1	**4**	3	2	3	1	2
score^4	6	6	4	4	4	3	3	3	1	2
score^5	6	6	6	6	6	**3**	**3**	**3**	2	2

Table 2.3: Scores in the Bucklin election (C,V) and the fallback election (C,V')

As stated above, (C,V) *can also be viewed as a fallback election in which all voters approve of all candidates. In that case, the Bucklin winners and the fallback winners always coincide. In order to illustrate the characteristics of a fallback election let us construct a second election* (C,V')*, where the candidate set remains the same, namely* $C = \{a,b,c,d,e\}$ *and the preferences given in V are altered by applying different approval strategies for the voters. This leads to the new voter list* V' *with the following ballots.*

$\mathbf{v_1}$:	$b > c \mid \{a,d,e\}$	$\mathbf{v_4}$:	$e > b > a > d \mid \{c\}$
$\mathbf{v_2}$:	$a > d > c > b > e \mid \emptyset$	$\mathbf{v_5}$:	$c > a \mid \{b,d,e\}$
$\mathbf{v_3}$:	$e \mid \{a,b,c,d\}$	$\mathbf{v_6}$:	$\emptyset \mid \{a,b,c,d,e\}$

The right side of Table 2.3 shows the scores in this new election (C,V') *and we see that due to the disapprovals, there is no candidate reaching the majority threshold of* 4*, thus the approval winners, namely candidates a, b, and c tie for winning and are all fallback winners in* (C,V').

2.3.3 Copeland, Schulze, and Cup Voting

We now turn to voting systems that use the concept of pairwise comparisons between candidates to determine the winner. For a given election (C,V) and two candidates $a,b \in C$, let $D_V(a,b)$ denote the number of votes in V that prefer a to b minus the number of votes in V that prefer b to a, that is

$$D_V(a,b) = \|\{v \in V \mid a > b \text{ in } v\}\| - \|\{v \in V \mid b > a \text{ in } v\}\|.$$

Whenever the voter list V is clear from the context, we will omit the subscript. If $D_V(a,b) > 0$, we say that *a (strictly) beats b in pairwise comparison*. Clearly, $D_V(a,b) = -D_V(b,a)$ holds by definition. If $D_V(a,b) = D_V(b,a) = 0$ we say that there is a *tie* between the candidates a and b.

One of the most prominent voting based on this concept is the *Condorcet system*, which goes back to Condorcet [Con85]. The so-called *Condorcet winner* of an election is the unique candidate strictly beating all other candidates. Unfortunately, the Condorcet winner does not always exist, as the following small example shows: Consider the election over the candidates a, b, and c with the three voters $a > b > c$, $b > c > a$, and $c > a > b$. We see that candidate a strictly beats b, while b strictly beats c, but c in turn beats candidate a. This election exemplifies the famous *Condorcet paradox*. A voting system $\mathscr{E}$ for which (whenever one exists) the Condorcet winner is always an $\mathscr{E}$ winner is called *Condorcet consistent*. Each of the three voting systems that we define below fulfills this property.

We start with defining the family of *Copeland*$^{\alpha}$ that was (in this generalization) defined by Faliszewski et al. [FHH^{+}09b]. For a rational number α with $0 \leq \alpha \leq 1$ and a given election (C,V), $D_V(a,b)$ is determined for every pair $(a,b) \in C \times C$. Each candidate a receives one point for every pairwise comparison she (strictly) wins and gets α points for every tie. All candidates with the highest score are the *Copeland*$^{\alpha}$ *winners* of (C,V).[5]

Turning to *Schulze voting*, that was introduced by Schulze [Sch11], we define for a given election (C,V) the *weighted majority graph*, denoted by $\mathrm{WMG}(C,V)$, to be a weighted, complete directed graph G with vertex set C, where the weight of an edge (a,b) is defined to be $D_V(a,b)$ (recall Section 2.1 for the graph-theoretic definitions). For each pair (a,b) of candidates, $P(a,b)$ denotes the *strength of a strongest path from a to b* (i.e., of a path with the greatest minimum edge weight among all paths from a to b). All candidates $a \in C$ with $P(a,b) \geq P(b,a)$ for all $b \in C \smallsetminus \{a\}$ are the *Schulze winners* of (C,V). Note that a candidate $a \in C$ is the unique Schulze winner of (C,V) if and only if $P(a,b) > P(b,a)$ for all $b \in C \smallsetminus \{a\}$.

In *cup* (or *sequential majority*) elections, (see [Mou88, CSL07]), an election is defined by specifying

- a candidate set C, a list of voters V, and additionally,
- a complete binary rooted tree T with as many leaves as there are candidates in C, called a *voting tree* (where we assume that C contains enough dummy candidates so as to satisfy $\|C\| = 2^k$ for some k and all dummy candidates are ranked worst in V), and
- a schedule that assigns the candidates to the leaves of T.

For determining the cup winner we compute the value of $D_V(a,b)$ for each pair of candidates, a and b, that are siblings in the tree and the winner of this pairwise comparison is assigned to the corresponding parent-node. This procedure is continued until the *cup winner* is assigned to the root. The schedule is known beforehand and whenever ties occur, they are broken by a beforehand fixed tie-breaking rule.

Example 2.24 *Recall the election (C,V) from Example 2.21. Table 2.4 shows the pairwise comparisons between the candidates in C and the Copeland$^{\alpha}$ scores for $\alpha \in \{0, 0.5, 1\}$.*

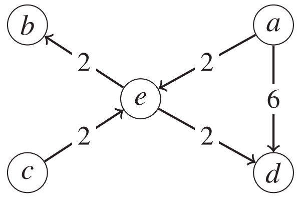
The scores in boldface indicate the Copeland$^{\alpha}$ winners for the corresponding α. We see that candidate a is a winner for all chosen values and is the unique winner for $\alpha = 0.5$ while candidate e is a co-winner for $\alpha = 0$ and c is a co-winner for $\alpha = 1$.

Figure 2.4 shows a subgraph of $\mathrm{WMG}(C,V)$ *in which all edges with negative or zero weight are omitted, and the strengths of the strongest paths in the weighted majority graph.*

[5] Note that originally the Copeland system was defined by Copeland [Cop51], which in the notation above is equivalent to Copeland$^{0.5}$. The system Copeland^{1} is also known as *Llull voting*.

	a	**b**	**c**	**d**	**e**	$\alpha = 0$	$\alpha = 0.5$	$\alpha = 1$
a	–	3 : 3	3 : 3	6 : 0	4 : 2	**2**	**3**	**4**
b	3 : 3	–	3 : 3	3 : 3	2 : 4	0	1.5	3
c	3 : 3	3 : 3	–	3 : 3	4 : 2	1	2.5	**4**
d	0 : 6	3 : 3	3 : 3	–	2 : 4	0	1	2
e	2 : 4	4 : 2	2 : 4	4 : 2	–	**2**	2	2

Table 2.4: Pairwise comparisons and Copeland$^\alpha$ scores for $\alpha \in \{0, 0.5, 1\}$



(a) Subgraph of WMG(C,V)

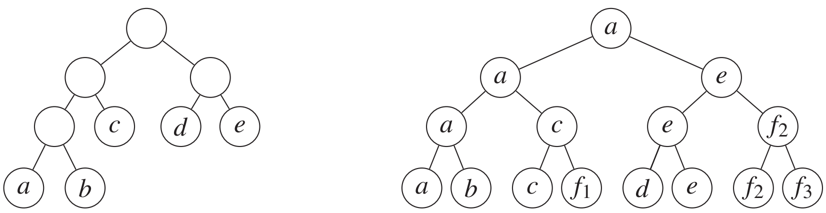
	a	**b**	**c**	**d**	**e**
a	–	2	0	6	2
b	0	–	0	0	0
c	0	2	–	2	2
d	0	0	0	–	0
e	0	2	0	2	–

(b) Strengths of the strongest paths

Figure 2.4: Subgraph of WMG(C,V) and the strenghts of the strongest paths

The first column of the table in Figure 2.4b gives us the values $P(x,a) = 0$ for $x \in C - \{a\}$. Since there are no negative values in the first row, we know that $P(a,x) \geq P(x,a)$ for all $x \in C - \{a\}$, thus a is a Schulze winner. Furthermore we see that $P(a,b) > P(b,a)$, $P(a,d) > P(d,a)$, and $P(a,e) > P(e,a)$ which implies that neither b, d, nor e can win the election. This leaves candidate c as a possible second Schulze winner and since $P(c,x) \geq P(x,c)$ for all $x \in C - \{c\}$ we have that both candidates a and c are Schulze winners in this election.

Turning to cup voting, let $((a,b),c,(d,e))$ be the schedule and $a > b > c > d > e$ be the order in which ties are broken if they occur. Figure 2.5a shows the unbalanced voting tree corresponding to this schedule.



(a) Voting tree without dummy candidates **(b)** Voting tree with dummy candidates

Figure 2.5: Voting tree for the cup election (C,V)

To transform this tree into a legal voting tree, we have to add 3 dummy candidates $F = \{f_1, f_2, f_3\}$ to the election. The new election is (C', V') with candidate set $C' = C \cup F$ and the new voter list V' below, where we fix the ordering of the dummy candidates in F at the bottom *of each ballot to be lexicographically, that is $f_1 > f_2 > f_3$ in each vote.*

$\mathbf{v_1}$:	$b > c > a > d > e > F$	$\mathbf{v_3}, \mathbf{v_4}$:	$e > b > a > d > c > F$
$\mathbf{v_2}$:	$a > d > c > b > e > F$	$\mathbf{v_5}, \mathbf{v_6}$:	$c > a > e > b > d > F$

Having the voting tree shown in Figure 2.5b, we can now determine the cup winner of the election starting with the first pairwise comparisons in the pairs of leaves (a,b), (c, f_1), (d,e), and (f_2, f_3). From Table 2.4 we know that $D_{V'}(a,b) = 0$ and $D_{V'}(d,e) = -2$, and since all dummy candidates are always ranked last in lexicographic order we have that $D_{V'}(c, f_1) = 6$ and $D_{V'}(f_2, f_3) = 6$. Thus, candidate a moves on to the next round because of the tie-breaking order, and c, e, and f_2 due to their win in the first round. On the second level of the tree we have the pairs (a,c) and (e, f_2) which are won by a and e, respectively, leading to the final head-to-head contest between a and e. Since $D_{V'}(a,e) = 2$ candidate a is assigned to the root and is the cup winner of the election.

2.4 Problems From Voting Theory and their Connections

Much research in the field of computational social choice focuses on the computational complexity of voting problems. In this section we want to shortly and informally describe those voting problems that will be studied in the Chapters 3, 4, and 5, and give an overview of their connections with respect to their computational complexity in Figure 2.6.

The different variants of *manipulation* (introduced by Bartholdi et al. [BTT89] and Conitzer et al. [CSL07]) are formally defined in Section 3.1.1 (pp. 28) and model the scenario of a given election in which there is a coalition of manipulative voters trying to make a certain candidate win the election by reporting insincere preferences. Formally, this problem is called CONSTRUCTIVE COALITIONAL UNWEIGHTED MANIPULATION, CCUM for short.[6] If the manipulators try to prevent a certain candidate from being a winner we are in the so-called destructive scenario (DCUM). When the manipulators try to make their favorite candidate the unique winner of the election or, in the destructive case, try to prevent a certain candidate from being a unique winner, we denote the problems by uCCUM and uDCUM (this is the so-called *unique-winner model*). If the given election is weighted, the corresponding problems are defined analogously and are denoted by CCWM, DCWM, uCCWM, and uDCWM. Trivial reductions between these different cases are presented in Observation 3.3 on page 28.

Electoral control ([BTT92, HHR07]), defined in Section 3.2.1, is a way of influencing an election by actions such as adding, deleting, or partitioning the set of candidates or the list of voters. Such control actions are exerted by an external actor with either the aim of making a candidate win the resulting election (constructive control) or preventing a candidate from winning (destructive control). Since there are no significant relations to the other voting problems, we omit the control problems in Figure 2.6.

[6] Note that in the formal definitions, the voting system $\mathscr{E}$ is always specified as a prefix of the problem name. For the sake of readability, we omit it in this section and especially in Figure 2.6.

By asking for a given election whether a given candidate can be made a winner by changing at most k of the given votes, the *bribery* variant CONSTRUCTIVE UNWEIGHTED BRIBERY (CUB) is defined (see [FHH09]). This and further variants are formally stated in Section 3.3.1 (pp. 78): CUB-$ denotes the problem in which each voter has a different price for changing her vote and the briber's action is limited by a given budget. CWB and CWB-$ denote the above defined problems in weighted elections, whereas the destructive cases of these four problems are DUB, DWB, DUB-$, and DWB-$. Just as for the manipulation problems, we denote these problems with uCUB, uCWB, uCUB-$, uCWB-$, uDUB, uDWB, uDUB-$, and uDWB-$ if they are stated in the unique-winner model. Trivial connections between these problems that directly follow from their definition are stated in Observation 3.33 on page 79. Faliszewski et al. [FHH09] furthermore point out that priced bribery can be reduced to the corresponding coalitional manipulation problem, which is stated in Proposition 3.34 on page 80.

The notion of SWAP BRIBERY (introduced by Elkind et al. [EFS09]) assumes that a briber cannot pay or persuade a voter and then change the entire ballot freely, but has to pay for each swap of adjacent candidates in a vote separately. Each voter may have a different price for each possible swap and the briber has to find a bribing action within a given budget. The unweighted versions are CUSB, DUSB, uCUSB, and uDUSB while the weighted cases are denoted by CWSB, DWSB, uCWSB, uDWSB. Besides the trivial reductions between these problems, it holds that for elections with exactly two candidates, thus $m = 2$, swap bribery and priced bribery are equivalent. Observation 3.36 on page 81 summarizes these connections.

Elkind et al. [EFS09] point out that CUSB can be seen as a generalization of yet another voting problem, namely the so-called POSSIBLE WINNER (PW) problem ([KL05]). In contrast to the previously defined voting problems, here we have given an election with a list of voters containing possibly partial votes, which means that not all ballots are complete linear orders and the question is, whether for a given candidate there exists an extension of these partial votes to complete linear orders such that the candidate is a winner of this new election with the extended votes. The problem NECESSARY WINNER (NW) asks for the same input whether the designated candidate is a winner for all possible extensions of the partial votes to linear orders. Again, we denote the problems by uPW and uNW if the unique-winner model is considered. Proposition 3.38 on page 82 due to [EFS09] and Proposition 3.40 on page 82 due to [SYE13] show the relations between constructive swap bribery and the possible winner problem, and the destructive cases and the complement of the necessary winner problem co(NW). Furthermore, CCUM can be seen as a special case of the PW problem, stated in Proposition 3.39 on page 82 which was shown by Xia and Conitzer [XC11b].

Another interesting and well-studied notion is the so-called *margin of victory* that we study in Chapter 4. For a given election, the margin of victory denotes the smallest number of votes that has to be changed in order to alter the election's winner set. The corresponding decision problem, called MoV, asks for a given election and a given bound ℓ whether the margin of victory is at most ℓ. We also define a more refined variant, called the *swap margin of victory* which is the smallest number of swaps needed to change the winner set of an election. swMoV denotes the corresponding decision problem. Both variants are highly related to destructive bribery in the unique winner model. The relations are formally specified in Proposition 4.2 that is due to [Xia12] and Corollary 4.3 (pp. 101) and Corollary 4.7 on page 104.

Figure 2.6 summarizes the relations given in the above mentioned propositions and observations. A dashed directed edge from problem A to problem B indicates that there exists a polynomial-time Turing reduction from A to B, that is, $A \leq_{T}^{p} B$. For example we have that for a fixed voting system UDUSB $\leq_{T}^{p}$ CUSB. Recalling Section 2.2, this observation may be very useful as in this case a P membership result for CUSB would directly transfer to UDUSB. A continuous directed edge from problem A to B specifies that $A \leq_{m}^{p} B$ holds, thus a hardness result for A implies hardness for B, as well. The four undirected edges between the problem pairs UCWSB and UCWB-\$, CWSB and CWB-\$, UDWSB and UDWB-\$, and finally DWSB and DWB-\$ state that if the number of candidates $\mathfrak{m}$ in the given election is fixed to 2, these problems are equivalent. The dotted line between the problems DCWM and DWB illustrates the connection between these two problems that was found in the analysis of these problems in Bucklin elections: Algorithm 3.3 in Section 3.3.2 solving Bucklin-DWB uses Algorithm 3.1 from Section 3.1.2 that solves Bucklin-DCWM. The edge labeled with "UC" from UDUSB to SWMOV indicates that the reduction holds for the case that the voters in the UDUSB instance all have unit costs for each possible swap.

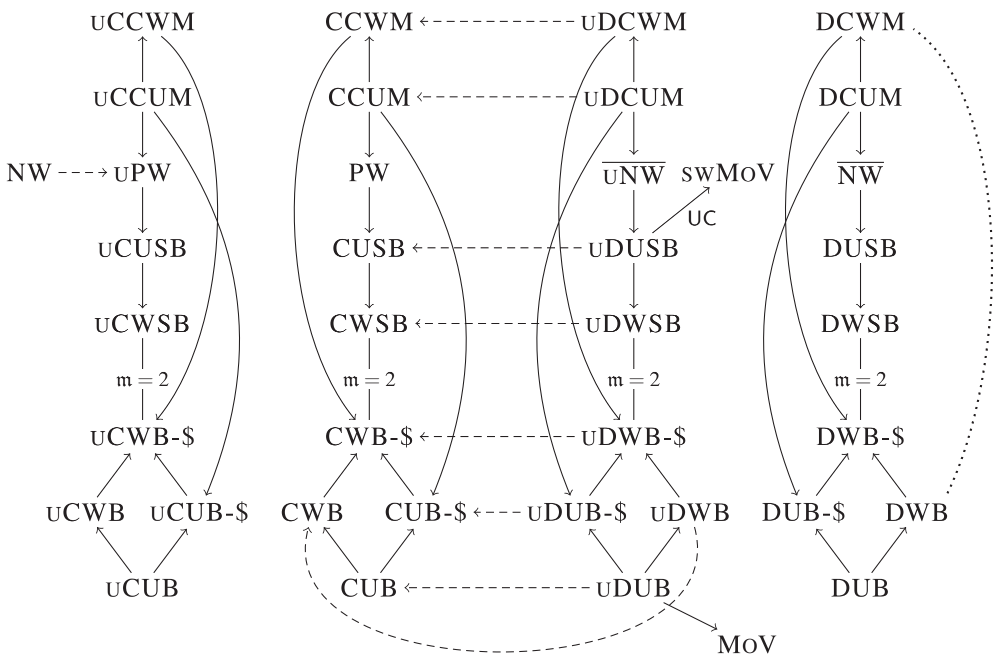


Figure 2.6: Overview of relations between voting problems

Note that we omit the possible winner problem with uncertain weights in this discussion and in Figure 2.6 as the goal of this section is to provide a compact overview of the most important connections.

3 Manipulative Attacks in Bucklin and Fallback Elections

This chapter is dedicated to a thorough analysis of the voting systems Bucklin and fallback voting with respect to the computational complexity of voting problems modelling manipulative attacks on elections. This line of research was triggered by the famous Gibbard-Satterthwaite theorem, independently shown by Gibbard [Gib73] and Satterthwaite [Sat75], stating that all reasonable voting systems can be manipulated in the sense that voters can alter an election's outcome to their benefit by reporting insincere preferences.

Theorem 3.1 (Gibbard [Gib73], Satterthwaite [Sat75]) *For elections with at least three candidates, there is no preference-based voting system $\mathscr{E}$ fulfilling the following four properties simultaneously.*

1. *There is no single voter $v \in V$ amongst other voters such that the outcome of an election solely depends on v's preference ($\mathscr{E}$ is* non-dictatorial*).*
2. *$\mathscr{E}$ always determines a unique winner ($\mathscr{E}$ is* resolute*).*
3. *For each candidate c there exists a set of voters making c an $\mathscr{E}$ winner ($\mathscr{E}$ fulfills* citizen's souvereignity*).*
4. *The voters can not alter an election's outcome to their benefit by reporting insincere preferences ($\mathscr{E}$ is* strategy-proof*).*

This result was extended to irresolute voting procedures by the work of Gärdenfors [Gär76] and Duggan and Schwartz [DS00].

The scenario of strategical behavior of voters is called *manipulation* and was firstly formally defined by Bartholdi et al. [BTT89] (and extended by Conitzer et al. [CSL07]) who introduced the path-breaking idea of studying the computational complexity of this voting problem. The idea behind this approach is, intuitively speaking, that even though all voting systems are manipulable in general, finding a successful manipulation may be hard to compute. Meaning that for a given election, the fact that a voter could not decide easily *how* to change her vote to achieve her goal, would be protection enough.

Besides manipulation, Bartholdi et al. [BTT92] introduced the notion of *electoral control* in which an external actor, called the *chair*, tries to alter an election's outcome by actions as adding or deleting candidates or voters or changing the entire structure of the election by partitioning the candidates or voters. Their model was extended by Hemaspaandra et al. [HHR07].

In the same line of research, Faliszewski et al. [FHH09, FHH+09b], introduced *bribery* modelling situations in which an external actor tries to change certain votes by bribing the voters. A refinement of bribery can be seen in the model of *campaign management* defined in the work of Elkind et al. [EFS09] and Elkind and Faliszewski [EF10a]. Here, the briber can change specific aspects of a vote such as swapping candidates but each single change has to be paid separately.

The worst-case complexity of these voting problems has been intensely studied for various voting systems. Their analysis has been extended to other aspects reaching from considering restricted domains (such as single-peakedness), approximability of their optimization variants to typical-case analyses and experimental studies. See the the surveys by Faliszewski et al. [FHH+09a], Faliszewski and Procaccia [FP10], Conitzer [Con10], and Rothe and Schend [RS13], and the book chapters by Brandt et al. [BCE13], Conitzer and Walsh [CW16], and Faliszewski and Rothe [FR16] for a comprehensive overview. Furthermore see also the related work we provide in the following sections for detailed information.

The analysis of Bucklin and fallback voting, however, was unsatisfyingly incomplete. This thesis, together with previous publications, closes this gap.

Basic Assumption – Full Knowledge of Preferences We assume that the manipulative participants in our voting problems (that is, the manipulators, the chair, or the briber, respectively) have complete knowledge of the voters' preferences and can construct the manipulative action based on this knowledge. Regarding real-life settings, there are scenarios in which this assumption is unrealistic, but for examples as small-scale elections among humans or large-scale elections among software agents it can be assumed that the voters or even an outside individuum can know how the voters will cast their votes. From a theoretical point of view, considering that we focus on a worst-case analysis of these voting problems, the assumption is reasonable in the sense that if a voting problem is hard to solve assuming that all preferences are known, restricting the knowledge of these preferences does not simplify the task for the manipulative participants. Hemaspaandra et al. [HHR07] provide a detailed discussion on this point.

Organization of this Chapter Each section of this chapter is dedicated to one type of manipulative attacks on elections. First the formal definitions are introduced and after giving a comprehensive overview of related work and known results, our results are presented. Starting with manipulation in Section 3.1, we turn to the study of electoral control in Section 3.2, where the analysis consists of a theoretical and an experimental part. In Section 3.3 we present our study on bribery and campaign management. The presented results were published in [FRR+14, FRR+15, EFR+15a, EFR+15b, RS12a]. At the end of the chapter we summarize our results and discuss future directions for further research.

3.1 Manipulation

The concept of manipulation as defined by Bartholdi et al. [BTT89] and Conitzer et al. [CSL07] models situations in which a group of manipulative voters (which may also consist of only one voter) reports insincere preferences in order to make a certain candidate win or prevent a certain candidate from winning.

One example where such behavior occurs in real-world elections is the election of the German *Bundestag*. Here, each voter has two votes: the so-called *first vote* and the *second vote*. The former is given directly to a candidate from a list of local candidates registered in the voter's electoral district (only one candidate can be selected). These candidates may be members of a political party, but that is not mandatory. From each district the candidate with a simple majority of votes is elected directly to the *Bundestag*. The districts are usually won by candidates from the two big parties the *Sozialdemokratische Partei Deutschlands* (SPD) and the *Christliche Demokratische Union Deutschlands/Christlich-Soziale Union in Bayern* (CDU/CSU); in relation to the number of seats, only a small part is won by candidates from the smaller parties. The second vote is given directly to one of the registered parties and these votes determine the percentage of seats each party obtains in the *Bundestag*. In the year 2013 the party *Freie Demokratische Partei* (FDP) campaigned with the slogan *"Second Vote FDP,"* calling the voters of their potential coalition partner CDU on voting strategically: Instead of giving their second vote to the CDU, the voters should vote for the FDP to ensure that the FDP would have enough seats for being a feasible coalition partner, and thus preventing the grand coalition consisting of the CDU and SPD. Interestingly, 2013 was the first year since the establishment of the Federal Republic in 1949 that the FDP missed the quota of 5% of second votes that is needed to be part of the *Bundestag*.

As we will see, this example does not fit the formal definition of the manipulation problem as we will study it in this section, but it nevertheless captures the idea of strategic behavior. Complementing the above example, we show that Bucklin and fallback voting are manipulable in the formal sense.

Example 3.2 *Recall the election (C,V) from Example 2.23 in Section 2.3.2. Candidate a is the unique level 3 Bucklin winner in this election. Consider the voters v_5 and v_6 who both have the preference $c > a > e > b > d$. Their favorite candidate is c and in order to make her a winner of the election, these voters could submit an untrue preference of the form $c > d > e > b > a$ leading to the scores below.*

	a	b	c	d	e
$score^1$	1	1	2	0	2
$score^2$	1	3	3	3	2
$score^3$	**4**	3	**4**	3	**4**

Table 3.1: Scores in the manipulated Bucklin election

As the majority threshold is 4, the strategic behavior of these two voters makes their preferred candidate c, amongst others, a level 3 Bucklin winner of the manipulated election.

3.1.1 Basic Definitions and Related Work

The original manipulation problem introduced by Bartholdi et al. [BTT89] models the scenario of a single manipulator trying to influence a given unweighted election. Conitzer et al. [CSL07] extended this definition by allowing a coalition of manipulators. Furthermore, they also defined a weighted variant for elections in which different voters may have different weights:

$\mathscr{E}$-CONSTRUCTIVE COALITIONAL WEIGHTED MANIPULATION ($\mathscr{E}$-CCWM)	
Given:	A set C of candidates, a list V of nonmanipulative votes over C, a list W_V of weights of the voters in V, a nonnegative integer k, a list W_S of the weights of k manipulators in S with $V \cap S = \emptyset$, and a designated candidate $c \in C$.
Question:	Can the votes in S be set such that c is an $\mathscr{E}$ winner of $(C, V \cup S)$?

The unweighted variant, denoted by $\mathscr{E}$-CCUM, can be obtained by setting all weights in both lists W_V and W_S to the unit weight of 1. By furthermore setting the number of manipulators to 1, the original definition given in [BTT89] can be obtained.

The destructive variants of the weighted and unweighted variants are denoted by $\mathscr{E}$-DCWM and $\mathscr{E}$-DCUM, respectively. They are defined analogously to their constructive counterparts except that the question has to be changed to whether the votes in S can be set such that the designated candidate is *not* an $\mathscr{E}$ winner of the resulting election $(C, V \cup S)$.

We stated the above problem in the so-called *co-winner model* by asking in the constructive case whether the designated candidate can be made *an* $\mathscr{E}$ winner. To apply the *unique-winner model*, the question has to be changed to whether the designated candidate can be made *the unique* $\mathscr{E}$ winner (can be prevented from being a unique $\mathscr{E}$ winner in the destructive cases). To give a comprehensive overview of the relationships between the different manipulation problems in Observation 3.3 below, we introduce the following notation for the four manipulation problems defined in the unique-winner model: $\mathscr{E}$-UCCWM, $\mathscr{E}$-UCCUM, $\mathscr{E}$-UDCWM, and $\mathscr{E}$-UDCUM. In the remaining course of this thesis, however, we will refrain from distinguishing between the two notations and ensure that it is always clear from the context which winner model is used.

We omit the proof of the following observation as the reductions directly follow from the definitions stated above. Note that the last part of this observation is often informally described as "the destructive case is never harder than the constructive case unless P equals NP," but we want to stress that it is important to carefully distinguish which winner model is used.

Observation 3.3 *Let $\mathscr{E}$ be a voting system, then the following holds.*

1. $\mathscr{E}$-CCUM $\leq_m^p$ $\mathscr{E}$-CCWM *and* $\mathscr{E}$-UCCUM $\leq_m^p$ $\mathscr{E}$-UCCWM.
2. $\mathscr{E}$-DCUM $\leq_m^p$ $\mathscr{E}$-DCWM *and* $\mathscr{E}$-UDCUM $\leq_m^p$ $\mathscr{E}$-UDCWM.
3. $\mathscr{E}$-UDCUM $\leq_T^p$ $\mathscr{E}$-CCUM *and* $\mathscr{E}$-UDCWM $\leq_T^p$ $\mathscr{E}$-CCWM.

Related Work and State of the Art The basic manipulation problems that we just defined have been intensely studied for a variety of voting systems. The first of such papers following the seminal paper by Bartholdi et al. [BTT89] is the work of Bartholdi and Orlin [BO91], who investigated the complexity of manipulation in single transferable vote. Table 3.2 surveys results from the literature for those voting systems that are studied besides Bucklin and fallback voting within this thesis.

Voting Rule	$\mathscr{E}$-CCUM	$\mathscr{E}$-DCUM	$\mathscr{E}$-CCWM	$\mathscr{E}$-DCWM
family of scoring rules $\vec{\alpha}^*$ [1]	NP-complete [2]	P [3]	NP-complete [2]	P [3]
family of scoring rules $\vec{\alpha}'$ [4]	NP-complete [5]	P [3]	NP-complete [5]	P [3]
plurality	P [3,5]	P [3]	P [3,5]	P [3]
veto	P [3,5]	P [3]	P [3,5]	P [3]
Borda	NP-complete [7]	P [3]	NP-complete [5]	P [3]
Copeland$^\alpha$, $\alpha \in [0,1] - \{0.5\}$	NP-complete [6]	P [3]	NP-complete [3]	P [3]
Copeland$^{0.5}$	?	P [3]	NP-complete [3]	P [3]
cup	P [3]	P [3]	P [3]	P [3]
Schulze	P [8]	P [8]	?	?

[1] $\vec{\alpha}^*$ from [XCP10, p. 8]
[2] due to [XCP10]
[3] due to [CSL07]
[4] $\vec{\alpha}'$ from [HH07, p. 12]
[5] due to [HH07]
[6] due to [FHS08, FHS10]
[7] shown independently in [BNW11, DKN+11]
[8] due to [GKN+13, PX12]

Table 3.2: Selection of known results regarding the complexity of manipulation

As NP-hardness establishes the worst-case complexity of the manipulation problem, the next natural step for the analysis of a voting system's manipulability is the question whether it is also hard to manipulate in practice. The various approaches reach from *experimental analyses* (see the work of Walsh [Wal10, Wal09], Davies et al. [DKN+11, DKN+10], and Narodytska et al. [NWX11]) and *approximation algorithms* (studied, for example, by Zuckerman et al. [ZPR09] and Xia et al. [XCP10]), to the study of parameterized complexity by Betzler et al. [BHN09, BNW11], Yang [Yan14], and Dey et al. [DMN15] (see also the survey of Betzler et al. [BBC+12]).

Rothe and Schend [RS13] provide an overview of further approaches challenging NP-hardness shields against manipulation, including the complexity of manipulation on restricted domains such as *single-peaked electorates*. See, for example, the work of Faliszewski et al. [FHH+11], Brandt et al. [BBH+10], and Faliszewski et al. [FHH14], and the book chapter by Hemaspaandra et al. [HHR15].

Hemaspaandra et al. [HHR14] study *online manipulation* in *online elections* in which the voters do not cast their votes simultaneously, but sequentially. The setting introduced in this work is closely related to noncooperative game theory.

Tie-breaking is a very important aspect when studying the complexity of manipulation. How breaking ties at random can influence the complexity of manipulation is studied by Obraztsova et al. [OEH11] and Obraztsova and Elkind [OE11], and more recently in the work of Aziz et al. [AGM+13].

Another fruitful line of work focuses on elections with incomplete information, see Chapter 5 for more background regarding such settings. In the model introduced by Conitzer et al. [CWX11] the manipulators only have partial information about the preferences the nonmanipulative voters cast. Elkind and Erdélyi [EE12] study manipulation when the uncertainty lies in the voting system itself. They consider the setting in which a list of voting systems is specified before the votes have to be cast and the winner will be determined based on one of the voting systems from this given list. Manipulation is also of great concern in other areas of computational social choice, such as judgment argumentation, fair division, and cooperative game theory, see, for instance, the recent work of Baumeister et al. [BEE+15, BEE+13, BRS15], Nguyen et al. [NBR15], and Rey and Rothe [RR14].

The study of manipulation and its variants in different setting is a quickly growing line of research and surveying it to a full extent would go beyond the scope of this thesis. For further related work regarding manipulation in voting we refer to the surveys by Faliszewski and Procaccia [FP10], Faliszewski et al. [FHH10, FHH+09a], Mossel and Rácz [MR12b], and Conitzer [Con10], and the book chapters by Baumeister and Rothe [BR15] and Conitzer and Walsh [CW16].

3.1.2 Complexity Results

The results presented in this section, see Table 3.3 for an overview, were published in [FRR+14, FRR+15]. All results hold in both winner models (recall their definition from Section 3.1.1).

	Bucklin Voting		**Fallback Voting**	
Problem	complexity	reference	complexity	reference
$\mathscr{E}$-CCWM	NP-complete	Thm. 3.5	P	Thm. 3.4
$\mathscr{E}$-DCWM	P	Thm. 3.7	P	Thm. 3.4
$\mathscr{E}$-CCUM	P	Thm. 3.9	P	Cor. 3.8
$\mathscr{E}$-DCUM	P	Cor. 3.10	P	Cor. 3.8

Table 3.3: Overview of results for manipulation in Bucklin and fallback voting

Results for Weighted Manipulation We start with analyzing the complexity of weighted manipulation in Bucklin and fallback voting.

Theorem 3.4 *Fallback-*CCWM *and fallback-*DCWM *are in* P*, each in both winner models.*

Proof Sketch. Manipulation in fallback voting can be tackled in a straightforward manner and we will informally describe the strategy manipulators can follow.

Given the designated candidate p in the constructive cases, the manipulators approve only of p and disapprove of the remaining candidates. This ensures that no other candidate than p gains any points on any level from the manipulators' votes. As this is clearly an optimal way of setting the votes with respect to the goal of making p a (unique) winner, there is no other way to set the manipulators' votes if this manipulation attempt does not succeed. Obviously, this holds for both winner models.

Observation 3.3 allows us to follow that also destructive weighted manipulation for fallback voting is in P for both winner models. ❑

In contrast to this easy result in fallback elections, constructive coalitional manipulation in weighted Bucklin voting confronts the manipulators with a far more difficult task.

Theorem 3.5 *For elections with at least three candidates, Bucklin-*CCWM *is* NP*-complete in both winner models.*

Proof Sketch. It is easy to see that Bucklin-CCWM is in NP in both winner models and for all numbers of candidates. Since the winner of any Bucklin election can be determined in deterministic polynomial time, it suffices to guess the manipulators' votes and check whether this manipulation is successful.

We present how NP-hardness can be established by a reduction from PARTITION. We will not provide the full proof of the above claim, but rather show how a Bucklin election can be constructed to show the co-winner case for an odd number of candidates ($\mathfrak{m} \geq 3$). Let an instance of PARTITION be given by $A = \{1, \ldots, k\}$ and $(a_1, \ldots, a_k)$ with $\sum_{i=1}^{k} a_i = 2K$. Let $C = \{c_1, c_2, \ldots, c_{\mathfrak{m}-1}\} \cup \{p\}$, be the set of candidates, where $\mathfrak{m} \geq 3$ is an odd number (the desired number of candidates in the constructed election). To simplify the description of the votes, we will use the following interval-like notation:

$$C[i,j] = \begin{cases} c_i > c_{i+1} > \cdots > c_j & \text{if } i < j, \\ c_i > c_{i+1} > \cdots > c_{\mathfrak{m}-1} > c_1 > \cdots > c_j & \text{otherwise.} \end{cases}$$

For example, by writing $C[1,4] > p > \cdots$ we mean a preference order described by $c_1 > c_2 > c_3 > c_4 > p > \cdots$ (i.e., we rank candidates c_1, c_2, c_3, and c_4 first, then p, and then all the remaining candidates in some arbitrary-but-easy-to-compute order). Similarly, $p > C[\mathfrak{m}-2, 2] > \cdots$ would mean a preference order of the form $p > c_{\mathfrak{m}-2} > c_{\mathfrak{m}-1} > c_1 > c_2 > \cdots$.

We construct a Bucklin election (C,V), where the candidate set C is as already specified, and where the voter list is as given in Table 3.4. Note that the overall weight of the voters in V is $2(\mathfrak{m}-1)K$.

Let there be k manipulators in S with weights $a_1, a_2, \ldots, a_k$. While in the original election we have a majority threshold of $(\mathfrak{m}-1)K+1$ points, the majority threshold is reached with $\mathfrak{m}K+1$ points in the election with the manipulators.

Since p receives no points at all in (C,V) before level $\frac{m-1}{2}+1$ and has fewer points than any other candidate on this level (see Table 3.5a), p is not a Bucklin winner of the original election (C,V).

It can be shown that $(A, (a_1, a_2, \ldots, a_k)) \in$ PARTITION if and only if p can be made a Bucklin winner in $(C, V \cup S)$. ❑

The following lemma will be useful to prove the upcoming result in Theorem 3.7 and we state it without proof.

Group	Preference	Weight
(1)	$C\left[1, \frac{\mathfrak{m}-1}{2}\right] > p > \cdots$	$\frac{\mathfrak{m}-1}{2}K$
(2)	$C\left[\frac{\mathfrak{m}-1}{2}+1, \mathfrak{m}-1\right] > p > \cdots$	$\frac{\mathfrak{m}-1}{2}K$
(3)	$C[1, \mathfrak{m}-1] > p$	K
	$C[2,1] > p$	K
	$\vdots$	$\vdots$
	$C[\mathfrak{m}-1, \mathfrak{m}-2] > p$	K

Table 3.4: Voter list V in the proof of Theorem 3.5 for an odd number $\mathfrak{m} \geq 3$ of candidates

	$c \in C - \{p\}$	p
$score^{\frac{\mathfrak{m}-1}{2}}$	$(\mathfrak{m}-1)K$	0
$score^{\frac{\mathfrak{m}-1}{2}+1}$	$\mathfrak{m}K$	$(\mathfrak{m}-1)K$

(a) Original election (C,V)

	$c \in C - \{p\}$	p
$score^{\frac{\mathfrak{m}-1}{2}}$	$\leq \mathfrak{m}K$	$2K$
$score^{\frac{\mathfrak{m}-1}{2}+1}$	$\mathfrak{m}K + K$	$\mathfrak{m}K + K$

(b) Manipulated election $(C, V \cup S)$

Table 3.5: Level i-scores, $i \in \left\{\frac{\mathfrak{m}-1}{2}, \frac{\mathfrak{m}-1}{2}+1\right\}$, of the candidates in C for odd $\mathfrak{m} \geq 3$

Lemma 3.6 *Let (C,V) be a weighted Bucklin election with total weight W and let $c, p \in C$. Then the following holds.*

1. *Assume that c is not a (unique) Bucklin winner in (C,V) and that the votes in V are changed such that the position of c is made worse in some votes, all else being equal.*[1] *Then c is still not a (unique) Bucklin winner.*
2. *Assume that c is a (unique) Bucklin winner of the election and that the votes in V are changed such that the position of c is improved in some votes, all else being equal. Then c remains a (unique) Bucklin winner.*
3. *Assume that c is a (unique) Bucklin winner of the election and that p is not a (unique) Bucklin winner. If in some votes the positions of candidates are swapped without changing the positions of c and p, all else being equal, then p is still not a (unique) Bucklin winner.*

The destructive case of coalitional manipulation in weighted Bucklin elections can be solved in deterministic polynomial time by a straightforward approach presented in Algorithm 3.1: Letting p be the current winner in the unmanipulated election, the algorithm tests for every

[1] By "all else being equal" we tacitly mean that all other candidates remain in the same position in each vote, except those candidates that improve their position by one due to shifting c toward the bottom. An analogous comment applies to the cases where c's position is improved in the second statement of this lemma and where other candidates are swapped in the third statement of this lemma.

candidate c other than p whether this candidate can beat p by setting all manipulators votes to $c > \cdots > p$, where the remaining candidates in C are arbitrarily positioned in the middle part of the preference. The algorithm accepts if one such candidate has been found.

Algorithm 3.1: Algorithm for Bucklin-DCWM

input : C set of candidates
V list of voters
W_V weights of the voters
W_S weights of the manipulators
p designated candidate

output: "YES" if $(C,V,W_V,W_S,p) \in$ Bucklin-DCWM
"NO" if $(C,V,W_V,W_S,p) \notin$ Bucklin-DCWM

1 **if** $\sum_{w \in W_S} w > \sum_{w \in W_V} w$ **then**
2   return "YES";
3 **foreach** $c \in C - \{p\}$ **do**
4   put p in the last position in the manipulators' votes;
5   put c in the first position in the manipulators' votes;
6   fill the remaining positions in the manipulators' votes arbitrarily;
7   let S be the list of the manipulators' votes
8   **if** $($*p is not a Bucklin winner of* $(C, V \cup S)$ *with weights* $W_V \cup W_S)$ **then**
9   return "YES";
10 return "NO";

We show the following theorem in detail as Algorithm 3.1 will be used to solve variants of destructive bribery in Bucklin elections in Section 3.3.2 (see Algorithm 3.3 on page 90).

Theorem 3.7 *In both winner models, Bucklin-*DCWM *can be decided in time* $\mathcal{O}(\mathfrak{m}^2(\mathfrak{n} + \|W_S\|))$*, where* W_S *is the list of the manipulators' weights by Algorithm 3.1.*

Proof. We begin with analyzing the runtime of Algorithm 3.1. The input of the algorithm is the set of $\mathfrak{m}$ candidates in C, the list of $\mathfrak{n}$ voters V each represented by a preference over $\mathfrak{m}$ candidates, the list of $\mathfrak{n}$ weights in W_V, the list of $\|W_S\|$ weights in W_S, and the designated candidate p. Thus the input size is in $\mathcal{O}(\mathfrak{m} + \mathfrak{n}\mathfrak{m} + \mathfrak{n} + \|W_S\| + 1) = \mathcal{O}(\mathfrak{n}\mathfrak{m} + \|W_S\|)$. Obviously, the algorithm always terminates and the most costly part of the algorithm is the for-loop. To construct the manipulators' votes, $\mathcal{O}(\|W_S\|\mathfrak{m})$ steps are needed. The winner-determination procedure for Bucklin can be implemented with a runtime of $\mathcal{O}(\mathfrak{n}\mathfrak{m})$, so the if-statement in line 8 can be computed in time $\mathcal{O}(\mathfrak{m}(\mathfrak{n} + \|W_S\|))$. Thus, the whole for-loop runs in time $\mathcal{O}(\mathfrak{m}^2(\mathfrak{n} + \|W_S\|))$.

To prove the correctness of the algorithm, we show that it gives the output "YES" if and only if $(C,V,W_V,W_S,p) \in$ Bucklin-DCWM. (Note that by changing the condition of the if-statement in line 8 to "(p is not a unique Bucklin winner of $(C, V \cup S)$ with weights $W_V \cup W_S$)," the algorithm solves Bucklin-DCWM in the unique-winner model which can be shown with an analogous argumentation as below.)

Only if: If the algorithm outputs "YES" in line 2, then we have $\sum_{w \in W_S} w > \sum_{w \in W_V} w$, i.e., the sum of the manipulators' weights is greater than the sum of the weights of the nonmanipulative voters. In this case, any of the candidates $c \neq p$ can be made a unique level 1 Bucklin winner in $(C, V \cup S)$ by putting c in the first position of all the manipulators' votes and filling the remaining positions arbitrarily. Hence, $(C, V, W_V, W_S, p) \in$ Bucklin-DCWM. If the algorithm outputs "YES" in line 9, the manipulators' votes have been constructed such that p is not a Bucklin winner in $(C, V \cup S)$. Thus, we have that (C, V, W_V, W_S, p) is a yes-instance of Bucklin-DCWM.

If: Assume that $(C, V, W_V, W_S, p) \in$ Bucklin-DCWM. If $\sum_{w \in W_S} w > \sum_{w \in W_V} w$, then the algorithm correctly outputs "YES." Otherwise, the following holds: Since the given instance is a yes-instance of Bucklin-DCWM, the votes of the manipulators in S can be set such that p is not a Bucklin winner of the election $(C, V \cup S)$. We know from Lemma 3.6 that successively swapping p with her neighbor until p is in the last position in all votes in S does not change the fact that p is not a Bucklin winner in $(C, V \cup S')$ (where S' are the new manipulative votes with p in the last position). Assume that $c \in C - \{p\}$ is a Bucklin winner in $(C, V \cup S)$. Then swap her position successively with her neighbor in the votes in S' until c is in the first position of all manipulative votes. Let S'' denote the accordingly changed list of manipulative votes. Again, from Lemma 3.6 we know that c still wins in $(C, V \cup S'')$. Let S''' be the list of manipulative votes that the algorithm constructs. We can transform S'' into S''' by swapping the corresponding candidates $c', c'' \in C - \{c, p\}$ accordingly. Since the positions of c and p remain unchanged, we have with Lemma 3.6 that p is still not a Bucklin winner in $(C, V \cup S''')$. Thus, the algorithm outputs "YES" in line 9. ❑

Results for Unweighted Manipulation We start with the unweighted cases in fallback voting which are handled in Corollary 3.8 and can be followed with Observation 3.3 from Theorem 3.4.

Corollary 3.8 *Fallback-*CCUM *and fallback-*DCUM *are in* P*, each in both winner models.*

This leads us to the last unresolved case: constructive manipulation in unweighted Bucklin elections. In contrast to the weighted variant which we showed to be NP-complete in Theorem 3.5, unweighted Bucklin elections can be efficiently be manipulated. Compared to fallback elections, however, the argumentation is much more involved as the manipulators do not have the possibility to generally preclude other candidates than the designated one from gaining points on relevant levels.

The following algorithm is an adaption of the corresponding algorithm for simplified Bucklin due to Xia et al. [XZP+09] and we will present a high-level description of the pseudocode presented in Algorithm 3.2 solving the problem in the co-winner case.

The given input consists of a Bucklin election (C, V) with the set of candidates C and the list of voters V with specified preferences. Candidate $p \in C$ is the candidate we want to

Algorithm 3.2: Algorithm for Bucklin-CCUM

input : C set of candidates
V list of voters
k number of manipulators
p designated candidate

output: "YES" if $(C,V,k,p) \in$ Bucklin-CCUM
"NO" if $(C,V,k,p) \notin$ Bucklin-CCUM

if $k > \|V\|$ **then**
  return "YES";
let $max\_scr^{\ell_{\min}}$, $max\_scr^{\ell_{\min}-1}$, $num^{\ell_{\min}}$, $num^{\ell_{\min}-1}$ be arrays of length $\mathfrak{m}$;
$maj = \lfloor \frac{\|V\|+k}{2} \rfloor + 1$;
$\ell_{\min} = \min\{i \mid score^{i}_{(C,V)}(p) + k \geq maj\}$;
foreach $c \in C - \{p\}$ **do**
  if $\min\{i \mid score^{i}_{(C,V)}(c) \geq maj\} < \ell_{\min}$ OR $score^{\ell_{\min}}_{(C,V)}(c) > score^{\ell_{\min}}_{(C,V)}(p) + k$ **then**
    return "NO";
  $max\_scr^{\ell_{\min}}[c] = score^{\ell_{\min}}_{(C,V)}(p) + k - score^{\ell_{\min}}_{(C,V)}(c)$;
  $max\_scr^{\ell_{\min}-1}[c] = maj - score^{\ell_{\min}-1}_{(C,V)}(c) - 1$;
  $num^{\ell_{\min}}[c] = \min\{max\_scr^{\ell_{\min}}[c], k\}$;
  $num^{\ell_{\min}-1}[c] = \min\{max\_scr^{\ell_{\min}-1}[c], max\_scr^{\ell_{\min}}[c], k\}$;
if $\sum_{c \in C-\{p\}} \min\{max\_scr^{\ell_{\min}-1}[c], max\_scr^{\ell_{\min}}[c], k\} < (\ell_{\min} - 2)k$ OR $\sum_{c \in C-\{p\}} \min\{max\_scr^{\ell_{\min}}[c], k\} < (\ell_{\min} - 1)k$ **then**
  return "NO";
return "YES";

make a winner of the resulting election by determining the yet unspecified preferences of k manipulators.

maj: Denotes the strict majority threshold in the final election counting both the number of regular voters and the k manipulators.

$\ell_{\min}$*:* Denotes the smallest level on which candidate p reaches the majority threshold maj in the manipulated election, assuming that all the manipulators position p on top. This means that if p is to win, p has to win at level $\ell_{\min}$, having $score^{\ell_{\min}}_{(C,V)}(p) + k$ points.

$max\_scr^{\ell_{\min}}$*:* This array indicates how many further points each candidate c can gain without having strictly more points than p on level $\ell_{\min}$.

$max\_scr^{\ell_{\min}-1}$*:* This array indicates how many further points each candidate c may gain without reaching or exceeding the majority threshold maj on one of the levels 1 through $\ell_{\min} - 1$.

$num^{\ell_{\min}-1}$*:* This array indicates the number of manipulators that may have candidate c in the first $\ell_{\min} - 1$ positions of their votes without preventing p from winning, that is, $num^{\ell_{\min}-1}[c] = \min\{max\_scr^{\ell_{\min}}[c], max\_scr^{\ell_{\min}-1}[c], k\}$.

$num^{\ell_{\min}}$: This array indicates the number of manipulators that can place c among their top $\ell_{\min}$ positions without preventing p from winning, that is, $num^{\ell_{\min}}[c] = \min\{max\_scr^{\ell_{\min}}[c], k\}$.

We have that for all $c \in C - \{p\}$, $max\_scr^{\ell_{\min}}$ and $max\_scr^{\ell_{\min}-1}$ contain positive numbers and that $num^{\ell_{\min}}[c] \geq num^{\ell_{\min}-1}[c]$.

The algorithm proceeds as follows. In a first step (in line 1) it is tested whether there are more manipulators than nonmanipulative voters which would lead to a trivial yes-instance. If this test fails, the algorithm proceeds and tests whether the given instance is a trivial no-instance in the sense that there is at least one other candidate that cannot be dethroned by p with the given number k of manipulators (in line 7). If no such candidate is found, the algorithm computes the necessary arrays described above and proceeds to line 13. In this final step, assuming that p is in the first position in every manipulator's preference, the algorithm checks whether the remaining positions in the preferences can be filled while still ensuring that no candidate $c \in C - \{p\}$ beats p.

The algorithm can easily be adapted to solve the unique-winner case by slightly modifying the definition of the array $max\_scr^{\ell_{\min}}$ (subtracting 1) and allowing equality in the second inequality in line 7.

We summarize this in the following theorem that we state without proof.

Theorem 3.9 *Bucklin-*CCUM$\in$ P *in both winner models.*

Similar to the case of fallback elections, Theorem 3.9 and Observation 3.3 give us the corollary below.

Corollary 3.10 *Bucklin-*DCUM *is in* P *in both winner models.*

3.2 Electoral Control

In the context of electoral control we are concerned with the situation that the chair of a given election tries to tamper with the election's result by altering the structure of the election itself, for example by adding or deleting candidates or voters. Bartholdi et al. [BTT92] introduced the constructive variants of control in which the chair's aim is to make a certain candidate the winner of the controlled election. Hemaspaandra et al. [HHR09] introduced the destructive variants in which the goal is to prevent the current winner from winning by conducting structural changes.

There is no analogon to Theorem 3.1 in the context of electoral control: there are voting systems that are immune to certain types of electoral control meaning that there do not exist any elections in which the given type of control can be exerted successfully. Two famous voting systems are approval voting and the Condorcet system that are immune to certain types of candidate control [BTT92, HHR09]. But no natural voting system with a deterministic polynomial-time winner determination has been found that is immune to all types of electoral control.

Thus, a rich line of research is concerned with finding a voting system with the two desired properties of having a deterministic polynomial winner determination procedure, that is, the winners of a given election can easily be determined, and is at the same time resistant to all control types, where resistance means that the corresponding control problems are NP-hard. Hemaspaandra et al. [HHR09] show that, indeed, such a voting system exists and prove that an impossibility theorem like "*For no election system whose winner complexity is in* P *are all types of control* NP-*hard"* does not hold in general. The voting system that is constructed and serves as a counterexample, however, is very artificial leaving the question open whether a natural voting system can be found that is resistant to all types of electoral control.

This research question is not resolved yet, but the work of Erdélyi et al. [ER10, EPR11] and Menton [Men13] came close to an answer: the voting systems fallback voting and *normalized range voting* are resistant to all types of electoral control except for two vulnerabilities (meaning the corresponding decision problem is in P). Bucklin voting behaves almost as well in terms of resistance, the only difference being that the complexity of one case of control by partition of voters is yet unresolved.

After giving the formal definitions of the different types of control, we provide a two-part analysis of control complexity in Bucklin and fallback voting: In a first step we extend the worst-case analysis by Erdélyi et al. [ER10, EPR11, EF10b] by analyzing the parameterized complexity of the various control problems with respect to natural parameters for Bucklin voting and extend previously known results to the co-winner model. In a second step we present the first experimental analysis of control complexity which is based on the approach introduced by Walsh [Wal10, Wal09] for the experimental analysis of manipulation problems. Note that this experimental analysis is also conducted for plurality voting.

3.2.1 Basic Definitions and Related Work

We now give the formal definitions of all considered types of control which were given by Hemaspaandra et al. [HHR07]. These definitions are based on those of Bartholdi et al. [BTT92], but provide refined definitions of the partition cases and furthermore introduce the destructive cases of control. We complement the definitions by real-life scenarios describing possible adaptions of the introduced models, define then the notions of resistance and vulnerability in terms of classical and parameterized complexity, and conclude this section by giving an overview of known results and interesting related work.

Adding and Deleting of Voters and Candidates Let us start with one of the most intuitive cases: Obviously, the outcome of an election can be influenced by deleting some of the votes that were originally cast. In real-life elections such deletions could be realized by illegally disposing of ballots or deleting votes when the election is held electronically. More subtle ways of voter deletion include preventing voters from participating in the election altogether by, for example, raising the voting age or leading voters to cast invalid votes when the design of the ballots is too complicated or misleading.

$\mathscr{E}$–Constructive Control by Deleting Voters ($\mathscr{E}$-CCDV)	
Given:	An $\mathscr{E}$ election (C,V), a designated candidate $c \in C$, and a nonnegative integer k.
Question:	Is there a sublist $V' \subseteq V$ with $\|V'\| \leq k$ such that c is an $\mathscr{E}$ winner of the election $(C, V - V')$?

The destructive variants of all control types can be obtained from the constructive cases by changing the question to whether there is a control action such that the designated candidate c is *not* an $\mathscr{E}$ winner of the resulting election. For this first type of control the question for the destructive variant thus has to be changed to whether there is a sublist $V' \subseteq V$ with $\|V'\| \leq k$ such that the designated candidate c is *not* an $\mathscr{E}$ winner of the election $(C, V - V')$, and we abbreviate the problem $\mathscr{E}$–Destructive Control by Deleting Voters by $\mathscr{E}$-DCDV.

Get-out-the-vote activities of political campaigns or lowering the voting age can increase the number of voters participating in an election which both can be modeled by control by adding voters. We define it in the constructive case and denote the destructive variant by $\mathscr{E}$-DCAV.

$\mathscr{E}$–Constructive Control by Adding Voters ($\mathscr{E}$-CCAV)	
Given:	An $\mathscr{E}$ election $(C, V \cup V')$ with $V \cap V' = \emptyset$, where V is a list of registered voters and V' is a list of unregistered voters from which additional votes can be added, a designated candidate $c \in C$, and a nonnegative integer k.
Question:	Is there a sublist $V'' \subseteq V'$ with $\|V'\| \leq k$ such that c is an $\mathscr{E}$ winner of the election $(C, V \cup V'')$?

Candidates can be forced to withdraw their candidacy by cutting the financial support of their campaign or by changing criteria in the requirements that have to be met for a valid candidacy, such as age limits. Amongst others, these scenarios are examples for control by deleting candidates, which we formally define as follows.

$\mathscr{E}$–Constructive Control by Deleting Candidates ($\mathscr{E}$-CCDC)	
Given:	An $\mathscr{E}$ election (C,V), a distinguished candidate $c \in C$, and a nonnegative integer k.
Question:	Is there a subset $C' \subseteq C$ with $\|C'\| \leq k$ such that c is an $\mathscr{E}$ winner of the election $(C - C', V)$?

Recall from Section 2.3 that for an election (C,V) and a subset of candidates $C' \subseteq C$ we define (C',V) to be the election where the list of voters V is restricted to the candidates in C', that is, in the voters' preferences the candidates in $C - C'$ are omitted.

The positive analogon to the examples for control by deleting candidates can be taken as examples for control by adding candidates: Dropping formerly determined requirements can enlarge the set of candidates, as well as actions that encourage candidates to participate in the election such as political endorsements or other forms of support for their campaign.

$\mathscr{E}$–CONSTR. CONTROL BY ADDING A LIMITED NUMBER OF CANDIDATES ($\mathscr{E}$-CCAC)	
Given:	An $\mathscr{E}$ election $(C \cup D, V)$, with $C \cap D = \emptyset$, where C is a set of qualified candidates and D is a set of spoiler candidates from which candidates can be added, a designated candidate c, and a nonnegative integer k.
Question:	Is there a subset $D' \subseteq D$ with $\|D'\| \leq k$ such that c is an $\mathscr{E}$ winner of the election $(C \cup D', V)$?

Originally, Bartholdi et al. [BTT92] defined control by adding candidates in a slightly different manner: In their definition there is no limit k on the number of candidates that may be added to the original election. To distinguish these two variants, the original definition is called CONSTRUCTIVE CONTROL BY ADDING AN UNLIMITED NUMBER OF CANDIDATES ($\mathscr{E}$-CCAUC). Faliszewski et al. [FHH+09b] show that depending on the voting system, these two problems might indeed strongly differ with respect to their complexity as, for example, for the Copeland^1 system (also called Llull voting), Copeland^1-CCAC is NP-hard whereas Copeland^1-CCAUC is in P. The destructive cases of these just defined three candidate control types will be denoted by $\mathscr{E}$-DCDC, $\mathscr{E}$-DCAC, and $\mathscr{E}$-DCAUC, respectively but we will not further investigate this latter type of control in this thesis.

Partitioning of Voters and Candidates In the formal model studied here, control by partitioning either the list of voters or the set of candidates results in a two-stage election consisting of one or two subelections in the first stage and a final election in which only the first-stage winners participate. So-called "tie-handling rules" determine the procedures when the subelections are not won uniquely, that is, when there is more than one winner in at least one of the subelections. We consider the two rules TP ("ties promote") and TE ("ties eliminate") that were defined by Hemaspaandra et al. [HHR07]. The TP-rule determines that all winners of the subelections participate in the final election, independently of whether they are unique winners or co-winners. The TE-rule, on the other hand, allows only unique winners to move forward to the final election stage; when there are multiple winners in a subelection, none of these candidates participates in the final round. Of course, this can lead to an empty candidate set in the final election. In this case, the two-stage election has no winners. Note that such a result is considered to be a successful control attempt for the destructive control types. We give the formal definitions of the considered partition cases for the TP-rule only since the corresponding control type in the TE-case can be defined analogously.

Considering to partition the list of voters in more than two sublists, can, in a wider sense, be seen as a formalization of *district gerrymandering*, see also the work of Erdélyi et al. [EHH15] for other variants of defining partition of voters. We, however, analyze the model originally proposed by Bartholdi et al. [BTT92] and Hemaspaandra et al. [HHR07].

$\mathscr{E}$–CONSTRUCTIVE CONTROL BY PARTITION OF VOTERS TP ($\mathscr{E}$-CCPV-TP)	
Given:	An $\mathscr{E}$ election (C, V) and a designated candidate $c \in C$.
Question:	Is it possible to partition V into two subsets V_1 and V_2 with $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$ such that c is an $\mathscr{E}$ winner of the two-stage election $(W_1 \cup W_2, V)$, where W_i, $i \in \{1,2\}$, is the set of $\mathscr{E}$ winners of subelection (C, V_i)?

The following example gives a motivation for partitioning the set of candidates.

Example 3.11 *The student representatives of the computer science department want to organize a movie night for the students of their department and it has to be decided which movie to show. All students coming to movie night should have a say in what movie will be watched, so the student representatives organize an election the following way: First, they group the different movies according to categories like "action," "comedy," "romance," "thriller," and so on, and the students determine the best movies in each category according to their tastes. Then, when the winners in the different categories are known, the students can decide what kind of movie they want to see, so in the final election, they vote over the winning movies from the categories. By limiting the number of categories to two, this situation can be modeled by partition of candidates with run-off.*

$\mathscr{E}$–CONSTR. CONTROL BY RUN-OFF PARTITION OF CANDIDATES TP ($\mathscr{E}$-CCRPC-TP)	
Given:	An $\mathscr{E}$ election (C,V) and a distinguished candidate $c \in C$.
Question:	Is it possible to partition C into two subsets C_1 and C_2 with $C = C_1 \cup C_2$ and $C_1 \cap C_2 = \emptyset$ such that c is an $\mathscr{E}$ winner of the two-stage election $(W_1 \cup W_2, V)$, where W_i, $i \in \{1,2\}$, is the set of $\mathscr{E}$ winners of subelection (C_i,V)?

The election in Example 3.11 can be modified as follows: To do justice to classic movies that always deserve to be shown, the students' representatives first choose some classic movies, and then they let the students elect their favorite one among a number of recently released movies. In the final election, the students vote over all the classic movies and the subelection winners among the recent released movies. This models partition of candidates without a run-off.

$\mathscr{E}$–CONSTRUCTIVE CONTROL BY PARTITION OF CANDIDATES TP ($\mathscr{E}$-CCPC-TP)	
Given:	An $\mathscr{E}$ election (C,V) and a distinguished candidate $c \in C$.
Question:	Is it possible to partition C into two subsets C_1 and C_2 with $C = C_1 \cup C_2$ and $C_1 \cap C_2 = \emptyset$ such that c is an $\mathscr{E}$ winner of the two-stage election $(W_1 \cup C_2, V)$, where W_1 is the set of $\mathscr{E}$ winners of subelection (C_1,V)?

We denote the corresponding problems using the TE-rule with $\mathscr{E}$-CCPV-TE, $\mathscr{E}$-CCRPC-TE, and $\mathscr{E}$-CCPC-TE and we abbreviate the destructive cases with $\mathscr{E}$-DCPV-TP, $\mathscr{E}$-DCPV-TE, $\mathscr{E}$-DCRPC-TP, $\mathscr{E}$-DCRPC-TE, $\mathscr{E}$-DCPC-TP, and $\mathscr{E}$-DCPC-TE, respectively.

The winner model in which we just defined our control problems is the so-called *co-winner model* as we always ask whether the designated candidate can be made or be prevented from being *an $\mathscr{E}$ winner*. The *unique-winner model* is defined by asking whether the designated candidate can be made or be prevented from being *a unique winner* of the resulting election. We analyze the stated problems for both winner models, see Section 3.2.2. Distinguishing between these two models is important as Hemaspaandra et al. [HHM13] have shown: In fact, in the unique-winner model the above defined 22 control types collapse to 21 control types since $\mathscr{E}$-DCRPC-TE = $\mathscr{E}$-DCPC-TE holds. In the co-winner model, there are only 20 different types of control as $\mathscr{E}$-DCRPC-TP = $\mathscr{E}$-DCPC-TP and $\mathscr{E}$-DCRPC-TE = $\mathscr{E}$-DCPC-TE holds.

Notions of Resistance and Vulnerability After having defined the numerous variations of electoral control we can now turn to the analysis of their complexity. Bartholdi et al. [BTT92] introduced the following notions of immunity and vulnerability to, and resistance against electoral control.

Definition 3.12 (Immunity, Susceptibility) *Let $\mathscr{E}$ be a voting system and let $\mathscr{C}$ be one of the control types defined above. We say that $\mathscr{E}$ is* immune to *$\mathscr{C}$ if it is impossible for the chair to successfully execute the given type of control. If $\mathscr{E}$ is not immune to $\mathscr{C}$, then we call $\mathscr{E}$* susceptible to *$\mathscr{C}$.*

In other words, for a voting system to be immune against, say CONSTRUCTIVE CONTROL BY DELETING CANDIDATES, it has to be shown that there is no election and no designated candidate that can be made a (unique) winner by deleting any number of candidates. If just one election can be found in which a successful control action can be achieved, then the voting system is proven to be susceptible to this type of control. If so, the following definition distinguishes further in terms of computational complexity.

Definition 3.13 (Vulnerability, Resistance) *Let $\mathscr{C}$ be one of the control types defined above and let $\mathscr{E}$ be susceptible to $\mathscr{C}$. We say that $\mathscr{E}$ is* vulnerable to *$\mathscr{C}$ if the corresponding decision problem $\mathscr{E}$-$\mathscr{C}$ is in* P. *If $\mathscr{E}$-$\mathscr{C}$ is* NP*-completeness, then we call $\mathscr{E}$ resistant to $\mathscr{C}$.*

Note that for voting systems with a winner determination procedure that is complete or hard for a complexity class above NP, such as the Dodgson system, this definition of resistance is not applicable. To address this problem when dealing with such voting systems, Hemaspaandra et al. [HHR09] suggest to drop the upper bound and redefine resistance as NP-hardness. All voting systems studied in this thesis indeed have a polynomial-time winner determination, which is why we use the original definition.

As we aim at analyzing control complexity beyond NP-hardness shields, we introduce a notion of parameterized resistance.

Definition 3.14 (Parameterized Resistance) *Let $\mathscr{C}$ be one of the control types defined above and let the voting system $\mathscr{E}$ be susceptible to $\mathscr{C}$. We say that $\mathscr{E}$ is* parameterized resistant to *$\mathscr{C}$ with respect to a fixed parameter k if the corresponding parameterized decision problem, denoted by k-$\mathscr{E}$-$\mathscr{C}$, is* W[1]*-hard.*

Since every Bucklin election is a special fallback election and electoral control does not involve any changes in the preferences other than removing candidates or adding candidates at a predetermined rank in a preference, the following useful lemma obviously holds.

Lemma 3.15 *Let $\mathscr{C}$ be one of the control types defined above. If Bucklin voting is known to be resistant to $\mathscr{C}$, then also fallback voting is resistant to $\mathscr{C}$. Furthermore if fallback voting is vulnerable against control $\mathscr{C}$, then Bucklin voting is vulnerable against control $\mathscr{C}$, as well.*

Related Work and State of the Art Bartholdi et al. [BTT92] started the study of constructive control for plurality voting (PV) and the Condorcet system. Destructive control was introduced by Hemaspaandra et al. [HHR07], who completed the study of plurality and Condorcet and also studied electoral control in approval voting.

Table 3.6 gives an overview of the results in terms of classical resistance for some chosen well-studied voting systems. We focus on those voting systems that are studied in this thesis and complement the overview with the results for *normalized range voting (NRV)*, studied by Menton and Singh [MS13], which shows the same number of resistances against electoral control as fallback voting.

							Copeland$^\alpha$					
	PV		**Borda**		**Schulze**		$\alpha \in \{0,1\}$		$\alpha \in (0,1)$		**NRV**	
Control	C	D	C	D	C	D	C	D	C	D	C	D
CAC	R [1]	R [2]	R [8,9]	V [10]	R [3]	S [3]	V [5]	V [5]	R [5]	V [5]	R [6]	R [6]
CAUC	R [1]	R [2]	?	?	R [4]	S [3]	R [5]	V [5]	R [5]	V [5]	R [6]	R [6]
CDC	R [1]	R [2]	R [8]	V [10]	R [4]	S [3]	R [5]	V [5]	R [5]	V [5]	R [6]	R [6]
CPC-TE	R [2]	R [2]	?	?	R [4]	V [4]	R [5]	V [5]	R [5]	V [5]	R [6]	R [6]
CRPC-TE	R [2]		?		R [4]		R [5]		R [5]		R [6]	
CPC-TP	R [2]	R [2]	?	?	R [4]	V [4]	R [5]	V [5]	R [5]	V [5]	R [6]	R [6]
CRPC-TP	R [2]		?		R [4]		R [5]		R [5]		R [6]	
CAV	V [1]	V [2]	R [7]	V [7]	R [3]	R [3]	R [5]	R [5]	R [5]	R [5]	R [6]	V [6]
CDV	V [1]	V [2]	?	V [7]	R [3]	R [3]	R [5]	R [5]	R [5]	R [5]	R [6]	V [6]
CPV-TE	V [2]	V [2]	?	V [7]	R [4]	R [4]	R [5]	R [5]	R [5]	R [5]	R [6]	R [6]
CPV-TP	R [2]	R [2]	?	?	R [4]	R [4]	R [5]	R [5]	R [5]	R [5]	R [6]	R [6]

[1] [BTT92] [4] [MS13] [7] [Rus07] [10] [LNR+15]
[2] [HHR07] [5] [FHH+09b] [8] [CFN+15] Key: C = constructive, D = destructive,
[3] [PX12] [6] [Men13] [9] [EFS11] S = susceptible, R = resistant, V = vulnerable

Table 3.6: Selection of known results regarding control complexity

A comprehensive study of approval voting and its variants can be found in the book chapter of Baumeister et al. [BEH+10]. Lin [Lin12] studies the control complexity in k-approval elections and also provides results for control when the voters have weights. Further results for control in weighted elections can be found in the work of Russell [Rus07] and Faliszewski et al. [FHH15]. Hemaspaandra et al. [HHS14] study constructive control by adding voters for the family of pure scoring rules and provide the first dichotomy result regarding electoral control.

The parameterized complexity of control problems has found much attention. Betzler and Uhlmann [BU09] study control by adding and deleting candidates in Copeland$^\alpha$ and plurality elections, while the work of Faliszewski et al. [FHH+09b] focuses on the family of Copeland$^\alpha$. Extending this line of research, Liu et al. [LFZ+09] focus on plurality, Condorcet, and approval elections. They consider natural parameters such as the number of deleted voters in control by deleting voters. Similar studies for the maximin rule, Schulze elections, and ranked pairs can be found in the work of Liu and Zhu [LZ10] and Hemaspaandra et al. [HLM13]. Recent results for the complexity of candidate control parameterized by the number of voters are due to Chen et al. [CFN+15] and are especially interesting as they capture the complexity of candidate control in elections with few voters. Yang [Yan14] studies, amongst others, control problems that are parameterized by the number of candidates participating in the election. Bredereck et al. [BFN+15a] consider the same parameter, but they analyze *priced control* problems, a model that was introduced by Miąsko and Faliszewski [MF]. The theoretical study in [BFN+15a] is complemented by an *experimental evaluation*, analyzing the running time of the presented FPT algorithms.

The complexity of electoral control in elections with special structures such as *single-peaked* or *single-crossing preferences* has been studied in the work of Faliszewski et al. [FHH14, FHH+11], Brandt et al. [BBH+10], and Magiera and Faliszewski [MF14]. See also the book chapter by Hemaspaandra et al. [HHR15] for an overview.

The basic model of control has been extended in various ways. Faliszewski et al. [FHH11] study notions of *multimode control*, where several control actions can be performed simultaneously. *Control by replacing candidates* has recently been introduced by Loreggia et al. [LNR+15], who study its destructive variant for positional scoring rules. Their theoretical analysis is complemented by an empirical evaluation on real-world data sets. In Loreggia et al. [LNR+15], the study is extended to *control by replacing voters* and the constructive case of control by replacing candidates. Erdélyi et al. [EHH15] focus on control by partition and introduce new variants of this type of control.

Recent research includes the study of control by adding voters when the set of unregistered voters has a combinatorial structure (see the work of [BCF+15]) and the question whether elections can be controlled by breaking ties (studied by Mattei et al. [MNW14]). Perek et al. [PFP+13] consider the setting of voting in parliaments and analyze the question of how many parliamentarians have to deviate from their party's vote to alter the outcome of the election. The notion of *online control* in the setting of *sequential elections* is defined and studied by [HHR12a, HHR12b]. Wojtas and Faliszewski [WF12] study the complexity counting versions of control by adding/deleting candidates or voters.

For a comprehensive overview of further research related to electoral control, we refer to the survey by Faliszewski et al. [FHH+09a] and to the recent book chapters by Faliszewski and Rothe [FR16] and Baumeister and Rothe [BR15].

3.2.2 A Worst-Case Analysis

The study of control complexity in fallback voting was initiated by Erdélyi and Rothe [ER10] and was continued by several follow up papers. Table 3.7 gives an overview of the results that were obtained since then.

Erdélyi and Rothe [ER10] proved the complexity of all previously defined control problems except three cases of voter partition in fallback elections, namely constructive control by partition of voters in the TP-model and destructive control by partition of voters in both tie-handling models. Erdélyi et al. [EPR11] solve these three open problems and complement the previous work by analyzing the complexity of control in Bucklin elections leaving one open problem: the complexity of destructive control by partition of voters in the TP-model.

	Fallback Voting		**Bucklin Voting**	
Control by	Const.	Dest.	Const.	Dest.
Adding Voters	P-**R** [1,3]	V [1]	P-**R** [3]	V [1]
Deleting Voters	P-**R** [1,3]	V [1]	P-**R** [3]	V [1]
Adding Candidates	P-**R** [1,3]	P-**R** [1,3]	P-**R** [3]	P-**R** [3]
Deleting Candidates	P-**R** [1,3]	P-**R** [1,3]	P-**R** [3]	P-**R** [3]
Partition of Voters - TE	R [1]	**R** [2]	R [2]	**R** [2]
Partition of Voters - TP	R [2]	**R** [2]	R [2]	S [2]
Partition of Candidates - TE	R [1]	R [1]	R [2]	R [2]
Run-off Partition of Candidates - TE	R [1]		R [2]	
Partition of Candidates - TP	R [1]	R [1]	R [2]	R [2]
Run-off Partition of Candidates - TP	R [1]		R [2]	

[1] shown in [ER10]
[2] shown in [EPR11]
[3] shown in [EF10b]

Key: S = susceptible, R = resistant, V = vulnerable, P-R = parameterizedly resistant, TE = ties eliminate, and TP = ties promote. **Boldface** results are shown in this thesis.

Table 3.7: Overview of classical and parameterized complexity results for control in Bucklin and fallback voting, all results hold in both the co-winner and the unique-winner model

In [EF10b] Erdélyi and Fellows started the study of parameterized complexity in Bucklin and fallback elections and investigated control by adding and deleting candidates or voters with respect to natural parameters (the number of voters or candidates that are added or deleted). All three of these previous studies focus on the unique-winner model, only.

The results that we present in the context of this thesis are published in [EFR+15a] which supersedes and complements the previous research mentioned above.

In this section we present the proofs of the results marked in boldface font in Table 3.7 in detail. The corresponding theorems are shown in Table 3.8.

Reduction from	**to**	**Reference**	**parameterized?**
k-DS	BV-CCDC	Theorem 3.18	yes
	BV-DCDC	Theorem 3.19	
	BV-CCAV	Theorem 3.24	
	BV-CCDV	Theorem 3.25	
	BV-DCPV-TE	Construction 3.26 and Theorem 3.28	no
RHS	FV-DCPV-TP	Construction 3.30 and Theorem 3.32	
	BV-CCAC BV-DCAC	Construction 3.21 and Theorem 3.23	yes

Table 3.8: Overview of the reductions used to prove the results in Table 3.7

The latter table also gives an overview of the problems we reduce from to show the respective hardness result, provides pointers to the corresponding theorems or constructions, and also states in the last column whether the reduction is parameterized or not. The prefixes “BV” and “FV” stand for “Bucklin voting” and “fallback voting,” respectively. Our hardness results for Bucklin voting imply the same results for fallback voting due to Lemma 3.15. The only proof that is presented explicitly for fallback elections is the case of destructive control by partition of voters in model TP. The complexity of this control problem is still not known for Bucklin elections.

Susceptibility Erdélyi et al. [EPR11] show that each of the defined control types is possible in both Bucklin and fallback elections in the unique-winner model. These examples can be straightforwardly adapted to cover the co-winner case, as well. Nevertheless, we provide a small example to formally illustrate how, for example, voter control can be executed in Bucklin elections.

Example 3.16 *Let (C,V) be a Bucklin election with four candidates $C = \{a,b,c,d\}$ and six voters in $V = (v_1, v_2, \dots, v_6)$ with the following preferences.*

$\mathbf{v_1}$:	$c > a > b > d$	$\mathbf{v_3}$:	$b > a > d > c$	$\mathbf{v_5}$:	$d > b > a > c$
$\mathbf{v_2}$:	$a > b > c > d$	$\mathbf{v_4}$:	$a > c > b > d$	$\mathbf{v_6}$:	$c > b > d > a$

We have a strict majority with 4 voters in (C,V) and the scores are shown in Table 3.9.

	(C,V)				(C,V_1)				(C,V_2)				$(\{b,c\},V)$	
	a	b	c	d	a	b	c	d	a	b	c	d	b	c
$score^1$	2	1	2	1	0	0	**1**	0	2	1	1	1	3	3
$score^2$	**4**	**4**	3	1	1	0	1	0	3	**4**	2	1	**6**	**6**

Table 3.9: Scores in the Bucklin elections $(C,V), (C,V_1), (C,V_2), ((\{b,c\},V))$

The candidates a and b are both level 2 winners in (C,V). Let (V_1,V_2) with $V_1 = (v_1)$ and $V_2 = (v_2, v_3, \dots, v_6)$ be a partition of V. We see in Table 3.9 that c is the level 1 Bucklin winner in election (C,V_1), while b is the level 2 Bucklin winner in (C,V_2).

So taking election (C,V), c can be made a Bucklin winner while a can be prevented from being a Bucklin winner by deleting the five voters $v_2, v_3, \dots, v_6$ or partitioning the voter list into V_1 and V_2. Since there are only unique winners in the subelections, in both tie-breaking models, the final election is $(\{b,c\},V)$. Furthermore starting from (C,V_2), candidate a can be made a Bucklin winner by adding v_1. Starting from (C,V_1), c can be prevented from being a Bucklin winner by adding the voters $v_2, v_3, \dots, v_6$.

Example 3.16 shows that Bucklin voting is susceptible to all types of voter control in the co-winner model. Since each Bucklin election is a special fallback election, this susceptibility result directly transfers to fallback voting, as well. It is easy to see that both Bucklin and fallback voting are also susceptible to all variants of candidate control we introduced. The following lemma summarizes our findings on susceptibility and we state it without proof.

Lemma 3.17 *Bucklin and fallback voting are both susceptible to each control type defined in Section 3.2.1, in both winner models.*

Now that we have stated that Bucklin and fallback voting can be controlled by any of the introduced control scenarios, we turn to the complexity of the corresponding control problems. Note that we assume that the above result is known and refrain from explicitly stating Lemma 3.17 in every upcoming resistance proof. Note further that, for the sake of readability, we will use the shorthand $a\,b\,c$ for a preference $a > b > c$ in our proofs.

Candidate Control We start with the results for the candidate control cases. To show that Bucklin voting is parameterizedly resistant to constructive and destructive control by deleting candidates when the parameter is the number of deleted candidates, we give reductions from the problem k-DOMINATING SET. Recall the definition from Section 2.2 on page 14.

Theorem 3.18 *Bucklin voting is parameterizedly resistant to constructive control by deleting candidates, when this control problem is parameterized by the number of candidates deleted, in both winner models.*

Proof. We show the parameterized resistance for both winner models starting with the unique-winner model. To this end, let $((G,k),k)$ with the undirected graph $G = (B,E)$ be a given instance of k-DOMINATING SET. Without loss of generality, we may assume that $k < n = \|B\|$, since the set B of all vertices is a trivial dominating set in G.

Define the election (C,V), where $C = B \cup D \cup \{w\} \cup X \cup Y$ is the set of candidates, w is the designated candidate, D is a set of "co-winners" (see below), and X and Y are sets of *padding candidates*.[2]

Co-winners in D: D is a set of $k+1$ candidates that tie with w. These candidates prevent that deleting up to k co-winners of election (C,V) makes w the unique winner.

Padding candidates in X: X is a set of $n(n+k) - \sum_{i=1}^{n} \|\mathcal{N}[b_i]\|$ candidates such that for each i, $1 \leq i \leq n$, we can find a subset $X_i \subseteq X$ with $n+k-\|\mathcal{N}[b_i]\|$ elements such that $X_r \cap X_s = \emptyset$ for all $r,s \in \{1,\dots,n\}$ with $r \neq s$. These subsets ensure that w is always placed at the $(n+k+1)^{st}$ position in the first voter group of V below.

Padding candidates in Y: Y is a set of $n(k+1)$ candidates such that for each j, $1 \leq j \leq k+1$, we can find a subset $Y_j \subseteq Y$ with n elements such that $Y_r \cap Y_s = \emptyset$ for all $r,s \in \{1,\dots,k+1\}$ with $r \neq s$. These subsets ensure that each $d_j \in D$ is always placed at the $(n+k+1)^{st}$ position in the second voter group of V below.

V is the following collection of $2n+1$ voters in Table 3.10, so that we have a strict majority with $n+1$ votes:

Note that when up to k candidates are deleted (no matter which ones), the candidates from D can never be among the top $n+k$ candidates in the votes of the first voter group. Table 3.11 shows the scores on the relevant levels of the relevant candidates in election (C,V).

[2]Note that in this construction as well as in later constructions, the subsets of padding candidates are always constructed so as to ensure that, at least up to a certain level, no padding candidate scores enough points to be

Group	For each ...	# of votes	preference
(1)	$i \in \{1,\dots,n\}$	1	$\mathscr{N}[b_i]\, X_i\, w\, ((B-\mathscr{N}[b_i]) \cup (X-X_i) \cup Y)\, D$
(2)	$j \in \{1,\dots,k+1\}$	1	$Y_j\, (D-\{d_j\})\, d_j\, (B \cup X \cup (Y-Y_j) \cup \{w\})$
(3)		$n-k-1$	$D\, (X \cup Y)\, w\, B$
(4)		1	$D\, w\, (X \cup Y)\, B$

Table 3.10: Voter list V in the proof of Theorem 3.18

	$b_i \in B$	w	$d_j \in D$
$score^{k+1}$	$\leq n$	0	$n-k$
$score^{k+2}$	$\leq n$	1	$\leq n-1$
$score^{n+k}$	$\leq n$	1	n
$score^{n+k+1}$	$\leq n$	$\mathbf{n+1}$	$\mathbf{n+1}$

Table 3.11: Level i scores in (C,V) for $i \in \{k+1, k+2, n+k, n+k+1\}$ and the candidates in $C-(X \cup Y)$

The candidates in D and candidate w reach a strict majority on level $n+k+1$, denoted by the boldfaced entries in Table 3.11. Since there is no other candidate reaching a strict majority of $n+1$ votes or more on any level up to $n+k+1$, w and the candidates in D are the Bucklin winners in the election.

We claim that G has a dominating set of size k if and only if w can be made the unique Bucklin winner by deleting at most k candidates.

Only if: Suppose G has a dominating set $B' \subseteq B$ of size k. Delete the corresponding candidates from C. Since B' is a dominating set in G (i.e., $B = \mathscr{N}[B']$), every $b_i \in B$ has a neighbor in B' or is itself in B', which means that in election $(C-B',V)$ candidate w gets pushed at least one position to the left in each of the n votes in the first voter group. So w reaches a strict majority already on level $n+k$ with a score of $n+1$. Since no other candidate does so (in particular, no candidate in D), it follows that w is the unique level $n+k$ Bucklin winner of election $(C-B',V)$.

If: Suppose w can be made the unique Bucklin winner of the election by deleting at most k candidates. Since there are $k+1$ candidates other than w (namely, those in D) having a strict majority on level $n+k+1$ in election (C,V), deleting k candidates from D is not sufficient for making w the unique Bucklin winner of the resulting election. So by deleting at most k candidates, w must become the unique Bucklin winner on a level lower than or equal to $n+k$. This is possible only if w is pushed at least one position to the left in all votes from the first voter group. This, however, implies that the $k' \leq k$ deleted candidates either are

- all contained in B and correspond to a dominating set of size k' for G, or
- are in $B \cup X$.

relevant for the outcome of the election. So in the following argument the padding candidates are mainly ignored and their scores are not listed in the overview tables.

Note that not all deleted candidates can be contained in X, since $k < n$ and the sets X_i, $1 \leq i \leq n$, are pairwise disjoint. If some of the k' deleted candidates are in X, say $\ell < k'$ of them, let B' be the set containing the $k' - \ell$ other candidates that have been deleted. For each i, $1 \leq i \leq n$, if in the i^{th} voter of the first group no candidate from $\mathscr{N}[b_i]$ was deleted but a candidate x_j from X_i, add an arbitrary candidate from $\mathscr{N}[b_i]$ to B' instead of x_j. This yields again a dominating set of size k' for G. In both cases, if $k' < k$ then by adding $k - k'$ further candidates from B (which is possible due to $k < n$) we obtain a dominating set of size k for G.

Note that this polynomial-time reduction is parameterized, as the given parameter k of k-Dominating Set is the same parameter k that bounds the number of candidates allowed to be deleted in the control problem.

For showing resistance in the co-winner model, the definition of the padding candidates in X has to be changed such that in each voter of the first group one more padding candidate is ranked ahead of w. This ensures that w is not a winner in the election (C,V) and, furthermore, reaches a strict majority one level later than the candidates in D. Thus, the remaining argumentation can be adapted straightforwardly. ❑

Theorem 3.19 *Bucklin voting is parameterizedly resistant to destructive control by deleting candidates, when this control problem is parameterized by the number of candidates deleted, in both winner models.*

Proof. Again, we show parameterized resistance for this control case for both winner models starting with the unique-winner case. Let $((G,k),k)$ with the graph $G = (B,E)$ be a given instance of k-Dominating Set. Define the election (C,V), where $C = B \cup \{c,w\} \cup M_1 \cup M_2 \cup M_3 \cup X \cup Y \cup Z$ is the candidate set, c is the designated candidate, and M_1, M_2, M_3, X, Y, and Z are sets of padding candidates (recall Footnote 2 on page 47).

Padding candidates in M_1, M_2, and M_3: M_1, M_2, and M_3 are three pairwise disjoint sets, where each is a set of k candidates that are positioned in the votes so as to ensure that no other candidate besides w and c can reach a strict majority up to level $n+k$.

Padding candidates in X: X is a set of $n^2 - \sum_{i=1}^{n} \|\mathscr{N}[b_i]\|$ candidates such that for each i, $1 \leq i \leq n$, we can find a subset $X_i \subseteq X$ with $n - \|\mathscr{N}[b_i]\|$ elements such that $X_r \cap X_s = \emptyset$ for all $r,s \in \{1,\ldots,n\}$ with $r \neq s$. These subsets ensure that w is always placed at the $(n+1)^{st}$ position in the first voter group of V below.

Padding candidates in Y: Y is a set of $n-1$ padding candidates ensuring that c is at position n in the votes of the second voter group of V below.

Padding candidates in Z: Z is a set of $n-2$ padding candidates ensuring that w is at position $n-1$ and c is at position n in the vote of the third voter group of V below.

Table 3.12 gives the collection V of $2n+1$ voters, so we have a strict majority threshold of $n+1$. Note that in the first voter group, candidate c has to be on the last position in every vote, which is why all candidates in $(B - \mathscr{N}[b_i]) \cup M_2 \cup M_3 \cup (X - X_i) \cup Y \cup Z$ have to be (in an arbitrary order) ranked before c.

Table 3.13 gives an overview of the scores on the relevant levels of the relevant candidates in election (C,V). Note that candidate c is the unique level n Bucklin winner of election (C,V),

Group	For each ...	# of votes	preference
(1)	$i \in \{1, \dots, n\}$	1	$\mathcal{N}[b_i]\, X_i\, w\, M_1\, ((B - \mathcal{N}[b_i]) \cup \dots$ $\dots \cup M_2 \cup M_3 \cup (X - X_i) \cup Y \cup Z)\, c$
(2)		n	$Y\, c\, M_2\, (B \cup M_1 \cup M_3 \cup X \cup Z \cup \{w\})$
(3)		1	$Z\, w\, c\, M_3\, (B \cup M_1 \cup M_2 \cup X \cup Y)$

Table 3.12: Voter list V in the proof of Theorem 3.19

since c is the first candidate reaching a strict majority of votes (namely, $n+1$ points on level n, as indicated by a boldfaced entry).

	$b_i \in B$	w	c
$score^{n-1}$	$\leq n$	1	0
$score^{n}$	$\leq n$	1	$\mathbf{n+1}$
$score^{n+1}$	$\leq n$	$n+1$	$n+1$

Table 3.13: Level i scores in (C,V) for $i \in \{n-1, n, n+1\}$ and the candidates in $B \cup \{c, w\}$

We claim that G has a dominating set of size k if and only if c can be prevented from being a unique Bucklin winner by deleting at most k candidates.

Only if: Suppose G has a dominating set $B' \subseteq B$ of size k. Delete the corresponding candidates. Now candidate w moves at least one position to the left in each of the n votes in the first voter group. Since candidate c reaches a strict majority no earlier than on level n and $score^{n}_{(C-B',V)}(w) = n+1 = score^{n}_{(C-B',V)}(c)$, candidate c is no longer a unique Bucklin winner of the resulting election.

If: Suppose c can be prevented from being a unique Bucklin winner of the election by deleting at most k candidates. Note that deleting one candidate from an election can move the strict majority level of another candidate at most one level to the left. Observe that only candidate w can prevent c from winning the election, since w is the only candidate other than c who reaches a strict majority of votes until level $n+k$. In election (C,V), candidate w reaches this majority no earlier than on level $n+1$, and candidate c not before level n. Thus w can prevent c from being a unique winner only by scoring at least as many points as c no later than on level n. This is possible only if w is pushed at least one position to the left in all votes of the first voter group. By an argument analogous to that given in the constructive case for this control type (see the proof of Theorem 3.18), this implies that G has a dominating set of size k.

Note that this polynomial-time reduction is parameterized, as the given parameter k of k-DOMINATING SET is the same parameter k that bounds the number of candidates allowed to be deleted in the control problem.

To handle the co-winner model, one additional voter has to be added to V, with the preference $w\,(M_1 \cup M_2 \cup M_3 \cup X \cup Y \cup Z)\,(B \cup \{c\})$. This ensures that on level $n+1$ candidate w has one point more than c and can thus beat c strictly if and only if there is a dominating set of size at most k (which can be shown with an analogous argument to the one presented above). ❑

The remaining cases of candidate control can be shown with one central construction of an election from a given RESTRICTED HITTING SET instance, a variant of the HITTING SET (HS) problem, see [GJ79], which we define as follows.

RESTRICTED HITTING SET (RHS)	
Given:	A set $B = \{b_1, b_2, \dots, b_m\}$, a collection $\mathscr{S} = \{S_1, S_2, \dots, S_n\}$ of nonempty subsets $S_i \subseteq B$ such that $n > m$, and a positive integer k with $1 < k < m$.
Question:	Does $\mathscr{S}$ have a hitting set of size at most k, i.e., is there a set $B' \subseteq B$ with $\|B'\| \leq k$ such that for each i, $S_i \cap B' \neq \emptyset$?

Lemma 3.20 establishes NP-completeness of the just defined problem and W[2]-hardness when the problem is parameterized by the solution size k. We denote the latter problem with k-RHS.

Lemma 3.20 (Erdélyi et al. [EPR11]) RHS *is* NP*-complete and* k-RHS *is* W[2]*-hard.*

Proof. It is immediate that RHS is in NP. To show NP-hardness of RHS, we reduce the (general) HS problem to RHS.

Let $(\hat{B}, \hat{\mathscr{S}}, \hat{k})$ be a given instance of HS, where $\hat{B} = \{b_1, b_2, \dots, b_{\hat{m}}\}$ is a set, we have a collection $\hat{\mathscr{S}} = \{S_1, S_2, \dots, S_{\hat{n}}\}$ of nonempty subsets of $\hat{B}$, and $\hat{k} \leq \hat{m}$ is a positive integer. If $\hat{k} = \hat{m}$ or $\hat{k} = 1$, $(\hat{B}, \hat{\mathscr{S}}, \hat{k})$ is trivially in HS, so we may assume that $1 < \hat{k} < \hat{m}$.

Define the following instance $(B, \mathscr{S}, k)$ of RHS:

$$(B, \mathscr{S}, k) = \begin{cases} (\hat{B} \cup \{a\}, \hat{\mathscr{S}} \cup \{S_{\hat{n}+1}, S_{\hat{n}+2}, \dots, S_{\hat{m}+2}\}, \hat{k}+1) & \text{if } \hat{n} \leq \hat{m} \\ (\hat{B}, \hat{\mathscr{S}}, \hat{k}) & \text{if } \hat{n} > \hat{m}, \end{cases}$$

where $S_{\hat{n}+1} = S_{\hat{n}+2} = \cdots = S_{\hat{m}+2} = \{a\}$.

Let n be the number of members of $\mathscr{S}$ and m be the number of elements of B. Since $1 < \hat{k} < \hat{m}$, we have $1 < k < m$. Note that if $\hat{n} > \hat{m}$ then $(B, \mathscr{S}, k) = (\hat{B}, \hat{\mathscr{S}}, \hat{k})$, so $n = \hat{n} > \hat{m} = m$; and if $\hat{n} \leq \hat{m}$ then $n = \hat{m} + 2 > \hat{m} + 1 = m$. Thus, in both cases $(B, \mathscr{S}, k)$ fulfills the restriction of RHS.

It is easy to see that $\hat{\mathscr{S}}$ has a hitting set of size at most $\hat{k}$ if and only if $\mathscr{S}$ has a hitting set of size at most k. In particular, assuming $\hat{n} \leq \hat{m}$, if $\hat{\mathscr{S}}$ has a hitting set B' of size at most $\hat{k}$ then $B' \cup \{a\}$ is a hitting set of size at most $k = \hat{k} + 1$ for $\mathscr{S}$; and if $\hat{\mathscr{S}}$ has no hitting set of size at most $\hat{k}$ then $\mathscr{S}$ can have no hitting set of size at most $k = \hat{k} + 1$ (because $a \notin \hat{B}$, so $\{a\} \cap S_i = \emptyset$ for each i, $1 \leq i \leq \hat{n}$). Thus, RHS is NP-hard.

In the above reduction we have that the HS instance has solution size $\hat{k}$ and the constructed RHS instance has solution size $k \in \{\hat{k}, \hat{k}+1\}$. By parameterizing both HS and RHS with the solution size (denoted by k-HS and k-RHS), we see that the above reduction also establishes that k-HS parameterizedly reduces to k-RHS, as the reduction is parameter preserving ($\hat{k} \leq k$ and k solely depends on $\hat{k}$). Since k-HS is known to be W[2]-hard, this implies W[2]-hardness of k-RHS, as well. ❑

Construction 3.21, which is due to Erdélyi et al. [EPR11], adapts Construction 4.28 of Hemaspaandra et al. [HHR07], which they used to handle certain candidate control cases for plurality voting.[3] Note that the only adaption we need to make regarding Construction 3.21 is that we construct the election from a given instance of the parameterized decision problem k-RHS, which technically, does not change the construction.

Construction 3.21 (Erdélyi et al. [EPR11]) *Let $((B,\mathscr{S},k),k)$ be an instance of k-RHS, with $B=\{b_1,b_2,\ldots,b_m\}$ a set, $\mathscr{S}=\{S_1,S_2,\ldots,S_n\}$ a collection of nonempty subsets $S_i\subseteq B$ such that $n>m$, and $k<m$ a positive integer. (Thus, $n>m>k>1$.)*

Define the election (C,V), where $C=B\cup\{c,d,w\}$ is the candidate set and where V consists of the following $6n(k+1)+4m+11$ voters in Table 3.14.

Group	For each …	# of voters	preference
(1)		$2m+1$	$c\,d\,B\,w$
(2)		$2n+2k(n-1)+3$	$c\,w\,d\,B$
(3)		$2n(k+1)+5$	$w\,c\,d\,B$
(4)	$i\in\{1,\ldots,n\}$	$2(k+1)$	$d\,S_i\,c\,w\,(B-S_i)$
(5)	$j\in\{1,\ldots,m\}$	2	$d\,b_j\,w\,c\,(B-\{b_j\})$
(6)		$2(k+1)$	$d\,w\,c\,B$

Table 3.14: Voter list V in Construction 3.21

We will make use of the following adaption of a lemma from [EPR11].

Lemma 3.22 (Erdélyi et al. [EPR11]) *Consider the election (C,V) constructed according to Construction 3.21 from a k-RHS instance $((B,\mathscr{S},k),k)$.*

1. *c is the unique level 2 Bucklin winner of election $(\{c,d,w\},V)$.*
2. *If $\mathscr{S}$ has a hitting set B' of size k, then w is the unique Bucklin winner of election $(B'\cup\{c,d,w\},V)$.*
3. *Let $D\subseteq B\cup\{d,w\}$. If c is not a unique Bucklin winner of election $(D\cup\{c\},V)$, then there exists a set $B'\subseteq B$ such that*
   a) *$D=B'\cup\{d,w\}$,*
   b) *w is a level 2 Bucklin winner of election $(B'\cup\{c,d,w\},V)$, and*
   c) *B' is a hitting set for $\mathscr{S}$ of size at most k.*
4. *Let $D\subseteq B\cup\{d,w\}$. If c is not a Bucklin winner of election $(D\cup\{c\},V)$, then there exists a set $B'\subseteq B$ such that*
   a) *$D=B'\cup\{d,w\}$,*

[3] Their construction was also useful in the proofs of most candidate control results for SP-AV [ENR09], so the structure of the constructions and the arguments in the proofs of Lemma 3.22 and Theorem 3.23 are adaptations of those by Hemaspaandra et al. [HHR07] and Erdélyi et al. [ENR09], tailored here to Bucklin voting.

b) w is the unique level 2 Bucklin winner of election $(B' \cup \{c,d,w\},V)$, and
c) B' is a hitting set for $\mathscr{S}$ of size at most k.

Proof. We only prove the last part as the first three are already shown in [EPR11]. Let $D \subseteq B \cup \{d,w\}$. Suppose c is not a Bucklin winner of election $(D \cup \{c\},V)$.

(4a) Besides c, only w has a strict majority of votes on the second level and only w can beat c in $(D \cup \{c\},V)$. Thus, w is clearly in D. In $(D \cup \{c\},V)$, candidate w has no level 1 strict majority and candidate c has already on level 2 a strict majority. Thus, w must beat c on level 2. For a contradiction, suppose $d \notin D$. Then

$$score^2_{(D\cup\{c\},V)}(c) \geq 4n(k+1)+2m+11;$$
$$score^2_{(D\cup\{c\},V)}(w) = 4n(k+1)+2m+10,$$

which contradicts the above observation that w beats c on level 2. Thus, $D = B' \cup \{d,w\}$, where $B' \subseteq B$.

(4b) This part follows immediately from the proof of part (4a).

(4c) Let ℓ be the number of sets in $\mathscr{S}$ not hit by B'. We have that

$$score^2_{(B'\cup\{c,d,w\},V)}(w) = 4n(k+1)+10+2(m-\|B'\|),$$
$$score^2_{(B'\cup\{c,d,w\},V)}(c) = 2(m-k)+4n(k+1)+9+2(k+1)\ell.$$

From part (4b) we know that $score^2_{(B'\cup\{c,d,w\},V)}(w) > score^2_{(B'\cup\{c,d,w\},V)}(c)$, so

$$\begin{aligned} score^2_{(B'\cup\{c,d,w\},V)}(w) &> score^2_{(B'\cup\{c,d,w\},V)}(c)\\ 4n(k+1)+10+2(m-\|B'\|) &\geq 2(m-k)+4n(k+1)+9+2(k+1)\ell+1\\ 2(m-\|B'\|) &\geq 2(m-k)+2(k+1)\ell\\ \|B'\| &\leq k+(k+1)\ell \end{aligned}$$

Thus, $\ell = 0$ has to hold and we have that B' is a hitting set of size at most k.

This completes the proof. ❑

Now we can show the following results.

Theorem 3.23 *Bucklin voting is parameterizedly resistant to constructive and destructive control by adding a limited number of candidates when these two control problems are parameterized by the number of candidates added. Each result holds in both winner models.*

Proof. We show parameterized resistance of Bucklin voting to both cases of control by adding candidates in both winner models by a reduction from k-RHS. Let $((B,\mathscr{S},k),k)$ be an instance of k-RHS and construct election (C,V) according to Construction 3.21. Let then $\{c,d,w\}$ be the set of original candidates and let B be the set of spoiler candidates. From (1) in Lemma 3.22 we know that c is the unique winner in the original election.

For the constructive case we assume candidate w to be the designated candidate. If there is a hitting set B' of size at most k for $\mathscr{S}$, we know with (2) from Lemma 3.22 that w is a unique (and thus a) Bucklin winner of the election in which the at most k candidates from B' have been added. Now we assume that w is a Bucklin winner in an election where at most k candidates from B have been added. This implies that c is not a unique winner in this election, thus with (3) from Lemma 3.22 we know that the given k-RHS instance is a yes-instance. If, on the other hand, we assume that w is a unique winner in such an election, c cannot be a winner and we directly have with (4) from Lemma 3.22 that the given k-RHS instance is a yes-instance.

For the destructive cases we set c to be the designated candidate. If there is a hitting set of size at most k, we again, know in both winner models with (2) from Lemma 3.22 that the control instance is a yes instance. With parts (3) and (4) the other direction of the reduction follows directly. ❑

Voter Control Regarding the various cases of voter control we will present the proofs in detail for constructive control by deleting and adding voters and destructive control by partition of voters. We start with the former control cases.

Theorem 3.24 *Bucklin voting is parameterizedly resistant to constructive control by adding voters, when this control problem is parameterized by the number of voters added, in both winner models.*

Proof. We show parameterized resistance in both winner models with one reduction from k-DOMINATING SET. To do so, let $((G,k),k)$ with an undirected graph $G=(B,E)$ be a given k-DS instance. We define the election $(C, V \cup U)$, where $C = B \cup \{c,w\} \cup X \cup Y$ is the set of candidates, w is the designated candidate, and X and Y are sets of padding candidates (recall Footnote 2 on page 47).

Padding candidates in X: X is a set of $\sum_{i=1}^{n} \|\mathscr{N}[b_i]\|$ candidates such that for each i, $1 \leq i \leq n$, we can find a subset $X_i \subseteq X$ with $\|\mathscr{N}[b_i]\|$ elements such that $X_r \cap X_s = \emptyset$ for all $r,s \in \{1,\dots,n\}$ with $r \neq s$. These subsets ensure that w is always placed at the $(n+1)^{st}$ position in the votes of the unregistered voters in U below.

Padding candidates in Y: Y is a set of n padding candidates ensuring that none of the candidates in B is ranked among the first n candidates in the votes of the registered voters in V below.

V is the collection of registered and U is the collection of unregistered voters. $V \cup U$ consists of the following $n+k-1$ voters shown in Table 3.15.

Clearly, c is the unique level 1 Bucklin winner of election (C,V).

We claim that G has a dominating set of size k if and only if w can be made the unique Bucklin winner by adding at most k voters from U.

Only if: Suppose G has a dominating set B' of size k. Add the corresponding voters from U to the election (i.e., each voter u_i for which $b_i \in B'$). Now we have an election with $2k-1$

Voter list	For each ...	# of voters	preference
V		$k-1$	$c\,Y\,B\,w\,X$
U	$i \in \{1,\ldots,n\}$	1	$(B-\mathscr{N}[b_i])\,X_i\,w\,c\,(\mathscr{N}[b_i]\cup(X-X_i)\cup Y)$

Table 3.15: Voter lists V and U in the proof of Theorem 3.24

voters, so the strict majority threshold is k. Since B' is a dominating set, we have $B = \mathscr{N}[B']$, so for each $b_j \in B$ there is at least one of the added voters u_i such that $b_j \in \mathscr{N}[b_i]$, which means that b_j is ranked worse than w in these k added votes. It follows that up to level $n+1$ only candidate w will reach this new threshold of k, hence w is the unique Bucklin winner of this election.

If: Suppose w can be made the unique Bucklin winner by adding at most k voters from U. Denote the set of these voters by U' and note that $\|U'\| \leq k$. Note further that we have $score^1_{(C,V\cup U')}(c) = score^1_{(C,V)}(c) = k-1$, that is, c reaches a score of $k-1$ already on the first level (with or without adding U'). However, if any candidate has a strict majority already on the first level, then he or she is the unique Bucklin winner of the election. As w is the unique Bucklin winner of election $(C,V\cup U')$, the strict majority threshold for $V\cup U'$ must be greater than $k-1$. This, in turn, implies $\|U'\| \geq k$, so $\|U'\| = k$ and the strict majority threshold for $V\cup U'$ is exactly k. Note that $score^{n+1}_{(C,V\cup U')}(w) = k > k-1 = score^{n+1}_{(C,V\cup U')}(x)$ and $score^{n}_{(C,V\cup U')}(w) = 0$. Moreover, since adding the voters from U' to the election has made w the unique Bucklin winner of election $(C,V\cup U')$, none of the candidates in B can be ranked among the first n candidates by each voter in U'; otherwise (i.e., if some candidate $b_j \in B$ would be ranked among the first n candidates by each voter in U'), we would have $score^{n}_{(C,V\cup U')}(b_j) = k$, i.e., b_j would reach a strict majority in $(C,V\cup U')$ earlier than w, a contradiction. But this means that the voters in U' correspond to a dominating set of size k in G.

Note that this polynomial-time reduction is parameterized, as the given parameter k of k-DOMINATING SET is the same parameter k that bounds the number of voters allowed to be added in this control problem.

The above proof works in the unique-winner and the co-winner model. ❑

Theorem 3.25 *Bucklin voting is parameterizedly resistant to constructive control by deleting voters, when this control problem is parameterized by the number of voters deleted, in both winner models.*

Proof. We start with proving parameterized resistance in the unique-winner model with a reduction from k-DOMINATING SET.

Let $((G,k),k)$ with an undirected graph $G = (B,E)$ be a given instance of k-DS and we define the election (C,V), where $C = B\cup\{c,w\}\cup X\cup Y\cup Z$ is the set of candidates, w is the designated candidate, and X, Y, and Z are sets of padding candidates (recall Footnote 2).

Padding candidates in X: X is a set of $\sum_{i=1}^{n} \|B - \mathscr{N}[b_i]\|$ candidates such that for each i, $1 \leq i \leq n$, we can find a subset $X_i \subseteq X$ with $n - \|\mathscr{N}[b_i]\|$ elements such that $X_r \cap X_s = \emptyset$ for all $r, s \in \{1, \dots, n\}$ with $r \neq s$. These subsets ensure that c is always placed among the top $(n+1)$ positions in the first voter group of V below.

Padding candidates in Y: Y is a set of $\sum_{i=1}^{n} \|\mathscr{N}[b_i]\|$ candidates such that for each i, $1 \leq i \leq n$, we can find a subset $Y_i \subseteq Y$ with $\|\mathscr{N}[b_i]\|$ elements such that $Y_r \cap Y_s = \emptyset$ for all $r, s \in \{1, \dots, n\}$ with $r \neq s$. These subsets ensure that w is always placed at the $(n+1)^{st}$ position in the second voter group of V below.

Padding candidates in Z: Z is a set of $(k-1)(n+1)$ candidates such that for each j, $1 \leq j \leq k-1$, we can find a subset $Z_j \subseteq Z$ with $n+1$ elements such that $Z_r \cap Z_s = \emptyset$ for all $r, s \in \{1, \dots, k-1\}$ with $r \neq s$. These subsets ensure that no other candidate besides c and the candidates in Z_j gain any points up to the $(n+2)^{nd}$ level in the third voter group of V below.

V is the following collection of $2n+k-1$ voters listed in Table 3.16.

Group	For each ...	# of votes	preference
(1)	$i \in \{1, \dots, n\}$	1	$\mathscr{N}[b_i]\, c\, X_i\, ((B - \mathscr{N}[b_i]) \cup (X - X_i) \cup Y \cup Z)\, w$
(2)	$i \in \{1, \dots, n\}$	1	$(B - \mathscr{N}[b_i])\, Y_i\, w \dots$ $\dots (\mathscr{N}[b_i] \cup X \cup (Y - Y_i) \cup Z \cup \{c\})$
(3)	$j \in \{1, \dots, k-1\}$	1	$c\, Z_j\, (B \cup X \cup Y \cup (Z - Z_j))\, w$

Table 3.16: Voter list V in the proof of Theorem 3.25

It holds that $n+k-1 > maj(V) > n$ and we see the relevant scores in (C, V) in Table 3.17.

	c	w	$b_j \in B$
$score^1$	$k-1$	0	$\leq n$
$score^{n+1}$	$\mathbf{n+k-1}$	n	n

Table 3.17: Level i scores in (C, V) for $i \in \{1, n+1\}$ and the candidates in $B \cup \{c, w\}$

Since candidate w reaches a strict majority only on the last level but c does so no later than on the $(n+1)^{st}$ level, w is not a unique Bucklin winner of this election.

We claim that G has a dominating set of size k if and only if w can be made the unique Bucklin winner by deleting at most k voters.

Only if: Suppose G has a dominating set B' of size k. Delete the corresponding voters from the first voter group (i.e., each voter v_i for which $b_i \in B'$). Let V' denote the resulting list of voters and note that $\|V'\| = 2n-1$. Now, in election (C, V') we have on level $n+1$:

- $score^{n+1}_{(C,V')}(b_j) \leq n-1$ for each $b_j \in B$ (from the first and second voter groups; no b_j can have a score of n on level $n+1$, since B' is a dominating set in G, so $B = \mathscr{N}[B']$, and all voters v_i corresponding to members b_i of B' have been deleted),

- $score^{n+1}_{(C,V')}(c) = (n-k)+(k-1) = n-1$ (from the first and third voter groups),
- $score^{n+1}_{(C,V')}(x_i) = 1$ for each $x_i \in X$ (from the first voter group),
- $score^{n+1}_{(C,V')}(y_i) = 1$ for each $y_i \in Y$ (from the second voter group),
- $score^{n+1}_{(C,V')}(z_i) = 1$ for each $z_i \in Z$ (from the third voter group), and
- $score^{n+1}_{(C,V')}(w) = n$ (from the second voter group).

That is, only candidate w reaches a strict majority on level $n+1$ in (C,V'), thus w is the unique Bucklin winner of this election.

If: Suppose w can be made the unique Bucklin winner by deleting at most k voters. Let V' be the set of remaining voters. Observe that deleting less than k voters would make it impossible for candidate w to be a unique Bucklin winner of the election. Indeed, if less than k voters are deleted from V, the strict majority threshold for the set V' of remaining voters would exceed n. However, since w is ranked last place in all votes except the n votes from the second voter group, w would reach a strict majority no earlier than on the last level and thus would not be a unique Bucklin winner of this election. Clearly, w has to win election (C,V') on level $n+1$. Since

$$score^{n+1}_{(C,V)}(b_i) = n = score^{n+1}_{(C,V)}(w)$$

for all i with $1 \leq i \leq n$, by deleting these k votes from V each b_i has to lose at least one point on the first $n+1$ levels. Obviously, no voters from the second voter group can be deleted, for otherwise candidate w would not reach the strict majority threshold on level $n+1$. Similarly, deleting voters from the third voter group does not make any $b_i \in B$ lose any points up to level $n+1$. So at least part of the deleted voters have to be from the first voter group, let us say we delete $k' \leq k$. Since *every* candidate $b_i \in B$ has to lose at least one point up to level $n+1$, the k' deleted voters in $V - V'$ correspond to a dominating set in G. If $k' < k$, we can delete voters arbitrarily from the first and/or third voter group until the total allowed number of k deleted voters is reached (that is needed to ensure the right majority threshold in the new election).

Note that this polynomial-time reduction is parameterized, as the given parameter k of k-DOMINATING SET is the same parameter k that bounds the number of voters that may be deleted in this control problem.

To show parameterized resistance also in the co-winner model, the voter list has to be slightly adapted:

- One voter of group 3 has to be deleted (thus, $j \in \{1,\dots,k-2\}$).
- One voter (the only one in a new, fourth group) with preference $B\,c\,(X \cup Y \cup Z)$ has to be added.

With this new voter list, the above argumentation can be adapted straightforwardly. ❑

Now we turn to the partition cases of voter control and begin with the analysis of destructive control by partition of voters in model TE in Bucklin elections. We will show this control problem to be NP-complete in both winner models, as stated in the upcoming Theorem 3.28, but we will only explicitly present the proof in the co-winner model.

Construction 3.26 *Let (G,k) be a given instance of* Dominating Set, *where $G=(B,E)$ is an undirected graph. We define the election (C,V) with candidate set $C = B\cup\{c,u,v,w,x,y\}\cup D\cup F\cup H\cup M$, where c is the designated candidate, D, F, H, M, and $\{u,v\}$ are sets of padding candidates (recall Footnote 2 on page 47), and y is a "partition-enforcing" candidate (see below).*

Padding candidates in D: D is a set of $(k-1)(n+4)$ candidates such that for each j, $1\le j\le k-1$, we can find a subset $D_j\subseteq D$ with $n+4$ elements such that $D_r\cap D_s=\emptyset$ for all $r,s\in\{1,\dots,k-1\}$ with $r\neq s$. These subsets ensure that no other candidate besides x gains more than one point up to level $n+5$ in the third voter group of V below.

Padding candidates in F: F is a set of $3n$ candidates such that for each i, $1\le i\le n$, we can find a subset $F_i\subseteq F$ with three elements such that $F_r\cap F_s=\emptyset$ for all $r,s\in\{1,\dots,n\}$ with $r\neq s$. These subsets ensure that the candidates in B do not gain any points up to the fourth level in the first voter group of V below.

Padding candidates in H: H is a set of n^2 candidates such that for each i, $1\le i\le n$, we can find a subset $H_i\subseteq H$ with $\|\mathscr{N}[b_i]\|$ elements such that $H_r\cap H_s=\emptyset$ for all $r,s\in\{1,\dots,n\}$ with $r\neq s$. These subsets ensure that w does not gain any points up to level $n+5$ in the first voter group of V below.

Padding candidates in M: M is a set of $2(k+n)$ candidates such that for each l, $1\le l\le k+n$, we can find a subset $M_l\subseteq M$ with two elements and it holds that $M_r\cap M_s=\emptyset$ for all $r,s\in\{1,\dots,k+n\}$ with $r\neq s$. These subsets ensure that x and y do not gain any points up to the fourth level in the fourth voter group of V below.

Padding candidates u and v: These two candidates ensure that the other padding candidates are not among the top $n+5$ positions in the second voter group of V below.

Partition-enforcing candidate y: This candidate ensures that the voter from the second voter group of V below has to be in the subelection candidate w wins to finally beat c in the final election.

V consists of the following $2k+2n+1$ votes that can be arranged in five groups displayed in Table 3.18.

Group	For each …	# of votes	preference
(1)	$i\in\{1,\dots,n\}$	1	$F_i\,(B-\mathscr{N}[b_i])\,H_i\,y\,w\,(\mathscr{N}[b_i]\cup D\cup E\cup(F-F_i)\cup(H-H_i))\,u\,v\,c\,x$
(2)		1	$x\,w\,c\,B\,u\,v\,(D\cup E\cup F\cup H)\,y$
(3)	$j\in\{1,\dots,k-1\}$	1	$x\,D_j\,(B\cup(D-D_j)\cup E\cup F\cup H)\,u\,v\,y\,w\,c$
(4)	$l\in\{1,\dots,k+n\}$	1	$c\,E_l\,x\,y\,(B\cup D\cup(E-E_l)\cup F\cup H)\,u\,v\,w$
(5)		1	$H\,D\,F\,M\,B\,y\,v\,x\,w\,c$

Table 3.18: Voter list V in Construction 3.26

Table 3.19 shows the scores of c, w, and x on the first three levels. None of the other candidates scores more than one point up to the third level. Note that c reaches a strict majority on this level and thus is the unique level 3 BV winner in this election.

The thus constructed election (C,V) has the following useful property.

	c	w	x
$score^1$	$k+n$	0	k
$score^2$	$k+n$	1	k
$score^3$	$\mathbf{k+n+1}$	1	k

Table 3.19: Level i scores of c, w, and x in (C,V) for $i \in \{1,2,3\}$

Lemma 3.27 *In the election (C,V) from Construction 3.26, for every partition of V into V_1 and V_2, candidate c is the unique BV winner of at least one of the subelections, (C,V_1) and (C,V_2).*

Proof. For a contradiction, we assume that in both subelections, (C,V_1) and (C,V_2), candidate c is not a unique BV winner. Table 3.19 shows that $\lfloor \|V\|/2 \rfloor$ voters in V place c on the first position in their votes. For our assumption to hold, the sizes of the partitions $V_i, i \in \{1,2\}$ have to be set such that in each V_i there are at most $\|V_i\|/2$ voters positioning c first (these voters have to be from the fourth voter group). Otherwise, c would be a Bucklin winner already on the first level. Without loss of generality, we assume that $\|V_1\| = \|V_2\| + 1$ and we have $\lceil (n+k)/2 \rceil$ voters from the fourth group in V_1 while the remaining voters from the fourth group are in V_2. (For other relations $\|V_1\| = \|V_2\| + \ell$ for $\ell \neq 1$ simply partition the voters from the fourth voter group to ensure that there are not more than $\|V_i\|/2$ in each V_i and allocate the remaining voters arbitrarily to obtain the wanted cardinalities.)

We start with the case that $n+k$ is an even number. Then we have that both V_i each contain $(n+k)/2$ voters from group (4) while V_1 furthermore contains $(n+k)/2+1$ other voters and the remaining $n+k/2$ voters are in V_2. In both subelections we have the same majority threshold $maj(V_i) = (n+k)/2+1$. Let the voter from the second voter group be in V_1. Then we know that c reaches the strict majority in (C,V_1) on the third level. Only x can possibly beat or tie with c on the second or third level in (C,V_1). However, since x does not score more than k points in total until the fourth level, c is the unique level 3 BV winner in subelection (C,V_1), a contradiction.

For the case that $n+k$ is an odd number, simply assume that from (4) there are $(n+k+1)/2$ voters in V_1, thus $n+k-1/2$ voters are in V_2, and the remaining voters are split such that the above assumed cardinalities do hold. The majority thresholds in V_i are $maj(V_i) = (n+k+1)/2+1$ and the same argumentation as above can be used to contradict the main assumption. This completes the proof. ❑

Theorem 3.28 *Bucklin voting is resistant to destructive control by partition of voters in model TE for both winner models.*

Proof. NP membership can be shown by guessing a partition of the voter list and checking in deterministic polynomial time (the winner problem is in P), whether the designated candidate has been prevented from being a Bucklin winner. To prove NP-hardness in the co-winner case, we construct an election (C,V) from a given DOMINATING SET instance (G,k) according to Construction 3.26.

We claim that $G = (B, E)$ has a dominating set B' of size at most k if and only if candidate c can be prevented from being a unique BV winner by partition of voters in model TE.

Only if: Let B' be a dominating set for G of size at most k. Partition V into V_1 and V_2 as follows. Let V_1 consist of the following $2k+1$ voters:

- The k voters of the first voter group corresponding to the dominating set,[4] i.e., for those i with $b_i \in B'$, we have one voter of the form:

$$F_i\,(B - \mathscr{N}[b_i])\,H_i\,y\,w\,(\mathscr{N}[b_i] \cup D \cup E \cup (F - F_i) \cup (H - H_i))\,u\,v\,c\,x,$$

- the one voter from the second group: $x\,w\,c\,B\,u\,v\,(D \cup E \cup F \cup H)\,y$,
- the entire third voter group, i.e., for each j, $1 \leq j \leq k-1$, there is one voter of the form:

$$x\,D_j\,(B \cup (D - D_j) \cup E \cup F \cup H \cup)\,u\,v\,y\,w\,c, \text{ and}$$

- the one voter from the fifth group: $H\,D\,F\,M\,B\,y\,v\,x\,w\,c$.

Let $V_2 = V - V_1$. Note that the strict majority threshold in V_1 is $maj(V_1) = k+1$. Again, since the candidates in D, F, H, and M do not score more than one point, respectively two points, up to level $n+5$, their level $n+5$ scores are not shown in Table 3.20. The level $n+5$ scores of the remaining candidates are shown in this table. Note that w reaches a strict majority of $k+1$ on this level (and no other candidate reaches a strict majority on this or an earlier level). Hence, w is the unique level $n+5$ BV winner in subelection (C, V_1) and thus participates in the final round.

	c	w	x	y	$b_i \in B$
$score^{n+5}$	1	$\mathbf{k+1}$	k	k	$\leq k$

Table 3.20: Level $n+5$ scores in (C, V_1)

From Lemma 3.27 it follows that candidate c is the unique winner in subelection (C, V_2). So the final-stage election is $(\{c, w\}, V)$ and we have that w has the following level 1 score:

$$score^1_{(\{c,w\},V)}(w) = k + n + 1$$

Thus, w is the unique level 1 winner of the final election and c has been successfully prevented from being a Bucklin winner by partition of voters in model TE.

If: Assume that c can be prevented from being a Bucklin winner by partition of voters in model TE. From Lemma 3.27 we know that candidate c must participate in the final-stage election. Since we are in model TE, at most two candidates participate in the final run-off. To prevent c from being a Bucklin winner of the final election, there must be another finalist beating c in a two-candidate election and w is the only candidate capable of that. Let us say

[4] If $\|B'\| < k$, add arbitrarily chosen voters from the first group besides those corresponding to B' such that we have in total k voters from this group.

that c is the unique winner of subelection (C,V_2) and w is the unique winner of subelection (C,V_1). For w to be the unique winner of subelection (C,V_1), V_1 has to contain voters from the first voter group and w can win only on the $(n+5)^{th}$ level: In particular, x is placed before w in all voter groups except the first, so w can win in (C,V_1) only via voters from the first voter group participating in (C,V_1). Moreover, since w is placed in the last or second-to-last position in all voters from the third and fourth groups, and since there is only one voter in the second group, w can win only on the $(n+5)^{th}$ level (which is w's position in the votes from the first voter group).

Let $I \subseteq \{1,\dots,n\}$ be the set of indices i such that first-group voter

$$F_i \;\; (B-\mathcal{N}[b_i]) \;\; H_i \;\; y \;\; w \;\; (\mathcal{N}[b_i] \cup D \cup E \cup (F-F_i) \cup (H-H_i)) \;\; u \;\; v \;\; c \;\; x$$

belongs to V_1. Let $\ell = \|I\|$. Since w is the unique level $n+5$ BV winner of subelection (C,V_1) but y is placed before w in every vote in the first group, the one voter from the second group (which is the only voter who prefers w to y) must belong to V_1. Thus we know that

$$score^{n+5}_{(C,V_1)}(w) = \ell+1 \quad \text{and} \quad score^{n+4}_{(C,V_1)}(y) = score^{n+5}_{(C,V_1)}(y) = \ell.$$

For the candidates in B, we have

$$score^{n+4}_{(C,V_1)}(b_j) = score^{n+5}_{(C,V_1)}(b_j) = 1 + \|\{b_i \,|\, i \in I \text{ and } b_j \notin \mathcal{N}[b_i]\}\|,$$

since each b_j scores one point up to the $(n+4)^{th}$ level from the voter in the second group and one point from the first group for every b_i with $i \in I$ such that $b_j \notin \mathcal{N}[b_i]$ in graph G. Again, since w is the unique level $n+5$ BV winner of subelection (C,V_1), no $b_j \in B$ can score a point in *each* of the ℓ votes from the first voter group that belong to V_2. This implies that for each $b_j \in B$ there has to be at least one b_i with $i \in I$ that is adjacent to b_j in G. Thus, the set B' of candidates b_i with $i \in I$ corresponds to a dominating set in G.

Recall that $score^{n+5}_{(C,V_1)}(w) = \ell+1$ and $score^{n+4}_{(C,V_1)}(y) = \ell$ and furthermore, for $1 \le j \le n$, $score^{n+4}_{(C,V_1)}(b_j) \le \ell$ holds. Since w needs a strict majority to be a BV winner in subelection (C,V_1), it must hold that $maj(V_1) \le \ell+1$. y and the $b_j \in B$ have a score of ℓ already one level earlier than w, so it must hold that $maj(V_1) = \ell+1$, which implies $\|V_1\| = 2\ell$ or $\|V_1\| = 2\ell+1$. To ensure this cardinality of V_1, other votes have to be added. Since y must not gain additional points from these votes up to the $(n+5)^{th}$, they cannot come from the fourth voter group. The remaining votes from the third and fifth voter group total up to k. Thus, since w is the unique Bucklin winner in subelection (C,V_1), it must hold that $\ell \le k$ and we have $\|B'\| = \ell \le k$. ❑

This directly leads, together with Lemma 3.15, to the following corollary.

Corollary 3.29 *Fallback voting is resistant to destructive control by partition of voters in model TE for both winner models.*

Finally, we come to the destructive case of voter partition in model TP. We will show that fallback voting is resistant to this type of control in both winner models. Unfortunately, for

Bucklin elections, the complexity of this problem is still unknown.

Construction 3.30 *Let $(B, \mathscr{S}, k)$ be a given instance of* RHS*, with the set $B = \{b_1, b_2, \dots, b_m\}$, the collection $\mathscr{S} = \{S_1, S_2, \dots, S_n\}$ of nonempty subsets $S_i \subseteq B$ such that $n > m$, and an integer k with $1 < k < m$. Define the election (C, V), where $C = B \cup \{c, f, w\} \cup D \cup E$ is the candidate set with padding candidates $D = \{d_1, \dots, d_{2(m+1)}\}$ and $E = \{e_1, \dots, e_{2(m-1)}\}$(recall Footnote 2 on page 47). The candidates in D ensure that w is always placed at the third position in the votes of the fourth voter group of V below. The collection of voters V consists of the following $2n(k+1) + 4m + 2mk$ voters displayed in Table 3.21. For the sake of readability we will state the preference of the voters by only giving the ranking of the approved candidates.*

Group	For each ...	# of voters	preference
(1)	$i \in \{1, \dots, n\}$	$k+1$	$w\, S_i\, c$
(2)	$j \in \{1, \dots, m\}$	1	$c\, b_j\, w$
(3)	$j \in \{1, \dots, m\}$	$k-1$	b_j
(4)	$p \in \{1, \dots, m+1\}$	1	$d_{2(p-1)+1}\, d_{2p}\, w$
(5)	$r \in \{1, \dots, 2(m-1)\}$	1	e_r
(6)		$n(k+1) + m - k$	c
(7)		$mk + k - 1$	$c\, w$
(8)		1	$w\, c$
(9)		1	$c\, f\, w$

Table 3.21: Voter list V in Construction 3.30

The strict majority threshold for V is $maj(V) = n(k+1) + 2m + mk + 1$. In election (C, V), only the two candidates c and w reach a strict majority, w on the third level and c on the second level (see Table 3.22). Thus c is the unique level 2 fallback winner of election (C, V).

	c	w	$b_j \in B$
score^1	$n(k+1) + 2m + mk$	$n(k+1) + 1$	$k-1$
score^2	$\mathbf{n(k+1) + 2m + mk + 1}$	$n(k+1) + mk + k$	$\leq k + n(k+1)$
score^3	$\leq 2n(k+1) + 2m + mk + 1$	$n(k+1) + 2m + mk + k + 2$	$\leq k + n(k+1)$
score^{m+2}	$2n(k+1) + 2m + mk + 1$	$n(k+1) + 2m + mk + k + 2$	$\leq k + n(k+1)$

Table 3.22: Level i scores for $i \in \{1, 2, m+2\}$ in the election $(C - (D \cup E), V)$ from Construction 3.30

Lemma 3.31 will be used in the proof of Theorem 3.32.

Lemma 3.31 *In the election (C, V) from Construction 3.30, for every partition of V into V_1 and V_2, candidate c is a fallback winner of election (C, V_1) or (C, V_2).*

Proof. For a contradiction, suppose that in both subelections, (C, V_1) and (C, V_2), candidate c is not a fallback winner. Since $score^1_{(C,V)}(c) = \|V\|/2$, the two subelections must satisfy that

- both $\|V_1\|$ and $\|V_2\|$ are even numbers, and
- $score^1_{(C,V_1)}(c) = \|V_1\|/2$ and $score^1_{(C,V_2)}(c) = \|V_2\|/2$.

Otherwise, c would have a strict majority already on the first level in one of the subelections and would win that subelection. For each $i \in \{1,2\}$, c already on the first level has only one point less than the strict majority threshold $maj(V_i)$ in subelection (C,V_i), and c will get a strict majority in (C,V_i) no later than on the $(m+2)^{nd}$ level. Thus, for both $i=1$ and $i=2$, there must be candidates whose level $m+2$ scores in (C,V_i) are higher than the level $m+2$ score of c in (C,V_i). Table 3.22 shows the level $m+2$ scores of all candidates in (C,V). Only w and some $b_j \in B$ have a chance to beat c on that level in (C,V_i), $i \in \{1,2\}$.

Suppose that c is defeated in both subelections by two distinct candidates from B (say, b_x defeats c in (C,V_1) and b_y defeats c in (C,V_2)). Thus the following must hold:[5]

$$\begin{aligned} score^{m+2}_{(C,V_1)}(b_x) + score^{m+2}_{(C,V_2)}(b_y) &\geq score^{m+2}_{(C,V)}(c) + 2 \\ 2n(k+1) + 2k - n(k+1) &\geq 2n(k+1) + mk + 2m + 3 \\ 2k &\geq n(k+1) + mk + 2m + 3, \end{aligned}$$

which by our basic assumption $m > k > 1$ implies the following contradiction:

$$0 \geq n(k+1) + (m-2)k + 2m + 3 > n(k+1) + (k-2)k + 2k + 3 = n(k+1) + k^2 + 3 > 0.$$

Thus the only possibility for c to not win any of the two subelections is that c is defeated in one subelection, say (C,V_1), by a candidate from B, say b_x, and in the other subelection, (C,V_2), by candidate w. Then it must hold that (again, see Footnote 5 on page 62):

$$\begin{aligned} score^{m+2}_{(C,V_1)}(b_x) + score^{m+2}_{(C,V_2)}(w) &\geq score^{m+2}_{(C,V)}(c) + 2 \\ 2n(k+1) + 2k + 2m + mk + 2 - n(k+1) - 1 &\geq 2n(k+1) + mk + 2m + 3 \\ 2k + 1 &\geq n(k+1) + 3. \end{aligned}$$

Since $n > 1$, this cannot hold, so c must be a fallback winner in one of the two subelections. ❑

Theorem 3.32 *Fallback voting is resistant to destructive control by partition of voters in model TP in both winner models.*

Proof. Containment in NP can be shown by guessing a partition and checking in deterministic polynomial time whether the control action was successful. To prove NP-hardness in the co-winner model, we reduce RHS to our control problem. Consider the election (C,V) constructed according to Construction 3.30 from a given RHS instance $(B,\mathscr{S},k)$, where

[5] For the left-hand sides of the inequalities, note that each vote occurs in only one of the two subelections. To avoid double-counting those votes that give points to both candidates, we first sum up the overall number of points each candidate scores and then subtract the double-counted points.

$B = \{b_1, \ldots, b_m\}$ is a set, $\mathscr{S} = \{S_1, \ldots, S_n\}$ is a collection of nonempty subsets $S_i \subseteq B$, and k is an integer with $1 < k < m < n$.

We claim that $\mathscr{S}$ has a hitting set $B' \subseteq B$ of size k if and only if c can be prevented from being a fallback winner by partition of voters in model TP.

Only if: Suppose, $B' \subseteq B$ is a hitting set of size k for $\mathscr{S}$. Partition V into V_1 and V_2 the following way. Let V_1 consist of those voters of the second group where $b_j \in B'$ and of those voters of the third group where $b_j \in B'$. Let $V_2 = V - V_1$. In (C, V_1), no candidate reaches a strict majority (see Table 3.23), where $maj(V_1) = \lfloor k^2/2 \rfloor + 1$, and candidates c, w, and each $b_j \in B'$ win the election with an approval score of k.

	c	w	$b_j \in B'$	$b_j \notin B'$
score^1	k	0	$k-1$	0
score^2	k	0	k	0
score^3	$\mathbf{k}$	$\mathbf{k}$	$\mathbf{k}$	0

Table 3.23: Level i scores in (C, V_1) for $i \in \{1, 2, 3\}$ and the candidates in $B \cup \{c, w\}$

The level i scores in election (C, V_2) for $i \in \{1, 2, 3\}$ and the candidates in $B \cup \{c, w\}$ are shown in Table 3.24.

	c	w	$b_j \notin B'$	$b_j \in B'$
score^1	$n(k+1)+2m-k+mk$	$n(k+1)+1$	$k-1$	0
score^2	$n(k+1)+2m-k+mk+1$	$n(k+1)+mk+k$	$\leq k+n(k+1)$	$\leq n(k+1)$
score^3	$\geq n(k+1)+2m-k+mk+1$	$n(k+1)+mk+2m+2$	$\leq k+n(k+1)$	$\leq n(k+1)$

Table 3.24: Level i scores in (C, V_2) for $i \in \{1, 2, 3\}$ and the candidates in $B \cup \{c, w\}$

Since in (C, V_2) no candidate from B wins, the candidates participating in the final round are $B' \cup \{c, w\}$. The scores in the final election $(B' \cup \{c, w\}, V)$ can be seen in Table 3.25 and we see that w is the unique level 2 fallback winner, thus candidate c has been prevented from being a fallback winner by partition of voters in model TP.

	c	w	$b_j \in B'$
score^1	$n(k+1)+2m+mk$	$n(k+1)+m+2$	$k-1$
score^2	$n(k+1)+2m+mk+1$	$\mathbf{n(k+1)+2m+mk+2}$	$\leq k+n(k+1)$

Table 3.25: Level i scores in the final-stage election $(B' \cup \{c, w\}, V)$ for $i \in \{1, 2\}$

If: Suppose candidate c can be prevented from being a fallback winner by partition of voters in model TP. From Lemma 3.31 it follows that candidate c participates in the final round. Since c has a strict majority of approvals, c has to lose against another candidate by a strict majority at some level. Only candidate w has a strict majority of approvals, so w has to beat c at some level in the final round. Because of the low scores of f and the candidates in D and E we may assume that only candidates from B are participating in the final round besides c and w. Let $B' \subseteq B$ be the set of candidates who also participate in the final round. Let ℓ be the number of sets in $\mathscr{S}$ not hit by B'. As w cannot reach a strict majority of approvals on the first level, we consider the level 2 scores of c and w:

$$\begin{aligned} score^2_{(B' \cup \{c,w\},V)}(c) &= n(k+1) + 2m + mk + 1 + \ell(k+1), \\ score^2_{(B' \cup \{c,w\},V)}(w) &= n(k+1) + 2m + mk + k - \|B'\| + 2. \end{aligned}$$

Since c has a strict majority already on the second level, w must beat c on this level, so the following has to hold:

$$\begin{aligned} score^2_{(B' \cup \{c,w\},V)}(c) - score^2_{(B' \cup \{c,w\},V)}(w) + 1 &\leq 0 \\ n(k+1) + 2m + mk + 1 + \ell(k+1) - n(k+1) - 2m - mk - k + \|B'\| - 2 + 1 &\leq 0 \\ \|B'\| - k + \ell(k+1) &\leq 0. \end{aligned}$$

This is possible only if $\ell = 0$ (i.e., all sets in $\mathscr{S}$ are hit by B'), which implies $\|B'\| \leq k$. ❑

3.2.3 An Experimental Analysis in Bucklin, Fallback, and Plurality Elections

In this section we present our experimental analysis of standard control problems in Bucklin and fallback voting which we complement with results we obtained for plurality voting. Note that our experimental study focuses on the control problems in the unique-winner model. After giving the general experimental setup, the results will be presented in general. The following discussion focuses on the most significant findings. The complete documentation of all results can be found in the 370-page technical report by Rothe and Schend [RS12b]. We conclude the analysis with a discussion on the confinements the experiments underly and possibilities of extending and improving the taken approach.

Experimental Setup To fully describe the setup of the conducted experiments we have to introduce the three main parts:

1. The chosen parameters confining the experiments to allow their realization within a reasonable time frame.
2. The sampling of the randomly generated elections that serve as an input for the algorithms.
3. The algorithms used to solve the given control problems.

Coping with NP-hard problems, the design of our experiments aims at optimizing the tradeoff between expressiveness and feasibility with respect to time constraints. The approach taken in tackling this task (see the high-level description of the algorithm later on for more detail) is to exhaustively test every possible control action up to a predefined size.

For the case of adding/deleting candidates or voters, this predefined size is $k = \lfloor \mathfrak{m}/3 \rfloor$ and $k = \lfloor \mathfrak{n}/3 \rfloor$, respectively, where $\mathfrak{m}$ is the number of candidates and $\mathfrak{n}$ is the number of voters of the given election, and k is the parameter given in the control instance (recall the definitions of the control problems from Section 3.2). On the positive side, we have that a yes-instance for a given k is also a yes-instance for each $k' \geq k$, so the number of yes-instances found in our experiments for smaller k directly transfers to instances with bigger values of k. On the negative side, if no successful control actions could be found for a given k, we cannot make conclusions for the same election with a bigger value of k.

Furthermore we implement a limit of 600 seconds stopping the computation when the limit is exceeded. Instances for which the computation is aborted have the specified output "timeout" separating them from instances in which all possibilities have been tested and no successful control action has been found. In our experiments we implemented the same timeout value for all investigated types of control. As our results will show, the different control types react differently to this constant timeout threshold, so a tuning of the timeout-parameter would be an interesting issue for further experiments. Also, varying the timeout value depending on the election size at hand might be an interesting approach.

The randomly generated elections (C,V) serving as an input for our algorithms have $\mathfrak{m} = \|C\|$ candidates and $\mathfrak{n} = \|V\|$ voters, where the values are chosen from $\{4, 8, 16, 32, 64, 128\}$. In the adding-candidates and adding-voters scenarios, the spoiler sets D and V' have the same size as the set of registered candidates and voters, respectively; i.e., $\|D\| = \|C\|$ and $\|V'\| = \|V\|$, and they are generated with the same distribution model as the registered voters. Each combination of $\mathfrak{n}$ and $\mathfrak{m}$ is one data point for which we evaluated 500 of these elections, trying to determine for each given election whether or not control is possible, and if it is possible, we say that this election is *controllable*. This restriction to 500 elections per data point, again, results from practical issues balancing out manageability and informative value of the experiments conducted.

The algorithms and data-generation programs are implemented in *Octave 3.2* and the experiments were run on a 2,67 GHz Core-I5 750 with 8GB RAM.

Election generation and distribution of voters: There are various ways of generating collections of votes, see, for instance, the work of Berg [Ber85], Mallows [Mal57], and Luce [Luc05]. In our experimental approach, we will define an adaption of the so-called *general Pólya–Eggenberger urn model* (*PE model*) mentioned by [Ber85]. To this end we first have to specify how random votes can be cast depending on the voting system at hand and how many different votes can exist. Since Bucklin and plurality voting both expect the voters to provide a complete ranking of the candidates, we can use the same generation model for both voting systems: Assuming that the generated election has $\mathfrak{m}$ candidates, in Bucklin voting a random vote can be obtained by generating a random permutation over the $\mathfrak{m}$ different candidates, so the overall number of different votes in Bucklin elections is $\mathfrak{m}!$. In fallback voting, random votes can be generated as follows:

- randomly draw a preference p from all $\mathfrak{m}!$ possible preferences with $\mathfrak{m}$ candidates;

- randomly draw a number, say $\ell \in \{0, 1, \ldots, \mathfrak{m}\}$, of approved candidates;
- the generated vote consists of the first ℓ candidates in p.

Thus, there can be $\sum_{\ell=0}^{\mathfrak{m}} \binom{\mathfrak{m}}{\ell} \ell!$ different votes in fallback elections with $\mathfrak{m}$ candidates.

In the PE model, a set of votes is sampled in the following way: Assume that we have an urn containing all possible votes that can be cast given a certain voting system and let the number of different votes be denoted by t. For Bucklin voting, for example, $t = \mathfrak{m}!$, while for fallback voting we have that $t = \sum_{\ell=0}^{\mathfrak{m}} \binom{\mathfrak{m}}{\ell} \ell!$, as explained above. To sample an electorate consisting of $\mathfrak{n}$ votes, we proceed in the following way for a fixed parameter b:

- randomly draw one preference from the given urn—this is the first of the $\mathfrak{n}$ votes that shall be sampled,
- put the preference back into the urn along with b additional copies of it,
- randomly draw the second vote from the new urn,
- put the second vote back into the urn along with b additional copies of it,

  ⋮
- randomly draw the $(\mathfrak{n}-1)^{st}$ vote from the new urn,
- put the $(\mathfrak{n}-1)^{st}$ vote back into the urn along with b additional copies of it,
- randomly draw the $\mathfrak{n}^{th}$ vote from the new urn.

The correlation of the sampled votes strongly depends on the parameter b. By setting $b = 0$, we obtain the *Impartial Culture model* (*IC model*) which samples uniformly distributed votes out of all possible preferences since in each step the just drawn preference is put back into the urn without adding any more preferences.

To sample correlated votes, the usual approach (also employed by, e.g., Walsh [Wal10]) is to use the above model with the parameter $b = t$. This means that when the first preference is drawn from the urn, it is put back into the urn along with t additional copies, leading to a probability of 0.5 that the second preference will be the same as the first one and, depending on the preferences that are drawn in each step, this effect can be intensified during the sampling process. Thus, there is a relatively high probability that many (or even all) sampled votes can be identical. In the setting of manipulation, where the preferences of the manipulators can be freely set independently of the nonmanipulators' preferences, this effect has less impact while in the control scenarios considered in this work, identical preferences of the voters (including e.g., unregistered votes that may be added), would artificially make control impossible or easy to find and thus would trivialize the problem.

Therefore, we introduce the *Two Mainstreams model* (*TM model*), which is the following adaption of the PE model:

- depending on the voting system, randomly draw two preferences out of an urn containing all possible, say t, preferences—recall that either $t = \mathfrak{m}!$ (for Bucklin) or $t = \sum_{\ell=0}^{\mathfrak{m}} \binom{\mathfrak{m}}{\ell} \ell!$ (for fallback);
- put each preference back into the urn with t additional copies;
- draw the votes out of this urn independently at random with replacement.

Each of the two preferences drawn in the first step can be interpreted as a representative of one "mainstream" in society (e.g., liberal and conservative).

High-level description of the algorithms: Our algorithms are greedy heuristics, designed so as to test the most "promising" cases (depending on the control type at hand) first, by using appropriate preorderings. We only provide a high-level description. All algorithms for the different types of control share the same essential method of testing various subsets, and they differ only in the type of preordering and internal testing. Before actually searching for a successful sublist of voters or subset of candidates, the algorithms check conditions that, if true, indicate that the given instance is a no-instance. Let c be the designated candidate in the control problems defined in Section 3.2. Depending on the control type, some of the following conditions are tested:

Condition 1 (applied to all constructive cases): The designated candidate is ranked last (for Bucklin), or is ranked last or disapproved (for fallback), in every vote.

Condition 2 (applied to control by deleting voters): For each $k' \leq k$, determine the smallest i and j such that

$$score^{i}_{(C,V)}(c') \geq \lfloor (\|V\|-k')/2 \rfloor + 1 + k' \text{ and } score^{j}_{(C,V)}(c) \geq \lfloor (\|V\|-k')/2 \rfloor + 1$$

hold for $c' \in C - \{c\}$. Note that $i \leq j-1$ for all $k' \leq k$.

Condition 3 (applied to control by adding voters): For each $k' \leq k$ determine the smallest i and j such that

$$score^{i}_{(C,V)}(c') \geq \lfloor (\|V\|+k')/2 \rfloor + 1 \text{ and } score^{j}_{(C,V)}(c) \geq \lfloor (\|V\|+k')/2 \rfloor + 1 - k'$$

hold for $c' \in C - \{c\}$. Note that $i \leq j-1$ for all $k' \leq k$.

Condition 4 (applied to all destructive cases): In the given election, the winner has a strict majority on the first level already.

For both constructive control by adding and deleting voters, Condition 1 is tested. Note that for the adding voter cases this condition has to hold for both voter lists, the registered voters and the unregistered voters.

For constructive control by deleting voters, Condition 2 is additionally tested. If Condition 2 holds, c cannot be made a unique winner by deleting at most k voters because even if all k voters would harm the strongest rival c' of c the most and c not at all, the rival would still reach a strict majority on a smaller level than c.

For constructive control by adding voters, Condition 1 and 3 are tested. If Condition 3 holds for the given election and the given distinguished candidate c, then even if all added voters helped only c on the smallest level, there would still be at least one other candidate reaching a strict majority on a level smaller than c.

For the voter-partition cases, we have Condition 4 indicating that control is not possible for both the constructive and destructive case, namely that in the given election there is a unique winner on the first level. It is easy to see that for every possible partition (V_1, V_2) of V a level 1 winner is also a level 1 winner in at least one of the subelections. Since level 1 winners are always unique, independent of the tie-handling model, this candidate always participates in the run-off and will therefore always be the unique level 1 winner of the resulting two-stage election. So no distinguished candidate can ever be made the unique winner by partitioning the voters. So the algorithms for destructive and constructive control by partition of voters first check Condition 4 where the latter checks Condition 1 as well.

The algorithms for the candidate control scenarios test Condition 1 in the constructive cases except where the candidates are partitioned. For the destructive cases, on the other hand, Condition 4 is always tested. Note that for the adding candidates cases both conditions must hold in the election over both the registered and the unregistered candidates.

After having excluded these trivial cases, each of the algorithms searches for a successful sublist/subset of preordered versions of V or C. Let us describe this procedure only for constructive control by deleting voters in detail. In this case, the voters are preordered ascendingly for c; that is, after the preordering v_1 is a voter ranking c worst and $v_\mathfrak{n}$ is a voter ranking c best among all voters. (In fallback voting, the "worst" position for a candidate is to be not approved at all.) The algorithm now starts with deleting those votes c benefits least of. It follows the procedure of a depth-first search on a tree of height k that is structured as shown in Figure 3.1. In each node, it is tested whether deleting the votes on the path is a successful control action. For example, on path $s \to 1 \to 2 \to 3$ the algorithm tests the sublists $(v_1),(v_1,v_2),(v_1,v_2,v_3)$ and then tracks back testing the sublists $(v_1,v_2,v_4),(v_1,v_2,v_5)$, $(v_1,v_3),(v_1,v_3,v_4)$, and so on. The branches on the left side are visited first and, due to the preordering of the votes, these are the votes c benefits least of.

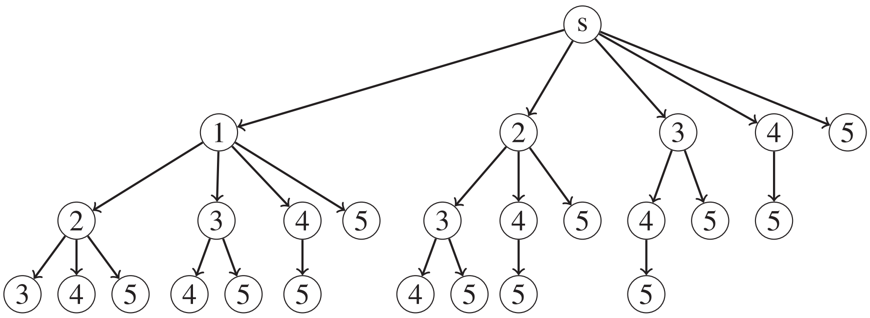


Figure 3.1: Tree for $\mathfrak{n} = 5$ voters where up to $k = 3$ voters may be deleted, a node i corresponds to voter v_i after the preordering

For the adding-voters cases, the unregistered voters are ordered in a descending order for the designated candidate, and the algorithm proceeds similarly as the algorithm for the deleting-voters cases. With this preordering, the algorithm first tests those voters the designated candidate can benefit most from when these are added to the voter list.

For the partition-of-voters cases, the algorithm considers every possible sublist of the voter list up to size $k = \lfloor \mathfrak{n}/2 \rfloor$ as V_1, sets $V_2 = V - V_1$, and tests whether this is a successful control action or not. For the constructive cases, the voters are preordered descendingly with respect to the designated candidate, whereas for the destructive control cases no preordering is implemented.

In the candidate control scenarios, the candidates are also ordered with respect to the designated candidate, where a descending order here means that the first candidate has the most voters ranking him or her before the designated candidate and the last candidate has the fewest voters doing so. An ascending order is defined analogously. Again, in the adding-candidates case, the votes over all candidates (including the spoiler candidates) are considered. A descending ordering is used for finding control actions for constructive control by deleting candidates and for destructive control by adding candidates, whereas for the remaining candidate control cases an ascending order is used.

In the worst case, our algorithms check all possible subsets of size k, so they have a worst-case running time of $\sum_{\ell=1}^{k} \binom{n}{\ell}$ for voter control and $\sum_{\ell=1}^{k} \binom{m}{\ell}$ for candidate control. Finally, note that for each yes-answer, our algorithms also provide the corresponding successful control action.

Summary of Experimental Results Table 3.26 summarizes our experimental results on control in Bucklin, fallback, and plurality voting. We investigated the three voting systems only for those control types they are not known to be vulnerable to, which is indicated by an R*-, R-, or an S-entry in Tables 3.7 and 3.6. That is, destructive control by adding and by deleting voters (DCAV and DCDV) are omitted in Table 3.26 and furthermore the constructive control by deleting and adding voters (CCAV and CCDV) and control by partition of voters in model TE (CCPV-TE and DCPV-TE) is omitted for plurality voting. Also, since our algorithms use the parameter k bounding the number of candidates to be added, constructive and destructive control by adding an unlimited number of candidates (CCAUC and DCAUC) are not considered either.

For both Bucklin and fallback voting we tested for each combination of any of the remaining 18 control types and any of the two distribution models (IC and TM), a total of $18,000 = 36 \cdot 500$ elections, varying over the 36 data points with different values for m and n, as explained above. This gives a total of $1,296,000 = 18 \cdot 4 \cdot 18,000$ generated and tested elections. For plurality voting we tested a total of 14 control types, each in both distribution models for 36 different election sizes, which results in $504,000 = 14 \cdot 2 \cdot 500 \cdot 36$ tested elections.

The tables give an overview of the percentage of timeouts for each such combination of control type/voting system/distribution model, and also the minimal and maximal percentage of yes-instances observed. We do not discuss the results for all these cases in detail here, but focus on those with the most interesting findings. For those cases that we discuss in detail, we provide plots giving the percentage of yes-instances, timeouts, and average computational costs for all different election sizes that were tested. Note that a comprehensive presentation of all results containing the above information for all cases (showing 168 plots of experiments in total) can be found in the appendix of the 370-page technical report by Rothe and Schend [RS12b]. Note that Table 3.26 lists the results separately for destructive control by partition of candidates as the result by Hemaspaandra et al. [HHM13] establishing $\mathscr{E}$-DCRPC-TE $=$ $\mathscr{E}$-DCPC-TE in the unique-winner model was published after the experiments were conducted.

Results for Adding and Deleting Voters We start with constructive control by adding and deleting voters. As plurality voting is vulnerable to these types of control, only Bucklin and fallback elections are tested. We discuss the results for the deleting voters case only, since

	FV			BV			PV		
Problem	**min**	**max**	**to**	**min**	**max**	**to**	**min**	**max**	**to**
CCAC	1 / **0**	11 / **7**	51 / **50**	0 / **0**	23 / **11**	50 / **49**	0 / **0**	20 / **3**	50 / **34**
DCAC	53 / **39**	92 / **71**	11 / **14**	71 / **42**	99 / **77**	6 / **12**	70 / **47**	99 / **60**	7 / **25**
CCDC	13 / **15**	33 / **36**	37 / **37**	13 / **17**	58 / **45**	34 / **37**	5 / **22**	66 / **40**	37 / **35**
DCDC	8 / **12**	78 / **63**	15 / **22**	48 / **25**	99 / **77**	7 / **18**	7 / **4**	99 / **50**	10 / **35**
CCPC-TE	0 / **0**	19 / **18**	62 / **64**	1 / **0**	57 / **37**	57 / **62**	0 / **0**	60 / **21**	58 / **65**
DCPC-TE	8 / **16**	88 / **65**	18 / **29**	49 / **29**	100 / **78**	10 / **23**	1 / **2**	100 / **59**	22 / **41**
CCPC-TP	1 / **0**	17 / **17**	62 / **64**	1 / **0**	60 / **38**	57 / **61**	0 / **0**	64 / **24**	58 / **65**
DCPC-TP	8 / **16**	87 / **61**	18 / **29**	49 / **29**	100 / **82**	9 / **23**	1 / **3**	100 / **59**	22 / **44**
CCRPC-TE	1 / **1**	18 / **14**	62 / **63**	1 / **0**	60 / **45**	57 / **62**	0 / **0**	65 / **19**	50 / **63**
DCRPC-TE	8 / **16**	86 / **68**	20 / **29**	46 / **29**	100 / **84**	9 / **23**	25 / **14**	100 / **61**	12 / **37**
CCRPC-TP	1 / **0**	19 / **14**	62 / **63**	1 / **1**	56 / **25**	53 / **61**	0 / **0**	65 / **21**	50 / **61**
DCRPC-TP	8 / **16**	85 / **68**	21 / **29**	45 / **27**	100 / **81**	10 / **23**	23 / **14**	100 / **61**	13 / **35**
CCPV-TP	1 / **1**	53 / **20**	40 / **50**	1 / **0**	72 / **23**	31 / **48**	0 / **0**	54 / **13**	24 / **23**
DCPV-TP	37 / **27**	100 / **87**	6 / **17**	60 / **39**	100 / **88**	3 / **10**	55 / **15**	100 / **59**	4 / **35**
CCPV-TE	2 / **0**	97 / **34**	9 / **45**	2 / **0**	98 / **32**	8 / **44**	n.i.	n.i.	n.i.
DCPV-TE	50 / **34**	100 / **88**	4 / **16**	64 / **40**	100 / **89**	4 / **10**	n.i.	n.i.	n.i.
CCDV	2 / **1**	97 / **39**	16 / **12**	2 / **1**	100 / **42**	11 / **7**	n.i.	n.i.	n.i.
CCAV	4 / **1**	99 / **41**	13 / **13**	2 / **1**	99 / **41**	11 / **6**	n.i.	n.i.	n.i.

- **min** and **max**: minimal and maximal percentage of yes-instances observed in all tested instances for the given control type, including those elections where timeouts occurred;
- **to**: percentage of timeouts that occurred for the total of 18,000 elections tested in this control case;
- **numbers in boldface**: elections generated in the TM model;
- **n.i.**: not investigated.

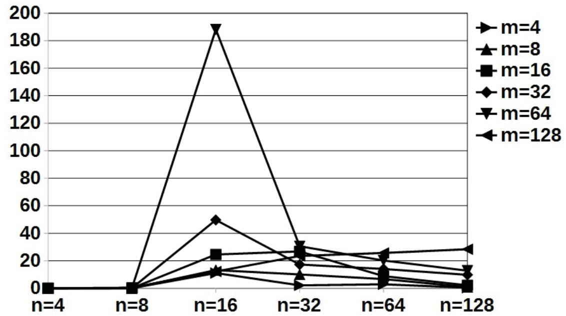
Table 3.26: Overview of experimental results on control in Bucklin and fallback voting

those for control by adding voters are very similar, in both Bucklin and fallback voting. Figure 3.2 shows the results for control by deleting voters for Bucklin voting in the IC model and in detail we have the percentage of yes-instances in Figure 3.2b, where the highest percentage of 100% and the lowest percentage of 2% can also be seen in the “max” and “min” column in Table 3.26. Figure 3.2c gives the detailed occurrence of timeouts for the different election sizes and Figure 3.2a shows the average time needed to determine whether a given Bucklin election generated under the IC model can be controlled by deleting voters or not. Note that in the latter figure the average values do not consider those elections where the algorithm exceeded the time limit of 600 seconds.

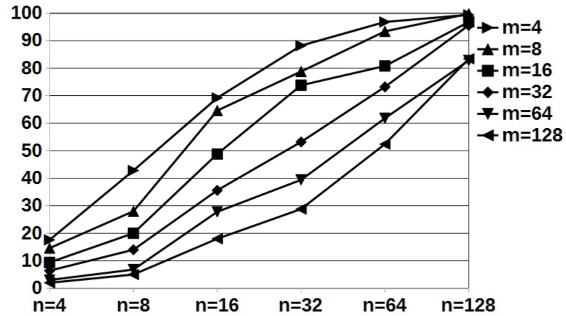
In the IC model, increasing the number of candidates decreases the number of yes-instances in the generated Bucklin elections. On the other hand, the number of yes-instances increases as the number of voters grows. In the TM model, the same correlations can be observed but here, again, the total number and percentage of yes-instances is smaller than in the IC model.

Fallback voting behaves very similarly, so for both distributions and both voting systems increasing the number of candidates makes successful control actions by deleting voters less likely.

In both voting systems and in both distribution models, timeouts occur whenever the number



(a) Average time (seconds) the algorithm needs to give a definite output, timeout-instances excluded.



(b) Percentage of yes-instances.

$m \backslash n$	4	8	16	32	64	128
4	0	0	0	10	3	1
8	0	0	0	20	7	0
16	0	0	0	23	19	3
32	0	0	0	39	27	4
64	0	0	1	49	38	17
128	0	0	31	53	47	17

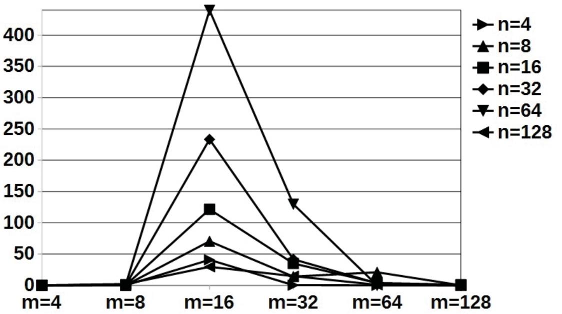
(c) Percentage of timeouts.

Figure 3.2: Bucklin voting in the IC model for CCDV, n is the number of votes, m is the number of candidates

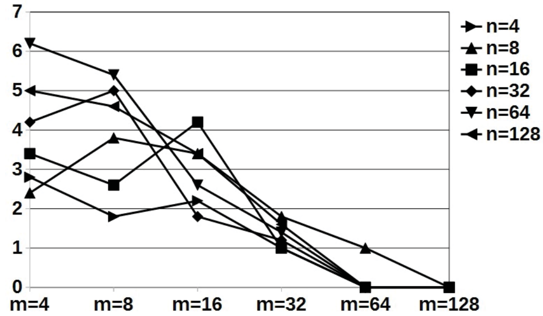
of voters exceeds 32. If the number of candidates is 128, we have timeouts already with 16 voters. This can also be seen in the development of the computational costs shown in Figure 3.2a after the peak for $n = 16$. For larger electorates, the average computational costs drop, since the number of timeouts increases as the number of no-instances diminishes.

Results for Adding and Deleting Candidates Among these four control types, constructive control by adding candidates has the most timeouts. For those election sizes where no timeouts occur (i.e., where the determination of yes- or no-instances is successful), we have that not many elections can be controlled successfully in either of the two voting systems. In Figure 3.3, we see the results for fallback voting in the TM model, exemplifying in Figure 3.3b the low number of yes-instances for this type of control. For example, in the "max" column in Table 3.26, the highest percentage of controllable fallback elections is 11% in the IC model and only 7% in the TM model. Figure 3.3d gives the detailed occurrence of timeouts for the different election sizes and Figure 3.3a shows the average time needed to determine whether a given fallback election generated under the TM model can be controlled by adding candidates or not. Remember that in the latter figure the average values do not consider those elections where the algorithm exceeded the time limit of 600 seconds.

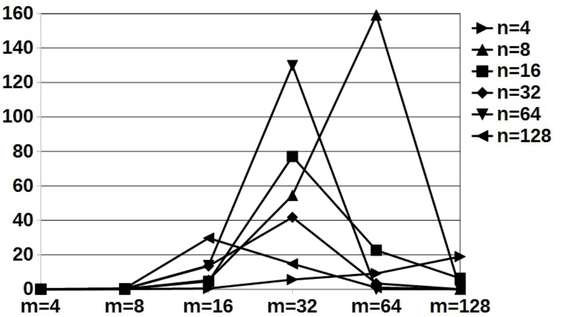
Together with the timeout table we can see in Figure 3.3b that in elections with up to 16 candidates the number of non-controllable elections is very high and increases as the number of candidates increases. When more than 16 candidates participate in an election the number of



(a) Average time (seconds) the algorithm needs to give a definite output, timeout-instances excluded.



(b) Percentage of yes-instances.



(c) Average time (seconds) the algorithm needs to find a yes-instance.

$n \backslash m$	4	8	16	32	64	128
4	0	0	0	83	81	81
8	0	0	0	93	92	93
16	0	0	0	98	99	98
32	0	0	0	99	99	100
64	0	0	0	99	100	100
128	0	0	97	98	100	100

(d) Percentage of timeouts.

Figure 3.3: Fallback voting in the TM model for CCAC, n is the number of votes, m is the number of candidates

no-instances diminishes as drastically as the timeout rate grows. Looking at the computational costs in Figure 3.3a we can see this in the peaks for $m = 16$. Since by design our algorithm needs generally more time to determine that an instance is a no-instance than finding a yes-instance, the high number of no-instances for 16 candidates inflates the average computing time. Knowing that the average computing time for finding yes-instances is not particularly high for this type of control, see Figure 3.3c, we might conjecture that for the bigger election sizes the instances where no distinction could be made by our algorithm are no-instances rather than yes-instances. This indicates that this control type presumably has the lowest overall number of yes-instances. Thus, even for small election sizes, this type of control seems to be hard to exert successfully.

Bucklin elections behave similarly, but yield more yes-instances: Up to 23% of the elections are controllable in the IC model and up to 11% in the TM model. In general, comparing the results in the IC model with those in the TM model, we see similar tendencies in both models, but the overall number and percentage of controllable elections is lower in the TM than in the IC model, for both Bucklin and fallback voting. Plurality voting shows similar results as fallback voting with at most 20% yes-instances in the IC model and less than 4% in the TM model for those election sizes where no timeouts occur.

Constructive control by deleting candidates can be exerted successfully in at most one third of the generated fallback elections, independently of the distribution model the elections are generated with. In Bucklin voting, on the other hand, the overall number and fraction of controllable elections is again higher than in fallback voting, in both distribution models. Up to 58% of the tested elections are yes-instances in the IC model. In the TM model, however, we have with at most 45% of controllable elections fewer than in the IC model. Plurality voting shows a similar percentage of timeouts as Bucklin and fallback voting while the maximum numbers of controllable elections are higher than in the other two voting systems. For $\mathfrak{m} = 16$ they reach a peak where two thirds of the instances are yes-instances.

The results for destructive control by adding or deleting candidates show a quite different picture than in the constructive cases, in both voting systems. The total number and fraction of controllable elections is considerably higher than in the constructive analogue, where Bucklin elections generated with the IC model show the highest number (and a percentage of 99%) of controllable elections, and a percentage of 77% in the TM model, for both control by adding and by deleting candidates. Fallback voting has again fewer yes-instances than Bucklin voting, up to 78% in the IC model and up to 63% in the TM model for destructive control by deleting candidates, and up to 92% (IC) and 71% (TM) for destructive control by adding candidates.

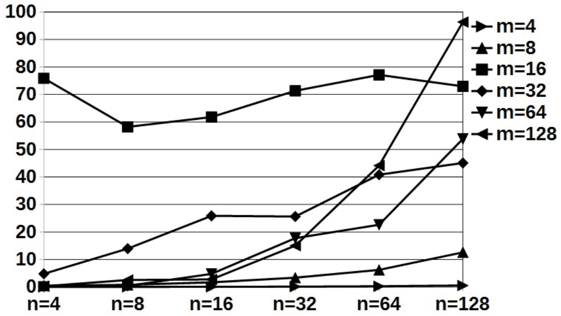
In the tested plurality elections generated with the IC model, similarly to Bucklin voting, more than 70% and nearly up to 100% are controllable. In the TM model, roughly between 50% and 60% of yes-instances are found for those election sizes where no timeouts occur, so between 40% and 50% of these plurality elections are not controllable. In this control scenario, for about 46% of the elections no definite output is given in the constructive case, whereas in only about 8% of the elections timeouts occur in the destructive case.

Results for Partition of Candidates The four cases of constructive control by partition of candidates show the highest number of timeouts out of all control cases. For all three election systems the percentage of timeouts per data point drastically increases from formerly 0% to values higher than 50% when the number of candidates reaches 16, and grows even up to 100% for $\mathfrak{m} \geq 64$. These values do not allow us to draw conclusions from these results. We conjecture that the taken greedy approach is not suitable for these control cases.

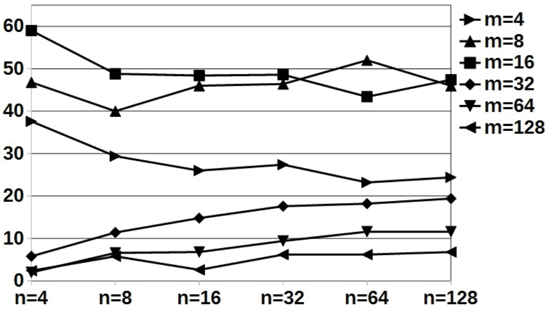
For the destructive cases, on the other hand, the number of timeouts is still higher than for other destructive control types, but low enough such that further analyzing these results is of interest. Summarizing the results briefly, we see that in Bucklin and plurality voting in the IC model, up to 100% of elections of a given size are controllable by our algorithms while fallback voting shows a smaller number of controllable elections. Figure 3.4 shows the results in plurality voting in the TM model.

Here we see that, as for other control types, the overall numbers of yes-instances is lower than in the IC model, but still up to nearly 60% of the tested elections are controllable.

Note that, as we stated in Section 3.2.1, in the unique-winner model $\mathscr{E}$-DCRPC-TE = $\mathscr{E}$-DCPC-TE holds. This result was published after the experiments were conducted, so we have separate results for these control types. Table 3.26, however, shows strikingly similar results for these two cases.



(a) Average time (seconds) the algorithm needs to give a definite output, timeout-instances excluded.



(b) Percentage of yes-instances.

$\mathfrak{m}\backslash\mathfrak{n}$	4	8	16	32	64	128
4	0	0	9	56	64	62
8	0	0	21	60	69	73
16	0	0	31	72	78	89
32	0	0	37	74	85	93
64	0	0	49	78	86	93
128	0	0	45	77	88	93

(c) Percentage of timeouts.

Figure 3.4: Plurality voting in the TM model for DCPC-TP, $\mathfrak{n}$ is the number of votes, $\mathfrak{m}$ is the number of candidates

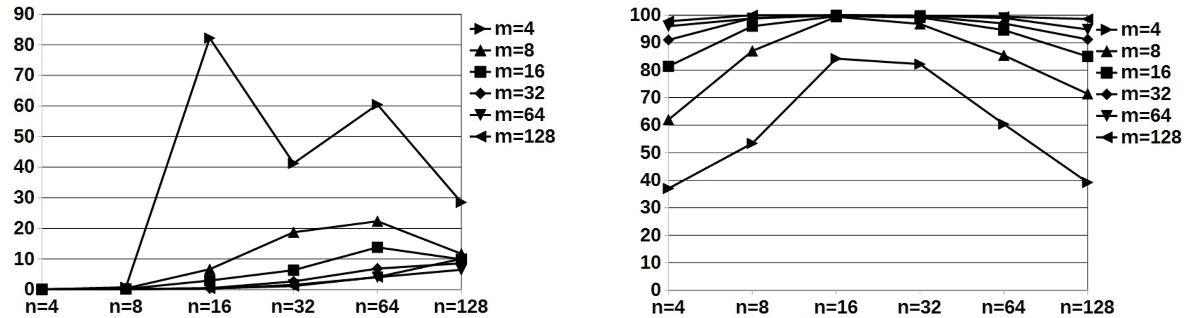
Results for Partition of Voters As mentioned in Section 3.2 control by partition of voters comes in four problem variants, where each case must be investigated separately. We very briefly discuss some observations made for these control types. For constructive control by partition of voters in model TP we made the following observations:

Similarly to control by deleting or by adding voters, the number of controllable elections increases with the number of voters increasing, which was also observed for the partition cases in Bucklin and fallback voting. We have seen that in at most 10% of the tested plurality elections in the TM model a successful control action can be found. Note that no timeouts occur for up to 32 candidates, so more than 90% of the elections tested are demonstrably not controllable in these cases. For both distribution models, plurality elections produce fewer timeouts than the corresponding fallback or Bucklin elections. This suggests that the control problem for plurality voting is easier to solve on average than for fallback or Bucklin voting.

Using the tie-handling model TE instead of TP, in both Bucklin and fallback voting an increase of yes-instances in the constructive cases is evident. By contrast, in the destructive counterparts no significant difference can be observed with respect to the tie-handling rule.

The most striking results are those obtained for the destructive cases. Here we have that, for both tie-handling models in the TM model, the average number of controllable elections is very high; and in the IC model, control is almost always possible, see Figure 3.5.

In light of the fact that for these cases the resistance proofs of Theorems 3.28 and 3.32 tend to be the most involved ones (yielding the most complex instances for showing NP-hardness),



(a) Average time (seconds) the algorithm needs to give a definite output, timeout-instances excluded.

(b) Percentage of yes-instances.

$\mathfrak{m}\backslash\mathfrak{n}$	4	8	16	32	64	128
4	0	0	0	18	40	61
8	0	0	0	3	15	29
16	0	0	0	1	5	15
32	0	0	0	0	3	9
64	0	0	0	0	1	5
128	0	0	0	0	1	1

(c) Percentage of timeouts.

Figure 3.5: Fallback voting in the IC model for DCPV-TP, $\mathfrak{n}$ is the number of votes, $\mathfrak{m}$ is the number of candidates

these results might be surprising at first glance. However, one explanation for the observed results can be found in exactly this fact: The elections constructed in these reductions have a very complex structure which seems to be unlikely to occur in randomly generated elections (at least in elections generated under the distribution models discussed in this paper). Another explanation is that the problems used to reduce from in these proofs tend to be easy to solve for small input sizes, but due to the complexity of the reduction, the resulting elections have many voters/candidates compared to the elections generated for the conducted experiments.

In the destructive cases, the number of timeouts is for all three voting systems the lowest among all control types investigated. In Bucklin elections with uniformly distributed votes and for destructive control by partition of voters in model TP, for only 3.32% of the elections no decision can be made within the time limit. As can be seen in the table, timeouts begin to occur for those elections where the number of voters exceeds 16. But, again, we have to emphasize that these values are very low compared to other types of control. This explains the plateaus all graphs show. On the one hand, increasing the number of voters increases the number of yes-instances. But on the other hand, for more than 16 voters timeouts begin to diminish the fraction of observed yes-instances. Also, the average running time of the algorithm for those instances where the time limit is not exceeded is rather low, compared to other types of control, see Figure 3.5a. The highest computational costs occur for those election sizes where the most no-instances were observed. As expected, in the corresponding constructive cases the number of timeouts is significantly higher and so are the average computational costs.

Discussion of Results Finally, we summarize the main findings of our experiments, which allow a more fine-grained analysis and comparison—across various control scenarios, vote distribution models, and voting systems—than merely stating NP-hardness for all these problems. Obviously, our findings are limited by the experimental setup and, of course, the fact that exponential time seems unavoidable for these problems unless P = NP; thus, our conclusions cannot be generalized unconditionally.

Discussion of Distribution Models: IC versus TM: Comparing the results for the different distribution models, we see that in every voting system for all control types studied (except fallback voting in constructive control by deleting candidates) the overall number of yes-instances is higher in the IC than in the TM model. This may result from the fact that in elections with uniformly distributed votes, all candidates are likely to be approximately equally preferred by the voters. So both constructive and destructive control actions are easier to find by our greedy algorithms. This also explains the observation that the IC model produces fewer timeouts.

Discussion of Constructive versus Destructive Control: For all investigated types of control where both constructive and destructive control was investigated, we found that the destructive control types are experimentally much easier than their constructive counterparts, culminating in almost 100% of controllable elections for certain control types in the IC model. Compare this with the theoretical insight of Hemaspaandra et al. [HHR07] that (unique-winner) destructive control problems disjunctively truth-table-reduce to their (co-winner) constructive counterparts and thus are never harder to solve, up to a polynomial factor (see also the corresponding observation of Conitzer et al. [CSL07] regarding manipulation): In fact, the destructive control cases tend to be even *much* easier in our experiments than their constructive counterparts.

Comparison across Voting Systems: For constructive control, we have seen that fallback and Bucklin voting show similar tendencies and numbers of yes-instances regarding voter control. We also observed that their constructive voter control problems are in general harder to solve than those for plurality voting. In all three voting systems, constructive control by partition of candidates seems to be the hardest control problem investigated, at least for our algorithms, as the most timeouts have occurred in these cases.

Adding Candidates/Voters versus Deleting Candidates/Voters: For fallback and Bucklin voting, we have seen that the results for control by adding voters do not differ significantly from those observed for control by deleting voters, suggesting that both types of control are roughly equally hard. By contrast, comparing control by adding candidates to control by deleting candidates in the constructive case leads to different findings. In both voting systems and both control types, we have small numbers of yes-instances. In the constructive case, however, we observed that the number of yes-instances for control by deleting candidates is significantly higher. These findings are perhaps not overly surprising, since in the voting systems considered here adding candidates to an election can only worsen the position of the designated candidate in the votes. That is, constructive control can be exerted successfully only if by adding candidates rivals of the designated candidate lose enough points so as to get defeated by him or her. This, in turn, can happen only if the designated candidate was already a highly preferred candidate in the original election.

Constructive Voter versus Candidate Control: For fallback and Bucklin voting, we can also compare constructive candidate and voter control directly. In both voting systems and both distribution models, the number of yes-instances for constructive control by adding voters is around four times higher than the number of yes-instances in the corresponding candidate control type, which confirms the argument above, saying that adding candidates cannot push the designated candidate directly. Constructive control by deleting voters can be successfully exerted more frequently when votes are less correlated, whereas the proportion of successful control actions for deleting candidates is about the same for both considered distribution models. The observed differences between these types of voter and candidate control may result from the fact that adding or deleting candidates only shifts the position of the designated candidate, which may not influence the outcome of the election as directly as increasing or decreasing the candidates' scores by adding or deleting voters does. This may explain why voter control can be tackled more easily than candidate control by greedy approaches such as ours.

Concluding Remarks: Reviewing the results obtained from our experiments, we can roughly group the investigated control types in three different categories:

1. For all destructive control cases, we could show that for instances randomly generated with either of the voter distribution models considered here, the control problems are easily solvable by our greedy approach. This suggests that the NP- and W[2]-hardness results from Section 3.2.2 for these cases describe solely the worst-case behavior and do not give information about the complexity for typical instances, assuming the used voter distribution models do give "typical" instances.
   For constructive voter control by adding, deleting, or partitioning in model TE, we have to distinguish between the two types of input instances. For uniformly distributed electorates, we have seen that these control actions can also be easily computed by our greedy approach, whereas for instances with correlated votes the problems become harder to solve. So the complexity of these problems in practice depends immensely on the given instance's structure. These problems cannot be grouped into some specific category, as they fall somewhere between the first and the second category.
2. The second category classifies those problems that are at least for small election sizes efficiently solvable in our setting. This category contains the constructive cases of control by deleting candidates and partition of voters in model TP. For these problems our experiments show that for very small instances the problems are in practice easy to solve, but the worst case that is reflected by the theoretical hardness results is likely to occur even for random instances (according to IC and TM) when their size increases.
3. The remaining cases of constructive candidate control (namely, adding candidates and all variants of partition of candidates) form the third and last category in which we collect those control problems that are hard to solve by our algorithms for all considered instance sizes and structures. For these problems, our experiments may allow the conclusion that these problems seem to be hard to solve even in practice and on random instances (according to IC and TM).

3.3 Bribery and Campaign Management

Every political election is preceded by extensive election battles. Long before the election is actually held, the streets are plastered with election posters, TV-spots are aired, candidates participate in talk shows and engage in political debates, volunteers go from door to door advertising the candidate or party they support, and online campaigns are launched.

Clearly, the management of such campaigns is an important part of politics – strategists and pollsters solely focus on tackling the task of analyzing the voters' behavior in order to decide how the campaign can maximize the influence on them within the given budget. From a voting-theoretical point of view, this idea is formally closely related to bribery: Candidates or parties invest a resource such as money or time into actions that hopefully persuade opposing voters to vote differently. Formulating *campaign management* and *bribery* as voting problems (as Faliszewski et al. [FHH09] and Elkind et al. [EFS09] have done in their work) these two scenarios differ on the possibilities the *briber* has of changing the bribed voters' ballots: While in the standard bribery scenario we assume that, once a voter is bribed, her entire ballot can be changed at the briber's will, the model of campaign management allows the briber to alter certain aspects only, where each aspect can have a different cost depending on the voter.

The next part of this section formally introduces the voting problems formalizing different variants of bribery and campaign management. In Sections 3.3.2 and 3.3.3 we present the results regarding the complexity of these problems in Bucklin and fallback elections.

3.3.1 Basic Definitions and Related Work

The formal description of bribery in elections as decision problems with the aim of studying their computational complexity was firstly introduced by Faliszewski et al. [FHH09] (see also [FHH+09b]).

$\mathscr{E}$-CONSTRUCTIVE UNWEIGHTED BRIBERY ($\mathscr{E}$-CUB)	
Given:	An $\mathscr{E}$ election (C,V), a designated candidate p, and a nonnegative integer k.
Question:	Is it possible to make p an $\mathscr{E}$ winner by changing the votes of at most k voters?

This standard scenario can be extended by allowing the voters to have different prices for changing their votes.

$\mathscr{E}$-CONSTRUCTIVE UNWEIGHTED PRICED BRIBERY ($\mathscr{E}$-CUB-\$)	
Given:	An $\mathscr{E}$ election (C,V) with $\mathfrak{n}$ voters of which each voter $v_i \in V$ has a nonnegative integer price π_i, $1 \leq i \leq \mathfrak{n}$, a designated candidate p, and a positive integer k.
Question:	Is there a set $I \subseteq \{1,\ldots,\mathfrak{n}\}$ such that $\sum_{i \in I} \pi_i \leq k$ and the voters v_i with $i \in I$ can be bribed so that p is an $\mathscr{E}$ winner of the resulting election?

Both of these problems can also be defined in weighted elections which implies that each voter has an integer weight. This leads to the final two types of standard bribery.

$\mathscr{E}$-CONSTRUCTIVE WEIGHTED BRIBERY ($\mathscr{E}$-CWB)	
Given:	An $\mathscr{E}$ election (C,V) with each voter $v_i \in V$ having a nonnegative integer weight w_i, a designated candidate p, and a positive integer k.
Question:	Is it possible to make p an $\mathscr{E}$ winner by changing the votes of at most k voters?

$\mathscr{E}$-CONSTRUCTIVE WEIGHTED PRICED BRIBERY ($\mathscr{E}$-CWB-\$)	
Given:	An $\mathscr{E}$ election (C,V) with $\mathfrak{n}$ voters of which each voter $v_i \in V$ has nonnegative integer weight w_i and price π_i, $1 \leq i \leq \mathfrak{n}$, a designated candidate p, and a positive integer k.
Question:	Is there a set $I \subseteq \{1,\ldots,\mathfrak{n}\}$ such that $\sum_{i \in I} \pi_i \leq k$ and the voters v_i with $i \in I$ can be bribed so that p is an $\mathscr{E}$ winner of the resulting election?

We have stated these scenarios in the *constructive* variant. By changing the questions in the above four problems to whether the designated candidate can be prevented from being an $\mathscr{E}$ winner, we obtain the *destructive* variants which we will denote by $\mathscr{E}$-DUB, $\mathscr{E}$-DUB-\$, $\mathscr{E}$-DWB, and $\mathscr{E}$-DWB-\$.

To define these eight problems in the *unique-winner model* instead of the *co-winner model* as we did above, we have to ask whether p can be made or be prevented from being a unique $\mathscr{E}$ winner. We denote the problems in the unique-winner model by $\mathscr{E}$-UCUB, $\mathscr{E}$-UCUB-\$, $\mathscr{E}$-UCWB, $\mathscr{E}$-UCWB-\$. $\mathscr{E}$-UDUB, $\mathscr{E}$-UDUB-\$, $\mathscr{E}$-UDWB, and $\mathscr{E}$-UDWB-\$ to present Observation 3.33. Note that the first four observations below directly follow from the fact that the weighted and priced variants are general cases of the unweighted and/or unpriced variants. To illustrate the notation, take the first observation:

$\mathscr{E}$-CUB $\leq_{\mathrm{m}}^{\mathrm{p}}$ {$\mathscr{E}$-CWB, $\mathscr{E}$-CUB-\$} $\leq_{\mathrm{m}}^{\mathrm{p}}$ $\mathscr{E}$-CWB-\$ is supposed to be a shorthand for the relations (1) $\mathscr{E}$-CUB $\leq_{\mathrm{m}}^{\mathrm{p}}$ $\mathscr{E}$-CWB, (2) $\mathscr{E}$-CUB-\$ $\leq_{\mathrm{m}}^{\mathrm{p}}$ $\mathscr{E}$-CWB-\$, (3) $\mathscr{E}$-CUB $\leq_{\mathrm{m}}^{\mathrm{p}}$ $\mathscr{E}$-CUB-\$, (4) $\mathscr{E}$-CWB $\leq_{\mathrm{m}}^{\mathrm{p}}$ $\mathscr{E}$-CWB-\$, (5) $\mathscr{E}$-CUB $\leq_{\mathrm{m}}^{\mathrm{p}}$ $\mathscr{E}$-CWB-\$, and where the last relation follows from the transitivity of $\leq_{\mathrm{m}}^{\mathrm{p}}$. For the fifth observation we refer the reader to the explanation of the corresponding observation for manipulation scenarios, Observation 3.3 on page 28.

Observation 3.33 *Let $\mathscr{E}$ be a voting system, then the following holds.*

1. $\mathscr{E}$-CUB $\leq_{\mathrm{m}}^{\mathrm{p}}$ {$\mathscr{E}$-CWB, $\mathscr{E}$-CUB-\$} $\leq_{\mathrm{m}}^{\mathrm{p}}$ $\mathscr{E}$-CWB-\$.
2. $\mathscr{E}$-UCUB $\leq_{\mathrm{m}}^{\mathrm{p}}$ {$\mathscr{E}$-UCWB, $\mathscr{E}$-UCUB-\$} $\leq_{\mathrm{m}}^{\mathrm{p}}$ $\mathscr{E}$-UCWB-\$.
3. $\mathscr{E}$-DUB $\leq_{\mathrm{m}}^{\mathrm{p}}$ {$\mathscr{E}$-DWB, $\mathscr{E}$-DUB-\$} $\leq_{\mathrm{m}}^{\mathrm{p}}$ $\mathscr{E}$-DWB-\$.
4. $\mathscr{E}$-UDUB $\leq_{\mathrm{m}}^{\mathrm{p}}$ {$\mathscr{E}$-UDWB, $\mathscr{E}$-UDUB-\$} $\leq_{\mathrm{m}}^{\mathrm{p}}$ $\mathscr{E}$-UDWB-\$.
5. $\mathscr{E}$-UDUB $\leq_{\mathrm{T}}^{\mathrm{p}}$ $\mathscr{E}$-CUB, $\mathscr{E}$-UDUB-\$ $\leq_{\mathrm{T}}^{\mathrm{p}}$ $\mathscr{E}$-CUB-\$, *and* $\mathscr{E}$-UDWB-\$ $\leq_{\mathrm{T}}^{\mathrm{p}}$ $\mathscr{E}$-CWB-\$.

Essentially, a briber has to tackle two tasks at the same time: choosing the voters to be bribed and determining how exactly to cast the preferences of these voters. The second part is closely related to the coalitional manipulation problem, see Section 3.1 for the exact definition, in which for a given set of manipulators previously undefined preferences have to be specified. We will see in the proof of Theorem 3.48 establishing tractability of Bucklin-DWB (and also of Bucklin-DUB-\$) that we can use a known algorithm for destructive coalitional manipulation in weighted Bucklin elections (Algorithm 3.1 on page 33) to solve the bribery problem at hand. Faliszewski et al. [FHH09] further analyzed the connections between manipulation and bribery problems and established a very strong connection between coalitional manipulation and priced bribery when both problems are given in the constructive case. Since their argumentation can be adapted straightforwardly to the destructive case, we state the following.

Proposition 3.34 (Faliszewski et al. [FHH09]) *Let $\mathscr{E}$ be a voting system, then it holds that*

1. $\mathscr{E}$*-CCUM* $\leq_{\mathrm{m}}^{\mathrm{p}}$ $\mathscr{E}$*-CUB-\$ and* $\mathscr{E}$*-CCWM* $\leq_{\mathrm{m}}^{\mathrm{p}}$ $\mathscr{E}$*-CWB-\$, and*
2. $\mathscr{E}$*-DCUM* $\leq_{\mathrm{m}}^{\mathrm{p}}$ $\mathscr{E}$*-DUB-\$ and* $\mathscr{E}$*-DCWM* $\leq_{\mathrm{m}}^{\mathrm{p}}$ $\mathscr{E}$*-DWB-\$.*

In the standard bribery scenario defined above we assume that once a voter accepted a bribe, she changes the entire ballot as the briber requests. This assumption can be seen as a very restricting one as it might be the case that voters would be willing to change some aspects of their ballots, but insist on certain orderings: For instance, a voter might be willing to change the positions of her mid-ranked candidates (she might be to some extent indifferent to all of them) but is very certain about (and thus unwilling to change) the order of her top-ranked or most despised candidates.

The concept of *campaign management* takes this into account and allows a more fine-grained definition of bribing voters. Elkind et al. [EFS09] propose SWAP BRIBERY as a refinement of standard bribery that allows the voters to have different prices for different changes in their vote which are defined as swaps between two adjacent candidates. Formally, the price functions of the voters are defined as follows.

Definition 3.35 (Elkind et al. [EFS09]) *A* swap-bribery price function *for voter v_i is a function $\pi_i : C \times C \to \mathbb{N}$ that specifies for each ordered pair (c_r, c_s) of candidates the price for changing v_i's preference order from $\cdots > c_r > c_s > \cdots$ to $\cdots > c_s > c_r > \cdots$. Only candidates that are adjacent in a vote can be swapped.*

We state the constructive case of SWAP BRIBERY directly for weighted elections as this is the most general case.

$\mathscr{E}$-CONSTRUCTIVE WEIGHTED SWAP BRIBERY ($\mathscr{E}$-CWSB)	
Given:	An $\mathscr{E}$ election (C,V), where $V = (v_1, \ldots, v_n)$, a designated candidate p, a list $(\pi_1, \ldots, \pi_n)$ of swap bribery price functions, a list of weights $(w_1, \ldots, w_n)$, and a non-negative integer k.
Question:	Can p be made an $\mathscr{E}$ winner of an election resulting from the input election by conducting a sequence of swaps of adjacent candidates in the voters' ballots such that the total cost of the swaps does not exceed the budget k?

The unweighted variant $\mathscr{E}$-CUSB can be obtained by assigning unit weights to all voters. The destructive variants $\mathscr{E}$-DWSB and $\mathscr{E}$-DUSB are defined in the standard way and we denote the four counterparts in the unique-winner model by $\mathscr{E}$-uCWSB, $\mathscr{E}$-uCUSB, $\mathscr{E}$-uDWSB, and $\mathscr{E}$-uDUSB.

Observe the following relations between the different swap bribery cases which can be obtained directly from their definitions.

Observation 3.36 *Let $\mathscr{E}$ be a voting system, then the following holds.*

1. $\mathscr{E}$-CUSB $\leq_m^p$ $\mathscr{E}$-CWSB *and* $\mathscr{E}$-uCUSB $\leq_m^p$ $\mathscr{E}$-uCUSB.
2. $\mathscr{E}$-DUSB $\leq_m^p$ $\mathscr{E}$-DWSB *and* $\mathscr{E}$-uDUSB $\leq_m^p$ $\mathscr{E}$-uCDSB.
3. $\mathscr{E}$-uDUSB $\leq_T^p$ $\mathscr{E}$-CUSB *and* $\mathscr{E}$-uDWSB $\leq_T^p$ $\mathscr{E}$-CWSB.
4. *For elections with a fixed number of 2 candidates, the following problems are equivalent:* $\mathscr{E}$-CWSB *and* $\mathscr{E}$-CWB-\$, $\mathscr{E}$-uCWSB *and* $\mathscr{E}$-uCWB-\$, $\mathscr{E}$-DWSB *and* $\mathscr{E}$-DWB-\$, *and* $\mathscr{E}$-uDWSB *and* $\mathscr{E}$-uDWB-\$, *respectively.*

Recalling the definition of fallback voting in Section 2.3.2, we know that the ballots in a fallback election consist of two parts: the approved candidates that are ranked and the unranked set of disapproved candidates. This compels us to further specify how a vote in a fallback election can be changed in the course of a swap bribery action: Swaps of candidates are only allowed in the approved parts of the votes. With this definition (and only this definition) of swap bribery in fallback elections, we can use the fact that Bucklin elections are special fallback elections, to state the following result.

Lemma 3.37 *Let $\mathscr{E}$-SB be any of the above defined swap bribery scenarios. Then it holds that Bucklin-SB$\leq_m^p$ fallback-SB.*

A special case of the Swap Bribery problem that is also introduced by Elkind et al. [EFS09] is the Shift Bribery problem in which it is only allowed to perform swaps involving the designated candidate. This means, that the designated candidate can be shifted upwards or downwards in the rankings, but no other changes are allowed. We give the definition of the problem for unit costs as we will need this variant in Section 4.4.

$\mathscr{E}$-Constructive Shift Bribery ($\mathscr{E}$-CShB)	
Given:	An $\mathscr{E}$ election (C,V), where $V = (v_1, \dots, v_n)$, a designated candidate p, and a nonnegative integer k.
Question:	Can p be made an $\mathscr{E}$ winner of an election resulting from the input election by conducting in total at most k shifts of p in the votes?

Swap bribery in the unweighted scenario is a generalization of the so-called Possible Winner problem ($\mathscr{E}$-PW) defined by Konczak and Lang [KL05] which asks for a given $\mathscr{E}$ election with possibly incomplete preferences whether there is an extension of the incomplete orders to linear order such that a given candidate is an $\mathscr{E}$ winner of the resulting election (see also Chapter 5 on page 122 for the formal definition definitions and further discussions and results). For bribery this connection is important since a lower bound (such as NP-hardness) for the $\mathscr{E}$-PW problem for a fixed voting system $\mathscr{E}$ also holds for the more general case of $\mathscr{E}$-CUSB which is formally stated in the following proposition.

Proposition 3.38 (Elkind et al. [EFS09]) *Let $\mathscr{E}$ be a voting system, then* $\mathscr{E}$-PW $\leq_{\rm m}^{\rm p}$ $\mathscr{E}$-CUSB *and* $\mathscr{E}$-uPW $\leq_{\rm m}^{\rm p}$ $\mathscr{E}$-uCUSB.

Combining the above result and Proposition 3.39 shown byXia and Conitzer [XC11b], it can be established that $\mathscr{E}$-CCUM $\leq_{\rm m}^{\rm p}$ $\mathscr{E}$-CUSB and $\mathscr{E}$-uCCUM $\leq_{\rm m}^{\rm p}$ $\mathscr{E}$-uCUSB hold.

Proposition 3.39 (Xia and Conitzer [XC11b]) *Let $\mathscr{E}$ be a voting system, then it holds that* $\mathscr{E}$-CCUM $\leq_{\rm m}^{\rm p}$ $\mathscr{E}$-PW *and* $\mathscr{E}$-uCCUM $\leq_{\rm m}^{\rm p}$ $\mathscr{E}$-uPW.

Closely related to the possible winner problem is the necessary winner problem ($\mathscr{E}$-NW) asking for the same input whether a given candidate is an $\mathscr{E}$ winner for every extension of the incomplete votes to linear orders. Shiryaev et al. [SYE13] show that destructive unweighted swap bribery in fact is a generalization of the complement of this problem.

Proposition 3.40 (Shiryaev et al. [SYE13]) *Let $\mathscr{E}$ be a voting system, then the following relations holds:* $\overline{\mathscr{E}\text{-NW}}$ $\leq_{\rm m}^{\rm p}$ $\mathscr{E}$-DUSB *and* $\overline{\mathscr{E}\text{-uNW}}$ $\leq_{\rm m}^{\rm p}$ $\mathscr{E}$-uDUSB.

In fallback elections, we restricted the allowed changes in swap bribery actions to the approved part of the votes only. A natural next step is to consider problem variants in which the approved part of the vote can be changed, for example, by adding formerly disapproved candidates to it. Elkind et al. [EFS09] defined the notion of *mixed bribery* for the voting system sincere-strategy approval voting (SP-AV), a hybrid voting system in which the votes also have an approved and a disapproved part.[6] Mixed bribery allows both changing the size of the approved parts of the votes and swaps of candidates therein.

We will follow the approach of Schlotter et al. [SFE11] and analyze the complexity of modifying the approved parts of the votes separately. To this end, we will make use of so-called *extension bribery functions* that are due to Baumeister et al. [BFL+12].

Definition 3.41 (Baumeister et al. [BFL+12]) *The* extension bribery price function $\delta_i : \mathbb{N} \to \mathbb{N}$ *of a voter v_i defines the price for extending the approved part of v_i's vote with a given number of so-far-disapproved candidates (these new candidates are ranked below the previously-approved candidates, but among themselves are ranked as the briber requests).*

We define the weighted variant of EXTENSION BRIBERY in fallback elections (FV-CWEB) formally below.

FV-CONSTRUCTIVE WEIGHTED EXTENSION BRIBERY (FV-CWEB)	
Given:	A fallback election (C,V), where $V = (v_1,\dots,v_n)$, a list of weights $(w_1,\dots,w_n)$), a designated candidate p, a list $(\delta_1,\dots,\delta_n)$ of extension bribery price functions, and a nonnegative integer k.
Question:	Can p be made a fallback winner by extending the approved parts of the voters' ballots without exceeding the budget k?

The unweighted variant FV-CUEB and the destructive variants FV-DWEB and FV-DUEB can be obtained from this version as we have seen before for other problem variants.

[6] This voting system was originally introduced in the work of Brams and Sanver [BS06b] and then modified by Erdélyi et al. [ENR09]. A thorough analysis of SP-AV and other variants of approval voting can be found in the book chapter by Baumeister et al. [BEH+10].

Related Work After having provided the formal definitions of the problems considered in this section, we shortly survey the state of the art regarding the complexity of bribery and campaign management.

We start by giving known results from the literature for the standard bribery scenarios in those voting systems that are studied within this thesis in Table 3.27.

Problem	PV	Borda	Schulze	Copeland$^\alpha$	Cup
$\mathscr{E}$-CUB	P [1]	NP-complete [4]	NP-complete [5]	NP-complete [6]	NP-complete [3]
$\mathscr{E}$-DUB	P [2]	P [2]	NP-complete [5]	NP-complete [6]	NP-complete [7]
$\mathscr{E}$-CUB-$	P [1]	NP-complete [4]	NP-complete [5]	NP-complete [6]	NP-complete [3]
$\mathscr{E}$-DUB-$	P [3]	P [3]	NP-complete [5]	NP-complete [6]	NP-complete [7]
$\mathscr{E}$-CWB	P [1]	NP-complete [4]	NP-complete [5]	NP-complete [6]	NP-complete [3]
$\mathscr{E}$-DWB	P [3]	P [3]	NP-complete [5]	NP-complete [6]	NP-complete [7]
$\mathscr{E}$-CWB-$	NP-complete [1]	NP-complete [4]	NP-complete [5]	NP-complete [6]	NP-complete [3]
$\mathscr{E}$-DWB-$	NP-complete [3]	NP-complete [3]	NP-complete [5]	NP-complete [6]	NP-complete [7]

[1] due to [FHH09] [3] due to [Rei07] [5] due to [PX12] [7] proof of Thm. 4.8
[2] due to [Xia12] [4] due to [BFH+08] [6] due to [FHH+09b]

Table 3.27: Selection of known results regarding the complexity of bribery

In Table 3.28 we present complexity results regarding swap briber that have been shown in the literature or can be followed from known results. We see that for two very prominent voting rules, Borda and Schulze, the complexity of unweighted swap bribery is not completely resolved.

Problem	PV	Borda	Schulze	Copeland$^\alpha$	Cup
$\mathscr{E}$-CUSB	P [1]	NP-complete [5]	?	NP-complete [7]	NP-complete [7]
$\mathscr{E}$-DUSB	P [2]	?	?	NP-complete [8]	NP-complete [8]
$\mathscr{E}$-CWSB	NP-complete [3]	NP-complete [5]	NP-complete [6]	NP-complete [7]	NP-complete [7]
$\mathscr{E}$-DWSB	NP-complete [4]	NP-complete [4]	NP-complete [6]	NP-complete [8]	NP-complete [8]

[1] due to [EFS09] [4] due to [Rei07] with Obs. 3.36 [7] due to [XC11b] with Prop. 3.38
[2] due to [SYE13] [5] due to [BD10] and Prop 3.38 [8] due to [XC11b] with Prop. 3.40
[3] due to [FHH09] with Obs. 3.36 [6] due to [PX12] with Obs. 3.36

Table 3.28: Selection of known results regarding the complexity of swap bribery

Constructive bribery is studied by Lin [Lin12] for k-approval and k-veto voting while in the work of Xia [Xia12] results for bribery in STV and ranked pairs elections are proven and destructive bribery is studied. Yang [Yan14] and Bredereck et al. [BFN+15a] provide complexity results for priced standard bribery parameterized by the number of candidates.

Besides those results given in the table above, Elkind et al. [EFS09] also study swap bribery in k-approval, SP-AV, and maximin elections and extend their study to the complexity of shift bribery for these voting systems. Their study is extended by Schlotter et al. [SFE11], who show tractability of shift bribery in (simplified) Bucklin voting and fallback voting. They furthermore introduce the notion of *support bribery*, another interesting variant of campaign management and study both the classical and parameterized complexity of this problem. Shiryaev et al. [SYE13] focus on destructive swap bribery and furthermore define a measure of robustness of elections with respect to incorrect swaps in votes, see also Chapter 4 of this thesis.

Much attention has been paid to natural parameterizations of campaign management problems. Dorn and Schlotter [DS12] provide a detailed study on the complexity of swap bribery in k-approval elections for different parameters. Different parameterizations of the shift bribery problem are studied by Bredereck et al. [BCF+14b]. The work of Elkind and Faliszewski [EF10a] focuses on the approximability of the shift bribery problem for a selection of common voting systems.

Mattei et al. [MPV+13] and Bredereck et al. [BFN+15b] focus on bribery problems tailored to elections with certain combinatorial structures in the electorate. Other restrictions regarding the given votes such as truncated ballots and single-peaked preferences have been addressed in the work of Baumeister et al. [BFL+12], Brandt et al. [BBH+10], Faliszewski et al. [FHH14], and Hemaspaandra et al. [HHR15].

Some interesting studies of bribery beyond the context of voting are the following: Bribery in path-disruption games has been studied by Rey and Rothe [RR11] and Marple et al. [MRR14], while Baumeister et al. [BEE+15] define and study bribery scenarios for judgment aggregation settings. The closely related *lobbying problem*, which was introduced by Christian et al. [CFR+07], has found much attention in recent research, see for example the work of Bredereck et al. [BCH+14] and Binkele-Raible et al. [BEF+14].

For a comprehensive overview, we refer to the survey by Faliszewski et al. [FHH+09a] and the book chapters of Baumeister and Rothe [BR15] and Faliszewski and Rothe [FR16].

3.3.2 Complexity of Bribery

Table 3.29 shows the result that were published in [FRR+14, FRR+15] and that we will present in this section.

Problem	**Bucklin Voting**		**Fallback Voting**	
	complexity	reference	complexity	reference
$\mathscr{E}$-CUB	NP-complete	Thm. 3.43	NP-complete	Thm. 3.45
$\mathscr{E}$-DUB	P	Cor. 3.49	P	Thm. 3.50
$\mathscr{E}$-CUB-$	NP-complete	Cor. 3.44	NP-complete	Cor. 3.46
$\mathscr{E}$-DUB-$	P	Thm. 3.48	P	Thm. 3.50
$\mathscr{E}$-CWB	NP-complete	Cor. 3.44	NP-complete	Cor. 3.46
$\mathscr{E}$-DWB	P	Thm. 3.48	P	Thm. 3.50
$\mathscr{E}$-CWB-$	NP-complete	Cor. 3.44	NP-complete	Cor. 3.46
$\mathscr{E}$-DWB-$	NP-complete	Thm. 3.47	NP-complete	Thm. 3.47

Table 3.29: Overview of results for bribery in Bucklin and fallback voting

Before we present the results that are given in the table, we state the following lemma and show NP-hardness only in the upcoming NP-completeness proofs.

Lemma 3.42 *For Bucklin and fallback voting each of the problems corresponding to the standard bribery scenarios is contained in* NP.

The claim directly follows from P membership of the winner problems of Bucklin and fallback since this allows us to check in deterministic polynomial time whether a guessed bribery action is successful or not.

Theorem 3.43 CUB *is* NP*-complete for Bucklin voting in both winner models.*

Proof Sketch. The following proof applies to both winner models with the exact same argumentation, no adaptions are needed. We show how a Bucklin-CUB instance $((C,V),p,k)$ can be constructed from an X3C instance $(B,\mathscr{S})$ such that $(B,\mathscr{S}) \in$ X3C if and only if $((C,V),p,k) \in$ Bucklin-CUB. But we will not show the equivalence.

Let $(B,\mathscr{S})$ be an instance of X3C with $B = \{b_1,b_2,\ldots,b_{3m}\}$ and $\mathscr{S} = \{S_1,S_2,\ldots,S_n\}$. Without loss of generality, we may assume that $n \geq 2m$. We construct a Bucklin-CUB instance $((C,V),p,k)$, where (C,V) is a Bucklin election with the candidates $C = B \cup \{c,d\} \cup G \cup \{p\}$, p is the designated candidate, and $k = m$. The set G is a set of "padding candidates," which are used to ensure that certain candidates do not gain points up to a certain level. Padding candidates are positioned in the votes such that, up to a certain level, they themselves do not gain enough points to be relevant for the central argument of the proof. (Specifically, we will ensure that up to a given level, each padding candidate gets at most one point.) Thus, their scores are not listed in the tables giving the scores of the relevant candidates.

For every $b_j \in B$, define ℓ_j to be the number of sets $S_i \in \mathscr{S}$ candidate b_j is contained in. V consists of the following $2n$ voters (i.e., a strict majority is reached with $n+1$ votes):

- The first voter group consists of n voters. For each i, $1 \leq i \leq n$, we have one voter of the form
$$c > d > S_i > G_i > \{C - (\{c,d\} \cup S_i \cup G_i)\},$$
where $G_i \subseteq G$ is a set of $3m-3$ padding candidates. When a set X of candidates is given in such a ranking, the order of the candidates from X can be fixed arbitrarily in this ranking.
- The second voter group consists of n voters as well. We will present the preferences level by level from the first to the $(3m+2)^{nd}$ position in Table 3.30, indicating the number (by #) of occurrences of each candidate in these positions. The first (3×3)-block in the left side of the table can be read as follows: m of the n voters position c on the first place, m of the n voters have candidate d on the first position while the remaining $n-2m$ voters each position a different candidate from G on their top position. The (2×3)-block below indicates that $n+1-\ell_1$ of the n voters position candidate b_1 on the second place and the remaining $\ell_1 - 1$ voters in this group each have a different candidate from G on the second position in their ballot. The remaining blocks can be read analogously until in the last block in the right side of the table, position $3m+2$ is shown. Note that

the notation for the padding candidates has been chosen to keep the table as readable as possible while still emphasizing that each candidate from G is only positioned once within the top $3m+2$ positions in this voter group and thus only gains at most 1 point up to level $3m+2$. The G'_r-sets are disjoint subsets of G each containing exactly as many candidates as voters are denoted by # in the respective block.

position	# voters	candidate	position	# voters	candidate
1	m m $n-2m$	c d $g \in G'_1$	⋮	⋮	⋮
2	$n+1-\ell_1$ $\ell_1 - 1$	b_1 $g \in G'_2$	$3m$	$n+1-\ell_{3m-1}$ $\ell_{3m-1}-1$	b_{3m-1} $g \in G'_{3m}$
3	$n+1-\ell_2$ $\ell_2 - 1$	b_2 $g \in G'_3$	$3m+1$	$n+1-\ell_{3m}$ $\ell_{3m}-1$	b_{3m} $g \in G'_{3m+1}$
⋮	⋮	⋮	$3m+2$	$n-m+1$ $m-1$	p $g \in G'_{3m+2}$

Table 3.30: Preferences of the voters in the second voter group in V in the proof of Theorem 3.43

Table 3.31a shows the scores of the relevant candidates in (C,V) (namely, c, d, p, and each $b_j \in B$) for the relevant levels (namely, 1, 2, $3m$, $3m+1$, and $3m+2$). In particular, one can see that c is the unique level 1 Bucklin winner in (C,V).

	$b_j \in B$	c	d	p
$score^1$	0	$n+m$	m	0
$score^2$	$\leq n+1$	$n+m$	$m+n$	0
$score^{3m}$	$\leq n+1$	$n+m$	$m+n$	0
$score^{3m+1}$	$\leq n+1$	$n+m$	$m+n$	0
$score^{3m+2}$	$n+1$	$n+m$	$m+n$	$n-m+1$

(a) Original election (C,V)

	$b_j \in B$	c	d	p
$score^1$	0	n	m	m
$score^2$	$\leq n$	n	n	m
$score^{3m}$	$\leq n$	n	n	m
$score^{3m+1}$	$\leq n$	n	n	m
$score^{3m+2}$	$\leq n$	n	n	$n+1$

(b) Modified election (C,V')

Table 3.31: Level i scores for $i \in \{1,2,3m,3m+1,3m+2\}$ and the candidates in $C-G$

It can now be shown that $\mathscr{S}$ has an exact cover $\mathscr{S}'$ for B if and only if p can be made a Bucklin winner by changing at most m votes in V. ❑

With Observation 3.33 we immediately obtain the following corollary.

Corollary 3.44 *In Bucklin elections,* CWB, CUB-\$, *and* CWB-\$ *are* NP-*complete, each in both winner models.*

Note that the result for Bucklin-CUB-\$ follows from Proposition 3.34 and the hardness result for Bucklin-CCWM, see Theorem 3.5 on page 31. The upcoming two results show that also in fallback elections, a briber is faced with an NP-hard task trying to make a certain candidate a winner by applying one of the four standard bribery variants. The following proof establishing NP-hardness for the unweighted and unpriced case is based on the proof of the corresponding bribery problem in approval voting due to Faliszewski et al. [FHH09].

Theorem 3.45 CUB *is* NP-*complete for fallback voting in both winner models.*

Proof. We show NP-hardness in the co-winner model by a reduction from X3C. Let $(B,\mathscr{S})$ be an instance of X3C with $B=\{b_1,b_2,\dots,b_{3m}\}$ and $\mathscr{S}=\{S_1,S_2,\dots,S_n\}$. (We assume that $n>m$; otherwise, it would be immediate to check if $(B,\mathscr{S})$ is a yes-instance of X3C or not.) We define the fallback election (C,V) with the candidate set $C=B\cup E\cup\{p\}$, where p is the designated candidate and E is a set of $n+m$ padding candidates. For every $j\in\{1,\dots,3m\}$, we again define ℓ_j as the number of subsets $S_i\in\mathscr{S}$ candidate $b_j\in B$ is contained in. Using this notation, we define the subsets $B_i=\{b_j\in B \mid i\le n-\ell_j\}$ for $i\in\{1,\dots,n\}$. V consists of the $4n-1$ voters whose preferences are given in Table 3.32.

Group	For each ...	# of votes	preference
(1)	$i\in\{1,\dots,n\}$	1	$S_i \mid (B-S_i)\cup E\cup\{p\}$
(2)	$i\in\{1,\dots,n\}$	1	$B_i \mid (B-B_i)\cup E\cup\{p\}$
(3)		$n-m-1$	$p \mid B\cup E$
(4)	$\ell\in\{1,\dots,n+m\}$	1	$e_\ell \mid B\cup(E-\{e_\ell\}\cup\{p\}$

Table 3.32: Voter list V in the proof of Theorem 3.45

There is no candidate reaching a strict majority on any level. With the approval scores in Table 3.33a we see that all candidates $b_j\in B$ are fallback winners in (C,V).

	$b_j\in B$	p	$e_l\in E$
score	n	$n-m-1$	1

(a) Original election (C,V)

	$b_j\in B$	p	$e_l\in E$
score	$n-1$	$n-1$	1

(b) Modified election (C,V')

Table 3.33: Overall scores of the candidates in C

We claim that $\mathscr{S}$ has an exact cover $\mathscr{S}'$ for B if and only if p can be made a fallback winner by bribing at most m voters.

Only if: Suppose that $\mathscr{S}$ has an exact cover $\mathscr{S}'$ for B. We change the votes of those voters in the first voter group where $S_i\in\mathscr{S}'$ from $S_i \mid (B-S_i)\cup E\cup\{p\}$ to $p \mid B\cup E$. In the resulting election (C,V'), only the scores of the candidates in B and the score of p change: p gains m points, whereas each $b_j\in B$ loses exactly one point. Thus, with an overall score of $n-1$, candidate p wins the election together with the candidates in B, see Table 3.33b.

If: Suppose that p can be made a fallback winner by changing at most m votes in V. That means that p can gain at most m points, so the maximum overall score that p can reach is $n-1$. Since each $b_j \in B$ has an overall score of n, every candidate in B has to lose at least one point by changing at most m votes (otherwise, there would be at least one candidate in B who beats p). This is possible only if in m votes of the first voter group the candidates in S_i are removed from the set of approved candidates such that these m sets S_i form an exact cover for B.

For the unique-winner model, simply change the third voter group in V to contain $n-m$ voters. ❑

Again, with Observation 3.33 we directly obtain NP-completeness for the other constructive bribery cases.

Corollary 3.46 *In fallback elections,* CWB, CUB-\$*, and* CWB-\$ *are* NP*-complete, each in both winner models.*

For destructive bribery we can establish NP-hardness in both Bucklin and fallback voting when the voters both have weights and prices. We show this for both voting systems simultaneously as the constructed elections has only two candidates.

Theorem 3.47 *Both Bucklin-*DWB-\$ *and fallback-*DWB-\$ *are* NP*-complete, each in both winner models.*

Proof. We show NP-hardness by a reduction from PARTITION. (The same reduction works for both problems.) Let $(A,(a_1,\dots,a_k))$ with $A=\{1,\dots,k\}$ and $\sum_{i=1}^{k} a_i = 2K$ be an instance of PARTITION. We construct the following Bucklin (fallback) election (C,V) with $C=\{c,p\}$ and $k+1$ votes in V: For each $i \in \{1,\dots,k\}$, we have one voter v_i with weight $w_i = a_i$, price $\pi_i = a_i$, and preference $p > c$, and we have one voter v_{k+1} with weight $w_{k+1} = 1$, price $\pi_{k+1} = K+1$, and preference $c > p$ (for fallback, all voters approve of both candidates).

The total weight of the voters in (C,V) is $2K+1$, so $maj(V) = K+1$. Let K be the budget that may not be exceeded and let p be the designated candidate. Obviously, p is the unique level 1 Bucklin (fallback) winner in (C,V).

We claim that $(A,(a_1,\dots,a_k)) \in$ PARTITION if and only if p can be prevented from being a Bucklin (fallback) winner by changing votes in V without exceeding the budget K.

Only if: Let $(A,(a_1,\dots,a_k)) \in$ PARTITION with $A' \subseteq A$ such that $\sum_{i\in A'} a_i = K$. Change the votes of those voters with weight $w_i = a_i$ for $i \in A'$ from $p > c$ to $c > p$. With these changes we have that on the first level, p has K points and c has $K+1$ points, so c is the new level 1 Bucklin (fallback) winner and p has been prevented from winning.

If: Assume that p is not a Bucklin (fallback) winner in the bribed election. Since there are only two levels, c has to win on the first level to prevent p from winning. Changing the vote of voter v_{k+1} would provide no gain (and would be too expensive), so only the votes of $v_1,\dots,v_k$ may be changed. For each of the voters, the price equals the weight, so voters with a total weight of K can be changed. Candidate c has one point on the first level in the original election, so it is only possible to make c a unique level 1 Bucklin winner by fully exhausting the budget and changing the votes with a total weight of K from $p > c$ to $c > p$ (or, for the case of fallback, to approve of c only, which gives the same effect). Thus, there is a subset $A' \subseteq A$ such that $\sum_{i \in A'} a_i = K$, so $(A, (a_1, \dots, a_k)) \in$ PARTITION.

For the unique-winner model, simply omit voter v_{k+1} in the voter list. ❑

The remaining destructive bribery cases can be solved in deterministic polynomial time. The algorithms that we will present are based on an algorithm presented by Xia [Xia12] for destructive unweighted bribery in elections held under the simplified Bucklin rule. Algorithm 3.3 adapts Xia's approach to Bucklin elections and is a procedure to solve the problem Bucklin-DWB. The main idea is to use Algorithm 3.1, which was designed in Section 3.1.2 on page 33 to solve the destructive coalitional weighted manipulation problem for Bucklin elections, Bucklin-DCWM. The main difference between a bribery and a manipulation instance is that in the latter only the preferences of the manipulators have to be found, whereas in the former both the votes that will be bribed and the new preferences for these voters have to be found. If we have the set of votes we want to change, we can use the algorithm for the manipulation problem to construct the preferences. Thus, for the runtime of the algorithm the determination of these voter sets is crucial, and we show that in Bucklin elections the number of voter sets whose modification might actually lead to a successful bribery is bounded by a polynomial in both the number of voters $\mathfrak{n}$ and the number of candidates $\mathfrak{m}$.

Theorem 3.48 *In Bucklin elections,* DWB *and* DUB-\$ *are in* P*, each in both winner models.*

Proof Sketch. Consider Algorithm 3.3 and a given input (C, V, W_V, p, k) to it. In particular, p is the designated candidate that we want to prevent from winning and assume that we have a yes-instance, i.e., our bribery action is successful. We denote by (C, V'') the election resulting from (C, V) where the k votes that can be changed have already been changed. Then there is a candidate $c \in C - \{p\}$ that reaches a strict majority on level i, and it holds that $score^i_{(C,V'')}(c) > score^i_{(C,V'')}(p)$, which means that p is not a Bucklin winner in (C, V''). To achieve that, for each $i < m$, there are only five types of preferences that might have been changed in V, and they can be grouped into the following subsets $T_{i,j} \subseteq V$, $1 \le j \le 5$:

$T_{i,1}$: p is among the top $i-1$ positions and c is among the top i positions (when changing: p loses points, c does neither lose nor win points up to level i).

$T_{i,2}$: p is among the top $i-1$ positions and c is not among the top i positions (when changing: p loses points, c wins points up to level i).

$T_{i,3}$: p is on position i and c is among the top $i-1$ positions (when changing: p loses points, c does neither lose nor win points up to level i).

$T_{i,4}$: p is on position i and c is not among the top $i-1$ positions (when changing: p loses points, c wins points up to level i).

$T_{i,5}$: both p and c are not among the top i positions (when changing: p does neither lose nor win points, c wins points up to level i).

For a sublist of voters $V' \subseteq V$, denote their total weight by W'_V. Algorithm 3.3 for Bucklin-DWB works as follows.

It is easy to see that Algorithm 3.3 runs in deterministic polynomial time: the two outer for-loops iterate up to $\mathfrak{m}$ times, whereas the inner loop tests up to k^5 variations of the vector

Algorithm 3.3: Algorithm for Bucklin-DWB

input : C set of candidates
V list of voters
W_V list of weights of voters
k number of votes that may be changed
p designated candidate
output: "YES" if $(C,V,W_V,k,p) \in$ Bucklin-DWB
"NO" if $(C,V,W_V,k,p) \notin$ Bucklin-DWB

1 let $A = \{(a_1,a_2,\ldots,a_5) \mid a_i \in \{0,1,\ldots,k\}\}$, $V' = \emptyset$;
2 **foreach** $c \in C - \{p\}$ **do**
3   **foreach** $i < \mathfrak{m}$ **do**
4   **foreach** $(a_1,a_2,\ldots,a_5) \in A$ **do**
5   **if** $\sum_{\ell=1}^{5} a_\ell \leq k$ **then**
6   **foreach** $j \in \{1,2,\ldots,5\}$ **do**
7   add the a_j heaviest votes in $T_{i,j}$ to V';
8   run Algorithm 3.1 on input $(C, V-V', W_{V-V'}, W_{V'}, p)$;
9   **if** Bucklin-DCWM$(C, V-V', W_{V-V'}, W_{V'}, p)$= *"YES"* **then**
10   return "YES";
11 return "NO";

$(a_1,a_2,\ldots,a_5)$. Since $k \leq \mathfrak{n}$, we have that the number of executions of Algorithm 3.1 is in $\mathcal{O}(\mathfrak{m}^2\mathfrak{n}^5)$. We omit the proof of correctness.

For the Bucklin-DUB-\$ problem the same algorithm can be used. The only difference is that all weights have to be set to one, the cheapest instead of the heaviest votes are added to V' in line 7 (i.e., in line 7 we add the votes with the lowest price instead of the ones with the greatest weight), and it has to be tested whether the sum of the prices of the chosen votes does not exceed the budget. For the unique-winner case, run the algorithm solving the unique-winner variant of Bucklin-DCWM in line 8. ❑

With Observation 3.33 we directly obtain the following corollary.

Corollary 3.49 *In Bucklin elections,* DUB *is in* P *in both winner models.*

The same approach can be taken for fallback voting:

- Change "$i < \mathfrak{m}$" in line 3 to "$i \leq \mathfrak{m}$,"
- use the fallback analogue of Algorithm 3.1 in line 8, and
- change "Bucklin-DCWM" in line 9 to "fallback-DCWM,"

Theorem 3.50 *In fallback elections,* DWB, DUB*, and* DUB-\$ *are in* P*, each in both winner models.*

3.3.3 Complexity of Campaign Management

In this section we present our results on the complexity of the campaign management problems that were defined in Section 3.3.1. Table 3.34 gives an overview of the results published in [FRR+14, FRR+15] and furthermore provides the references of the corresponding theorems and corollaries within this thesis.

	Bucklin voting		fallback voting	
Problem	complexity	reference	complexity	reference
$\mathscr{E}$-CUSB	NP-complete	Thm. 3.52	NP-complete	Cor. 3.53
$\mathscr{E}$-DUSB	NP-complete	Thm. 3.52	NP-complete	Cor. 3.53
$\mathscr{E}$-CWSB	NP-complete	Cor. 3.53	NP-complete	Cor. 3.53
$\mathscr{E}$-DWSB	NP-complete	Cor. 3.53	NP-complete	Cor. 3.53
FV-CUEB	–		P	Thm. 3.56
FV-DUEB	–		P	Thm. 3.56
FV-CWEB	–		NP-complete	Thm. 3.55
FV-DWEB	–		NP-complete	Thm. 3.55

Table 3.34: Overview of results for swap bribery and extension bribery in Bucklin and fallback voting; The dashes "–" indicate that extension bribery is not applicable to Bucklin voting

We state an analogon to Lemma 3.42 in Section 3.3.2 in the context of campaign management.

Lemma 3.51 *For Bucklin and fallback voting each of the problems corresponding to the campaign management scenarios is contained in* NP.

Swap Bribery We start with the analysis of swap bribery, which we conduct for both voting systems. We show that swap bribery in all its variants is NP-complete with a reduction from the following swap bribery variant introduced by Elkind et al. [EFS09].

Single-Vote Swap Bribery	
Given:	A vote v (expressed as a preference order over some candidate set C), a swap-bribery price function π for v, a designated candidate $p \in C$, and two nonnegative integers ℓ and k.
Question:	Is there a sequence of swaps of adjacent candidates, of total cost at most k, that ensure that p is ranked among the top ℓ positions in v?

Elkind et al. [EFS09, Theorem 6] show this problem to be NP-complete and we use it to show that the above claimed NP-completeness of all swap bribery variants even holds in elections with only two voters.

Theorem 3.52 BV-CUSB *and* BV-DUSB *are* NP*-complete each in both winner models, even for elections with only two voters.*

Proof. We show NP-hardness by a reduction from SINGLE-VOTE SWAP BRIBERY.

Let $I = (C, v, \pi, p, \ell, k)$ be an instance of SINGLE-VOTE SWAP BRIBERY, where $\|C\| = m$. We form a Bucklin election $E = (A, V)$ as follows. Let C' be a collection of $m-1$ dummy candidates with $C \cap C' = \emptyset$. We set $A = C \cup C' \cup \{d\}$. We partition C' into two sets, C'_1 and C'_2, such that $\|C'_1\| = \ell - 1$ and $\|C'_2\| = \|C'\| - (\ell - 1) = m - \ell$. (We pick any easily computable partition.) We let V be a collection of two voters, v_1 and v_2, with price functions π_1 and π_2:

1. v_1 has preference order $d > v > C'$ (i.e., v_1 ranks d on the top position, then all the candidates from C in the same order as v, and then all the candidates from C', in some arbitrary-but-easy-to-compute order). For each two candidates $x, y \in A$, if both x and y are in C then we set $\pi_1(x,y) = \pi(x,y)$, and otherwise we set $\pi_1(x,y) = k+1$.
2. v_2 has preference order $p > C'_1 > d > C'_2 > C - \{p\}$ (that is, v_2 ranks p first, then the $\ell - 1$ candidates from C'_1 followed by d, followed by the remaining candidates from C', which are then followed by the candidates from $C - \{p\}$). For each two candidates $x, y \in A$, we set $\pi_2(x,y) = k+1$.

Note that in our election $maj(V) = 2$. Further, the only two candidates that are ranked among the top $m+1$ positions of both voters are p and d. Candidate d has Bucklin score $\ell + 1$ and, thus, we have the following situation:

1. If p is ranked among the top ℓ positions in v_1, then p is the unique Bucklin winner of the election.
2. If p is ranked in the $(\ell+1)^{st}$ position by v_1, then both p and d are Bucklin winners.
3. If p is ranked in a position worse than the $(\ell+1)^{st}$ position by v_1, then d is the unique Bucklin winner.

We claim that p can become a Bucklin winner of election E through a swap bribery of cost at most k if and only if I is a yes-instance of SINGLE-VOTE SWAP BRIBERY.

If: Assume that I is a yes-instance of SINGLE-VOTE SWAP BRIBERY. This means that there is a sequence of swaps within v after which p is ranked among the top ℓ positions in v. Applying the same swaps to v_1 would cost the same and would put p among the top $\ell + 1$ positions in v_1, making p a Bucklin winner.

Only if: Assume that there is a cost-at-most-k sequence of swaps within V that make p a Bucklin winner. Since any swap that is not in the v part of v_1 costs $k+1$, we have that d's Bucklin score is still $\ell + 1$, and, thus, after the swaps, p's Bucklin score is in $\{2, \dots, \ell+1\}$. Executing the same swaps within v shows that I is a yes-instance of SINGLE-VOTE SWAP BRIBERY.

For the unique-winner model, simply move d one position lower in v_2.

To establish that BV-DUSB in the co-winner model also is NP-complete for the case of two voters, it suffices to use the unique-winner construction from the proof of Theorem 3.52, but with the goal to prevent candidate d from being a Bucklin winner (the reader can see that p is the only candidate who can threaten d without exceeding the given budget). For the unique-winner destructive case, it suffices to use the BV-CUSB co-winner constructive construction, but with the goal to prevent candidate d from being a unique Bucklin winner. ❑

With Observation 3.36 and Lemma 3.37 we obtain the complexity of the remaining cases in Bucklin elections and all results for fallback voting.

Corollary 3.53 *The problems* BV-CWSB, BV-DWSB, FV-CUSB, FV-DUSB, FV-CWSB, *and* FV-DWSB *are* NP*-complete, each in both winner models, even for elections with only two voters.*

Extension Bribery Now we turn to the analysis of extension bribery in fallback elections. To simplify our proofs, observe the following.

Observation 3.54 *In (constructive) extension bribery problems for the fallback rule it is never profitable to extend any vote in any other way than by asking the voter to include the designated candidate on the last unranked position.*

This allows us to give the extension bribery price functions by only specifying the cost of extending the vote by one candidate. We call the cost of it the *extension cost* of the vote.

Theorem 3.55 *For elections with at least three candidates, both* FV-CWEB *and* FV-DWEB *are* NP*-complete, each in both winner models.*

Proof. To show NP-hardness, we use a reduction from PARTITION. Note that our reduction, which has three candidates, can be modified so that an election with any number $\mathfrak{m} \geq 3$ of candidates will be constructed: Simply add the needed number of candidates to C and let all voters disapprove of the newly added candidates. Let $(A, (a_1, \dots, a_k))$ with $A = \{1, \dots, k\}$ and $\sum_{i=1}^{k} a_i = 2K$ be an instance of PARTITION. We define the fallback election (C, V) with the candidate set $C = \{b, c, p\}$, the designated candidate is p, and we let V consist of $k+2$ voters in Table 3.35.

Group	For each ...	# voters	preference	weight	extension cost
(1)		1	$p \mid \{b, c\}$	K	$K+1$
(2)	$i \in \{1, 2, \dots, k\}$	1	$c \mid \{b, p\}$	a_i	a_i
(3)		1	$b \mid \{c, p\}$	K	$K+1$

Table 3.35: Voter list V in the proof of Theorem 3.55

The total sum of the voters' weights in this election is $4K$, thus $maj(V) = 2K + 1$. The weighted scores of the candidates in (C, V) are shown in Table 3.36a. As no candidate reaches the majority threshold, candidate c wins by approval score and is the unique fallback winner in (C, V).

We claim that there is a set $A' \subseteq A$ such that $\sum_{i \in A'} a_i = \sum_{i \notin A'} a_i = K$ if and only if p can be made a fallback winner by extension-bribing some of the voters without exceeding the budget K.

Only if: We assume that there is a set $A' \subseteq A$ such that $\sum_{i \in A'} a_i = \sum_{i \notin A'} a_i = K$. We can change the votes $c \mid \{b, p\}$ to $c > p \mid \{b\}$ from those voters in the second voter group where

	b	c	p
$score$	K	**2K**	K

(a) Scores in (C,V)

	b	c	p
$score^1$	K	$2K$	K
$score^2$	K	**2K**	**2K**

(b) Scores in (C,V')

Table 3.36: Scores in the election constructed in the proof of Theorem 3.55

$i \in A'$. Each of these changes costs a_i, so the overall sum of the costs is K. The candidates have the weighted scores in the resulting election (C,V') that are shown in Table 3.36b. So p can be made a fallback winner by extension-bribing voters in V without exceeding the budget K.

If: We assume that p is a fallback winner in election (C,V'), where V' is the changed voter list and the costs for the changes are at most K. Since the cost limit is K the only changes that can be made, and that are profitable for p, are adding p to the approval strategies of some of the voters in the second voter group. The weighted score of candidate c cannot be decreased, so p has to gain K points to tie with candidate c. Hence, there has to be a set $A' \subseteq A$ such that $\sum_{i \in A'} a_i = \sum_{i \notin A'} a_i = K$ and p has to be added to the approval strategies of the voters from the second voter group where $i \in A'$.

For the unique-winner case of FV-CWEB, only the weight of the voter in the first voter group has to be changed to $K+1$ in the above election.

To show the result for the destructive case, for the co-winner model it suffices to use the same construction as for the constructive unique-winner case, with the goal to prevent c from winning (it can be accomplished either by p or by b). Similarly, for the unique-winner destructive case, we use the same construction as for the co-winner constructive case, with the goal to prevent c from being a unique winner (again, either p or b can be used for this purpose). ❑

Turning to unweighted fallback elections, we see that the extension bribery problem becomes tractable.

Theorem 3.56 FV-CUEB *and* FV-DUEB *are in* P*, each in both winner models.*

Proof Sketch. Algorithm 3.4 solves the problem in polynomial time. The algorithm considers each level s in which p could possibly become a fallback winner and tries the cheapest bribery that might achieve this.

The algorithm clearly runs in polynomial time and its correctness follows from Observation 3.54. Furthermore, it can easily be adapted to solve the problem in the unique-winner case. With that and some small adaptions, also the destructive cases can be solved. ❑

Algorithm 3.4: Algorithm for fallback-CUEB

input : C set of candidates
V list of voters
$\Delta = (\delta_1, \ldots, \delta_n)$ list of extension bribery price functions
k budget
p designated candidate
output: “YES” if $(C, V, \Delta, k, p) \in$ fallback-CUEB
“NO” if $(C, V, \Delta, k, p) \notin$ fallback-CUEB

foreach $s \in \{1, \ldots, \|C\|\}$ **do**
  let $(v'_1, \ldots, v'_r)$ be a sublist of V containing voters that approve of at most $s-1$ candidates and do not approve of p, sorted by extension costs in ascending order;
  foreach $t \in \{0, \ldots, r\}$ **do**
    if *changing* $v'_1, \ldots v'_t$ *to approve* p *makes* p *a fallback winner* **then**
      if *the sum of extension costs of* $v'_1, \ldots, v'_t$ *is less than or equal to* k **then**
        return “YES”;
return “NO”;

3.4 Concluding Remarks and Future Work

In this chapter we have studied the complexity of manipulation, control, bribery, and campaign management problems in Bucklin and fallback elections.

Manipulation Regarding manipulation, we have provided a complete study of the classical worst-case complexity for constructive and destructive manipulation by a coalition of manipulators in weighted and unweighted elections. For both voting systems manipulation is tractable with the one exception of constructive coalitional manipulation in weighted Bucklin elections. The reduction for proving this latter result, however, starts from the problem PARTITION, which is NP-complete, but not in the strong sense. This means that it can be solved in pseudo-polynomial time, see the textbook by Garey and Johnson [GJ79] for further insight regarding this matter. Thus, aiming at developing such a pseudo-polynomial time algorithm tailored to the specific setting of constructive coalitional manipulation in weighted Bucklin elections (or proving that no such algorithm can exist unless P = NP) would be a natural next step.

Electoral Control In Section 3.2 we have given a comprehensive study of the computational complexity of all standard types of electoral control in Bucklin and fallback elections, extending the studies by Erdélyi and Rothe [ER10], Erdélyi et al. [EPR11], and Erdélyi and Fellows [EF10b]. With a total of only two vulnerabilities, fallback voting is, together with normalized range voting [Men13], the voting system with the broadest resistance to electoral control currently known to hold. Bucklin voting is a good candidate for a similar behavior if the unsatisfyingly unsettled complexity of DESTRUCTIVE CONTROL BY PARTITION OF VOT-

ERS IN MODEL TP can be resolved and the problem can be shown to be NP-hard. Our study of the parameterized variants of adding and deleting candidates and voters strengthens the known unparameterized resistance results as we could establish W[2]-hardness for all investigated parameterized control problems. Whether these problems are W[2]-complete remains as an interesting open question for future research as well as natural parameterizations for control by partitioning either the set of candidates or the list of voters.

Summarizing our experimental results, we have shown that for the considered election sizes and those voter generation models we used in our setting, destructive control could be efficiently solved by our greedy approach, while the results for the constructive cases are more complex. This leads to the conjecture that the natural parameterization by the number of deleted/added voters might not be fine-grained or expressive enough to give information about the behavior of instances actually occurring in practice. Even though parameterized complexity offers a more differentiated worst-case analysis with respect to the considered parameter than NP-hardness, we have seen that a further experimental analysis can provide further insights. The presented experimental analysis is a first step and future research can focus on improving the experimental setting. Just as Walsh [Wal09, Wal10] observes for manipulation in the veto rule and in STV, for all types of control investigated in our experiments, the curves do not show the typical phase transition known for "really hard" computational problems such as the satisfiability problem (see [GW95, CKT91] for a detailed discussion of this issue). These observations raise the question of how other distribution models influence the outcome of such experiments. For further such models see, for instance, the work of Berg [Ber85], Mallows [Mal57], and Luce [Luc05]. Furthermore, the algorithms implemented could be improved in terms of considering a higher number of elections per data point, increasing the election sizes, or allowing a higher number of voters or candidates to be deleted or added in the corresponding control scenarios. Also the tuning of the timeout-parameter could allow for further results either by simply increasing the value or varying the value depending on other parameters of the given instance such as the election size. Besides this, other voting systems can be analyzed since only their winner determination has to be implemented in addition to a few minor adjustments such as trivial-case checks for the investigated control scenarios tailored to the voting system at hand. Also empirical studies based on real-world election data instead of mere simulations of randomly generated elections can be considered for future research.

Bribery and Campaign Management We have shown that constructive standard bribery in Bucklin and fallback election is intractable while the destructive cases can be solved in deterministic polynomial time when the voters either are weighted or have prices, but not both at the same time. Combining both weights and prices leads to NP-completeness of the corresponding bribery problems also in the destructive case.

Swap bribery, on the other hand, turns out to be intractable in all considered scenarios for both voting systems. The complexity of extension bribery, which we tailored to fallback voting only, highly depends on whether weighted or unweighted votes are given. For the former case we established NP-completeness and showed that in the latter case the problem is tractable in the destructive scenario as well as in the constructive case.

Directions for Future Work Summarizing our findings, we provide a complete picture of the classical worst-case complexity of all standard manipulative attacks on Bucklin and fallback elections.

Regarding electoral control, we were able to further differentiate our analysis by studying the complexity of parameterized problem variants and also conducting an experimental evaluation. Such studies certainly are also of high interest in the context of bribery and campaign management.

For all considered voting problems, a complexity-theoretic analysis of the intractable cases parameterized by the number of voters or candidates participating in an election is worth pursuing. These parameterizations are especially interesting as they allow us to analyze the behavior of the given NP-hardness shield in small-scale elections, see the work of [CFN+15, Yan14]. Furthermore we suggest the development of heuristics and approximation algorithms, not only for those problems that turned out to be intractable. Especially for the manipulation scenarios, the study of distributed heuristics minimizing the cost of communication between the manipulators seems to be a promising direction for future work.

Regarding bribery, campaign management, and control it would also be interesting and challenging to complement the existing worst-case study by a typical-case analysis as has been done for manipulation (see, for example, the work of Conitzer et al. [CSL07], Procaccia and Rosenschein [PR07], Friedgut et al. [FKN08], Isaksson et al. [IKM12], Mossel and Rácz [MR12a], and Xia and Conitzer [XC08b, XC08a]).

Finally, we suggest to address the unsettled problems for other voting rules that are shown in Tables 3.2, 3.6, and 3.28: The complexity of coalitional manipulation in weighted Schulze elections is yet unsettled in the constructive and the destructive cases while for weighted Copeland$^{0.5}$ elections only the constructive case is unresolved. Electoral control in Borda elections has only been partly studied, for the cup rule a study of control complexity is completely missing. Unweighted Schulze elections have not been analyzed with respect to the complexity of swap bribery and also the complexity of destructive unweighted swap bribery for the Borda rule remains as an open problem.

4 The Margin of Victory and Destructive Bribery

The respect for democratic processes such as elections highly depends on the reliability of the officially determined outcome. If voters can have reasonable doubt that the election results are correct, this might not only have an impact on their willingness to participate in future elections, but can also have legal consequences.

Clearly, errors in elections, for example regarding the tallying of the ballots can occur in practice. Reasons for such irregularities can reach from accidental miscounts of votes to intended illegal election fraud. Especially when electronic voting machines are used, the problem becomes obvious as there are many ways to alter an election's outcome by attacks on these machines, see for example the informative case study on the security of voting machines widely used in India in the work of Wolchok et al. [WWH+10], and the publication by Epstein [Eps15] addressing the security issues of a recently decertified voting machine that was used in the state of Virgina, United States of America. One way of addressing this issue is to equip voting machines such that so-called *verifiable paper records* are produced and whenever too many mismatches are found, the votes have to be recounted [NBH+07]. Clearly, this attempt only allows for spotting counting errors, but does not solve the problem of attacks on the software of the machines, see the detailed discussion on this point in [Nor06].

To avoid costly recounts of the complete set of ballots the so-called *margin of victory* of an election can be used. It is defined as the smallest number of votes that has to be changed to alter an election's outcome and thus defines a measure of robustness of elections. The larger the margin of victory of an election is, the robuster the outcome is against changes in the ballots. This notion can be used to define risk-limiting audit methods, as Stark [Sta09, Sta10] suggests, which sequentially test batches of randomly sampled votes from an election until either the tested votes statistically strongly suggest that the election's outcome is correct or all votes have been recounted in the process. Much effort has been put into the design of such audit methods tailored to specific voting systems such as scoring rules, approval voting, range voting, and instant run-off voting, see the work of Sarwate et al. [SCS13].

Considering the notion of *bribery* in elections defined as the situation in which a briber tries to change an elections outcome by changing a given number of votes [FHH09], we see that the margin of victory and bribery are closely related.

In this chapter we continue the study of Xia [Xia12] who introduced a decision problem corresponding to the margin of victory and studied its computational complexity for various voting systems. We furthermore expand the study's scope by defining further variants of the margin of victory problem.

Organization of this Chapter In the first section we introduce the three different notions of the margin of victory we want to study and present relations of these problems to certain cases of destructive bribery. We conclude the section with an overview of related work and the new results that will be presented in the following three sections: Section 4.2 provides our study of the complexity of the standard margin of victory problem in cup and fallback elections while in Section 4.3 the exact variant is studied for Schulze, Copeland, cup, and fallback elections. These results were published in [RRS14]. In Section 4.4 we present yet unpublished results regarding the complexity of the swap margin of victory problem in elections held under certain positional scoring rules and the cup rule. The last section gives some concluding remarks and pointers to potential future work.

4.1 Variants of the Margin of Victory: Definitions and Related Work

We now present the formal definitions of the different variants of the margin of victory and the corresponding decision problems. We furthermore explore the relations to destructive bribery in elections and expand the study Xia [Xia12] started. In particular, we show that for multi-winner voting systems, theoretically, destructive bribery can be easy while the margin of victory problem is intractable.

4.1.1 The Margin of Victory

We start with the standard variant of the margin of victory that has been studied in the related work we stated so far and that also Xia [Xia12] considers.

Definition 4.1 (Margin of Victory) *For a given voting system $\mathscr{E}$ and a given $\mathscr{E}$ election (C,V), we define the* margin of victory *to be the smallest nonnegative integer ℓ such that the winner set can be altered by changing ℓ votes in V, while the other votes remain unchanged. We will use the notation* $\textsc{MoV}(\mathscr{E},(C,V)) = \ell$.

The margin of victory gives us a measure of robustness for a given election. To study this measure of robustness for given $\mathscr{E}$ elections for a certain voting system $\mathscr{E}$ from a computational perspective, Xia [Xia12] introduced the decision problem called $\mathscr{E}$-MoV.

$\mathscr{E}$-Margin of Victory ($\mathscr{E}$-MoV)	
Given:	An $\mathscr{E}$ election (C,V) and a positive integer k.
Question:	Is $\textsc{MoV}(\mathscr{E},(C,V)) \leq k$?

In other words, in the $\mathscr{E}$-MoV problem we ask for a given election and a given k whether the set of winners can be altered by changing at most k votes. For successfully altering the winner set, the changes in the votes may lead to new winning candidates, can prevent former winners from winning, or both at the same time. It becomes apparent that the margin of victory and the standard bribery scenarios introduced by Faliszewski et al. [FHH09] are highly related; especially the destructive case of unweighted bribery in the unique winner model which is defined as follows (see also Section 3.3 for further discussion on this model and for the definitions of other bribery scenarios).

$\mathscr{E}$-UNIQUE DESTRUCTIVE UNWEIGHTED BRIBERY ($\mathscr{E}$-UDUB)	
Given:	An $\mathscr{E}$ election (C,V), a designated candidate $p \in C$, and a positive integer k.
Question:	Is it possible to prevent p from being a unique $\mathscr{E}$ winner by bribing at most k voters, i.e., changing their votes?

For elections with unique winners Xia [Xia12] established the following connection between the $\mathscr{E}$-MOV problem and destructive unweighted bribery in the unique winner model.

Proposition 4.2 ([Xia12]) *Let $\mathscr{E}$ be a voting system that always selects a unique winner of an election in deterministic polynomial time and satisfies $\mathscr{E}$-MOV $\neq \emptyset$. Then $\mathscr{E}$-MOV and $\mathscr{E}$-UDUB are $\leq_m^p$-equivalent, i.e., $\mathscr{E}$-MOV $\leq_m^p$ $\mathscr{E}$-UDUB and $\mathscr{E}$-UDUB $\leq_m^p$ $\mathscr{E}$-MOV.*

The equivalence of both problems does only hold when the voting system always selects unique winners. If more than one winner can occur, the following corollary holds.

Corollary 4.3 *Let $\mathscr{E}$ be a voting system with $\mathscr{E}$-MOV $\neq \emptyset$ and that determines the set of winners in deterministic polynomial time. Then $\mathscr{E}$-UDUB $\leq_m^p$ $\mathscr{E}$-MOV holds.*

Proof. Let (C,V,p,k) be an $\mathscr{E}$-UDUB instance, that is, we have the $\mathscr{E}$ election (C,V) and candidate p is the unique winner of (C,V). We map (C,V,p,k) to the $\mathscr{E}$-MOV instance (C,V,k) and claim that $(C,V,p,k) \in \mathscr{E}$-UDUB if and only if $(C,V,k) \in \mathscr{E}$-MOV.

Only if: We assume that p can be prevented from being the unique winner by changing at most k votes in V. This directly implies that $(C,V,k) \in \mathscr{E}$-MOV since the k changes in V successfully alter the winning set.

If: Assume that the set of winners of (C,V) can be changed by changing at most k votes. We call the new set of winners W and note that either $W \subseteq C - \{p\}$ holds (this is the case if p was prevented from being a winner by the changes in the votes) or $W = \{p\} \cup W'$ for $W' \subseteq C - \{p\}$ (this is the case if other candidates were made winners due to the changes in the votes). In both cases, p has been prevented from being a unique winner in the election by changing at most k votes, thus $(C,V,p,k) \in \mathscr{E}$-UDUB. ❑

To show that in the above case $\mathscr{E}$-MOV $\leq_m^p$ $\mathscr{E}$-UDUB in fact does not hold (unless it could be shown that P equals NP), we construct a nonneutral voting rule that serves as a counterexample.

Theorem 4.4 *There exists a voting system $\mathscr{K}$ such that $\mathscr{K}$-*UDUB $\in$ *P but $\mathscr{K}$*-MOV *is* NP-*complete.*

Proof. We prove this claim by constructing a voting system $\mathscr{K}$ that always outputs at least two winners if there are at least two candidates. For an election (C,V) with $C = \{p\} \cup C'$ winner determination in $\mathscr{K}$ proceeds as follows:

$$\mathscr{K}(C,V) = \begin{cases} p & \text{if } C = \{p\} \\ \{p\} \cup \text{cup}(C',V) & \text{otherwise.} \end{cases}$$

It is easy to see that $\mathscr{K}$-UDUB $\in$ P: If $C = \{p\}$, then p is always a unique winner, but due to the definition of the voting system, this cannot be changed. So we can easily decide that this is a no-instance of $\mathscr{K}$-UDUB. If there are more than two candidates we always have two winners which means that in this case we always have trivial yes-instances of $\mathscr{K}$-UDUB.

It remains to show that $\mathscr{K}$-MOV is NP-complete. Membership in NP is easy to see. By looking closely at the winner determination we can see that for elections with more than two candidates, the winner set can only be changed by changing the winner set of the cup election (C',V). Thus, we can easily construct a trivial reduction cup-MOV $\leq_{\text{m}}^{\text{p}}$ $\mathscr{K}$-MOV and follow NP-hardness of $\mathscr{K}$-MOV from NP-hardness of cup-MOV which we show in Theorem 4.8. ❑

Whether there exists a neutral voting rule satisfying the same properties as the voting rule $\mathscr{K}$ in the proof above is an interesting but yet unsolved open question.

4.1.2 The Exact Margin of Victory

The standard $\mathscr{E}$-MOV problem asks the question whether the margin of victory of an election is smaller than or equal to a given nonnegative integer. Thus, when given a yes-instance for a fixed k it is unclear, whether the margin of victory actually reaches this upper bound of k or whether it equals 1 (the smallest value possible).

One way of bypassing this fact is to define a variant that asks whether the margin of victory lies in a predetermined interval. Note that from a computational complexity point of view, the size of this interval is irrelevant in the sense that when DP-completeness can be shown for an interval consisting of only one integer, increasing the size of the interval does not change the problem's complexity, see the work of Wagner [Wag87]. We make use of this fact and define our exact variant of the margin of victory problem as follows.

$\mathscr{E}$-EXACT MARGIN OF VICTORY ($\mathscr{E}$-XMOV)	
Given:	An $\mathscr{E}$ election (C,V) and a positive integer k.
Question:	Is MOV$(\mathscr{E},(C,V)) = k$?

Clearly, $\mathscr{E}$-XMOV $= \mathscr{E}$-MOV $\cap \overline{\mathscr{E}\text{-MOV}}$, thus we can state the following remark.

Remark 4.5 *If $\mathscr{E}$*-MOV $\in$ P *holds for a voting system $\mathscr{E}$, then also $\mathscr{E}$*-XMOV $\in$ P *holds.*

4.1.3 The Swap Margin of Victory

When a voter is faced with the task of providing a complete ranking over a large list of candidates, there is a possibility that small errors can occur and faulty preferences are reported. Also when votes are stored electronically errors may occur during the process, leading to slightly changed votes. We want to address such errors on a smaller scale and analyze the effect of single changes in the votes on an election's outcome. As we are mostly concerned with elections, where the votes are linear orders, we define a single change to be a swap of two adjacent candidates.[1]

We base our formal definition on the model introduced by Shiryaev et al. [SYE13]. They define a measure of robustness restricted to elections with unique winners, the so-called *robustness radius* of election $E = (C,V)$ with respect to the winning candidate c, to be the smallest value δ such that there exists an election $E' = (C,V')$ resulting from E by conducting at most δ swaps in the votes where c is not a unique winner of E'.

We, however, also allow elections with more than one winner and analyze the number of changes needed to change this given winner set. We consider this definition to be a natural extension of the margin of victory introduced above to the scenario where the single changes in the votes are counted.

Definition 4.6 (Swap Margin of Victory) *For a given voting system $\mathscr{E}$ and a given $\mathscr{E}$ election (C,V), we define the* swap margin of victory *to be the smallest nonnegative integer ℓ such that the winner set can be changed by conducting a sequence of at most ℓ swaps in the votes in V, while no further changes are made. We will use the notation* $\text{SWMOV}(\mathscr{E},(C,V)) = \ell$.

Based on this definition, we will analyze the following decision problem in Section 4.4.

$\mathscr{E}$-SWAP MARGIN OF VICTORY ($\mathscr{E}$-SWMOV)	
Given:	An $\mathscr{E}$ election (C,V) and a positive integer k.
Question:	Is $\text{SWMOV}(\mathscr{E},(C,V)) \leq k$?

Just as the previously defined standard margin of victory, the swap margin of victory is closely related to a known destructive bribery scenario. Elkind et al. [EFS09] defined constructive swap bribery in elections by assigning a so-called *swap-bribery price function* $\delta_i : C \times C \to \mathbb{N}$ for every voter v_i as a function that specifies for each ordered pair (c_i, c_j) of candidates the price for changing v_i's preference order from $\cdots > c_i > c_j > \cdots$ to $\cdots > c_j > c_i > \cdots$. Only candidates that are adjacent in a vote can be swapped. By changing the question in their definition of CONSTRUCTIVE UNWEIGHTED SWAP BRIBERY to whether the designated candidate can be prevented from being the unique winner of the resulting election, we obtain the definition of DESTRUCTIVE UNWEIGHTED SWAP BRIBERY in the unique winner model. (See also Section 3.3 for further discussion and definitions of various bribery scenarios.) In particular, we will study the special case of the destructive swap bribery problem where each

[1] See Section 3.3.1 for a similar discussion in the context of bribery and swap bribery.

swap has unit costs, that is $\delta_i(c,d) = 1$ for all $v_i \in V$ and all $c,d \in C$ with $c \neq d$. Formally, this problem is defined as follows.

$\mathscr{E}$-UNIQUE DESTR. UNWEIGHTED SWAP BRIBERY WITH UNIT COSTS ($\mathscr{E}$-UDUSB-UC)	
Given:	An $\mathscr{E}$ election $E = (C,V)$, a distinguished candidate $c \in C$, and a budget k.
Question:	Is there a sequence of at most k swaps such that c is not a unique winner in the changed election?

Looking closely at the definitions of the $\mathscr{E}$-SWMOV and the $\mathscr{E}$-UDUSB-UC problem, it becomes clear that we have analog connections between both problems as we have stated for their standard variants in Proposition 4.2 and Corollary 4.3. We summarize both statements in one corollary of which we omit the proof since it is a straightforward adaption of the proofs of Corollary 4.3 and Proposition 4.2.

Corollary 4.7 *Let $\mathscr{E}$ be a voting system with $\mathscr{E}$-SWMOV $\neq \emptyset$ that determines the winners of an election in deterministic polynomial time. Then the following holds.*

1. $\mathscr{E}$-UDUSB-UC $\leq_m^p$ $\mathscr{E}$-SWMOV.
2. *If $\mathscr{E}$ always determines unique winners, then* $\mathscr{E}$-SWMOV $\leq_m^p$ $\mathscr{E}$-UDUSB-UC *also holds.*

Shiryaev et al. [SYE13] show that the NECESSARY WINNER problem in the unique-winner model ($\mathscr{E}$-UNW) polynomial-time many-one reduces to $\mathscr{E}$-UDUSB, see also Proposition 3.40 on page 82. This reduction, however, does not map to the special case of $\mathscr{E}$-UDUSB-UC as it requires that the costs of the votes can be chosen from $\{0,1\}$. Thus unfortunately, we cannot derive any results from this interesting connection in our setting.

4.1.4 Overview of Results and Related Work

After having introduced the basic definitions, we give an overview of the complexity of the different variants of the margin of victory problems in Table 4.1. It shows known results due to Xia [Xia12] as well as those results that we present in this thesis which were partly published in [RRS14].

The work of Xia [Xia12] provides a comprehensive study on the computational aspects of the margin of victory problem and its relations to destructive bribery focusing on the unique-winner model. Besides those voting systems that are displayed in Table 4.1, approval voting, STV, ranked pairs, plurality with run-off, and the maximin rule are studied. For the latter and for the family of Copeland$^\alpha$ elections, approximation algorithms are presented. Furthermore, for the large class of *continuous generalized scoring rules*, see also the work of Xia and Conitzer [XC08b], a dichotomy result regarding the typical size of the margin of victory is shown for elections, where votes are generated by randomly drawing the votes from all possible votes assuming that these are independent and identically distributed with respect to a given distribution.

	Cup	Scoring Rules	Fallback	Bucklin	Schulze	Copeland$^\alpha$
$\mathscr{E}$-MoV						
Compl.	NP-c.	P	P	P [1]	NP-c.	P
Ref.	Thm. 4.8	[Xia12]	Thm. 4.9	[Xia12]	[Xia12] [2]	[Xia12] [2]
$\mathscr{E}$-XMoV						
Compl.	DP-c.	P	P	P [1]	DP-c.	DP-c.
Ref.	Thm. 4.11	[Xia12] [2]	Thm. 4.12	[Xia12] [2]	Thm. 4.10	Thm. 4.11
$\mathscr{E}$-swMoV						
Compl.	NP-c.	P [3]	?	?	?	?
Ref.	Thm. 4.13	Thm. 4.15,4.14				

[1] simplified Bucklin [2] follows with results in [Xia12] [3] not for all scoring vectors
Key: NP-c. = NP-complete, DP-c. = DP-complete

Table 4.1: Overview of results for the margin of victory

Motivated by practical aspects, namely that exactly computing the margin of victory for a given election might be infeasible, Cary [Car11] and Magrino et al. [MRS$^+$11] study how the margin of victory can be estimated in elections held under instant run-off voting (IRV) and evaluate their findings empirically on real-world test data. See also the work by Sarwate et al. [SCS13], who present algorithms for computing upper and lower bounds for the margin of victory for several voting systems including IRV in order to develop further risk-limiting audit methods. Recently, Blom et al. [BST$^+$15] followed up on the approach taken by Magrino et al. [MRS$^+$11] and present improved algorithms for the estimation of the margin of victory in IRV elections.

Rather remotely related is the approach taken by Procaccia et al. [PRK07], who similarly to Shiryaev et al. [SYE13], focus on the error model considering single swaps in the given preferences but define robustness significantly different: In their setting, the robustness of a voting system is measured by the probability that the outcome of a given election alters when random swaps in the votes are performed. Their study covers many prominent voting rules, namely positional scoring rules, Copeland^0, maximin, simplified Bucklin, and plurality with run-off.

Note that, for the sake of readability, we abbreviate the given preferences and will use the shorthand abc for a preference $a > b > c$ in the upcoming proofs.

4.2 The Margin of Victory in Cup and Fallback Elections

We start with the NP-completeness result for cup-MoV which we obtain by showing the problem cup-uDUB NP-hard and by then applying Corollary 4.3. For the hardness proof we will construct a reduction from the well-known NP-complete problem Vertex Cover, see [GJ79].

VERTEX COVER (VC)	
Given:	An undirected graph $G = (V,E)$ and a positive integer k.
Question:	Is $\tau(G) \leq k$, i.e., is the size of a smallest vertex cover in G at most k?

Theorem 4.8 *For cup elections,* $\mathscr{E}$-MOV *is* NP-*complete.*

Proof. Membership to NP is easy to see, as the winner determination of the cup rule allows to check in deterministic polynomial time whether a change of k guessed votes alters the election's outcome. Thus, we have to show NP-hardness in detail. We do so by showing NP-hardness of cup-uDUB by a reduction from VERTEX COVER. In the proof we will use the so-called *UV technique* introduced by Faliszewski et al. [FHH+09b].

Let $G = (A,E)$ be an undirected graph with vertex set $A = \{a_1,a_2,\dots,a_n\}$ and edge set $E = \{e_1,e_2,\dots,e_m\}$, and let $k \in \mathbb{N}$. We construct the cup election (C,V) with $C = \{c,d\} \cup E \cup P \cup T$, where $P = \{p_1, p_2, \dots, p_m\}$ and T is a set of dummy candidates that will be used to ensure that the voting tree is balanced (we will come to that later). Let $S_a = \{e \in E \mid e \cap \{a\} \neq \emptyset\}$ be the set of edges incident to vertex $a \in A$.

V contains $2m(n+k-3)+6n+6k-3$ voters whose preferences are listed in Table 4.2. When a set of candidates, say $Z \subseteq C$, is given in a voter's preference, then we assume that the candidates in Z are ordered with respect to a (tacitly assumed) fixed order, while $\overleftarrow{Z}$ denotes that the candidates are ordered in reverse. In particular, we fix the order of the candidates in P to be $p_1 > p_2 > \cdots > p_m$.

Group	For each ...	# votes	Preference
(1)	$a \in A$	1	$c\ d\ S_a\ P\ (E-S_a)\ T$
	$a \in A$	1	$P\ c\ d\ \overleftarrow{(E-S_a)}\ \overleftarrow{S_a}\ T$
(2)		k votes	$c\ d\ P\ E\ T$
		k votes	$c\ d\ P\ \overleftarrow{E}\ T$
(3)		$2(n+k-2)$	$c\ E\ P\ d\ T$
		$2(n+k-2)$	$d\ \overleftarrow{E}\ P\ c\ T$
(4)	$i \in \{1,\dots,m\}$	$n+k-3$	$c\ (P-\{p_i\})\ p_i\ e_i\ (E-\{e_i\})\ d\ T$
	$i \in \{1,\dots,m\}$	$n+k-3$	$d\ \overleftarrow{(E-\{e_i\})}\ p_i\ e_i\ \overleftarrow{(P-\{p_i\})}\ c\ T$
(5)		1	$P\ c\ d\ E\ T$

Table 4.2: Voter list V in the proof of Theorem 4.8

The dummy candidates in T are always positioned at the bottom of each voter's preference, so they lose every pairwise comparison to the candidates in $C-T$. This implies that their position in the schedule is irrelevant, so we will omit them in Figure 4.1.

For the sake of readability and clarity, we will omit the dummy candidates in our further arguments, and we will use the voting tree and schedule shown in Figure 4.1. (To transform

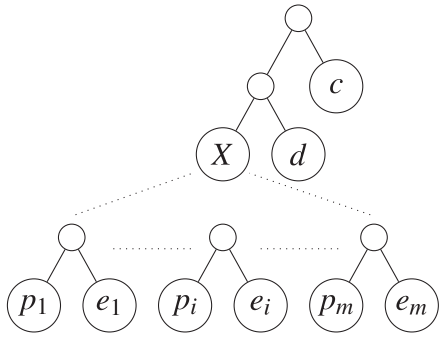


Figure 4.1: Voting tree of the cup election $(C-T,V)$ without dummies in the proof of Theorem 4.8

this tree into a complete binary tree (i.e., into a legal voting tree), the dummy candidates in T have to be added to the three subtrees with the roots d, c, and X, respectively.)

Since the height of the tree is in $\mathcal{O}(\log m)$, we have in total a polynomial number of leaves, which ensures that the reduction is in fact polynomial-time computable.

Table 4.3 shows the pairwise comparisons of the relevant candidates in $C-T$ and we see that c is the unique cup winner of this election.

$D_V(x,y)$	d	P	E
c	$> 4k$	$2k-1$	$2n+2k+1$
d	$-$	$2k-1$	$2n+2k+1$
P		$i<j: D_V(p_i,p_j) > 4k$	$i \neq j: D_V(p_i,e_j) = -2n-2k+5$
		$i \geq j: D_V(p_i,p_j) \leq 0$	$i = j: D_V(p_i,e_j) = -1$

Table 4.3: Pairwise comparisons of the candidates in $C-T$

We claim that G has a vertex cover of size at most k if and only if c can be prevented from being a unique cup winner by changing at most k votes.

Only if: Assume that $A' \subseteq A$ is a vertex cover of size k. Change the preferences of those k voters corresponding to A' in the first voter group from $c\,d\,S_a\,P\,(E-S_a)\,T$ to $P\,c\,d\,S_a\,(E-S_a)\,T$. Since A' is a vertex cover we have that due to these changes each $e_i \in E$ has one vote where she is positioned behind all candidates in P. So we have that each p_i wins her first pairwise comparison against e_i by one point. In the subelection corresponding to the subtree with root X (recall Figure 4.1), the relevant pairwise comparisons are among the candidates in P and due to the fixed ordering of these candidates in the votes, p_1 is the winner of this subelection. Both c and d have lost k votes in comparison to p_1 due to the bribe, so p_1 wins both pairwise comparisons and is thus the unique cup winner of this election. So c has been successfully prevented from winning.

If: Assume that c can be prevented from being a unique winner by bribing at most k voters. Due to the scores only candidates from P have a chance to prevent c from being a unique winner, so the following has to hold for the bribed election: A candidate from P, say p_1, has to be the winner of the subelection corresponding to the subtree with root X and p_1 has to win the pairwise comparisons against both d and c. For the latter to hold, all k bribed votes have to have p_1 positioned behind d and c (before the bribe). For the former to hold, no candidate in E may win her first contest, which implies that every $p_i \in P$ has to win the pairwise comparison against the corresponding candidate $e_i \in E$. So the votes that are bribed also have to rank the candidates in E better than those in P before the bribe is conducted. With this we see that the k bribed votes have to be from the first voter group and that the vertices corresponding to these votes have to form a vertex cover of size k to ensure that each $e_i \in E$ loses the first pairwise comparison. ❑

For fallback voting we can show a tractability result.

Theorem 4.9 *In fallback elections,* $\mathscr{E}$-MOV *is in* P.

Proof. Let $((C,V),k)$ be a given $\mathscr{E}$-MOV instance with a fallback election (C,V) and an integer k. We denote the set of fallback winners in (C,V) with W.

Assume that $\|W\| = 1$. To change W, we have to dethrone the current winner. To check whether this is possible by changing not more than k votes, we can use the known algorithm for fallback-UDUB, see Theorem 3.50 on page 90.

If $\|W\| \geq 2$ we can show that $\text{MOV}(\text{fallback},(C,V)) = 1$: Let there be two fallback winners $a,b \in C$, thus $W = \{a,b\}$, and let ℓ denote the level on which both candidates win the election. (Note that $\ell = \|C\|$ if they win by approval.) We know that $score^{\ell}_{(C,V)}(a) = score^{\ell}_{(C,V)}(b)$ and furthermore we can find a voter in V ranking candidate a among the top ℓ positions. By letting this voter disapprove of a while letting the rest of the vote remain unchanged, we can achieve that in this new election, a has one point less on level ℓ and is thus no longer a fallback winner. The same approach can be used if there are more than two winners. ❑

4.3 The Exact Margin of Victory in Schulze, Copeland, Cup, and Fallback Elections

In this section we present the complexity of the exact margin of victory in Schulze, Copeland, cup, and fallback elections. The former three results are published in [RRS14]. Furthermore we show the complexity of $\mathscr{E}$-XMOV in fallback elections.

We start with a DP-completeness result for the Schulze rule (recall its definition from Section 2.3).

Theorem 4.10 *For Schulze elections,* $\mathscr{E}$-XMOV *is* DP-*complete.*

Proof Sketch. Since Schulze-MOV is in NP, we know with

$$\begin{aligned}\text{Schulze-XMOV} &= \{(C,V,k) \mid \text{MOV}(\text{Schulze},C,V) = k\} \\ &= \{(C,V,k) \mid \text{MOV}(\text{Schulze},C,V) \leq k\} - \{(C,V,k) \mid \text{MOV}(\text{Schulze},C,V) \leq k-1\}\end{aligned}$$

that Schulze-XMOV $\in$ DP holds. For showing DP-hardness we provide a reduction from the DP-complete problem XVC (recall its definition from Section 2.2).

Let $G = (A, E)$ be an undirected graph with vertex set $A = \{a_1, a_2, \dots, a_n\}$ and edge set $E = \{e_1, e_2, \dots, e_m\}$, and let k be a positive integer. Without loss of generality, we assume that $6 \le k \le n$ and that $k - 1 \bmod 5 = 0$. Let $U = E_1 \cup E_2 \cup E_3$ be the marked union of three copies of E, which are denoted by $E_i = \{e_{i1}, e_{i2}, \dots, e_{im}\}$ for $i \in \{1,2,3\}$, and let $S_a = \{e_{ij} \mid e_j \cap \{a\} \neq \emptyset$ and $i \in \{1,2,3\}\}$ again denote the set of all edges in U that are incident to vertex $a \in A$.

We define the Schulze election (C, V), where $C = \{c,d,e,f,g,h,p\} \cup U$, and V is a list of $40n + 324k - 132$ voters, whose preferences are specified in Table 4.4. When a set of candidates, say $Z \subseteq C$, is given in a voter's preference, then we assume that the candidates in Z are ordered with respect to a (tacitly assumed) fixed order.

Figure 4.2 shows a subgraph of the weighted majority graph of this election in which, for the sake of readability, all those edges that are not relevant for the argumentation are omitted, namely edges with negative or zero weight and edges that are not relevant for determining the strengths of the strongest paths. Table 4.5 shows the strengths of the relevant strongest paths in (C, V). We can see that candidate c is the unique Schulze winner in this election.

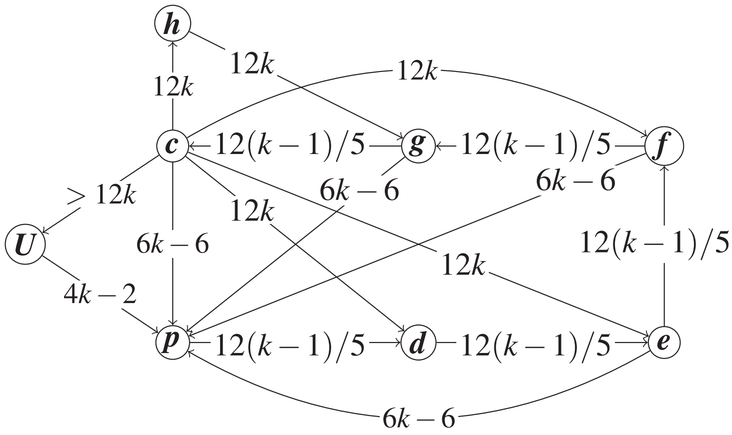


Figure 4.2: Subgraph of the WMG of the Schulze election (C, V)

The following properties of the constructed elections are useful for proving the correctness of the reduction: Since candidate c is the unique Schulze winner in the election, the winner set can only be changed by achieving $P(c,x) \le P(x,c)$ for at least one candidate $x \in C - \{c\}$. Further, since

$$P(c,x) - 2k \ge 12k - 2k > \frac{12(k-1)}{5} + 2k \ge P(x,c) + 2k$$

holds for all candidates $x \in C - \{c,p\}$, only p can tie with c when no more than k votes can be changed. So it suffices to focus on the paths leading from c to p, and vice versa. From p to c, the only reasonable path is $((p,d),(d,e),(e,f),(f,g),(g,c))$. From c we can reach p either directly, or via the candidates in U, or via a path having one of the candidates in $\{e,f,g\}$ as the second-to-last vertex.

The path $((p,d),(d,e),(e,f),(f,g),(g,c))$ consists of five edges and since there is no pref-

Group	**For each ...**	**# voters**	**Preference**	**Type**
(1)	$a \in A$	1	$h\ c\ g\ f\ e\ S_a\ p\ d\ (U - S_a)$	1
	$a \in A$	1	$c\ g\ e\ f\ d\ S_a\ p\ (U - S_a)\ h$	1
	$a \in A$	1	$h\ c\ g\ f\ d\ e\ S_a\ p\ (U - S_a)$	1
	$a \in A$	1	$c\ f\ g\ e\ d\ S_a\ p\ (U - S_a)\ h$	1
	$a \in A$	1	$h\ g\ c\ f\ e\ d\ S_a\ p\ (U - S_a)$	1
(2)		n	$d\ p\ e\ f\ g\ c\ U\ h$	2
		n	$h\ p\ d\ f\ e\ g\ c\ U$	2
		n	$p\ e\ d\ f\ g\ c\ U\ h$	2
		n	$h\ p\ d\ e\ g\ f\ c\ U$	2
		n	$p\ d\ e\ f\ c\ g\ U\ h$	2
(3)		$12(k-1)/10$	$h\ p\ d\ e\ f\ g\ c\ U$	2
		$12(k-1)/10$	$p\ d\ e\ f\ g\ c\ U\ h$	2
(4)		$3k - 3 + 12(k-1)/10$	$h\ e\ f\ g\ c\ p\ d\ U$	3
		$3k - 3 + 12(k-1)/10$	$d\ c\ g\ f\ e\ p\ U\ h$	3
(5)		$6k + 12(k-1)/10$	$h\ p\ g\ c\ f\ e\ d\ U$	2
		$6k + 12(k-1)/10$	$c\ d\ e\ f\ g\ p\ U\ h$	3
		$6k + 12(k-1)/10$	$h\ p\ c\ f\ g\ e\ d\ U$	2
		$6k + 12(k-1)/10$	$g\ d\ e\ f\ c\ p\ U\ h$	3
		$6k + 12(k-1)/10$	$h\ p\ g\ c\ e\ f\ d\ U$	2
		$6k + 12(k-1)/10$	$f\ d\ e\ c\ g\ p\ U\ h$	3
(6)		$6k$	$c\ h\ g\ e\ f\ p\ d\ U$	3
		$6k$	$d\ p\ f\ e\ g\ c\ h\ U$	2
		$6k$	$h\ g\ c\ e\ f\ p\ d\ U$	3
		$6k$	$d\ p\ f\ e\ c\ h\ g\ U$	2
(7)		$5n + 41k - 20$	$h\ d\ c\ g\ E_1\ E_2\ p\ E_3\ f\ e$	$4a$
		$5n + 41k - 20$	$e\ f\ E_2\ E_1\ p\ E_3\ g\ c\ d\ h$	$4b$
		$5n + 41k - 20$	$h\ d\ c\ f\ E_1\ E_3\ p\ E_2\ g\ e$	$5a$
		$5n + 41k - 20$	$e\ g\ E_3\ E_1\ p\ E_2\ f\ c\ d\ h$	$5b$
		$5n + 41k - 20$	$h\ d\ c\ e\ E_2\ E_3\ p\ E_1\ f\ g$	$6a$
		$5n + 41k - 20$	$g\ f\ E_3\ E_2\ p\ E_1\ e\ c\ d\ h$	$6b$

Table 4.4: Voter list V in the proof of Theorem 4.10

erence in V with the ordering $c\,g\,f\,e\,d\,p$, the weight of only four of these edges can be increased by 2 when one vote is changed. Thus, changing five votes can increase the strength of the path by at most 8 and the maximum value of 8 can only be achieved when the weight of each edge is increased by 8.

Generalizing this observation, by changing at most $k-1$ votes in V, we have that in the new

x	d	e	f	g	h	p	U
$P(c,x)$	$12k$	$12k$	$12k$	$12k$	$12k$	$6k-6$	$>12k$
$P(x,c)$	$\frac{12(k-1)}{5}$	$\frac{12(k-1)}{5}$	$\frac{12(k-1)}{5}$	$\frac{12(k-1)}{5}$	$\frac{12(k-1)}{5}$	$\frac{12(k-1)}{5}$	$\frac{12(k-1)}{5}$

Table 4.5: Strengths of the strongest paths in (C,V)

election $P(p,c) \leq \frac{12(k-1)}{5} + \frac{8(k-1)}{5} = 4(k-1)$ holds. This upper bound for $P(p,c)$ also holds when k changes are allowed.

By showing the following, the main argument for the reduction's correctness is given.

$$\text{MOV}(\text{Schulze},(C,V)) \begin{cases} = k-1 & \text{if } \tau(G) < k \\ = k & \text{if } \tau(G) = k \\ > k & \text{otherwise,} \end{cases} \tag{4.1}$$

where, recall, $\tau(G)$ denotes the size of a smallest vertex cover in G.

We show only the third part in detail since it is the most involved one. To this end let $\tau(G) > k$ hold. To construct a contradiction, we assume that $\text{MOV}(\text{Schulze},(C,V)) \leq k$. We know that $P(p,c) \leq 4(k-1)$ and by changing $k-1$ votes, $D_{V'}(x,p) \leq 4(k-1)$ for all $x \in \{c,e,f,g\}$ must be achieved, where V' denotes the election with the changed votes. This implies that p has to be positioned behind the candidates in $\{c,e,f,g\}$ in at least $k-1$ of the changed votes. Thus, we have that up to k votes of type 1 or type 3 might be changed, but at most two votes of type $4,5$, or 6, or at most one vote of type 2. Note that if two votes of the types $4,5$, or 6 are changed, both votes have to be of the same type, e.g., $4a$ and $4b$, or $6a$ and $6b$. Otherwise, when for example a type-$4b$ vote and a type-$6a$ vote are changed, candidate d would remain unaffected by this change and $D_{V'}(d,p) > 4k-4$ would hold, where again, V' denotes the changed voter list. Each other possible pairing of votes of different types leads to $D_{V'}(x,p) > 4k-4$ for at least one candidate $x \in \{c,e,f,g\}$.

Furthermore the weight of every edge from U to p has to be decreased by at least 2, so that all paths from c to p have a strength of at least $4k-4$. Let $V_1 \subseteq V$ denote the sublist of V of size at most k that have to be changed to ensure a different winner set. We distinguish the following two cases.

1. V_1 consists only of votes of type $1,2$, and 3: Those votes of type 1 have to induce a vertex cover for G since all e_{ij} have to be positioned better than p in at least one vote. So $\tau(G) \leq k$, which contradicts the assumption that $\tau(G) > k$.
2. V_1 consists only of votes of type $1,2,3$, and two votes of either type $4,5$, or 6: For the latter two votes (which have to be of the same type) we have that one of the sets E_i, $i \in \{1,2,3\}$, is positioned behind p. So the pairwise comparison between these candidates and p is not affected by the change. This, however, implies that those votes of type 1 that are changed, have to induce a vertex cover which contains less than k elements. This contradicts the assumption that $\tau(G) > k$.

With (4.1), the correctness of the reduction can be followed straightforwardly. ❑

We state the DP-completeness result for the XMOV problem in cup and Copeland$^{\alpha}$ elections without proof as the final argumentation is very similar to that in the proof of Theorem 4.10.

Theorem 4.11 *For cup elections and for Copeland$^{\alpha}$ elections the problem $\mathscr{E}$-XMOV is* DP-*complete.*

Finally we state the following result that directly follows with Remark 4.5 and Theorem 4.9.

Theorem 4.12 *For fallback elections the $\mathscr{E}$-XMOV problem is in* P.

4.4 The Swap Margin of Victory in Cup Elections and Positional Scoring Rules

In this section we present the results regarding the complexity of the newly defined problem $\mathscr{E}$-Swap Margin of Victory. These results are not published yet. We start with an NP-completeness result in cup elections. The reduction is an adaption of the proof of Theorem 4.8 showing that in cup elections the problems cup-UDUB and cup-MOV both are NP-complete. Similar to the proof of Theorem 4.8, we first show that cup-UDUSB-UC is NP-hard and follow with this result and Corollary 4.7 NP-hardness for cup-SWMOV.

Theorem 4.13 *In cup elections, the destructive swap bribery problem with unit prices is* NP-*complete. This implies that in cup elections $\mathscr{E}$-SWMOV is* NP-*complete, as well.*

Proof. Membership to NP can be easily established by guessing k changes in the votes and testing whether the changes alter the set of winners. The latter can be done in deterministic polynomial time since the winner determination of the cup rule has exactly this complexity.

To show NP-hardness, we will make use of the construction built to prove NP-hardness of destructive bribery in cup elections in the proof of Theorem 4.8. By adjusting the limit of the allowed changes and adding dummy candidates to make trivial changes impossible, the just mentioned reduction from VERTEX COVER can be used to show the claim at hand.

Let $G = (A, E)$ be an undirected graph with vertex set $A = \{a_1, a_2, \ldots, a_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_m\}$, and let k be a nonnegative integer. Without loss of generality, we may assume G to be a cubic graph, i.e., each vertex has exactly three neighbors—when restricted to such graphs, VERTEX COVER is still NP-complete [GJ79].

We construct the following instance (C, V, ℓ) of cup-SWMOV from (G, k) where the candidate set is $C = \{c, d, p', e'\} \cup E \cup P \cup R \cup T$ with $P = \{p_1, p_2, \ldots, p_m\}$ and dummy candidates in $R \cup T$. $R = \bigcup_{i=1}^{10} R_i$ with $\|R_i\| = 11k$ is a set of dummy candidates used to make certain changes impossible within the given swap limit which we define to be $\ell = 11k$. The dummy candidates in T are added to ensure that the voting tree is balanced and will, as well as the candidates in R, be omitted from the further a argumentation as they lose every pairwise comparison to the candidates in $C - (R \cup T)$.

Let $S_a = \{e \in E \mid e \cap \{a\} \neq \emptyset\}$ be the set of edges incident to vertex $a \in A$, and for each such set we define the corresponding set $P_a \subseteq P$ of candidates in P. If, for example, $S_a = \{e_1, e_2, e_3\}$, then $P_a = \{p_1, p_2, p_3\}$. Note that every edge is in exactly 2 of these defined sets and that, since G is a cubic graph, every S_a contains exactly three edges.

The preferences of the $2m(n+k-3)+6n+6k-3$ voters in V are shown in Table 4.6. When a set of candidates, say $Z \subseteq C$, is given in a voter's preference, then we assume that the candidates in Z are ordered with respect to a (tacitly assumed) fixed order, while $\overleftarrow{Z}$ denotes that the candidates are ordered in reverse. In particular, we fix the order of the candidates in P to be $p_1 > p_2 > \cdots > p_m$ and the order of the candidates in each S_a and P_a to also be lexicographical.

For each ...	**# votes**	**Preference**	**Type**
$a \in A$	1	$c\ d\ S_a\ p'\ P_a\ R_1\ (P-P_a)\ e'\ (E-S_a)\ (R-R_1)\ T$	1a
$a \in A$	1	$p'\ P\ c\ R_2\ d\ \overleftarrow{(E-S_a)}\ \overleftarrow{S_a}\ e'\ (R-R_2)\ T$	1b
	k	$c\ R_3\ d\ p'\ P\ e'\ E\ (R-R_3)\ T$	2a
	k	$c\ R_4\ d\ p'\ P\ \overleftarrow{E}\ e'\ (R-R_4)\ T$	2b
	$2(n+k-2)$	$c\ R_5\ E\ R_6\ p'\ P\ d\ e'\ (R-(R_5 \cup R_6))\ T$	3a
	$2(n+k-2)$	$d\ \overleftarrow{E}\ R_7\ p'\ P\ c\ e'\ (R-R_7)\ T$	3b
$i \in \{1,\dots,m\}$	$n+k-3$	$c\ R_8\ (P-\{p_i\})\ p_i\ e_i\ (E-\{e_i\})\ e'\ d\ (R-(R_8))\ T$	4a
$i \in \{1,\dots,m\}$	$n+k-3$	$d\ \overleftarrow{(E-\{e_i\})}\ R_9 p_i\ e_i\ \overleftarrow{(P-\{p_i\})}\ p'\ c\ e'\ (R-R_9)\ T$	4b
	1	$p'\ P\ c\ R_{10}\ d\ e'\ E\ (R-R_{10})\ T$	5

Table 4.6: Voter list V in the proof of Theorem 4.13

From the preferences we can obtain the following pairwise comparisons between the candidates in $C-(R \cup T)$:

- $D_V(c,d) = D_V(c,E) = D_V(d,E) = 2n+2k+1$,
- $D_V(c,P) = D_V(d,P) = D_V(c,p') = D_V(d,p') = 2k-1$,
- $D_V(p',P) = 2n+2k+4(n+k-2)+1$,
- $D_V(p',e') = 2m(n+k-3)+6n+6k-3$,
- $D_V(p_i,p_j) \begin{cases} > 0 & \text{if } i<j \\ \leq 0 & \text{if } i \geq j, \end{cases} \quad D_V(p_i,e_j) = \begin{cases} -2n-2k+5 & \text{if } i \neq j \\ -1 & \text{if } i = j. \end{cases}$

The voting tree without the dummy candidates is shown in Figure 4.3.

From the pairwise comparisons and the schedule given in Figure 4.3 we see that each e_i wins the first contest and moves on to the second level of the tree together with p' who wins her first pairwise comparison against e'. Candidate p' loses against e_1 in the second round, so the subtree with the root marked by X is won by a candidate from E. This candidate, however, loses to d in the next round, so we have the final contest between the candidates d and c, which is won by c, the cup winner of election (C,V).

To conclude the proof, we will show the following claim: The graph $G=(A,E)$ has a vertex cover of size at most k if and only if c can be prevented from being the cup winner in election (C,V) by conducting at most $11k$ swaps in the voters' preferences.

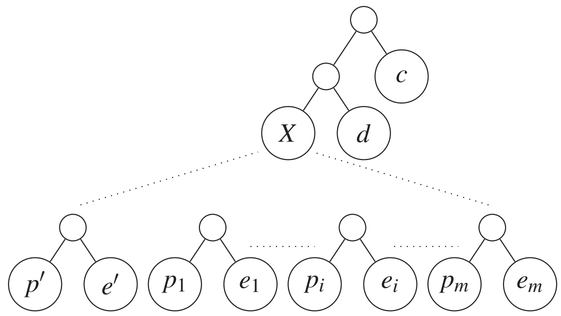


Figure 4.3: Voting tree of the cup election $(C-(R\cup T),V)$ without dummies in the proof of Theorem 4.13

Only if: Suppose there is a vertex cover $A' \subseteq A$ of size k. Take the corresponding k votes of type $1a$ of the form $c\ d\ e_h\ e_i\ e_j\ p'\ p_h\ p_i\ p_j\ R_1\ \cdots$ for $S_a = \{e_h, e_i, e_j\}$ and $P_a = \{p_h, p_i, p_j\}$, and change these votes to $p'\ c\ d\ p_h\ e_h\ p_i\ e_i\ p_j\ e_j\ R_1\ \cdots$. This change can be achieved by first swapping p' in all these k votes from position 6 to position 1, which needs $5k$ swaps in total. Then, we move p_h from position 7 to position 4 needing $3k$ swaps, then move p_i from position 8 to position 6 needing $2k$ swaps, and finally move p_j from position 9 to position 8 needing k swaps. In total, we have $6k$ swaps for moving the candidates in P_a, leading to a total number of $11k$ swaps, which is exactly the given limit. Let V' be the new list of votes after these swaps.

Since A' is a vertex cover, we have that for each $i \in \{1,2,\ldots,m\}$ candidate p_i is positioned before candidate e_i, leading to the new pairwise comparison $D_{V'}(p_i, e_i) = 1$. Thus all p_i win their first pairwise comparison and move on to the second round. The relations between p' and e' and between p' and the candidates in P are not affected by these changes, so we have that p' is assigned to the vertex marked by X in the voting tree shown in Figure 4.3. Candidate p' is now positioned better than c and d in k votes and thus beats both candidates, which makes her the new cup winner of the changed election.

If: Suppose that c can be prevented from being the cup winner by conducting at most $11k$ swaps in the voters' preferences. For this assumption to hold, there has to be a candidate beating c in the final contest and we will, step by step, check for each candidate if there exists a legal swap sequence that allows her to be this new winner.

We begin with candidate d and recall that in the unchanged election $D_V(c,d) = 2n+2k+1$ holds. So for d to beat c we have to find at least $n+k+1$ votes in which c is positioned better than d and change their positions without exceeding the swap limit of $11k$. The only votes in which c is positioned better than d without being separated by (too many) dummy candidates from R are those of type $1a$. Since there are only n of these votes in total, d cannot be the candidate preventing c from winning.

With the same argument we can rule out the candidates in E and also candidate e', which leaves the candidates in P and p'. Only the votes of type $1a$ allow reasonable swaps within the swap limit and due to the position of the dummy candidates in R_1, only p' and those candidates in P_a can be moved.

For both the candidates in P and candidate p' it holds that in order to beat c, the candidate at hand has to be moved to a better position than c in at least k of these n votes. Due to the definition of S_a and P_a each $p_i \in E$ is contained in P_a and P_b for exactly two vertices $a, b \in A$ in the graph. Thus, unless $k \leq 2$, we cannot find k votes in which we can move p_i to a better position than c. This implies that the candidate that has to finally beat c has to be p'. To achieve this, three conditions have to be fulfilled:

(i) p' has to be the winner of the subelection associated to the subtree with root X,
(ii) p' has to strictly beat d, and
(iii) p' has to strictly beat c.

Condition (i) can be achieved only by assuring that each p_i wins her pairwise contest against the corresponding e_i. For this to hold, each p_i has to be positioned better than e_i in exactly one vote and this vote can only be of type $1a$. Without loss of generality, we assume that we first change the positions of the (e_i, p_i) pairs and then move p' up. (For the reverse we can find a similar argument.) Thus we have that p' is positioned between the candidates in S_a and P_a. The cheapest way to position every p_i better than e_i (we mean cheap here with respect to the number of needed swaps) is to change the position of each (e_i, p_i) pair in one vote: The first change needs 4 swaps, the second one only 3, and the last one only 2, which gives a total of 9 swaps per vote. The corresponding votes would then be of the form $c\ d\ p'\ p_h\ e_h\ p_i\ e_i\ p_j\ e_j\ \cdots$ for $S_a = \{e_h, e_i, e_j\}$ and $P_a = \{p_h, p_i, p_j\}$. In each of these votes, only two further swaps are needed to move p' in front of both c and d, leading to a total number of 11 swaps per vote. Note that to not exceed the swap limit of $11k$, p' must be moved to the first position only in those votes where the (e_i, p_i) pairs have already been changed. Finally, for p' to beat c and d (and thus fulfilling conditions (ii) and (iii)), there have to be at least k votes that have been changed in this way and to ensure that every p_i wins her first contest, these k votes have to correspond to a vertex cover in G. ❑

With the next two results we are taking a first step towards characterizing the complexity of $\mathscr{E}$-SWMOV in elections held under positional scoring rules. We show that for certain scoring vectors $\vec{\alpha}$, the corresponding $\mathscr{E}$-SWMOV problem is solvable in deterministic polynomial time. To prove our claims, we will distinguish whether in the given $\vec{\alpha}$ election the set of winners contains several candidates or whether the election is won uniquely.

In the latter case we will make use of Corollary 4.7 from Section 4.1.3 by using an algorithm due to to Shiryaev et al. [SYE13, Theorem 4.1] that solves the destructive unweighted swap bribery problem with unit costs in the unique-winner model for positional scoring rules in deterministic polynomial time. In this context, $\vec{\alpha}$-UDUSB-UC$((C,V),\ell,c) =$"YES" denotes the case that the just mentioned algorithm identifies the instance $((C,V),\ell,c)$ as a yes instance, that is, the given candidate c can be prevented from being the unique $\vec{\alpha}$ winner in (C,V) by conducting at most ℓ swaps.

If, however, the given $\vec{\alpha}$ election (C,V) has more than one candidate in the set of winners $W \subseteq C$, we can distinguish the following three obvious options on how W can be changed:

(O1) Make at least one candidate $c \in W$ (but not all) gain at least one point.
(O2) Make at least one candidate $c \in W$ (but not all) lose at least one point.

(O3) Make at least one candidate $d \notin W$ gain points such that afterwards $d \in W$ holds.

Having this in mind, we now come to our first result regarding the family of strictly decreasing scoring vectors. Note that, amongst others, the famous Borda rule is contained in this family.

Theorem 4.14 *For $\vec{\alpha}$-elections with $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ and $\alpha_i > \alpha_j$ for all $i < j$, the problem $\mathscr{E}$-SWMOV is in* P.

Proof. Let (C, V, ℓ) be the given instance and denote the winner set of the election by W. Algorithm 4.1 determines in deterministic polynomial time whether the given instance is a yes instance or not.

Algorithm 4.1: Algorithm for SWMOV for strictly decreasing $\vec{\alpha}$

input : C set of candidates
V list of voters
number of votes that may be changed
output: "YES" if $(C, V, \ell) \in \vec{\alpha}$-SWMOV
"NO" if $(C, V, \ell) \notin \vec{\alpha}$-SWMOV

1 let W be the set of $\vec{\alpha}$ winners in (C, V);
2 **if** $W = \{c\}$ **then**
3   **if** $\vec{\alpha}$-UDUSB-UC$((C, V), \ell, c) =$ *"YES"* **then**
4   return "YES";
5   return "NO";
6 **if** $\ell > 0$ **then**
7   return "YES";

Clearly, we have that Algorithm 4.1 runs in polynomial time since the used algorithm in line 3 does. Furthermore it is clear that the algorithm works correctly for the case of $W = \{c\}$ (lines 2–5).

It remains to show that if $\|W\| \geq 2$ and $\ell > 0$, the winner set can always be changed by conducting at most ℓ swaps. To do so, we show that in this case SWMOV$(\vec{\alpha}, (C, V)) = 1$ always holds: If $\|W\| \geq 2$, we have at least two winners with exactly the same number of points. We have the three options (O1) – (O3) to change W and we can easily achieve one of the options by taking one arbitrary $c \in W$ and an arbitrary vote $v \in V$ and swap c in this vote with either her right or left neighbor. In the former case c loses at least one point and c's neighbor d gains a least one point because of the assumption that $\alpha_i > \alpha_j$ for $i < j$. If $d \notin W$, then the scores of the other winning candidates do not change due to the swap and we have the new winner set $W' = W - \{c\}$. If $d \in W$ then d is the new unique winner, thus the winning set changes to $W' = \{d\}$. If c is swapped with her left neighbor e, then c gains at least one point and e loses at least one point during the swap. Thus c has at least one point more than all the other candidates in W and is the new unique winner. In both cases we could successfully change the winner set by conducting only one swap. This concludes the proof. ❑

The set of strictly decreasing scoring vectors covers a large set of positional scoring rules but by far not all; the family of *k*-approval voting, for example, does not fit this description as for these particular scoring vectors only for $i = k$ and $j = k+1$ the condition $\alpha_i > \alpha_j$ does hold. For *k*-approval elections we will show separately that $\mathscr{E}$-SWMOV $\in$ P extending the approach used in the proof of Theorem 4.14. Besides the result from Shiryaev et al. [SYE13] we will, if needed, also use an algorithm for constructive shift bribery in *k*-approval elections due to Elkind et al. [EFS09, Theorem 4.2]. We will similarly denote the usage of this algorithm: k-AV-CSHB$((C,V),\ell,d) =$"YES" means that candidate d can be made a *k*-approval winner of (C,V) by performing at most ℓ shifts only involving d. For a formal definition of the shift bribery problem and further related work, we refer the reader to Section 3.3.1. Let us come to our result.

Theorem 4.15 *In k-approval elections with* $\mathfrak{m}$ *candidates, the problem* $\mathscr{E}$-SWMOV *is in* P *for all* $1 \leq k \leq \mathfrak{m}$.

Proof. Let (C,V,ℓ) be the given instance and denote the winner set of the election by W and let $\mathfrak{m} = \|C\|$.

If $k = \mathfrak{m}$ we have that all candidates are winners with a score of $\|V\|$ which cannot be changed by any sequence of swaps. Thus $\mathfrak{m}$-AV-SWMOV $= \emptyset$.

Assume now that $1 \leq k < \mathfrak{m}$. Algorithm 4.2 solves the problem in deterministic polynomial time as the used algorithms solving $\vec{\alpha}$-UDUSB-UC and k-AV-CSHB are P-algorithms.

For the proof of correctness we start with the case $W = \{c\}$ for a candidate $c \in C$. We can use the polynomial-time algorithm proposed by Shiryaev et al. [SYE13, Theorem 4.1] which solves the destructive swap bribery problem for positional scoring rules in the unique-winner model (lines 2–5).

If $\|W\| \geq 2$, the algorithm proceeds in line 6. In this first for-loop we try to change W according to (O1) with at most ℓ swaps. If none of the winning candidates can gain even one point without exceeding the number of swaps, we have that for each $c \in W$ all voters not contributing to c's score rank c on a position strictly worse than $k+\ell$. So for each $c \in W$ the set of voters is divided into those voters ranking c in their first k positions and those ranking c on positions $k+\ell+1$ to m.

With this in mind we try to change the winner set according to (O2) in the second for-loop starting in line 10. Note that because we already tested option (O1), it is not possible that any other candidate $c \neq d \in W$ gains a point due to the swaps performed in this loop since otherwise the algorithm would have returned "YES" already in the first loop for this candidate d. So we have that if the algorithm does not return "YES" in this second loop, the voters contributing to the winning candidates' scores position all of them in their top $k - \ell$ positions while the remaining positions have to be filled with candidates not in W.

Altogether we can conclude that if we reach line 14 of the algorithm, the scores of the winning candidates cannot be changed by conducting at most ℓ swaps. This leaves only option (O3): Trying to make a former non-winning candidate a winner, but in this case (and this is crucial) with the restriction that the scores of the winners cannot be changed. This restriction implies that with respect to our goal, namely making $d \notin W$ a winner, a swap is optimal if d's

Algorithm 4.2: Algorithm for SWMOV in k-AV with $1 \leq k < \mathfrak{m}$

input : C set of candidates
V list of voters
number of votes that may be changed
output: "YES" if $(C,V,\ell) \in k$-AV-SWMOV
"NO" if $(C,V,\ell) \notin k$-AV-SWMOV

1 let W be the set of k-AV winners in (C,V);
2 **if** $W = \{c\}$ **then**
3   **if** $\vec{\alpha}$-UDUSB-UC$((C,V),\ell,c) =$ *"YES"* **then**
4   return "YES";
5   return "NO";
6 **foreach** $c \in W$ **do**
7   **foreach** $v \in V$ **do**
8   **if** *c gains at least one point by swapping her ℓ positions forward in v* **then**
9   return "YES";
10 **foreach** $c \in W$ **do**
11   **foreach** $v \in V$ **do**
12   **if** *d looses one point by swapping her ℓ positions backward in v* **then**
13   return "YES";
14 **foreach** $d \notin W$ **do**
15   **if** k-AV-CSHB$((C,V),\ell,d) =$ *"YES"* **then**
16   return "YES";
17 return "NO";

position is improved due to the swap. Thus an optimal approach of making d a winner would be to find a sequence of shifts all involving d. Thus, we can use the P-algorithm introduced by Elkind et al. [EFS09, Theorem 4.2] for CONSTRUCTIVE SHIFT BRIBERY in k-approval elections with the budget ℓ, unit prices for each shift, and the designated candidate d, which the algorithm does in lines 14–16. If this approach does not change the winner set, then the algorithm returns "NO" as the given instance is indeed a no instance. ❑

4.5 Concluding Remarks and Future Work

We have studied three variants of the margin of victory for a selection of important voting rules and discussed their connections to the closely related setting of destructive bribery.

For the standard variant we have shown that for irresolute voting rules, the standard MARGIN OF VICTORY ($\mathscr{E}$-MOV) problem generalizes destructive unweighted and unpriced bribery in the unique-winner model. Furthermore we have seen that for the cup rule, $\mathscr{E}$-MOV is NP-hard, while in fallback elections the problem is tractable. The result for cup elections is obtained by proving destructive unweighted bribery to be NP-complete.

Extending this study, we introduced the EXACT MARGIN OF VICTORY problem and established DP-completeness for cup, the family of Copeland$^{\alpha}$, and the Schulze rule. Another variant that we introduced is the so-called SWAP MARGIN OF VICTORY ($\mathscr{E}$-SWMOV) problem which we studied for the cup rule and certain positional scoring rules. Analogously to the standard variant, we show that the swap margin of victory problem can be reduced to destructive swap bribery with unit costs in the unique-winner case. By adapting the proof of destructive bribery in cup elections to the model of swap bribery, we obtain NP-completeness for the cup rule, which in turns gives us NP-completeness of $\mathscr{E}$-SWMOV in cup elections. For k-approval voting and positional scoring rules with strictly decreasing scoring vectors, however, we provide tractability results. For future work we suggest to complete this study of $\mathscr{E}$-SWMOV in positional scoring rule, fallback and Bucklin voting as well as for Schulze and Copeland$^{\alpha}$ elections and thus fill in the missing results in Table 4.1.

In the context of the margin of victory, tractability of the corresponding decision problem is desired, for all variants we considered here. Thus, especially for the intractable problems found in this chapter, a further analysis with respect to their approximability is a promising and interesting direction for future work.

5 The Possible Winner Problem with Uncertain Weights

In political elections we assume that the voters fill in their ballots exactly as the voting system dictates, otherwise the ballots are declared void. For most preference-based voting systems this means that each voter has to provide a complete ranking of the running candidates for her ballot to be valid. Aside from the context of political elections, there are situations in which incomplete preferences are unavoidable or even desired: For instance elections with a large set of candidates making it infeasible for the voters to provide a complete ranking or candidate sets that change while parts of the electorate already cast their votes.

Given an election with incomplete preferences a natural question is whether there exists an extension of the partial orders to linear orders such that a certain candidate is a winner of the election. Such a candidate is called a *possible winner*. This notion was introduced in the work of Konczak and Lang [KL05], who also defined the *necessary winner* of an incomplete election to be a candidate who is a winner for every possible extension of the votes to linear orders.

The idea of possible and necessary winners can be applied to any conceivable variant of uncertainty in elections: The list of running candidates might be incomplete when the first votes are cast [CLM+12], it can be uncertain which candidates finally run or which voters actually cast their votes [WF12], or the voting system itself might not be determined when the election is held [BRR11].[1] Notions as the set of possible or necessary winners for a given election can be useful for related problems such as *vote elicitation*, where the aim is to determine the winner set of an election before all preferences are known, see [CL02, PRV+11]. For a comprehensive overview of related research regarding incomplete information in elections, see for example the book chapter by Boutilier and Rosenschein [BR16].

Considering the setting of weighted elections, we are interested in the scenario where not the voters' preferences are incomplete, but their assigned weights are not known beforehand. We call this problem POSSIBLE WINNER WITH UNCERTAIN WEIGHTS (PWUW) and we dedicate this chapter to a computational complexity-theoretic analysis of this problem and its defined variants.

Weighted elections can be found in many real-world examples such as stockholder meetings or committees consisting of representatives of different groups. While in stockholder meetings the weights are predetermined by the share each member holds, the representatives' weights in committees can be defined with respect to various criteria: the size or the importance the group

[1]Note that these chosen examples do not fully capture the variety of related problems; see Section 5.1 for a comprehensive overview of studied variants.

that is represented, or other aspects such as expertise or seniority. The chair of a committee (or any instance with similar authorization) could change these criteria to alter the weights of the participants. This leads to the question whether it is possible to define the criteria in order to ensure, for example, that a certain long-term policy will be pursued by the committee, assuming that the preferences of the representatives are known. This is exactly the scenario we model with our newly defined PWUW problem.

Organization of this Chapter In Section 5.1 we present related work by surveying known results for the original variant of the possible and necessary winner problem for some chosen voting rules. In addition, we shortly name other variants of these problems that have been studied so far. In Section 5.2 we introduce the new variant POSSIBLE WINNERS WITH UNCERTAIN WEIGHTS and its variations, elaborate on some interesting properties and show relations to other voting problems. Our results are presented in Section 5.3, where we provide a detailed discussion of the case of nonnegative rational weights and shortly survey results for the case of natural weights. We conclude the chapter with a summary of our findings and giving pointers to future work.

5.1 The Standard Possible Winner Problem and Related Work

The standard possible and necessary winner problems were formally introduced by Konczak and Lang [KL05] and they defined the former in the following manner.

$\mathscr{E}$-POSSIBLE WINNER ($\mathscr{E}$-PW)	
Given:	An $\mathscr{E}$ election (C,V) with possibly incomplete preferences and a designated candidate $c \in C$.
Question:	Can the partial votes in V be extended to linear orders such that c is a winner of the resulting election?

Analogously, the so-called necessary winner problem can be defined.

$\mathscr{E}$-NECESSARY WINNER ($\mathscr{E}$-NW)	
Given:	An $\mathscr{E}$ election (C,V) with possibly incomplete preferences and a designated candidate $c \in C$.
Question:	Is c a winner for all possible extensions of V to linear orders?

Let $\mathscr{E}$-UPW and $\mathscr{E}$-UNW denote the problems where unique winners are considered. Note that the addition *possibly incomplete* allows the two special cases of V consisting of (1) only partial preferences or (2) only linear orders. Looking closely at this formal definition, we see that the $\mathscr{E}$-PW problem is a generalization of the CONSTRUCTIVE COALITIONAL UNWEIGHTED MANIPULATION problem, see Section 3.1.1 for the formal definition of manipulation. We state this relation in Proposition 3.39 on page 82 which is due to Xia and Conitzer

[XC11b]. Furthermore, the problem of CONSTRUCTIVE UNWEIGHTED SWAP BRIBERY generalizes the $\mathscr{E}$-PW problem as we have stated in Proposition 3.38 on page 82. This result is shown in [EFS09]. Shiryaev et al. [SYE13] show that the destructive case of unweighted swap bribery is a generalization of the complement of the $\mathscr{E}$-NW problem, see Proposition 3.40 on page 82. These Propositions can be used to establish lower bounds or upper bounds for the $\mathscr{E}$-PW and $\mathscr{E}$-NW problem from known results for other voting systems (an clearly, also the other way around). For a comprehensive overview of such connections, we refer to Section 2.4, and especially to Figure 2.6 on page 24.

Table 5.1 provides an overview of known complexity results for the $\mathscr{E}$-PW and $\mathscr{E}$-NW problem for those voting system this thesis focuses on.

Voting System	**Possible Winner**	**Necessary Winner**
veto & plurality	P [1]	P [2]
remaining positional scoring rules	NP-complete [3]	
Bucklin	NP-complete [4]	P [4]
Copeland$^\alpha$	NP-complete [2]	coNP-complete [2]
cup		
fallback	?	?
Schulze		

[1] due to [BD10]
[2] due to [XC11b]
[3] due to [BD10, BR12, XC11b]
[4] due to [XC11b] for simplified BV

Table 5.1: Selection of known complexity results for the standard possible and necessary winner problem for chosen voting systems

Xia and Conitzer [XC11b] continue the study initiated by Konczak and Lang [KL05] and provide results for many commonly studied voting systems such as Copeland, maximin, ranked pairs, and simplified Bucklin. Betzler and Dorn [BD10] took a first step to fully characterize the complexity of the $\mathscr{E}$-PW problem for pure positional scoring rules. Their study was continued in the work of Baumeister and Rothe [BR12] who completed the dichotomy result that can be derived from Table 5.1. Barrot et al. [BGL+13] introduce the setting of possible and necessary winners and winner sets for approval elections and study inter alia the complexity of verifying whether a given set is a possible/necessary approval winner set. Lang et al. [LPR+12] and Pini et al. [PRV+11] expand the above study to elections held under voting trees and other multi round elections. Betzler et al. [BHN09] and Betzler [Bet10] study the parameterized complexity of the possible winner problem, where the latter work focuses on k-approval elections. This study is expanded by the recent work of Dey et al. [DMN15]. Bachrach et al. [BBF10] analyze the complexity of the counting version of the $\mathscr{E}$-PW problem and furthermore provide a randomized approximation algorithm for solving the standard manipulation and possible winner problem for voting rules with polynomial-time winner determination procedures.

Certain domain restrictions such as *single-peaked preferences* and *truncated ballots* are considered in the work of Walsh [Wal07] and Baumeister et al. [BFL$^+$12], while the complexity of computing the set of possible/necessary winners is addressed by Pini et al. [PRV$^+$11] and more recently by Gaspers et al. [GNN$^+$14].

Based on the initial definition of the possible winner problem, a variety of related problems have been introduced: Elections in which the set of candidates changes while the voters cast their votes have been studied by Chevaleyre et al. [CLM$^+$10]. Their approach has been followed up in the work of Xia et al. [XLM11] (see also [CLM$^+$12]) and of Baumeister et al. [BRR11], while the latter study also comprises the setting, where the uncertainty lies in the voting system. Possible and necessary winners in the context of parliamentary voting are studied by Bredereck et al. [BCN$^+$15].

Probabilistic approaches to determine possible winners from incomplete preferences have been taken, for example, by Xia and Conitzer [XC11a] and Hazon et al. [HAK$^+$12].

The book chapter by Boutilier and Rosenschein [BR16] provides a diversified survey on problems studied in the wider context of voting with incomplete information.

5.2 The Possible Winner Problem with Uncertain Weights – Basic Definitions

We introduce a new variant of the possible winner problem in weighted elections in which we do not assume that the uncertainty lies in the voters' preferences, but in the weights that are assigned to the voters. We introduce several variants of the problem allowing integer or rational weights and distinguishing between several restrictions on the assignments of weights.

For a given voting system $\mathscr{E}$ and $\mathbb{F} \in \{\mathbb{Q}_{\geq 0}, \mathbb{N}_0\}$ we define the unrestricted variant of our problem as follows.

	$\mathscr{E}$-POSSIBLE-WINNER-WITH-UNCERTAIN-WEIGHTS-$\mathbb{F}$ ($\mathscr{E}$-PWUW-$\mathbb{F}$)
Given:	An $\mathscr{E}$ election $(C, V_0 \cup V_1)$, $V_0 \cap V_1 = \emptyset$, where the weights of the voters in V_0 are not specified yet and weight zero is allowed for them, yet all voters in V_1 have weight one, and a designated candidate $c \in C$.
Question:	Is there an assignment of weights $w_i \in \mathbb{F}$ to the votes v_i in V_0 such that c is an $\mathscr{E}$ winner of election $(C, V_0 \cup V_1)$ when v_i's weight is w_i for $1 \leq i \leq \|V_0\|$?

We consider in total three restrictions of $\mathscr{E}$-PWUW-$\mathbb{F}$, where the third combines the first two:

- In $\mathscr{E}$-PWUW-RW-$\mathbb{F}$, an $\mathscr{E}$-PWUW-$\mathbb{F}$ instance and regions (i.e., intervals) $R_i \subseteq \mathbb{F}$, $1 \leq i \leq |V_0|$, are given, and the question is the same as in $\mathscr{E}$-PWUW-$\mathbb{F}$, except that each weight w_i must be chosen from R_i in addition.
- In $\mathscr{E}$-PWUW-BW-$\mathbb{F}$, an $\mathscr{E}$-PWUW-$\mathbb{F}$ instance and a positive bound $B \in \mathbb{F}$ is given, and the question is the same as in $\mathscr{E}$-PWUW-$\mathbb{F}$, except that $\sum_{i=1}^{|V_0|} w_i \leq B$ must hold in addition (i.e., the total weight that can be assigned must be bounded by B).

- In $\mathscr{E}$-PWUW-BW-RW-$\mathbb{F}$, an $\mathscr{E}$-PWUW-BW-$\mathbb{F}$ instance and regions (i.e., intervals) $R_i \subseteq \mathbb{F}$, $1 \leq i \leq |V_0|$, are given, and the question is the same as in $\mathscr{E}$-PWUW-BW-$\mathbb{F}$, except that each weight w_i must be chosen from R_i in addition.

Our problems model elections in which the list of voters is partitioned into those voters in V_1 each having unit weight and the voters in V_0 whose weights are not specified yet. The ballots of all voters are known and the set of candidates is fixed. Instances with $V_0 = \emptyset$ are allowed and for these inputs our problem simplifies to the winner problem for $\mathscr{E}$ in unweighted elections. Taking V_1 into the instances is motivated by the fact that the original PW problem also allows that complete preferences are part of an input. Setting their weight to 1, however, is an intended restriction that simplifies our proofs. By changing the question to whether c can be made *the unique winner* of the resulting election, we obtain the problem for the *unique-winner model*.

We summarize some obvious relations between the just defined variants in the following observation.

Observation 5.1 *For a fixed voting system $\mathscr{E}$ the following trivial reductions hold.*

1. PWUW-RW-$\mathbb{Q}_{\geq 0} \leq_m^p$ PWUW-BW-RW-$\mathbb{Q}_{\geq 0}$,
2. PWUW-RW-$\mathbb{N}_0 \leq_m^p$ PWUW-BW-RW-$\mathbb{N}_0$,
3. PWUW-BW-$\mathbb{Q}_{\geq 0} \leq_m^p$ PWUW-BW-RW-$\mathbb{Q}_{\geq 0}$, *and*
4. PWUW-BW-$\mathbb{N}_0 \leq_m^p$ PWUW-BW-RW-$\mathbb{N}_0$.

We stated that the standard $\mathscr{E}$-PW problem generalizes the manipulation problem when a group of manipulators tries to alter an election's outcome. Analogously, the problem variant $\mathscr{E}$-PWUW-BW-RW-$\mathbb{N}_0$ can be seen as a generalization of $\mathscr{E}$-CONSTRUCTIVE CONTROL BY ADDING VOTERS ($\mathscr{E}$-CCAV); recall Section 3.2.1 for its formal definition and further background.

Proposition 5.2 *Let $\mathscr{E}$ be a voting system. Then* $\mathscr{E}$-CCAV $\leq_m^p$ $\mathscr{E}$-PWUW-BW-RW-$\mathbb{N}_0$ *holds. If voters are represented succinctly, also* $\mathscr{E}$-PWUW-BW-RW-$\mathbb{N}_0 \leq_m^p$ $\mathscr{E}$-CCAV *holds.*

Proof Sketch. We only shortly describe the mapping of the instances and omit the rest of the proof.

For the first claim let $((C, V \cup V'), c, k)$ be a given instance of $\mathscr{E}$-CCAV, where V contains the registered and V' contains the unregistered voters. Define the $\mathscr{E}$-PWUW-BW-RW-$\mathbb{N}_0$ instance to be $V_0 = V'$, $V_1 = V$, $R_i = \{0, 1\}$ for each $v_i \in V_0$, and $B = k$.

The second claim assumes succinct representation of the voters, which allows to construct an $\mathscr{E}$-CCAV directly from an $\mathscr{E}$-PWUW-BW-RW-$\mathbb{N}_0$ instance, where V' contains the ballots corresponding to the voters in V_0 and the number of occurrences of each ballot is the maximal weight the voter in V_0 can be assigned to. The remaining parts of the instance are defined as in the reduction establishing the first claim. ❑

5.3 Complexity of the Possible Winner Problem with Uncertain Weights

Finally, we present in this section the results that were published in [BRR$^+$12]. For the variants with integer weights, we will survey the obtained results in Section 5.3.1, while the proofs for rational weights are presented in detail in Section 5.3.2.

5.3.1 Complexity for Integer Weights

We start with those results from [BRR$^+$12] covering the cases in which the weights can be nonnegative integers, which are shown in Table 5.2. We, again, present known results for all those voting systems studied throughout this thesis but we will not present the proofs in detail.

<table>
<tr><th>Voting Rule</th><th>PWUW-$\mathbb{N}_0$</th><th>PWUW-BW-$\mathbb{N}_0$</th><th>PWUW-RW-$\mathbb{N}_0$</th><th>PWUW-BW-RW-$\mathbb{N}_0$</th></tr>
<tr><td>Bucklin voting
fallback voting
Copeland$^\alpha$</td><td>NP-c.</td><td rowspan="2">NP-c.</td><td>NP-c.</td><td rowspan="2">NP-c.</td></tr>
<tr><td>k-AV, $k \geq 4$</td><td rowspan="3">P</td><td rowspan="3">P</td></tr>
<tr><td>veto
k-AV, $k \in \{1,2\}$</td><td>P</td><td>P</td></tr>
<tr><td>3-AV</td><td rowspan="3">?</td><td rowspan="2">?</td></tr>
<tr><td>cup
scoring rules</td><td rowspan="2">?</td><td rowspan="2">?</td></tr>
<tr><td>Schulze</td><td>NP-c. [1]</td></tr>
</table>

[1] follows from Proposition 5.2 and the result for CCAV in [PX12]. Key: NP-c. = NP-complete

Table 5.2: Overview of results for the complexity of the possible winner problem with uncertain weights when the weights can be positive integers; if not stated otherwise, the results are due to [BRR$^+$12]

5.3.2 Complexity for Rational Weights

In this section we present our results for the possible winner problem with uncertain weights when the weights can be rational numbers: all four variants of this problem are solvable in deterministic polynomial time for fallback voting, Bucklin voting, and the complete family of positional scoring rules.

We obtain our results from the central idea to formulate the given PWUW problem as a linear program with rational variables. The solution of the linear program gives us the weights that have to be assigned to achieve the given goal. We then make use of the fact that linear programs can be solved efficiently if the values are rational numbers, as has been shown by Hačijan [Hač79]. Note that this approach is not applicable to the case of integer weights, as linear programs with integer values cannot be solved efficiently unless P turns out to equal NP. Furthermore, a voting system has to fulfill certain properties for being expressible by a set of linear inequalities, which we will discuss in this section. We state our central idea in Theorem 5.4 and present the proofs tailored to the analyzed voting systems in the theorems listed in Table 5.3.

Voting Rule	Complexity	Reference
positional scoring rules	P	Theorems 5.4 and 5.5
Bucklin voting	P	Theorems 5.4 and 5.6
fallback voting	P	Theorems 5.4 and 5.7
Copeland$^{\alpha}$ Schulze cup	?	–

Table 5.3: Overview of results for the complexity of the possible winner problem with uncertain weights when the weights can be nonnegative rational numbers

The standard matrix-vector form of a linear program is usually defined as a minimization program. The maximization variant of the problem is then defined to be the dual of the corresponding linear program. In our setting we are only interested in finding maximal solutions, therefore we will slightly abuse notation and define a linear program directly in the maximization variant. For more background on different variants of definitions and notations for linear programs we refer the reader to the textbook by Dantzig and Thapa [DT97].

Definition 5.3 (Linear Program) *We define a* linear program *in the matrix-vector standard form to be a tuple* $(\vec{d},\vec{x},A,\vec{b})$ *with* $\vec{d},\vec{x} \in \mathbb{Q}^{s\times 1}$, $\vec{b} \in \mathbb{Q}^{r\times 1}$, *and* $A \in \mathbb{Q}^{r\times s}$, *where* $\vec{x}$ *is the vector of the* variables, $\vec{d}^T \cdot \vec{x}$ *is the* objective function *we want to maximize,*[2] *and* $A,\vec{b}$ *are the* constraints *that have to be fulfilled while maximizing the objective function. Shortly, a linear program can be defined as:*

$$\begin{array}{ll} \text{Maximize} & \vec{d}^T \cdot \vec{x} \\ \text{object to} & A \cdot \vec{x} \leq \vec{b}. \end{array}$$

For the sake of readability and to avoid overly formalized argumentations, we will present the constraints for the linear programs in the following proofs as sets of inequalities rather than explicitly defining their matrix representation.

To formulate the $\mathscr{E}$-PWUW problem as a linear program for a given voting system $\mathscr{E}$, we are in need of a tool to represent $\mathscr{E}$ in the linear program, as well. One way to do so is to represent the winner determination of the voting system as a system of linear inequalities. Chamberlin and Cohen [CC78] (see also the work of Faliszewski et al. [FHH11] and Dorn and Schlotter [DS12]) provide such a representation for various voting systems in unweighted elections.

[2] Note that usually the coefficients of the objective function are denoted by $\vec{c}$, but to avoid confusion with the designated candidate c in our voting problem, we denote the coefficients with $\vec{d}$.

To cover the case of weighted elections, the inequalities describing the winner determination have to incorporate the voters' weights and have further to be of a form that allows us to define the weights as the variables of the linear program. This is the case for voting systems having winner determination procedures that, firstly, chose the winners of the election based on scores and, secondly, these scores are independent of the voters' weights in the following sense: We call a scoring function *weight independent* if a candidate's score in a weighted election does not differ from the score she would get in a corresponding unweighted election except that in the former, her score is a weighted sum while in the latter it is a plain sum.

The Copeland$^\alpha$ scores, for example, are not weight-independent, but the scores computed in fallback and Bucklin elections, as well as in elections held under positional scoring rules clearly are.

We now come to the central theorem of this chapter. In our setting, we have given an $\mathscr{E}$ election (C,V) with $\mathfrak{m}$ candidates in C and $\mathfrak{n}$ voters $V = V_0 \cup V_1$, where $\mathfrak{n}_i = \|V_i\|$ for $i \in \{0,1\}$ and $V_0 = \{v_1, v_2, \dots, v_{\mathfrak{n}_0}\}$, $V_1 = \{v_{\mathfrak{n}_0+1}, \dots, v_{\mathfrak{n}}\}$. The voters in V_1 all have unit weight. The goal is to decide whether there is an assignment of the yet undefined weights $x_1, x_2, \dots, x_{\mathfrak{n}_0}$ of the voters in V_0 such that the designated candidate c is a winner of the resulting election. Furthermore we have given a bound $B \in \mathbb{Q}_{\geq 0}$, and regions $R_i \subseteq \mathbb{Q}_{\geq 0}$, $1 \leq i \leq \mathfrak{n}_0$.

Recall that by definition, we allow to assign weights of zero to the voters in V_0, but we will seek to find solutions containing positive weights (we will go into detail on how this can be achieved in the proof below).

Theorem 5.4 *Let $\mathscr{E}$ be a voting rule with a weight-independent scoring function that can be described by a system A of polynomially many linear inequalities. Then $\mathscr{E}$-*PWUW-$\mathbb{Q}_{\geq 0}$*, $\mathscr{E}$-*PWUW-BW-$\mathbb{Q}_{\geq 0}$*, $\mathscr{E}$-*PWUW-RW-$\mathbb{Q}_{\geq 0}$*, and $\mathscr{E}$-*PWUW-BW-RW-$\mathbb{Q}_{\geq 0}$ *are each in* P.

Proof. Let $x_1, x_2, \dots, x_{\mathfrak{n}}$ be the variables of the system A of polynomial many linear inequalities that describes $\mathscr{E}$ for an $\mathscr{E}$ election with $\mathfrak{n}$ voters.

We state a linear program for the problem variant $\mathscr{E}$-PWUW-BW-RW-$\mathbb{Q}_{\geq 0}$ and to this end let an instance of this problem be given: An election $(C, V_0 \cup V_1)$ with as yet unspecified weights in V_0, a designated candidate $c \in C$, a bound $B \in \mathbb{Q}_{\geq 0}$, and regions $R_i \subseteq \mathbb{Q}_{\geq 0}$, $1 \leq i \leq \mathfrak{n}_0$.

Based on this instance we construct the linear program with variables $\vec{x} = (x_1, x_2, \dots, x_{\mathfrak{n}_0}, \chi)$ for $x_i, \chi \in \mathbb{Q}$, and we maximize the objective function $\vec{d}^T \cdot \vec{x}$ with $\vec{d} = (0, 0, \dots, 0, 1)$ and the following set of constraints:

$$A \tag{5.1}$$

$$x_i - \chi \geq 0 \qquad \text{for } 1 \leq i \leq \mathfrak{n}_0 \tag{5.2}$$

$$\chi \geq 0 \tag{5.3}$$

$$\sum_{i=1}^{\mathfrak{n}_0} x_i \leq B \tag{5.4}$$

$$x_i \leq r_i \qquad \text{for } 1 \leq i \leq \mathfrak{n}_0 \tag{5.5}$$

$$-x_i \leq -\ell_i \quad \text{for } 1 \leq i \leq \mathfrak{n}_0 \tag{5.6}$$

Constraint (5.1) gives the linear inequalities that have to be fulfilled for the designated candidate c to win under $\mathscr{E}$. This condition ensures that a solution gives an assignment of the weights such that the designated candidate indeed is a winner of the resulting election. By maximizing the additional variable χ in the objective function we try to find solutions where the weights are positive, this is accomplished by constraint (5.2). Constraint (5.4) implements our given upper bound B for the total weight to be assigned and constraints (5.5) and (5.6) implement our given ranges $R_i = [\ell_i, r_i] \subseteq \mathbb{Q}$ for each weight.

To obtain a linear program for the remaining variants of our problem, it suffices to drop certain constraints that are not needed for the variant at hand. In more detail: Omit constraint (5.4) for $\mathscr{E}$-PWUW-RW-$\mathbb{Q}_{\geq 0}$, omit constraints (5.5) and (5.6) for $\mathscr{E}$-PWUW-BW-$\mathbb{Q}_{\geq 0}$, and finally omit constraints (5.4), (5.5), and (5.6) for $\mathscr{E}$-PWUW-$\mathbb{Q}_{\geq 0}$.

Obviously, the designated candidate c is a possible winner of the given election if and only if the above linear program has a feasible solution. The fact that a solution $\vec{x}$ with $x_i \in \mathbb{Q}$ for a linear program with polynomial bounded constraints can be found in deterministic polynomial time completes the proof. ❑

We now show the results stated in Table 5.3 by explicitly stating the systems of linear inequalities that are needed to describe the voting system at hand (recall Section 2.3 for the definitions of the voting systems) and then applying Theorem 5.4. For the following proofs we assume that the parameters of the $\mathscr{E}$-PWUW-BW-RW-$\mathbb{Q}_{\geq 0}$ problem are given as stated before Theorem 5.4.

We start with the result for positional scoring rules defined by a scoring vector $\vec{\alpha}$.

Theorem 5.5 *For each positional scoring rule with scoring vector $\vec{\alpha}$, each of the problems $\vec{\alpha}$-PWUW-$\mathbb{Q}_{\geq 0}$, $\vec{\alpha}$-PWUW-BW-$\mathbb{Q}_{\geq 0}$, $\vec{\alpha}$-PWUW-RW-$\mathbb{Q}_{\geq 0}$, and $\vec{\alpha}$-PWUW-BW-RW-$\mathbb{Q}_{\geq 0}$ is in* P *in both winner models.*

Proof. We are given an election with $\mathfrak{m}$ different candidates in C, where $c \in C$ is the distinguished candidate, and the scoring vector is $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{\mathfrak{m}})$. For each candidate $c \in C$ in the given election, we denote by $\rho_i^0(c)$ the position of c in the preference of voter v_i in V_0 for $1 \leq i \leq \mathfrak{n}_0$. Analogously, $\rho_j^1(c)$ denotes the position of candidate c in the preference of voter v_j in V_1 for $(\mathfrak{n}_0 + 1) \leq j \leq \mathfrak{n}$. Recall that $\alpha_{\rho_i^0(c)}$ denotes the number of points c gets for this position according to the scoring vector $\vec{\alpha}$.

Let $S_{V_1}(c)$ denote the number of points candidate c gains from the voters in V_1 (recall that those have all weight one). Then the distinguished candidate c is a winner if and only if for all candidates $c' \in C$ with $c' \neq c$, we have

$$S_{V_1}(c') + \sum_{i=1}^{\mathfrak{n}_0} x_i \alpha_{\rho_i^0(c')} \leq S_{V_1}(c) + \sum_{i=1}^{\mathfrak{n}_0} x_i \alpha_{\rho_i^0(c)},$$

where $x_1, x_2, \ldots, x_{\mathfrak{n}_0} \in \mathbb{Q}$ are the weights that will be assigned to the voters in V_0. Thus, the constraints ensuring that the designated candidate is a winner of the resulting election are fully described by the above inequality.

With this, the linear program for the $\vec{\alpha}$-PWUW-BW-RW-$\mathbb{Q}_{\geq 0}$ problem can be defined as follows: The variables are $\vec{x} = (x_1, x_2, \dots, x_{\mathfrak{n}_0}, \chi)$ with $x_i, \chi \in \mathbb{Q}$ and we maximize the objective function $\vec{d} \cdot \vec{x}^T$ with $\vec{d} = (0, 0, \dots, 0, 1)$ while the following constraints have to be fulfilled:

$$-\sum_{i=1}^{\mathfrak{n}_0} \left(\alpha_{\rho_i^0(c)} - \alpha_{\rho_i^0(c')} \right) x_i \leq S_{V_1}(c) - S_{V_1}(c') \ \forall\, c' \neq c \tag{5.7}$$

$$x_i - \chi \geq 0 \qquad \text{for } 1 \leq i \leq \mathfrak{n}_0 \tag{5.8}$$

$$\chi \geq 0 \tag{5.9}$$

$$\sum_{i=1}^{\mathfrak{n}_0} x_i \leq B \tag{5.10}$$

$$x_i \leq r_i \qquad \text{for } 1 \leq i \leq \mathfrak{n}_0 \tag{5.11}$$

$$-x_i \leq -\ell_i \quad \text{for } 1 \leq i \leq \mathfrak{n}_0 \tag{5.12}$$

Here again, constraints (5.10) to (5.12) can be dropped to obtain the linear programs corresponding to the other problem variants.

Since we have at most $(\mathfrak{m}-1)\mathfrak{n}_0 + 3\mathfrak{n}_0 + 2 = (\mathfrak{m}+2)\mathfrak{n}_0 + 2$ constraints, we can apply Theorem 5.4 and we have shown the claim.

Note that the above linear program can be adapted to also solve the unique winner case: To this end, the variable χ has to be added to the left-hand side of constraint (5.7) and whenever a solution is found with $\chi > 0$, candidate c is a possible unique winner of the election. ❑

For Bucklin and fallback voting we have to adapt the just presented approach to their level-based winner determination. Intuitively speaking, we check level by level whether the given candidate is a possible winner for the given level. We will make use of the representation of simplified Bucklin elections due to Dorn and Schlotter [DS12] and slightly adapt it to a representation of the unsimplified Bucklin rule.

Let (C,V) be an election as given in our PWUW instances. We define for the candidates $c' \in C$, each $\ell \in \{1, 2, \dots, \mathfrak{m}\}$, and each $i \in \{1, 2, \dots, \mathfrak{n}\}$ the value

$$lev_i^l(c') = \begin{cases} 0, & c' \text{ is not in top } l \text{ positions in the preference of } v_i, \\ 1, & \text{otherwise.} \end{cases}$$

Thus, the level ℓ score of a candidate $c' \in C$ in the election (C,V) is

$$score_{(C,V)}^{\ell}(c') = \sum_{i=1}^{\mathfrak{n}_0} lev_i^{\ell}(c') x_i + \sum_{i=\mathfrak{n}_0+1}^{\mathfrak{n}} lev_i^{\ell}(c').$$

This notation will be useful in the proofs of the two upcoming theorems.

Theorem 5.6 *In Bucklin elections, each of the problems* PWUW-$\mathbb{Q}_{\geq 0}$*,* PWUW-BW-$\mathbb{Q}_{\geq 0}$*,* PWUW-RW-$\mathbb{Q}_{\geq 0}$*, and* PWUW-BW-RW-$\mathbb{Q}_{\geq 0}$ *is in* P *in both winner models.*

Proof. Let an instance of Bucklin-PWUW-BW-RW-$\mathbb{Q}_{\geq 0}$ be given. We now define a linear program with variables $\vec{x} = (x_1, x_2, \ldots, x_{\|V_0\|}, \chi)$, where $x_i, \chi \in \mathbb{Q}$ and the objective function $\vec{d} \cdot \vec{x}$ with $\vec{d} = (0, 0, \ldots, 1)$. Furthermore for $\ell \in \{1, 2, \ldots, \mathfrak{m}\}$ we have the following constraints:

$$\sum_{i=1}^{\mathfrak{n}_0} lev_i^{\ell-1}(c')x_i \leq \frac{\mathfrak{n}_1 + w}{2} - \sum_{i=\mathfrak{n}_0+1}^{\mathfrak{n}} lev_i^{\ell-1}(c') \quad \forall\, c' \in C \tag{5.13}$$

$$-\sum_{i=1}^{\mathfrak{n}_0} lev_i^{\ell}(c)x_i + \chi \leq \frac{\mathfrak{n}_1 + w}{2} + \sum_{i=\mathfrak{n}_0+1}^{\mathfrak{n}} lev_i^{\ell}(c) \tag{5.14}$$

$$-\sum_{i=1}^{\mathfrak{n}_0} (lev_i^{l}(c) - lev_i^{l}(c'))x_i \leq \sum_{i=\mathfrak{n}_0+1}^{\mathfrak{n}_1} (lev_i^{l}(c) - lev_i^{l}(c')) \quad \forall\, c' \neq c \tag{5.15}$$

$$\sum_{i=1}^{\mathfrak{n}_0} x_i \leq w \tag{5.16}$$

$$-\sum_{i=1}^{\mathfrak{n}_0} x_i \leq -w \tag{5.17}$$

$$x_i - \chi \geq 0 \quad 1 \leq i \leq \mathfrak{n}_0 \tag{5.18}$$

$$\chi \geq 0 \tag{5.19}$$

$$\sum_{i=1}^{\mathfrak{n}_0} x_i \leq B \tag{5.20}$$

$$x_i \leq r_i \quad 1 \leq i \leq \mathfrak{n}_0 \tag{5.21}$$

$$-x_i \leq -\ell_i \; 1 \leq i \leq \mathfrak{n}_0 \tag{5.22}$$

Just as specified in the proof of Theorem 5.4, the constraints (5.20) to (5.22) have to be dropped for the other cases of the PWUW-problem.

To solve the given Bucklin-PWUW instance, we solve the above linear program for each $\ell \in \{1, 2, \ldots, \mathfrak{m}\}$, starting with $\ell = 1$. That is, we check for each level ℓ starting with level 1, whether c is a possible level ℓ Bucklin winner. If none of the $\mathfrak{m}$ different linear programs has a feasible solution, the given PWUW instance is a no-instance. Thus, there is no level on which c can be made a Bucklin winner by setting the weights of the voters in V_0 accordingly.

If on the other hand, we find a solution with $\chi > 0$ for an ℓ, we know that c is a possible winner of the election. For the case when a solution with $\chi = 0$ has been found for an ℓ, we first have to test whether c is really a level ℓ Bucklin winner because in this case in the constraint (5.14) it could hold that the left-hand side equals the right-hand side. This, however, implicates that c does not reach a strict majority on level ℓ. If c does not win the election for this solution, we have to move on to the next ℓ if possible.

For each $\ell \in \{1, 2, \ldots, \mathfrak{m}\}$ we have a linear program with at most $2\mathfrak{m} + 3\mathfrak{n}_0 + 4$ constraints. Thus, by Theorem 5.4 we have shown the claim.

By adding χ to the left-hand side of (5.15) a solution with a positive value for χ means that c is a possible unique winner for the given election. ❑

Due to the close relation of fallback and Bucklin winners, we can adapt the proof for Bucklin elections in a straightforward manner to prove the following theorem.

Theorem 5.7 *In fallback elections the problems* PWUW-$\mathbb{Q}_{\geq 0}$, PWUW-BW-$\mathbb{Q}_{\geq 0}$, PWUW-RW-$\mathbb{Q}_{\geq 0}$, *and* PWUW-BW-RW-$\mathbb{Q}_{\geq 0}$ *are all in* P *in both winner models.*

Proof. For a given fallback-PWUW-BW-RW-$\mathbb{Q}_{\geq 0}$ instance we denote by $\ell_{\mathfrak{m}}$ the maximal number of candidates any voter in V ranks, that is, $\ell_{\mathfrak{m}}$ is the maximal level on which a Bucklin winner may exist. (We know that $\ell_{\mathfrak{m}} < \mathfrak{m}$ can hold due to disapprovals.)

Using the representation of Bucklin elections, we first test whether the designated candidate c is a possible Bucklin winner on a level $\ell \in \{1, 2, \ldots, \ell_{\mathfrak{m}}\}$. If not, we check whether c is a possible fallback winner by approval. If this is neither the case, we know that the given fallback-PWUW-BW-RW-$\mathbb{Q}_{\geq 0}$ is a no-instance.

We know that candidate c is a fallback winner by approval if and only if the following holds for an assignment of the weights x_i to the voters in V_0:

$$\sum_{i=1}^{\mathfrak{n}_0} lev_i^{\ell_{\mathfrak{m}}}(c')x_i + \sum_{i=\mathfrak{n}_0+1}^{\mathfrak{n}_1} lev_i^{\ell_{\mathfrak{m}}}(c') \leq \sum_{i=1}^{\mathfrak{n}_0} lev_i^{\ell_{\mathfrak{m}}}(c)x_i + \sum_{i=\mathfrak{n}_0+1}^{\mathfrak{n}_1} lev_i^{\ell_{\mathfrak{m}}}(c)$$

To decide the given fallback-PWUW-BW-RW-$\mathbb{Q}_{\geq 0}$ instance, solve the linear program given in the proof of Theorem 5.6 for each $\ell \in \{1, 2, \ldots, \ell_{\mathfrak{m}}\}$. If no solution could be found in which c is a level ℓ Bucklin winner, we exchange constraints (5.13) to (5.17) in the linear program in the proof of Theorem 5.6 with (5.23) below

$$-\sum_{i=1}^{\mathfrak{n}_0} x_i(lev_i^{\ell_{\mathfrak{m}}}(c) - lev_i^{\ell_{\mathfrak{m}}}(c')) \leq \sum_{i=\mathfrak{n}_0+1}^{\mathfrak{n}_1} (lev_i^{\ell_{\mathfrak{m}}}(c) - lev_i^{\ell_{\mathfrak{m}}}(c')) \quad \forall c' \neq c \tag{5.23}$$

If this linear program also has no feasible solution, we know that the given fallback-PWUW-BW-RW-$\mathbb{Q}_{\geq 0}$ is a no-instance.

Since $\ell_{\mathfrak{m}} \leq \mathfrak{m}$ by definition, we know that we have to solve at most $\mathfrak{m}$ many linear programs in which each program has $2\mathfrak{m} + 3\mathfrak{n}_0 + 4$ constraints and additionally, if none of them had a solution, one linear program with $\mathfrak{m} + 3\mathfrak{n}_0 + 1$ constraints. Thus, with Theorem 5.4 we have shown that the claim holds.

The case of a possible unique winner can also be solved by the above approach. On levels 1 to $\ell_{\mathfrak{m}}$ the the linear program corresponding to the problem for Bucklin elections has to be adapted as mentioned at the end of the proof for Theorem 5.6. To do so for the approval stage, the same adjustment has to be made in constraint (5.23), namely adding the variable χ to the left-hand side of the constraint and search for a solution with $\chi > 0$. ❑

5.4 Concluding Remarks and Future Work

We introduced a new variant of the possible winner problem in weighted elections, where the uncertainty lies in the voters' weights while the preferences are completely specified. Our model distinguishes between two settings, one in which the weights can be arbitrary natural numbers ($\mathscr{E}$-PWUW-$\mathbb{N}_0$) and one in which we allow the weights to be nonnegative rational numbers ($\mathscr{E}$-PWUW-$\mathbb{Q}_{\geq 0}$). We have shown that all defined variants for rational weights can be solved in deterministic polynomial time when the given voting rule can be described by a system of polynomial many linear inequalities. This approach is applied to the family of positional scoring rules, Bucklin, and fallback voting. Whether it can be extended to other voting rules is left for future work. Particularly, the open problems displayed in Tables 5.2 and 5.3 could be a next step for further investigation: The study of the complexity of the problem $\mathscr{E}$-PWUW-$\mathbb{N}_0$ is incomplete for 3-approval voting, positional scoring rules as well as for the cup and the Schulze rule. For the latter two voting systems also $\mathscr{E}$-PWUW-$\mathbb{Q}_{\geq 0}$ has to be investigated.

The suggested new problems are defined in the constructive case aiming at making a designated candidate win the given election by assigning the weights appropriately. A natural next step would be to consider the destructive variants. Moreover, other restrictions regarding the weights can be introduced, for instance, by allowing sets of intervals the weights can be chosen from. Another interesting and more general variant can be considered for future research in which in addition to the undefined weights, also incomplete preferences are allowed.

6 Hedonic Games

"Play is the highest form of research" – Albert Einstein states what everyone sees when watching children play. Nothing compares to childlike curiosity and their enthusiasm for perseveringly searching for new challenges. To a lesser extent, most people keep this urge to play in their free time, engaging in team sports, playing poker and chess, just to name a few.

But also in a broader sense we encounter situations that can be seen as games. In salary negotiation, for example, we have two players, the employer and the employee, both strategically acting in order to maximize their gain. At a first glance, the goals of the two players seem obvious: the employer wants to pay as little as possible while the employee certainly aims at raising her income. There are, however, other aspects that come into play. If the employee's demand is too high, the employer might consider it unreasonable and may even decide to let the employee go. The employer, on the other hand, should have in mind that payment is a form of acknowledgment and that underpaid employees often underachieve and are less motivated. We see that even in a simple looking setting, the players' strategies can be rather involved.

Far more complicated scenarios can be found in the wider context of economics and microeconomics – the background on which the research area of *game theory* was build in the early 40s of the last century in the seminal work of von Neumann and Morgenstern [NM44]. Based on their approach, Nash [Nas50b, Nas51, Nas50a] developed a further theory of *noncooperative games* and defined a notion of a stable solution, today known as *Nash equilibrium*. In these games, the participating players choose their actions with the sole aim of maximizing their outcome. For an overview of research areas in the field of *noncooperative game theory*, we refer to the book chapter by Faliszewski et al. [FRR15], who also cover computational aspects. Further algorithmic aspects are discussed in the work of Nisan et al. [NRT+07].

In *cooperative game theory* the focus lies on games in which players strategically play together and form coalitions, see the textbook by Peleg and Sudhölter [PS03]. Such games are called *coalition formation games* and the solution of such a game is a partition of the players into disjoint coalitions such that the players' utilities are maximized. How these utilities are defined exactly depends on the context and there is a variety of different classes of cooperative games. For a computational point of view on cooperative game theory, and a comprehensive overview of different classes of games, we refer to the work of Elkind and Rothe [ER15] and Chalkiadakis et al. [CEW11].

In this chapter we focus on coalition formation games in which the players have preferences over the possible coalitions they can join and these preferences are *purely hedonic*. This concept was introduced by Drèze and Greenberg [DG80] and it describes the situation in which a player's evaluation of a coalition does only depend on the players contained in the coalition, thus inter-coalitional dependencies have no impact. The formal model that will be used in this chapter was independently introduced in the work of Bogomolnaia and Jackson [BJ02] and

Banerjee et al. [BKS01]. These games are particularly interesting as they combine aspects from cooperative game theory and voting theory: The players, which can be considered the voters, in some sense vote for the coalitions they consider worth joining by expressing their preferences. How happy the players are with a given solution can be measured with notions of stability. There are two major issues that are typically addressed in the context of hedonic game when computational aspects are in the focus of interest.

(1) Preference representation: How can the players' preferences over coalitions (which are exponentially many in the number of players for each player) be represented in a compact, thus, feasible way, while allowing the players to define their preferences as precisely and freely as possible?

(2) Verification and existence of stable solutions: Given a solution concept, how hard is it for a given game and a given solution to verify its stability? And furthermore, how hard is it to decide whether a stable solution exists at all for a given game?

Woeginger [Woe13b] provides a detailed survey on the study of computational aspects in hedonic games. See also the book chapters by Elkind and Rothe [ER15] and Aziz and Savani [AS16].

Organization of this Chapter In the first section of this chapter we give the basic definitions needed for our study: We define the concept of hedonic game and present preference representations from the literature as well as well-known solution concepts. We conclude the section with an overview of related work. In Section 6.2 we turn to the analysis of strictly core-stable coalition structures in enemy-based hedonic games and wonderfully stable partitions in corresponding graphs. We discuss known complexity results for the existence and verification problems and present an approach for pinpointing the exact complexity of these problems. In Section 6.3 we introduce a new class of hedonic games, so-called *FEN-hedonic games* in which ordinal preferences are combined with the notion of friends and enemies. We discuss how such preferences over players can be extended to preferences over coalitions and define the concept of *Borda-induced FEN-hedonic game*. For these games we study the computational complexity of various solution concepts. The chapter concludes with a summary of our results and pointers to interesting future work.

6.1 Hedonic Games and Stability Concepts

Following the formal concept of Banerjee et al. [BKS01] and Bogomolnaia and Jackson [BJ02], we define a hedonic game as follows.

Definition 6.1 (Hedonic Game) *A* hedonic game $\mathscr{G}$ *is a tuple* $(P,\succeq)$ *consisting of a set of players* $P = \{1,2,\dots,n\}$ *and a preference profile* $\succeq = (\succeq_1,\succeq_2,\dots,\succeq_n)$*, where* $\succeq_i$ *is the preference relation of player* $i \in P$*.*

A *coalition* C of players in a game $\mathscr{G} = (P,\succeq)$ is a subset of P. For each player $i \in P$ we denote with $\mathscr{P}_i$ the set of coalitions containing i, that is $\mathscr{P}_i = \{C \subseteq P \mid i \in C\}$. The preference relation $\succeq_i \in \mathscr{P}_i^2$ determines for a pair of coalitions containing i, which coalition is preferred by i. For two coalitions $A, B \in \mathscr{P}_i$, we say that *player i weakly prefers A to B* if $A \succeq_i B$, whereas *player i prefers A to B* if $A \succeq_i B$, but not $B \succeq_i A$, and we write $A \succ_i B$. If $A \succeq_i B$ and $B \succeq_i A$, *player i is indifferent between A and B*, which we denote with $A \sim_i B$.

We call a partition Γ of the players in P into $k \in \mathbb{N}$ coalitions a *coalition structure*, denoted by $\Gamma = \{C_1, C_2, \ldots, C_k\}$, where $\emptyset \neq C_r \subseteq P$ for each $1 \leq r \leq k$, $\bigcup_{r=1}^{k} C_r = P$, and $C_r \cap C_s = \emptyset$ for each $1 \leq r \neq s \leq k$ holds. We denote with $\Gamma(i)$ the unique coalition player i is assigned to in coalition structure Γ. A coalition structure of a given game is also called a *solution* of the game.

6.1.1 Preference Representations

Clearly, the cardinality of $\mathscr{P}_i$ is exponential in the number of players for each player i, thus expecting the players to provide a complete ranking over all possible coalitions would be impractical.

This leads to a central problem that has been addressed by various approaches: How can the preference of player i be given such that all coalitions are comparable while at the same time ensuring that the representation is compact and for two given coalitions their relation with respect to $\succeq_i$ can be determined in deterministic polynomial time in the number of players.

We start with introducing a class of hedonic games, namely hedonic games with *additively separable preferences*, which was introduced by Banerjee et al. [BKS01]. In these games the players assign values to the other players and the relation of two coalitions from player i's view only depends on the values she assigns to those players that are part of the two coalitions.

Definition 6.2 (Additively Separable Hedonic Game) *An additively separable hedonic game is a tuple $(P, \succeq^{AS})$, where $P = \{1, 2, \ldots, n\}$ is the set of players and $\succeq^{AS} = (\succeq_1^{AS}, \succeq_2^{AS}, \ldots, \succeq_n^{AS})$ gives the additively separable preference relations of the players in P. Each player $i \in P$ provides a value function $w_i : P - \{i\} \to \mathbb{Z}$ determining the value player i gains if player $j \neq i$ is contained in the same coalition as i. With this notion a players'* utility u_i for a coalition *$A \in \mathscr{P}_i$ is defined to be $u_i(A) = \sum_{j \in A - \{i\}} w_i(j)$. For two coalitions $A, B \in \mathscr{P}_i$ it holds that*

- *$A \succeq_i^{AS} B$ if and only if $u_i(A) \geq u_i(B)$,*
- *$A \succ_i^{AS} B$ if and only if $u_i(A) > u_i(B)$, and*
- *$A \sim_i^{AS} B$ if and only if $u_i(A) = u_i(B)$.*

Since the values the players assign to the other players fully describe their preference relation over the coalitions they are contained in, an additively separable hedonic game can also be given by (P, w) for $w = (w_1, w_2, \ldots, w_n)$.

Dimitrov et al. [DBH+06] have taken a different approach in tackling the problem of preference representation and introduced a class of hedonic games where each player segregates the other players into friends and enemies and the relation between two coalitions depends on the number of friends or enemies the coalitions contain. Two types of preference extension were defined in this context: one focuses on the appreciation of friends while the other lays the emphasis on the aversion to enemies. We will only define the latter in detail.

Definition 6.3 (Enemy-Based Hedonic Game) *An enemy-based hedonic game* $\mathscr{G} = (P, \succeq^E)$ *consists of a player set* $P = \{1, 2, \ldots, n\}$ *and the enemy-based preference relations of the players* $\succeq^E = (\succeq_1^E, \succeq_2^E, \ldots, \succeq_n^E)$*. Each player* $i \in P$ *has a set of friends* $F_i \subseteq P - \{i\}$ *and a set of enemies* $E_i = P - (F_i - \{i\})$*. For two coalitions* $A, B \in \mathscr{P}_i$ *it holds that* $A \succeq_i^E B$ *if either*

- $\|A \cap E_i\| < \|B \cap E_i\|$ *or*
- $\|A \cap E_i\| = \|B \cap E_i\|$ *and* $\|A \cap F_i\| \geq \|B \cap F_i\|$

holds. $A \succ_i^E B$ *holds if* $A \succeq_i^E B$*, but not* $B \succeq_i^E A$*, and we write* $A \sim_i^E B$ *whenever both* $A \succeq_i^E B$ *and* $B \succeq_i^E A$ *hold.*

As Dimitrov et al. [DBH[+]06] pointed out, every enemy-based hedonic game is a special additively separable hedonic game. We state this in the following remark.

Remark 6.4 *For a given enemy-based hedonic game* $\mathscr{G} = (P, \succeq^E)$ *an equivalent additively separable hedonic game* $\mathscr{G}' = (P, w)$ *can be obtained by defining the values of the players in* P *to be* $w_i(j) = 1$ *for* $j \in F_i$*, and* $w_i(j) = -\|P\|$ *for* $j \in E_i$*.*

Strictly following the definition of enemy-based hedonic games, the friendship relations between the players do not have to be symmetric, that is, it could be possible that for two players $i, j \in P$ player i considers j to be a friend ($j \in F_i$), but j on the other hand considers i to be an enemy ($i \notin F_j$). Woeginger [Woe13b], however, points out that when studying stability concepts the assumption that all friendship relations are indeed symmetric is a reasonable assumption; we will explain this in detail in Section 6.1.2. Thus in the following, we assume that for two players $i, j \in P$ it holds that $i \in F_j$ if and only if $j \in F_i$.

Assuming symmetric friendship relations furthermore allows to represent an enemy-based hedonic game $\mathscr{G} = (P, \succeq^E)$ as an undirected graph $G = (V, H)$[1] in which we have a vertex v_i for each $i \in P$ and an edge $\{v_i, v_j\} \in H$ if $i \in F_j$, thus two vertices in the graph are connected by an edge if the players in the game are friends of each other. We call this graph G the *network of friends* or *the graph associated with the game* $\mathscr{G}$ *(or graph representation of* $\mathscr{G}$*)*, and a coalition structure Γ in $\mathscr{G}$ corresponds to a partition Π of the vertices, where we denote the set in Π containing vertex v_i by $\Pi(v_i)$.

Note that for the sake of readability, in figures illustrating the graph representation of a hedonic game, we will denote the vertices with $1, 2, \ldots, n$ instead of $v_1, v_2, \ldots, v_n$.

Example 6.5 *We define the enemy-based hedonic game* $\mathscr{G} = (P, \succeq^E)$ *with* $P = \{1, 2, 3, 4, 5, 6\}$ *with* $F_1 = \{2, 4\}$*,* $F_2 = \{1, 3, 4, 5, 6\}$*,* $F_3 = \{2, 5, 6\}$*,* $F_4 = \{1, 2, 5\}$*,* $F_5 = \{2, 3, 4, 6\}$*, and finally* $F_6 = \{2, 3, 5\}$*. Graph* $G = (V, H)$ *showed in Figure 6.1 is the graph representation of* $\mathscr{G}$*.*

Based on this network of friends we can see that the relation between the two coalitions $A = \{1, 2, 4, 6\}$ *and* $B = \{1, 2, 5, 6\}$ *drastically varies depending on the player:* $A \succ_1^E B$*, but* $B \succ_6^E A$*, while* $A \sim_2^E B$*.*

[1] Note that in this chapter we will denote the set of edges in a graph with H instead of E as the latter is already used to denote the set of enemies.

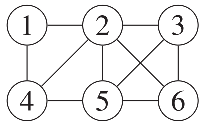


Figure 6.1: Graph representation of enemy-based hedonic game $\mathscr{G} = (\{1,2,3,4,5,6\}, \succeq^E)$

We conclude by giving a short overview of chosen encodings defined in the literature, which will, however, not be further studied in the scope of this thesis.

Dimitrov et al. [DBH+06] introduced a second type of extending preferences when a network of friends is given, where the focus lies on the appreciation of friends instead of the aversion to enemies. In the so-called *friend-based* preference extension model, a player prefers coalition A to another coalition B whenever there are either more friends in A than in B or, if the number of friends is equal, A contains less enemies than B.

In *fractional hedonic games*, introduced by Aziz et al. [ABH14], the players assign numerical values to their co-players (similar to additively separable hedonic games introduced above) and the value of a coalition is the average of those players' values who are part of the coalition.

The *anonymous encoding*, see the work of Ballester [Bal04], expects the players to provide a ranking solely over the sizes of coalitions, thus assuming that the identities of the players are not important for the value of a coalition.

In the *singleton encoding*, which is due to Cechlárová and Romero-Medina [CR01] and studied by Cechlárová and Hajduková [CH03, CH04], the players provide a ranking over the players and the relation of two coalitions is derived from this ranking depending on the worst ranked players in the two coalitions (the pessimistic extension) or the best ranked players in the two coalitions (the optimistic extension).

In contrast to the just defined compact representations, the *individually rational encoding*, see [Bal04], is an example of a representation with exponential size in the number of players. Here, each player provides a ranking over all coalitions she considers to be *acceptable*, that is, prefers to being alone.

As this list is by far not exhaustive, we refer the reader to the survey by Woeginger [Woe13b] and the book chapters by Elkind and Rothe [ER15] and Aziz and Savani [AS16] for further definitions and discussions.

6.1.2 Stability Concepts and Decision Problems

The solution of a given hedonic game is a partition of the players into disjoint coalitions and, depending on the context, there might be different properties such a coalition structure should preferably fulfill. We focus on notions of *stability* that aim at capturing the players' satisfaction with the partition the given coalition structure determines.

The considered stability concepts can be grouped in three categories: single-player deviations, deviations based on comparisons of coalitions, and deviations of entire groups. We start with two variants from the first category and the definition of perfectness.

Definition 6.6 (Stability Concepts – Part I) *Let $\mathscr{G} = (P, \succeq)$ be a hedonic game and let Γ be a coalition structure. We call Γ*

- individual rational *if $\Gamma(i) \succeq_i \{i\}$ for all $i \in P$;*
- perfect *if for all $i \in P$ we have that $\Gamma(i) \succeq_i B$ for each $B \in \mathscr{P}_i$;*
- Nash-stable *if for all $i \in P$ we have that $\Gamma(i) \succeq_i C \cup \{i\}$ for each $\Gamma(i) \neq C \in \Gamma \cup \{\emptyset\}$.*

In other words, individual rationality assures each player a coalition that she prefers to being alone. In some contexts, such coalitions, namely $C \in \mathscr{P}_i$ with $C \succeq_i \{i\}$ are called *acceptable for player i*, see for example the work of Darmann et al. [DEK+12], which implies that this can be seen as type of minimum satisfaction a coalition structure should provide. A perfect coalition structure, on the other hand, can be considered (as the name already suggests) to be an ideal partition into coalitions since each player considers her coalition to be one of the best of all possible coalitions she might be contained in. Needless to say, perfect coalition structures do not always exist. Nash stability somehow lies in between the two above defined types of stability: A Nash-stable coalition structure Γ guarantees for each player that she weakly prefers her own coalition to the remaining coalitions given in the structure, that is, there might be a coalition $B \in \mathscr{P}_i$ player i weakly prefers to $\Gamma(i)$ (which would already violate perfectness of Γ), but as long as $B - \{i\} \notin \Gamma$, Nash stability is still fulfilled (assuming of course that for the other players the criterion is fulfilled, as well). Furthermore, every Nash-stable coalition structure is clearly individually rational as by definition $\Gamma(i) \succeq_i \{i\} \cup \{\emptyset\} = \{i\}$ has to hold.

Here it becomes obvious why only symmetric friendship relations matter in the context of stability in enemy-based hedonic games. From the definition of enemy-based preferences it follows that as soon as an enemy of player i is in the same coalition, player i prefers being alone. Thus any coalition structure in which there is a coalition containing two players that are not mutual friends, the coalition structure is not even individually rational. This justifies the assumption of symmetric friendship relations.

Example 6.7 illustrates the just defined notions of individual rationality, perfectness, and Nash stability in an enemy-based hedonic game.

Example 6.7 *Recall the game $\mathscr{G} = (P, \succeq^E)$ from Example 6.5 and consider the coalition structure $\Gamma = \{\{1,4\},\{2,3,5,6\}\}$, which is indicated by the dashed lines in Figure 6.2 below.*

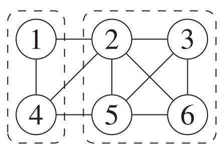


Figure 6.2: Graph G corresponding to $\mathscr{G}$ and $\Gamma = \{\{1,4\},\{2,3,5,6\}\}$

The coalition structure Γ is individually rational since every player $i \in P$ is in a coalition with at least one friend, thus prefers $\Gamma(i)$ to being alone in coalition $\{i\}$. It is clear that Γ is not perfect since no player is in a coalition with all of her friends which would be the most preferred coalition. Nash stability, on the other hand, is fulfilled: Players $1,4,3,5$, and 6 have *no incentive for moving to the respective other coalition since they would then be in a coalition containing at least one enemy. Only for player* 2 *the other coalition,* $C \neq \Gamma(2)$ *consists of only friends. But since in* $\{1,2,4\}$ *there are fewer friends than in* $\{2,3,5,6\}$*, player* 2 *does not want to change coalitions.*

The following stability concepts do not only consider the preference of player i wanting to leave her coalition $\Gamma(i)$ to another coalition C. They also take into account how the leaving of player i would impact those players contained in $\Gamma(i)$ and furthermore also check whether the players of coalition C want the deviating player i to join them.

Definition 6.8 (Stability Concepts – Part II) *Let* $\mathscr{G} = (P, \succeq)$ *be a hedonic game and let* Γ *be a coalition structure. We say that* Γ *is*

- individually stable *if for all* $i \in P$ *and all* $C \in \Gamma \cup \{\emptyset\}$ *it either holds that* $\Gamma(i) \succeq_i C \cup \{i\}$ *or there is a player* $j \in C$ *with* $C \succ_j C \cup \{i\}$*;*
- contractually individually stable *if for all* $i \in P$ *and all* $C \in \Gamma \cup \{\emptyset\}$ *it either holds that* $\Gamma(i) \succeq_i C \cup \{i\}$*, or there is a player* $j \in C$ *with* $C \succ_j C \cup \{i\}$*, or there is a player* $k \in \Gamma(i) - \{i\}$ *with* $\Gamma(i) \succ_k \Gamma(i) - \{i\}$.

Let Γ be a fixed coalition structure and let $i \in N$ be a player who may be willing to deviate since there is a coalition $C \in \Gamma$ with $C \succ_i \Gamma(i)$. Then Γ is *not* individually stable if all players $j \in C$ want player i to join them, that is $C \cup \{i\} \succeq_j C$ for all $j \in C$. Contractually individual stability also takes those players into account who are part of the coalition the deviating player wants to leave, meaning that if there is a player i with $C \succ_i \Gamma(i)$ and the players in C want i to join the coalition, the contractually individual stability of Γ is only violated if the players in $\Gamma(i)$ want player i to leave, thus $\Gamma(i) - \{i\} \succeq_k \Gamma(i)$ has to hold for all $k \in \Gamma(i) - \{i\}$.

Example 6.9 *Consider again the game* $\mathscr{G} = (P, \succeq^E)$ *from Example 6.5 and the coalition structure* $\Gamma = \{\{1,4\},\{2,5\},\{3,6\}\}$ *displayed in Figure 6.3 by the dashed lines.*

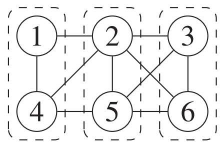


Figure 6.3: Graph G corresponding to $\mathscr{G}$ and $\Gamma = \{\{1,4\},\{2,5\},\{3,6\}\}$

Player 1 *does not want to deviate since in all other coalitions there are enemies of his; thus* $\Gamma(1) \succ_1^E C$ *for all* $\Gamma(1) \neq C \in \Gamma$*. Player* 2 *would prefer joining* $\{1,4\}$ *or* $\{3,6\}$*, and the players in both coalitions would want* 2 *to join them, but* 5 *prefers being with* 2 *to being alone. Similarly, players* 3, 4, 5*, and* 6 *would prefer joining at least one other coalition and the other players would welcome the new member. But for each of the four players, the other player that would be left alone by the deviation, would be worse off, since she considers the current coalition to be acceptable. Thus,* Γ *is individually stable, but not contractually individually stable.*

Now we turn to stability concepts that consider the deviation of entire groups of players for which we need the notion of blocking coalitions: Let Γ be a coalition structure in a hedonic game $\mathscr{G} = (P, \succeq)$. We say that a *coalition* $C \subseteq P$ *blocks* Γ if for all $i \in C$ it holds that $C \succ_i \Gamma(i)$. If, on the other hand, for each $i \in C$ we have that $C \succeq_i \Gamma(i)$ and there is at least one player $j \in C$ with $C \succ_j \Gamma(j)$, we say that this *coalition weakly blocks* Γ.

Definition 6.10 (Stability Concepts – Part III) *Let $\mathscr{G} = (P, \succeq)$ be a hedonic game and let Γ be a coalition structure. We say that Γ is*

1. core-stable *if there is no nonempty coalition $C \subseteq P$ that blocks Γ;*
2. strictly core-stable *if there is no coalition $C \subseteq P$ that weakly blocks Γ.*

Recalling the definition of enemy-based hedonic games, we can see that in such a game a core-stable coalition structure always corresponds to a partition into cliques in the corresponding graph.

Example 6.11 *Consider again the game $\mathscr{G} = (P, \succeq^E)$ from Example 6.5 and the coalition structures $\Gamma_1 = \{\{1,4\},\{2,3,5,6\}\}$ and $\Gamma_2 = \{\{1,4\},\{2,5\},\{3,6\}\}$, illustrated in Figure 6.4a and Figure 6.4b, respectively, by the dashed lines.*

(a) $\Gamma_1 = \{\{1,4\},\{2,3,5,6\}\}$ **(b)** $\Gamma_2 = \{\{1,4\},\{2,5\},\{3,6\}\}$

Figure 6.4: Graph G corresponding to $\mathscr{G}$ and two coalition structures Γ_1, Γ_2

Consider coalition structure Γ_1. We see that the players in coalition $\{2,3,5,6\}$ form the largest clique in G and this clique is a unique clique of size 4. Thus by definition, these players cannot be part of a blocking coalition (which implies that it is neither a weakly blocking coalition). This leaves the coalition $\{1,4\}$ as a possibly (weakly) blocking one, but since $\{1,4\} \in \Gamma_1$, this is not the case. So Γ is core-stable and also strictly core-stable.

Turning to Γ_2, we directly see with the argumentation above that $\{2,3,5,6\}$ is a blocking, and thus a weakly blocking coalition (illustrated in Figure 6.4b by the filled vertices). Γ_2 is therefore not (strictly) core-stable.

Figure 6.5 gives an overview of the relations between the just defined stability concepts. A directed edge from stability α to stability β indicates that if a coalition structure Γ for a given hedonic game $\mathscr{G}$ satisfies α, then Γ also satisfies β. For example, a core-stable coalition structure is always individually rational. Note that the relations are transitive, for example, each perfect coalition structure is also individually stable. For a detailed overview surveying also other stability concepts, see the work of Aziz et al. [ABS13].

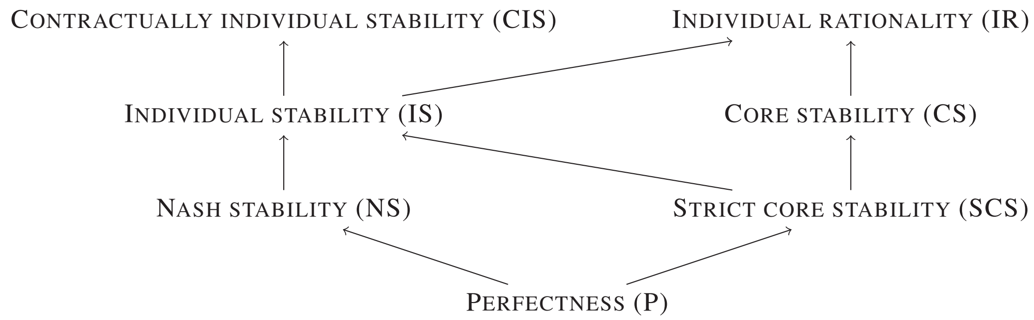


Figure 6.5: Overview of relations between stability concepts

There are two questions that naturally arise from a complexity-theoretic point of view in the context of stability in hedonic games: Given a hedonic game and a coalition structure, how hard is it to decide whether the coalition structure is stable in the sense of α in the given game, where α is a beforehand fixed stability concept. To answer this question, the so-called *verification problem* is defined and its complexity is analyzed.

α-VERIFICATION (αV)	
Given:	A hedonic game $(P, \succeq)$ and a coalition structure Γ.
Question:	Does Γ in $\mathscr{G} = (P, \succeq)$ satisfy α?

The second question aims at the problem of deciding whether a given game has a coalition structure which is stable in the sense of α-stable, which is formally stated in the so-called *existence problem*.

α-EXISTENCE (αE)	
Given:	A hedonic game $(P, \succeq)$.
Question:	Is there a coalition structure Γ in $(P, \succeq)$ satisfying α?

6.1.3 Related Work

Ballester [Bal04] initiated the complexity-theoretic study of the existence problem for Nash stability, core stability, and individual stability for hedonic games in the anonymous encoding and the individual rational encoding for both strict and weak preferences.

The class of additively separable hedonic games has been intensely studied. We provide an overview of known results in Table 6.1 for those stability concepts that will be further analyzed in the upcoming Section 6.3.

These results are complemented by further analyses regarding symmetric preferences and other stability concepts. Also the complexity of computing stable outcomes in additively separable hedonic games has found much attention, see also the work of Gairing and Savani [GS10]. For a comprehensive overview, we refer to the survey by Aziz et al. [ABS13]. Very

Stability	VERIFICATION	EXISTENCE
Perfectness		P [1]
Nash Stability Individual Stability	P [1]	NP-complete [2]
Contractually Individual Stability		trivial [3]
Core Stability	coNP-complete [4]	Σ_2^p-complete [5]
Strict Core Stability	coNP-complete [1]	Σ_2^p-complete [6]

[1] due to [ABS13] [3] due to [Bal04] [5] due to [Woe13a]
[2] due to [SD10] [4] due to [SD07] [6] due to [Woe13b, Pet15]

Table 6.1: Overview of known complexity results for chosen stability problems in additively separable hedonic games

recently, Peters [Pet15] resolved the most glaring open question for additively separable hedonic games regarding strict core stability by showing that the existence problem is indeed, as Woeginger [Woe13b] conjectured, Σ_2^p-complete. Prior to this result the best lower bound was known to be DP-hardness, which we show in Corollary 6.22 on page 150. The result is based on a proof for the corresponding problem in enemy-based hedonic games (established in Theorem 6.21 on page 150). Peters [Pet15] furthermore shows that STRICT CORE STABILITY EXISTENCE is also Σ_2^p-complete for *boolean hedonic games*.

Hedonic games with enemy-based and friend-based preference extensions are studied in the work of Dimitrov et al. [DBH+06] and Sung and Dimitrov [SD07]. They show that in these games, core-stable coalition structures always exist and that games with friend-based preference extensions also always have a strictly core-stable coalition structure. The exact complexity of the verification problem for core stability in friend-based hedonic games however, is still unresolved, but Woeginger [Woe13b] conjectures that it is solvable in deterministic polynomial time. Dimitrov et al. [DBH+06] show that verifying whether a given coalition structure in an enemy-based hedonic game is core-stable is NP-complete, while the best lower bound for STRICT CORE STABILITY EXISTENCE is DP-hardness, as we show in Theorem 6.21 on page 150.

Hedonic coalition nets, a representation of hedonic games using classical propositional logic, are introduced by Elkind and Wooldridge [EW09], who study the complexity of CORE STABILITY EXISTENCE and CORE STABILITY VERIFICATION in these games.

Aziz et al. [AHP12] study the complexity of individual and Nash stability in hedonic games with $\mathscr{B}$- and $\mathscr{W}$-preferences, respectively. They furthermore introduce and analyze so-called *B-hedonic games* and *W-hedonic games*, which are closely related but not equivalent to $\mathscr{B}$- and $\mathscr{W}$-hedonic games.

Roommate games and *marriage games* are hedonic games in which the size of the coalitions the players can join in a feasible solution is restricted to 2 and in the latter case, the set of players is the union of two types of, classically female and male, players. Finding stable solutions for these games is closely related to solving the *stable roommate problem* and the *stable marriage problem*, respectively, see the work of Gale and Shapley [GS62]. Recently, Aziz [Azi13] complemented known results on core stability from Gale and Shapley [GS62] and Irving [Irv85], while Munera et al. [MDA+15] present new algorithmic methods for solving these problems (see also the textbooks by Gusfield and Irving [GI89] and Roth and Sotomayor [RS92] and the book chapter by Klaus et al. [KMR16]).

Pareto-optimality and perfectness have been recently studied in the work of Aziz et al. [ABH13] for various classes of hedonic games.

Hedonic games can also be used to model problems of scheduling or selecting group activities, see the work of Darmann et al. [DEK+12] and Lee [Lee14], or to improve the experience of popular online games, see the work of Spradling et al. [SGX+13, Spr14, SG15].

The class of fractional hedonic games has been introduced by Aziz et al. [ABH14] and further studied in the work of Bilò et al. [BFF+15] and Brandl et al. [BBS15].

In their intriguing work, Peters and Elkind [PE15] establish connections between certain properties of preference extensions in hedonic games that imply NP-hardness of the existence problem for most of the above defined stability concepts. We survey and discuss some of their findings in Section 6.3.1.

Finally, we conclude by referring to the survey by Woeginger [Woe13b] and the book chapters by Elkind and Rothe [ER15] and Aziz and Savani [AS16] for related work beyond the just presented.

6.2 Wonderful and Strict Core Stability in Enemy-Based Hedonic Games

In this section we focus on hedonic games with enemy-based preferences which can be derived from the network of friends that the players' friendship relations define. These games are particularly interesting as graph-theoretic properties of the network can be exploited to find solutions or analyze the complexity of stability problems. The presented results are published in the work of Rey et al. [RRS+14, RRS+15].

6.2.1 Wonderful Stability – Definition and Relation to Strict Core Stability

For hedonic games with enemy-based preferences, Woeginger [Woe13b] suggests the notion of *wonderful stability*, a stability concept which is directly defined for the network of friends of a given game.

Definition 6.12 (Wonderful Stability) *Let $\mathcal{G} = (P, \succeq^E)$ be a hedonic game with enemy-based preferences and let $G = (V, H)$ be the corresponding graph representation. We denote a partition of the vertex set V by Π and the unique set containing vertex v_i by $\Pi(v_i)$. We say that a partition Π of V is* wonderfully stable *if all $C \in \Pi$ are cliques and $\|\Pi(v_i)\| = \omega_G(v_i)$ for all* *vertices $v_i \in V$. If there is a clique $C \subseteq V$ containing at least one vertex v_i with $\|C\| > \Pi(v_i)$, we call C a* blocking clique.

We continue our running example as follows.

Example 6.13 *Recall the game $\mathscr{G} = (P, \succeq^E)$ from Example 6.5 in Section 6.1 and the corresponding graph G. Clearly, there is no wonderfully stable partition in graph G as the maximal clique $\{2,3,5,6\}$ cannot be an element of Π without having players 1 and 4 violate the criterion for wonderful stability.*

Considering the game $\mathscr{G}'$ arising from deleting the friendship relation between players 2 and 4, we obtain the network G' and see that $\Pi = \{\{1,4\},\{2,3,5,6\}\}$ shown in Figure 6.6 is wonderfully stable.

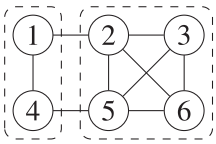


Figure 6.6: Graph G' corresponding to $\mathscr{G}'$ and the wonderfully stable partition $\Pi = \{\{1,4\},\{2,3,5,6\}\}$

As enemy-based hedonic games model preferences with the focus on enemy-aversion, a coalition is only acceptable for a player if it does not contain any enemies of his. Thus, any stable, and in particular strictly core-stable, coalition structure has to correspond to a partition of cliques in the corresponding graph. How closely wonderful stability and strict core stability are related becomes obvious with the following lemma.

Lemma 6.14 *Let $G = (V,H)$ be the network of friends in an enemy-based hedonic game $\mathscr{G}$. Let Π be a partition of V and let Γ be the corresponding coalition structure in $\mathscr{G}$.*

1. *If Π is a wonderfully stable partition for G, then Γ is a strictly core-stable coalition structure for $\mathscr{G}$.*
2. *If there is an integer $c \in \mathbb{N}$ such that $\omega_G(v) = c$ for all vertices $v \in V$ and Γ is a strictly core-stable coalition structure for $\mathscr{G}$, then Π is a wonderfully stable partition for G.*

Proof. The first implication holds by definition: If a coalition C weakly blocks a coalition structure that corresponds to a partition into cliques, C has to be a clique with a larger cardinality and hence blocks the partition.

Second, assume that there is a blocking clique C for Π, i.e., there exists some vertex $v_i \in C$ with $\omega_G(v_i) > \|\Pi(v_i)\|$. Since $\omega_G(v_i) = c$, there is a clique D with $C \subseteq D$ and $\|D\| = c$. Now, the corresponding coalition $\tilde{D} = \{i \mid v_i \in D\}$ is a weakly blocking coalition for Γ, because $\tilde{D} \succ_i^E \Gamma(i)$ and $\tilde{D} \succeq_j^E \Gamma(j)$ for each $j \in \tilde{D}$, which follows from the fact that the number of friends in $\Gamma(i)$ is at most $c-1$ and the number of friends in $\Gamma(j)$ is at most c, respectively. ❑

The following proposition yields a property of the SCSE and WSE problem that turns out to be useful for the analysis of their complexity.

Proposition 6.15 *In enemy-based hedonic games, the problems* SCSE *and* WSE *have* AND_ω *functions.*

Proof. Let n be a positive integer and let $\mathscr{G}_\ell$ with players P_ℓ for $\ell \in \{1,2,\dots,n\}$ be n different enemy-based hedonic games with a corresponding graph G_i. Define the game $\mathscr{G}$ with players $P = \bigcup_{\ell=1}^{n} P_\ell$ and the network of friends G which consists of the disjoint union of the graphs G_ℓ. In other words, G is a disconnected graph consisting of n independent components. Clearly, since the components are independent, it holds that G has a wonderfully stable partition of the vertices if and only if each of the components has such a partition. The same holds for strictly core-stable coalition structures. ❑

Observe that Proposition 6.15 can be generalized to hold for the SCSE problem in additively separable hedonic games. This is particularly interesting as the argumentation that will be presented in Section 6.2.3 also applies to additively separable hedonic games.

Proposition 6.16 *In additively separable hedonic games the problem* SCSE *has* AND_ω *functions.*

Proof. We show the needed reduction for $n = 2$. It is easy to see that the presented approach can be extended for arbitrary $n \geq 2$.

Let $\mathscr{G}_1 = (P_1, w^1)$ and $\mathscr{G}_2 = (P_2, w^2)$ be two additively separable hedonic games. We construct the additively separable hedonic game $\mathscr{G} = (P, w)$, where the set of players $P = P_1 \cup P_2$ is the marked union of the players in $\mathscr{G}_1$ and $\mathscr{G}_2$. To define the values in w we denote for each $p_i \in P$ with M_{p_i} the sum of all positive values player p_i assigns to the other players in her original game, formally:

$$M_{p_i} = \begin{cases} \sum\limits_{\{p_j \in P_1 \mid w^1_{p_i}(p_j) > 0\}} w^1_{p_i}(p_j), & \text{for } p_i \in P_1, \\ \sum\limits_{\{p_j \in P_2 \mid w^2_{p_i}(p_j) > 0\}} w^2_{p_i}(p_j), & \text{for } p_i \in P_2. \end{cases}$$

This allows us to define the values in w.

$$w_{p_i}(p_j) = \begin{cases} w^1_{p_i}(p_j), & \text{if } p_i, p_j \in P_1, \\ w^2_{p_i}(p_j), & \text{if } p_i, p_j \in P_2, \\ -(M_{p_i} + 1), & \text{otherwise.} \end{cases}$$

Thus we have that in the new game $\mathscr{G}$ any coalition C containing players from P_1 and P_2 has a negative value for all players contained in C.

We claim that $\mathscr{G}_1 \in$SCSE and $\mathscr{G}_2 \in$SCSE if and only if $\mathscr{G} \in$SCSE.

Only if: Assume that Γ_j are strictly core-stable coalition structures for the games $\mathscr{G}_j$ for $j \in \{1,2\}$. Then $\Gamma = \Gamma_1 \cup \Gamma_2$ is a strictly core-stable coalition structure for $\mathscr{G}$: Because of the assumption there cannot exist any weakly blocking coalition C containing only players from P_1 or only players from P_2. Any coalition C' containing players from both original games cannot be weakly blocking since all players contained in C prefer being alone to being in C'. Thus, there is no weakly blocking coalition in $\mathscr{G}$ and Γ is strictly core-stable for $\mathscr{G}$.

If: Assume that there exists a strictly core-stable coalition structure, for $\mathscr{G}$ and we denote it with Γ. We know from the definition of w that $\Gamma(p_i) \subseteq P_1$ has to hold for each $p_i \in P_1$ and $\Gamma(p_i) \subseteq P_2$ has to hold for each $p_i \in P_2$. Thus, $\Gamma = \{C_1, C_2, \dots, C_k\}$ with $C_\ell \subseteq P_1$ or $C_\ell \subseteq P_2$ for all $\ell \in \{1, 2, \dots, \ell\}$. Since Γ is strictly core-stable, there is no coalition $A \subseteq P_1$ weakly blocking Γ and there is also no coalition $B \subseteq P_2$ weakly blocking Γ. This implies that Γ_j containing those $C_\ell \in \Gamma$ with $C_\ell \in P_j$ is a strictly core-stable coalition structure for $\mathscr{G}_j$ for $j \in \{1, 2\}$. ❑

6.2.2 Complexity of Wonderful and Strict Core Stability Existence

Now we turn to the complexity of the verification and existence problems in the context of wonderful and strict core stability. For the former stability concept, we introduce these two problems formally and define them directly for the network of friends corresponding to the enemy-based hedonic game.

WONDERFUL STABILITY VERIFICATION (WSV)	
Given:	A graph $G = (V, H)$ and a partition Π of V into cliques.
Question:	Is the given partition Π wonderfully stable?

WONDERFUL STABILITY EXISTENCE (WSE)	
Given:	A graph $G = (V, H)$.
Question:	Does there exist a wonderfully stable partition for G?

For both wonderful stability and strict core stability, Woeginger [Woe13b] observes that the verification and existence problems can be stated in a compact quantified characterization, which allows to directly imply upper bounds. Note that Woeginger [Woe13b] defines the verification problem as the complement of our verification problem by asking whether there exists a (with respect to the stability concept) blocking coalition. In our context, the two problems for wonderful stability can be stated as follows.

$$(G, \Pi) \in \mathrm{WSV} \iff (\forall P)\,[\neg(P \text{ blocks } \Pi)], \tag{6.1}$$
$$G \in \mathrm{WSE} \iff (\exists \Pi)\,(\forall P)\,[\neg(P \text{ blocks } \Pi)]. \tag{6.2}$$

Clearly, the problems SCSV and SCSE can be characterized analogously.

Table 6.2 shows the complexity of the existence and verification problems for wonderful and strict core stability in enemy-based hedonic games.

The upper bounds can be followed directly from the above representation: Since testing whether a given subset of vertices in a graph is a clique can be done in polynomial-time,

<table>
<tr><th rowspan="2">Problem</th><th colspan="2">Complexity</th><th colspan="2">Reference</th></tr>
<tr><th>Upper bound</th><th>Lower Bound</th><th>Upper Bound</th><th>Lower Bound</th></tr>
<tr><td>SCSV</td><td colspan="2" rowspan="2">coNP-complete</td><td colspan="2">[SD07]</td></tr>
<tr><td>WSV</td><td rowspan="3">[Woe13b]</td><td>Theorem 6.17</td></tr>
<tr><td>WSE</td><td>$\in \Theta_2^p$</td><td rowspan="2">DP-hard</td><td>Theorems 6.18–6.20</td></tr>
<tr><td>SCSE</td><td>$\in \Sigma_2^p$</td><td>Theorem 6.21</td></tr>
</table>

Table 6.2: Overview of complexity results for wonderful and strict core stability in enemy-oriented hedonic games

WSV and SCSV belong to NP. This, however, implies that WSE and SCSE are contained in Σ_2^p due to (6.2) and the quantifier representation of Σ_2^p, recall Lemma 2.17 on page 13. In his survey, Woeginger [Woe13b] shows that the upper bound of WSE can be improved by arguing that the problem is contained in Θ_2^p which is believed to be a proper subset of Σ_2^p. We tackle the open problems regarding the unspecified lower bounds and start with the verification of wonderfully stable partition structures and we state the result without proof.

Theorem 6.17 *The problem* WSV *is* coNP-*complete.*

To establish the DP-hardness of WSE in Theorem 6.20 using Lemma 2.12, we need the following results and we state them without proof.

Theorem 6.18 *The problem* WSE *can be shown to be* coNP-*hard via a reduction establishing* CLIQUE$\leq_m^p \overline{\text{WSE}}$.

Woeginger [Woe13b] states NP-hardness of WSE in his survey, but omits the proof.

Theorem 6.19 (Woeginger [Woe13b]) *The problem* WSE *can be shown to be* NP-*hard by a reduction establishing* X3C$\leq_m^p$WSE.

The two above results allow us to show DP-hardness of WSE by applying Wagner's sufficient condition from Lemma 2.12 and making use of the fact that the problem WSE has AND_2 functions.

Theorem 6.20 WSE *is* DP-*hard.*

Proof. Consider the NP-hard problem X3C. Given two instances of X3C, $(B_1, \mathscr{S}_1)$ and $(B_2, \mathscr{S}_2)$, where $(B_2, \mathscr{S}_2) \in$ X3C implies $(B_1, \mathscr{S}_1) \in$ X3C, we construct the following graph $G = (V, H)$. The graph G consists of two disconnected subgraphs $G_1 = (V_1, H_1)$ and $G_2 = (V_2, H_2)$, that is, $G = (V_1 \cup V_2, H_1 \cup H_2)$. The graph G_1 is obtained from $(B_1, \mathscr{S}_1)$ by applying the reduction from Theorem 6.19. The graph G_2 is built in two steps. First, the X3C instance $(B_2, \mathscr{S}_2)$ is transformed into an instance of CLIQUE: For each set $S_i \in \mathscr{S}$, create a vertex v_i.

If two sets S_i and S_j are disjoint, connect the corresponding vertices by an edge $\{v_i, v_j\}$. Let $k = |B|/3$. In the second step, transform this graph by applying the reduction from Theorem 6.18 to it. This construction can obviously be done in polynomial time. Note that, again, the proof only works for $k \geq 3$. If $k \leq 2$, reduce to an appropriate trivial WSE instance.

We claim that $(B_1, \mathscr{S}_1) \in$ X3C and $(B_2, \mathscr{S}_2) \notin$ X3C if and only if there exists a wonderfully stable partition for G. Note that, since $(B_2, \mathscr{S}_2) \in$ X3C implies $(B_1, \mathscr{S}_1) \in$ X3C, this is enough to establish equivalence (2.1) in Lemma 2.12.

Only if: Suppose $(B_1, \mathscr{S}_1) \in$ X3C and $(B_2, \mathscr{S}_2) \notin$ X3C. Since $(B_1, \mathscr{S}_1)$ is in X3C, G_1 has a wonderfully stable partition by Theorem 6.19. Since additionally $(B_2, \mathscr{S}_2) \notin$ X3C, there are no $k = |B|/3$ pairwise disjoint sets in $\mathscr{S}$, thus there is no clique of size k in G. By Theorem 6.18, G_2 then also has a wonderfully stable partition. With Proposition 6.15 there has to be a wonderfully stable partition for G, as well.

If: We prove the contrapositive, i.e., if $(B_1, \mathscr{S}_1) \notin$ X3C or $(B_2, \mathscr{S}_2) \in$ X3C, then there is no wonderfully stable partition for G. Indeed, if $(B_1, \mathscr{S}_1) \notin$ X3C, then by Theorem 6.19, there is no wonderfully stable partition for G_1. On the other hand, if $(B_2, \mathscr{S}_2) \in$ X3C, there exists an exact cover of B in $\mathscr{S}$, that is, there are $k = |B|/3$ pairwise disjoint sets in $\mathscr{S}$. By construction, these sets are represented by k vertices in G_2, each connected to one another, thus forming a k-clique. By Theorem 6.18, it follows that there is no wonderfully stable partition for G_2. By construction, since there is no wonderfully stable partition for G_1 or G_2, there is no wonderfully stable partition for G either.

By Lemma 2.12, WSE is DP-hard. ❑

Following the same line of thoughts, Theorem 6.21 can be shown.

Theorem 6.21 *In enemy-based hedonic games, the problem* SCSE *is hard for the complexity classes* NP, coNP, *and* DP.

Enemy-based hedonic games can also be represented by additively separable hedonic games, recall Remark 6.4, which allows us to state the following interesting corollary.

Corollary 6.22 *The problem* SCSE *is* DP*-hard in additively separable hedonic games.*

Note that for additively separable hedonic games this lower bound was raised by Peters [Pet15] by establishing Σ_2^p-hardness. For enemy-based hedonic games, however, no further improvement of our DP-hardness bound is currently known.

6.2.3 Challenge: Toward Θ_2^p-Hardness of Existence

After having taken the first step toward pinpointing the exact complexity of the WSE and SCSE problem in enemy-based hedonic games, we try to improve the lower DP-hardness bound even further. Woeginger [Woe13b] conjectures the problem WSE to be Θ_2^p-hard and we will establish the interesting result that this hardness would follow immediately follow if the problem can be shown to be coDP-hard.

The fact that the problem WSE has AND_ω functions, as shown in Proposition 6.15, implies with Lemma 2.16(3) from Section 2.2.1 that it is either complete for DP or complete for Θ_2^p, or it is something completely different. Note that the same argumentation clearly also holds for the problem SCSE, but we will present our approach in detail for WSE only.

To show Θ_2^p-hardness of the problem, which seems to be the most likely result, a natural approach is to use Lemma 2.14 from Section 2.2.1. To this end, we have to generalize the construction that we defined for showing DP-hardness in the proof of Theorem 6.20.

Let $x_1, x_2, \ldots, x_{2k}$ be $2k$ given instances of an NP-hard problem $\mathscr{D}$, for example X3C or 3-SAT, and we construct a network of friends G consisting of $k+1$ independent components G_ℓ for $1 \leq \ell \leq k+1$. We know from Proposition 6.15 that G has a wonderfully stable partition if and only if each of the $k+1$ components has one. The components are constructed as follows: G_1 is constructed from the instance x_1, while G_{k+1} is constructed from instance x_{2k}. The remaining $k-1$ instances G_ℓ, $2 \leq \ell \leq k$ are constructed from pairs of $\mathscr{D}$ instances, namely $(x_{2\ell-2}, x_{2\ell-1})$. Figure 6.7 illustrates these constructions.

G_1	G_2	G_3	$\cdots$	G_{k-1}	G_k	G_{k+1}
$\uparrow$	$\uparrow$	$\uparrow$		$\uparrow$	$\uparrow$	$\uparrow$
x_1	(x_2, x_3)	(x_4, x_5)	$\cdots$	(x_{2k-4}, x_{2k-3})	(x_{2k-2}, x_{2k-1})	x_{2k}
$+$	$(+,+)$	$(+,+)$	$\cdots$	$(+,+)$	$(+,+)$	$+$
$+$	$(+,+)$	$(-,-)$	$\cdots$	$(-,-)$	$(-,-)$	$-$
$+$	$(+,+)$	$(+,+)$	$\cdots$	$(+,-)$	$(-,-)$	$-$
$-$	$(-,-)$	$(-,-)$	$\cdots$	$(-,-)$	$(-,-)$	$-$

Figure 6.7: Illustration of the reduction using Lemma 2.14. The last rows show possible cases of yes/no-instances due to the relation between the x_ℓ, "+" denotes a yes-instance, and "−" denotes a no-instance

The following properties have to hold for the just constructed graphs to apply Lemma 2.14 in the proof of Proposition 6.23 below.

Property 6.23 *Let $x_1, \ldots, x_{2k}$ be given instances of an* NP*-hard problem $\mathscr{D}$. Construct graphs $G_1, \ldots, G_{k+1}$ as follows:*

1. *Construct G_1 from x_1 such that $x_1 \in \mathscr{D} \iff G_1 \in$ WSE.*
2. *Construct G_ℓ, $2 \leq \ell \leq k$, from $x_{2\ell-2}$ and $x_{2\ell-1}$ such that*

$$(x_{2\ell-2}, x_{2\ell-1} \in \mathscr{D}) \text{ or } (x_{2\ell-2}, x_{2\ell-1} \notin \mathscr{D}) \iff G_\ell \in \text{WSE}.$$

3. *Construct G_{k+1} from x_{2k} such that $x_{2k} \in \mathscr{D} \iff G_{k+1} \notin$ WSE.*

Proposition 6.24 *Let $\mathscr{D}$ be an* NP*-hard problem and let $x_1, \ldots, x_{2k}$ be any $2k$ instances of $\mathscr{D}$ such that $x_j \in \mathscr{D}$ implies $x_\ell \in \mathscr{D}$ for $\ell < j$. If $G_1, \ldots, G_{k+1}$ are graphs that can be constructed from $x_1, \ldots, x_{2k}$ in polynomial time such that Property 6.23 is satisfied, then* WSE *is Θ_2^p-hard.*

Proof. Let f be a polynomial-time computable function such that $f(x_1, \ldots, x_{2k}) = G$, where G is the graph consisting of $k+1$ independent components $G_1, \ldots, G_{k+1}$ that satisfy Property 6.23. To apply Lemma 2.14, we have to show equivalence (2.2) stated in that lemma:

$$|\{x_\ell \,|\, x_\ell \in \mathscr{D}, 1 \le \ell \le 2k\}| \text{ is odd} \iff G \in \text{WSE}.$$

Only if: Assume that $|\{x_\ell \,|\, x_\ell \in \mathscr{D}, 1 \le \ell \le 2k\}|$ is odd. Since $x_j \in A$ implies that $x_\ell \in \mathscr{D}$ for $\ell < j$, neither $x_1 \notin \mathscr{D}$ nor $x_{2k} \in \mathscr{D}$ can hold.[2] By Property 6.23, we have that both G_1 and G_{k+1} have a wonderfully stable partition. Since $x_1 \in \mathscr{D}$ and $x_{2k} \notin \mathscr{D}$, there exists an index s (which we call the *separation index*) such that $x_\ell \in \mathscr{D}$ for $\ell \le s$, and $x_\ell \notin \mathscr{D}$ for $\ell > s$. Again, since $x_j \in \mathscr{D}$ implies that $x_\ell \in \mathscr{D}$ for $\ell < j$, only the following three cases can occur for each pair $(x_{2\ell-2}, x_{2\ell-1})$ of the remaining instances:

Case 1: both $x_{2\ell-2}$ and $x_{2\ell-1}$ are in $\mathscr{D}$,
Case 2: neither $x_{2\ell-2}$ nor $x_{2\ell-1}$ are in $\mathscr{D}$, or
Case 3: $x_{2\ell-2}$ is in $\mathscr{D}$, yet $x_{2\ell-1}$ is not.

Case 3 implies that the separation index is of the form $s = 2\ell - 2$ for some ℓ (see the third row of Figure 6.7), which leads to a contradiction, since that would mean that there is an even number of yes-instances. So all pairs have to be of the form stated in Case 1 or Case 2 (see the second row of Figure 6.7). By Property 6.23, each component G_ℓ, $2 \le \ell \le k$, has a wonderfully stable partition and so has G.

If: Assume that G has a wonderfully stable partition. This implies that every component G_ℓ, $1 \le \ell \le k+1$, does as well. By Property 6.23, we have that $x_1 \in \mathscr{D}$, $x_{2k} \notin \mathscr{D}$, and for all pairs $(x_{2\ell-2}, x_{2\ell-1})$, $2 \le \ell \le k$, either both $x_{2\ell-2}$ and $x_{2\ell-1}$ are in $\mathscr{D}$, or neither $x_{2\ell-2}$ nor $x_{2\ell-1}$ are in $\mathscr{D}$. In total, we have an odd number of yes-instances among $x_1, \ldots, x_{2k}$.

By Lemma 2.14, WSE is Θ_2^p-hard. ❑

The question remains how G can be constructed from $\mathscr{D}$ such that Property 6.23 is fulfilled. The first and the third statement can be fulfilled by constructing the graphs G_1 and G_{k+1} from X3C instances as in the DP-hardness proof in the proof of Theorem 6.20.

Thus, it remains to ensure the second statement. By letting the problem $\mathscr{D}$ be, for example, the well-known problem 3-SAT, we are searching for a polynomial-time reduction f fulfilling for two given 3-SAT instances φ_1, φ_2:

$$(\varphi_1, \varphi_2 \in \text{3-SAT}) \text{ or } (\varphi_1, \varphi_2 \notin \text{3-SAT}) \iff f(\varphi_1, \varphi_2) \in \text{WSE}$$
$$\iff$$
$$(\varphi_1, \varphi_2) \notin \text{SAT-UNSAT} \iff f(\varphi_1, \varphi_2) \in \text{WSE},$$

where SAT-UNSAT is the DP-complete problem defined in Section 2.2.1. Note that we assume that $\varphi_2 \in$ 3-SAT implies $\varphi_1 \in$ 3-SAT and this restricted version of SAT-UNSAT remains DP-complete. This leads to our final claim.

Theorem 6.25 WSE *is* Θ_2^p*-complete if and only if it is* coDP*-hard. The same holds for the problem* SCSE *in enemy-based hedonic games.*

[2] Indeed, looking at the top and the bottom row of Figure 6.7, we see that if either $x_{2k} \in \mathscr{D}$ or $x_1 \notin A$, then either all $x_1, \ldots, x_{2k}$ would be in $\mathscr{D}$ or none of them, contradicting the assumption that $|\{x_\ell \,|\, x_\ell \in \mathscr{D}, 1 \le \ell \le 2k\}|$ is odd.

6.3 Hedonic Games with Friends, Enemies, and Neutral Players

So far we have seen a variety of definitions of the players' preferences, each of which has its assets and drawbacks, recall Section 6.1.1.

In the enemy-based and friend-based representation of preferences, the players can express their like and dislike of the other players, but with the underlying assumption that within the set of friends and enemies, the players are indifferent. Ordinal preferences as given in the singleton encoding, on the other hand, allow player i to explicitly state a ranking $\triangleright_i$ over the remaining players, but how this ranking can be extended to a preference over coalitions containing i is not obvious: Consider the preference of player 1 given by $\triangleright_1 = 3 \triangleright_1 2 \triangleright_1 4$, we cannot deduce whether 1 prefers being in the coalition $\{1,3\}$ to being part of coalition $\{1,2,4\}$. In the individually rational encoding the players provide a ranking over all acceptable coalitions, but as there might be exponentially many of these in the number of players, this encoding is not compact.

Aiming at giving the players the possibility of providing more fine-grained preferences while still having a compact representation, we propose the model of *weak preferences with thresholds*. Such preferences combine ordinal preferences with the concept of friends and enemies. Furthermore we introduce the *set of neutral players* to which a player can assign those players she is indifferent about. Formally, we define these preferences as follows.

Definition 6.26 *Let $P = \{1,2,\ldots,n\}$ be the set of players. For each $i \in P$, a* weak preference (or ranking) with double thresholds*, denoted by $\trianglerighteq_i^{FEN}$, consists of a partition of $P - \{i\}$ into three sets:*

- *F_i (i's friends), together with a weak order $\trianglerighteq_i^F$ over F_i,*
- *E_i (i's enemies), together with a weak order $\trianglerighteq_i^E$ over E_i, and*
- *N_i (the neutral players, i.e., the players i does not care about).*

We also write $\trianglerighteq_i^{FEN}$ as $(\trianglerighteq_i^F \mid j_1 \ldots j_k \mid \trianglerighteq_i^E)$ for $N_i = \{j_1,\ldots,j_k\}$.

We do not provide an order over the neutral players in N_i as we want to capture the intuition that i is indifferent about all players in N_i. That is, we tacitly assume that $j_a \sim_i j_b$ for all $j_a, j_b \in N_i$. We also follow that player i strictly prefers all her friends in F_i to the players in N_i while the players in N_i are strictly preferred to i's enemies in E_i. So the weak order $\trianglerighteq_i$ of player i that is induced by $\trianglerighteq_i^{FEN}$ can be defined as $\trianglerighteq_i^F \triangleright_i N_i \triangleright_i \trianglerighteq_i^E$.

In Example 6.27 we illustrate the just defined notion. This example will be extended in the course of this section.

Example 6.27 *Let $P = \{1,2,3,4,5\}$ be a set of five players with the preferences shown in Table 6.3.*

Taking the preference of player 1, $\trianglerighteq_1^{FEN} = (5 \triangleright_1 3 \mid 2 \mid 4)$*: she has two friends* 3 *and* 5 *in* F_1 *and strictly prefers* 5 *to* 3*. Player* 1 *does not care about player* 2 *and considers* 4 *to be her only enemy. The weak preference of player* 1 *induced by* $\trianglerighteq_1^{FEN}$ *is* $5 \triangleright_1 3 \triangleright_1 2 \triangleright_1 4$*. The induced*

$i \in P$	$\trianglerighteq_i^F$	N_i	$\trianglerighteq_i^E$
1	$5 \triangleright_1 3$	2	4
2	$5 \triangleright_2 1 \sim_2 4$	$\emptyset$	3
3	$\emptyset$	$\emptyset$	$2 \triangleright_3 4 \triangleright_3 5 \triangleright_3 1$
4	5	$\emptyset$	$2 \triangleright_4 1 \sim_4 3$
5	$4 \triangleright_5 1 \sim_5 2 \triangleright_5 3$	$\emptyset$	$\emptyset$

Table 6.3: Ranking with double thresholds of the players in P in Example 6.27

weak preference of player 4*, for example, is* $5 \triangleright_4 2 \triangleright_4 1 \sim_4 3$*. As we see in the preferences of player* 2*,* 3*, and* 5*, each of the sets* F_i*,* N_i*, and* E_i *is allowed to be empty.*

As seen in Example 6.27, we sometimes slightly abuse notation and write "$\emptyset$" for an empty preference, that is, $\trianglerighteq_i^F = \emptyset$ or $\trianglerighteq_i^E = \emptyset$ if $F_i = \emptyset$ or $E_i = \emptyset$, respectively.

The preference $\trianglerighteq_i^{FEN}$ specifies the opinion player i has about the other players, but does not provide a preference relation over coalitions that i is contained in. To obtain such a relation, which is in turn needed for our final goal, namely to define a new class of hedonic games, we have to extend $\trianglerighteq_i^{FEN}$. We do so by using the generalized Bossong–Schweigert extension principle, see the work of Bossong and Schweigert [BS06a] and Delort et al. [DSW11].

Definition 6.28 *Let* $\trianglerighteq_i^{FEN}$ *be a weak ranking with double threshold for player i. The* extended order $\succeq_i^{FEN}$ *is defined as follows: For every* $A, B \subseteq P$*,* $A \succeq_i^{FEN} B$ *if and only if the following two conditions hold:*

1. *There is an injective function* σ *from* $B \cap F_i$ *to* $A \cap F_i$ *such that for every* $y \in B \cap F_i$*, we have* $\sigma(y) \trianglerighteq_i y$*.*
2. *There is an injective function* θ *from* $A \cap E_i$ *to* $B \cap E_i$ *such that for every* $x \in A \cap E_i$*, we have* $x \trianglerighteq_i \theta(x)$*.*

Finally, $A \succ_i^{FEN} B$ *if and only if* $A \succeq_i^{FEN} B$ *and not* $B \succeq_i^{FEN} A$*.*

The above definition intuitively means that for a given coalition $A \in \mathscr{P}_i$ the coalition $A \cup \{j\}$ with $j \in F_i$ is strictly better than A, thus $A \cup \{j\} \succ_i^{FEN} A$. Adding an enemy, on the other hand, does strictly decrease a coalition's value: $A \succ_i^{FEN} A \cup \{k\}$ for $k \in E_i$. For two friends $j, j' \in F_i$ the relation between the coalitions $A \cup \{j\}$ and $A \cup \{j'\}$ depends on the relation between j and j' from player i's view. The same holds for enemies $k, k' \in E_i$ and the relation between the coalitions $A \cup \{k\}$ and $A \cup \{k'\}$. For $j \in F_i$ and $k \in E_i$ the relation between the coalitions A and $A \cup \{j, k\}$ is not defined by $\succeq_i^{FEN}$. We will call such pairs of coalitions *incomparable with respect to* $\succeq_i^{FEN}$. Note that adding or removing players from N_i does not change the value of a coalition and due to that fact we omit the neutral players from the following explanations and definitions.

Before we formally characterize the relation $\succeq_i^{FEN}$ in Proposition 6.30, we illustrate how for a given preference $\trianglerighteq_i^{FEN}$ the generalized Bossong–Schweigert extension can be constructed:

We start with the coalition $F_i \cup \{i\}$ which, obviously, has to be the most valued coalition from i's point of view. By adding enemies, removing friends, or exchanging friends or enemies we can obtain all coalitions that are comparable to $F_i \cup \{i\}$ with respect to $\succeq_i^{FEN}$. This step is repeated starting from each of the just constructed coalitions until the least valued coalition is constructed, which is $E_i \cup \{i\}$. We illustrate this construction in Example 6.29.

Example 6.29 *Let $P = \{1,2,3,4,5\}$ be a set of players with the preferences $\trianglerighteq_i^{FEN}$ presented in Example 6.27. To extend these preferences over the players to preferences over coalitions in $\mathscr{P}_i$ for the players $i \in P$, we construct the generalized Bossong–Schweigert extensions of $\trianglerighteq_i^{FEN}$.*

The graphs in Figure 6.8 show the generalized Bossong–Schweigert extensions $\succeq_i^{FEN}$ of the preferences $\trianglerighteq_i^{FEN}$ for the players $i \in \{1,2,4\}$. In any of the three graphs, an edge (A,B) from coalition A to B implies that $A \succeq_i^{FEN} B$. Note that the relation is transitive: if there is a directed path from coalition A to coalition B, this also implies $A \succeq_i^{FEN} B$. Since player 1 has player 2 in N_1, each coalition A shown in the below graph is of the same value to 1 as $A \cup \{2\}$. For the sake of readability we omit these indifferences.

For players 3 and 5, the extension results in a graph consisting of a single path starting in the most preferred coalition and can thus be given as a complete ranking over all coalitions:

$\succeq_3^{FEN}$: $\{3\} \succ_3 \{2,3\} \succ_3 \{3,4\} \succ_3 \{3,5\} \succ_3 \{1,3\} \succ_3 \{2,3,4\} \succ_3 \{2,3,5\} \succ_3 \{1,2,3\} \succ_3 \{3,4,5\} \succ_3 \{1,3,4\} \succ_3 \{1,3,5\} \succ_3 \{2,3,4,5\} \succ_3 \{1,2,3,4\} \succ_3 \{1,2,3,5\} \succ_3 \{1,3,4,5\} \succ_3 \{1,2,3,4,5\}$

$\succeq_5^{FEN}$: $\{1,2,3,4,5\} \succ_5 \{1,2,4,5\} \succ_5 \{1,3,4,5\} \sim_5 \{2,3,4,5\} \succ_5 \{1,2,3,5\} \succ_5 \{1,4,5\} \sim_5 \{2,4,5\} \succ_5 \{3,4,5\} \succ_5 \{1,3,5\} \sim_5 \{2,3,5\} \succ_5 \{4,5\} \succ_5 \{1,5\} \sim_5 \{2,5\} \succ_5 \{3,5\} \succ_5 \{5\}$

Clearly, there are no incomparable coalitions in $\mathscr{P}_i$ with respect to $\succeq_i^{FEN}$ for $i \in \{3,5\}$. The players 1, 2, and 4, on the other hand, have coalitions that remain incomparable, see Table 6.4

player	pairs of incomparable coalitions
1	$(\{1,5\},\{1,3,4,5\});(\{1,3\},\{1,3,4,5\});(\{1,3\},\{1,4,5\});(\{1\},\{1,3,4,5\});(\{1\},\{1,4,5\})$
2	$(\{1,2,5\},\{1,2,3,4,5\});(\{2,5\},\{1,2,3,4,5\});(\{1,2,4\},\{1,2,3,4,5\});(\{1,2\},\{1,2,3,4,5\});$ $(\{2\},\{1,2,3,4,5\});(\{2,5\},\{1,2,4\});(\{2,5\},\{1,2,3,5\});(\{2,5\},\{1,2,3,4\});(\{1,2,4\},\{1,2,3,5\});$ $(\{1,2,4\},\{2,3,5\});(\{1,2,3,5\},\{1,2\});(\{1,2,3,5\},\{2\});(\{1,2,3,4\},\{1,2\});(\{1,2,3,4\},\{2\});$ $(\{1,2,3,4\},\{2,3,5\});(\{1,2\},\{2,3,5\});(\{2\},\{2,3,5\});(\{2\},\{1,2,3\})$
4	$(\{2,4\},\{1,4,5\});(\{4\},\{1,4,5\});(\{4\},\{2,4,5\})$

Table 6.4: Pairs of incomparable coalitions for players 1, 2, and 4 in Example 6.29

Inspired by Bouveret et al. [BEL10], who use the original Bossong–Schweigert extension to extend preferences over items to preferences over bundles of items in the context of fair division, we formally characterize the just defined preference relation $\succeq_i^{FEN}$.

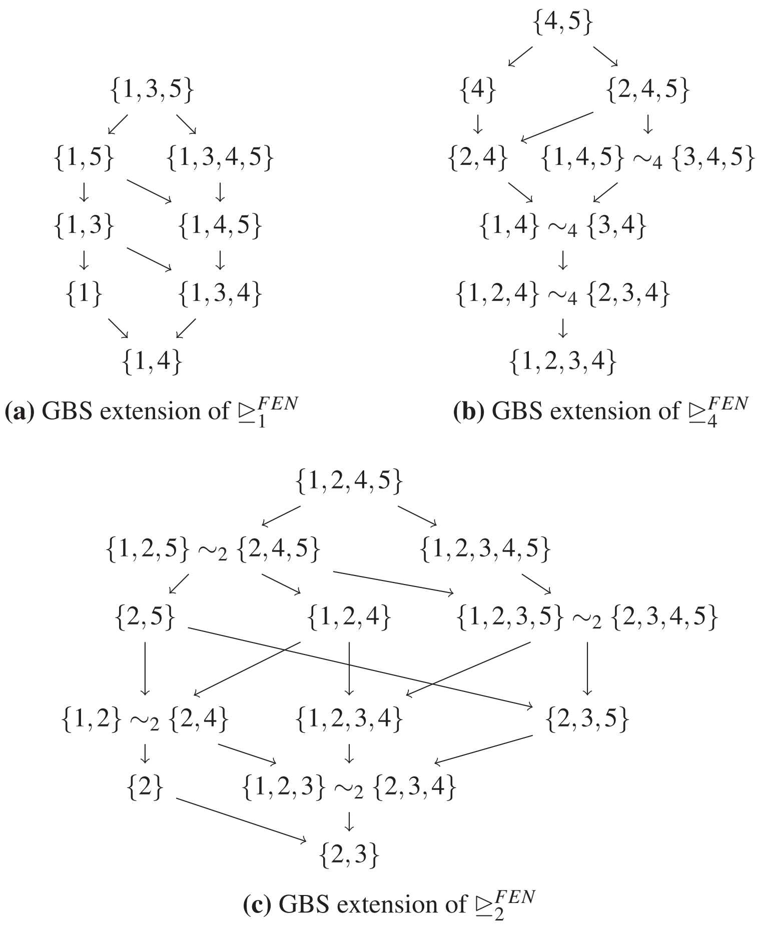


Figure 6.8: Bossong–Schweigert extensions for players 1, 2, and 4 in Example 6.29

Proposition 6.30 *Let $\trianglerighteq_i^{FEN}$ be a weak ranking with double threshold for player i, and let $A, B \in \mathscr{P}_i$ be two coalitions. Consider the orders $f_1 \trianglerighteq_i f_2 \trianglerighteq_i \cdots \trianglerighteq_i f_\mu$ with $\{f_1, f_2, \ldots, f_\mu\} = A \cap F_i$ and $f'_1 \trianglerighteq_i f'_2 \trianglerighteq_i \cdots \trianglerighteq_i f'_{\mu'}$ with $\{f'_1, f'_2, \ldots, f'_{\mu'}\} = B \cap F_i$, as well as $e_1 \trianglerighteq_i e_2 \trianglerighteq_i \cdots \trianglerighteq_i e_\nu$ with $\{e_1, e_2, \ldots, e_\nu\} = A \cap E_i$ and $e'_1 \trianglerighteq_i e'_2 \trianglerighteq_i \cdots \trianglerighteq_i e'_{\nu'}$ with $\{e'_1, e'_2, \ldots, e'_{\nu'}\} = B \cap E_i$. Then, $A \succeq_i^{FEN} B$ holds if and only if*

1. *$\mu \geq \mu'$ and $\nu \leq \nu'$,*
2. *for each k, $1 \leq k \leq \mu'$, it holds that $f_k \trianglerighteq_i f'_k$, and*
3. *for each ℓ, $1 \leq \ell \leq \nu$, it holds that $e_{\nu-\ell+1} \trianglerighteq_i e'_{\nu'-\ell+1}$.*

Proof. Obviously, if (a) to (c) hold, the two injective functions $\sigma : B \cap F_i \to A \cap F_i$, and $\theta : A \cap E_i \to B \cap E_i$ mapping $f'_k \mapsto f_k$ for each k, $1 \leq k \leq \mu'$, and $e_{\nu-\ell+1} \mapsto e'_{\nu'-\ell+1}$ for each ℓ, $1 \leq \ell \leq \nu$, satisfy $\sigma(f'_k) \trianglerighteq_i f'_k$ and $e_{\nu-\ell+1} \trianglerighteq_i \theta(e_{\nu-\ell+1})$, for the same range of k and ℓ. On the other hand, if there are two injective functions with the desired requirements, (a) holds. If there was a k with $f'_k \triangleright_i f_k$ (or an ℓ with $e'_{\nu'-\ell+1} \triangleright_i e_{\nu-\ell+1}$), this would imply $\sigma(f'_k) = f_j$ for a $j < k$ (or $\theta(e_{\nu-\ell+1}) = e'_{\nu-j+1}$ with $j > \ell$, respectively).

This, however, implies that either a requirement is violated for f_1' (or e_v), or that σ (or θ) is not injective, a contradiction. ❑

Finally, we conclude the introduction of our model in the following definition of FEN-hedonic games.

Definition 6.31 (FEN-Hedonic Game) *An* FEN-hedonic game *is a tuple* $\mathscr{G} = (P, \succeq^{FEN})$*, where* $P = \{1, 2, \ldots, n\}$ *is the set of players and* $\succeq_i^{FEN}$ *is the extended order of player i based on her preference with double thresholds* $\trianglerighteq_i^{FEN}$*. Since* $\succeq_i^{FEN}$ *can be directly obtained from* $\triangleright_i^{FEN}$*, an FEN-hedonic game can also be stated as* $\mathscr{G} = (P, \trianglerighteq^{FEN})$*.*

We are now interested in studying these games with respect to the complexity of stability problems. Any such study, however, has to consider the fact that the players' preference relations $\succeq_i^{FEN}$ might be incomplete.

Lang et al. [LRR⁺15] consider two possibilities to deal with incomparabilities of coalitions:

1. Leave the incomparabilities open and define notions such as *possible* and *necessary stability* concepts.
2. Determine the relation between incomparable coalitions by using an adaption of a voting rule or a similar mechanism while preserving those relations that are defined in $\succeq_i^{FEN}$.

The latter approach will be presented in detail in Sections 6.3.1 and 6.3.2 while the former will only be briefly discussed below. For both approaches, however, we need a notion of how a completion of these incomplete preference relation could be defined. A first step is the following definition of extensions.

Definition 6.32 (Extension) *A complete preference relation* $\succeq_i$ *over all coalitions containing i* extends $\succeq_i^{FEN}$ *if and only if it contains it; that is, if* $A \succeq_i^{FEN} B$ *implies* $A \succeq_i B$ *for all coalitions* A, B*. Let* $\mathrm{Ext}(\succeq_i^{FEN})$ *be the set of all complete preference relations extending* $\succeq_i^{FEN}$*.*

To shortly survey the obtained results in [LRR⁺15] for the case of leaving incomparabilities in the analysis of FEN-hedonic games open, we present Definition 6.33 that proposes two variants of stability in the context of incomplete preference relations.

Definition 6.33 (Possible and Necessary Stability) *Let* $\mathscr{G} = (P, \trianglerighteq^{FEN})$ *be an FEN-hedonic game and let* Γ *be a coalition structure. For any stability concept* α *defined in Section 6.1.2 we say that* Γ *satisfies*

- possible α *if and only if there exists a profile* $\succeq \in \times_{i=1}^{n} \mathrm{Ext}(\succeq_i^{FEN})$ *such that* Γ *is stable in the sense of* α *in the game* $\mathscr{G}' = (P, \succeq)$*;*
- necessary α *if and only if for all profiles* $\succeq \in \times_{i=1}^{n} \mathrm{Ext}(\succeq_i^{FEN})$ *it holds that* Γ *is stable in the sense of* α *in the game* $\mathscr{G}' = (P, \succeq)$*.*

To study the complexity of possible and necessary stability the verification and the existence problem can be defined analogously to those problems defined in Section 6.1.2.

POSSIBLE/NECESSARY α-VERIFICATION	
Given:	An FEN-hedonic game $\mathscr{G} = (P, \trianglerighteq^{FEN})$ and a coalition structure Γ.
Question:	Does Γ satisfy possible/necessary α in $\mathscr{G}$?

POSSIBLE/NECESSARY α-EXISTENCE	
Given:	An FEN-hedonic game $(P, \trianglerighteq^{FEN})$.
Question:	Is there a coalition structure Γ satisfying possible/necessary α in $\mathscr{G}$?

Table 6.5 provides an overview of selected results found and discussed in [LRR+15].

	VERIFICATION		EXISTENCE	
Stability	**Possible**	**Necessary**	**Possible**	**Necessary**
Perfectness Individual Rationality	P	P	P	P
Contractually Individual Stability Individual Stability	NP	P	NP	NP
Nash Stability			NP-c.	NP-c.
Core Stability Strict Core Stability	Σ_2^p, coNP-h.	coNP	Σ_2^p	Σ_2^p

Key: NP-c. = NP-complete, coNP-h. = coNP-hard, remaining entries indicate upper bounds

Table 6.5: Selected results for the complexity of necessary and possible verification and existence in FEN-hedonic games

6.3.1 Breaking Incomparabilities with Borda-Like Scoring Vectors

Now we present in detail the second approach proposed by Lang et al. [LRR+15] for dealing with incomplete preference relations in FEN-hedonic games. Our goal is to define functions that allow to compare those coalitions that are incomparable with respect to $\succeq_i^{FEN}$ while still preserving those relations that are determined by $\succeq_i^{FEN}$.

Our definition of such functions is inspired by voting theory. Based on player i's preference over the other players given in $\trianglerighteq_i^{FEN}$ we determine values that i assigns to the other players and aggregate these values to compute the values of coalitions in $\mathscr{P}_i$. We call such functions computing the value of coalitions *comparability functions*.

Proposition 6.34 gives a characterization of how such comparability functions can be defined such that those relations that are already determined by $\succeq_i^{FEN}$ are preserved. In other words, it provides a formal description of comparability functions that can be used to define an extension of $\succeq_i^{FEN}$.

Proposition 6.34 *Let $(P, \trianglerighteq^{FEN})$ be a given FEN-hedonic game and let $\succeq_i^{FEN}$ be the generalized Bossong–Schweigert extension of $\trianglerighteq_i^{FEN}$ of player $i \in P$. We say that a function $w_i : P \to \mathbb{R}$ is* compatible with $\trianglerighteq_i^{FEN}$ *if and only if*

- *for each $j \in F_i$, we have $w_i(j) > 0$;*
- *for each $j \in E_i$, we have $w_i(j) < 0$;*
- *for each $j \in N_i$, we have $w_i(j) = 0$; and*
- *for all $j,k \in F_i \cup E_i$, we have $j \triangleright_i k$ if and only if $w_i(j) > w_i(k)$.*

With this notion we can state that $A \succeq_i^{FEN} B$ if and only if for any w_i compatible with $\trianglerighteq_i^{FEN}$, we have $\sum_{j \in A} w_i(j) \geq \sum_{j' \in B} w_i(j')$.

Proof. Assume that $A \succeq_i^{FEN} B$. For the set of friends F_i, with $F = A \cap F_i$ and $F' = B \cap F_i$, it follows that there is an injective function $\sigma : F' \to F$ such that for each $y \in F'$, we have $\sigma(y) \trianglerighteq_i y$. Hence, for each compatible w_i, $w_i(\sigma(y)) \geq w_i(y)$. Thus, since σ is injective,

$$\begin{aligned} \sum_{j \in F} w_i(j) \quad &\geq \sum_{j \in \sigma(F') \subseteq F} w_i(j) \;=\; \sum_{j' \in F'} w_i(\sigma(j')) \\ &\geq \sum_{j' \in F'} w_i(j'). \end{aligned} \tag{6.3}$$

Similarly, for E_i, with $E = A \cap E_i$ and $E' = B \cap E_i$, and θ injective, it holds that

$$\begin{aligned} 0 \;&\geq\; \sum_{j \in E} w_i(j) \geq \sum_{j \in E} w_i(\theta(j)) \;=\; \sum_{j' \in \theta(E) \subseteq E'} w_i(j') \\ &\geq\; \sum_{j' \in E'} w_i(j'). \end{aligned} \tag{6.4}$$

For each player $j \in N_i$, we have $w_i(j) = 0$; therefore, in total,

$$\sum_{j \in A} w_j > \sum_{j' \in B} w_{j'}. \tag{6.5}$$

Now assume that for each compatible w_i, (6.5) holds. Thus,

$$\sum_{j \in F} w_i(j) - \sum_{j' \in E'} w_i(j') > \sum_{j' \in F'} w_i(j') - \sum_{j \in E} w_i(j).$$

Assume there were no injective function mapping from each summand from the right-hand side to one at least as large on the left hand side; then, there exists an assignment to the values of w_i compatible with $\trianglerighteq_i^{FEN}$ that does not satisfy the inequality, a contradiction. This completes the proof. ❑

Based on this characterization we define our comparability function as a function $w_i : P \to \mathbb{Z}$ with $w_i(i) = 0$. Clearly, $w_i(j) = 0$ has to hold for all $j \in N_i$. Using terminology from voting theory, we define so-called *scoring vectors*

$$\mathbf{f}_i \in \mathbb{Z}_{>0}^{\|F_i\|}, \quad \mathbf{e}_i \in \mathbb{Z}_{<0}^{\|E_i\|}$$

determining the values that are assigned to i's friends and enemies, respectively, and we define these values by using Borda-like scoring vectors (see Section 2.3.1 for the definition of this voting rule). Inspired by the work of Baumeister et al. [BFL⁺12] regarding modified Borda voting, we introduce several variants capturing the notions of "optimistic" and "pessimistic" assessments of friend or enemy relations.

Let $\trianglerighteq_i^{FEN}$ be the weak preference with double thresholds of player $i \in P$ with the following ordering of i's friends and enemies:

- $\trianglerighteq_i^F = F_{i,1} \triangleright_i^F F_{i,2} \triangleright_i^F \cdots \triangleright_i^F F_{i,\ell}$, where each $F_{i,j}$ contains friends player i is indifferent about, and
- $\trianglerighteq_i^E = E_{i,1} \triangleright_i^E E_{i,2} \triangleright_i^E \cdots \triangleright_i^E E_{i,m}$, where each $E_{i,j}$ contains enemies i is indifferent about.

With this we define the following variants of our Borda-like scoring vectors.

1. $\mathbf{f}_i$ can be one of the following four vectors:
   a) *Strongly friend-optimistic (***sfo***)*: Each player in $F_{i,1}$ gets n points, each in $F_{i,2}$ gets $n-1$ points, ..., each in $F_{i,\ell}$ gets $n-\ell+1$ points.
   b) *Friend-optimistic (***fo***)*: Each player in $F_{i,1}$ gets $\|F_i\|$ points, each in $F_{i,2}$ gets $\|F_i\|-1$ points, ..., each in $F_{i,\ell}$ gets $\|F_i\|-\ell+1$ points.
   c) *Strongly friend-pessimistic (***sfp***)*: Each player in $F_{i,\ell}$ gets 1 point, each in $F_{i,\ell-1}$ gets 2 points, ..., each in $F_{i,1}$ gets ℓ points.
   d) *Friend-pessimistic (***fp***)*: Each player in $F_{i,\ell}$ gets $n-\|F_i\|+1$ points, each in $F_{i,\ell-1}$ gets $n-\|F_i\|+2$ points, ..., each in $F_{i,1}$ gets $n-\|F_i\|+\ell$ points.
2. $\mathbf{e}_i$ can be one of the following four vectors:
   a) *Strongly enemy-optimistic (***seo***)*: Each player in $E_{i,1}$ gets -1 point, each in $E_{i,2}$ gets -2 points, ..., each in $E_{i,m}$ gets $-m$ points.
   b) *Enemy-optimistic (***eo***)*: Each player in $E_{i,1}$ gets $-(n-\|E_i\|+1)$ points, each in $E_{i,2}$ gets $-(n-\|E_i\|+2)$ points, ..., each in $E_{i,m}$ gets $-(n-\|E_i\|+m)$ points.
   c) *Strongly Enemy-pessimistic (***sep***)*: Each player in $E_{i,m}$ gets $-n$ points, each in $E_{i,m-1}$ gets $-n+1$ points, ..., each in $E_{i,1}$ gets $-(n-m+1)$ points.
   d) *Enemy-pessimistic (***ep***)*: Each player in $E_{i,m}$ gets $-\|E_i\|$ points, each in $E_{i,m-1}$ gets $-\|E_i\|+1$ points, ..., each in $E_{i,1}$ gets $-(\|E_i\|-m+1)$ points.

Each pair of scoring vectors $(\mathbf{f}_i, \mathbf{e}_i) \in \{\mathbf{sfo}, \mathbf{fo}, \mathbf{sfp}, \mathbf{fp}\} \times \{\mathbf{seo}, \mathbf{eo}, \mathbf{sep}, \mathbf{ep}\}$ defines a particular way of how the scores a player i assigns to the other players are derived from $\trianglerighteq_i^{FEN}$. The intuition behind these definitions and why it is reasonable to distinguish each of the four cases can be best seen assuming that player i is indifferent between all of his friends and all of his enemies. With the above notation, it holds that that $\ell = 1$ and $m = 1$ and the values shown in Table 6.6 are assigned to i's friends and enemies depending on $\mathbf{f}_i$ and $\mathbf{e}_i$, respectively.

We see that in the friend-optimistic case a bigger friend set implies higher values for the

	sfo	**fo**	**sfp**	**fp**	**seo**	**eo**	**sep**	**ep**
value	n	$\|F_i\|$	1	$n - \|F_i\| + 1$	-1	$-(n - \|E_i\| + 1)$	$-n$	$-\|E_i\|$

Table 6.6: Values that are derived from different choices for $\mathbf{f}_i, \mathbf{e}_i$, when there are only indifferences within $\trianglerighteq_i^F$ and $\trianglerighteq_i^E$

friends contained in it, while the opposite is the case in the friend-pessimistic case. The same holds for the comparison between the enemy-pessimistic and enemy-optimistic case with the difference that in the former case a bigger enemy set reduces the enemies' scores and in the latter case a bigger enemy set implies higher values.

On the other hand, when there are no indifferences within $\trianglerighteq_i^F$, for $\mathbf{f}_i \in \{\mathbf{sfo}, \mathbf{fp}\}$ and $\mathbf{f}_i \in \{\mathbf{sfp}, \mathbf{fo}\}$ the two scoring vectors from one set yield the same scores for player i's friends. The same holds for $\mathbf{e}_i \in \{\mathbf{seo}, \mathbf{ep}\}$ and $\mathbf{e}_i \in \{\mathbf{eo}, \mathbf{sep}\}$, whenever there are no indifferences in $\trianglerighteq_i^E$.

Analogously to the definition of positional scoring rules and having Proposition 6.34 in mind, we define the value of a coalition from player i's view as the sum of the values she assigns to the players in the coalition.

Definition 6.35 (Borda-Like CF) *Let $i \in P$ be a player. For a fixed choice of scoring vectors $\mathbf{f}_i$ and $\mathbf{e}_i$ defining the score function w_i we define the* Borda-like comparability function (CF)

$$f^i_{\text{Borda}} : \mathscr{P}_i \to \mathbb{Z}, \quad C \mapsto \sum_{j \in C - \{i\}} w_i(j)$$

to be a function mapping every coalition C containing i to the sum of the scores the players in $C - \{i\}$ obtain from w_i.

With this notion of comparability functions, we can derive a complete preference relation from given preferences with double thresholds; we call this relation *Borda-induced* and define it in Definition 6.36.

Definition 6.36 (Borda-Induced Preference Extension) *For an FEN-hedonic game $(P, \trianglerighteq^{FEN})$ with n players and a fixed choice of $\mathbf{f}_i$ and $\mathbf{e}_i$, let f^i_{Borda} be the Borda-like CF. For two coalitions $A, B \in \mathscr{P}_i$ it holds that*

- *$A \succeq_i^{FENb} B$ if and only if $f^i_{\text{Borda}}(A) \geq f^i_{\text{Borda}}(B)$,*
- *$A \succ_i^{FENb} B$ if and only if $f^i_{\text{Borda}}(A) > f^i_{\text{Borda}}(B)$, and*
- *$A \sim_i^{FENb} B$ if and only if $f^i_{\text{Borda}}(A) = f^i_{\text{Borda}}(B)$.*

Example 6.37 shows how for two choices of scoring vectors the incomparabilities in our running example are be broken.

Example 6.37 *Let $\mathscr{G} = (P, \trianglerighteq^{FEN})$ be the FEN-hedonic game from Example 6.29. Table 6.7 shows the values the players assign to each other for two choices of scoring vectors: Table 6.7a shows the values for $\mathbf{f}_i = \mathbf{sfp}$ and $\mathbf{e}_i = \mathbf{seo}$, while Table 6.7b presents the values for $\mathbf{f}_i = \mathbf{fo}$ and $\mathbf{e}_i = \mathbf{ep}$.*

i \ j	1	2	3	4	5
1	–	0	1	−1	2
2	1	–	−1	1	2
3	−4	−1	–	−2	−3
4	−2	−1	−2	–	1
5	2	2	1	3	–

(a) $\mathbf{f}_i = \mathbf{sfp}$, $\mathbf{e}_i = \mathbf{seo}$

i \ j	1	2	3	4	5
1	–	0	1	−1	2
2	2	–	−1	2	3
3	−4	−1	–	−2	−3
4	−3	−2	−3	–	1
5	3	3	2	4	–

(b) $\mathbf{f}_i = \mathbf{fo}$, $\mathbf{e}_i = \mathbf{ep}$

Table 6.7: Values $w_i(j)$ for different choices of $\mathbf{f}_i, \mathbf{e}_i$ in the game in Example 6.37

We see that for both choices of $\mathbf{f}_i$ *and* $\mathbf{e}_i$*, the values of player* 1 *and player* 4 *do not differ since these players do not have any indifferences within their ordering of friends and enemies.*

Recalling Example 6.29, the preference relations $\succeq_i^{FEN}$ *are incomplete for the players* $i \in \{1,2,4\}$*, Table 6.8 shows the values that these players assign to (a selection of) their incomparable coalitions on* f^i_{Borda} *for the scoring vectors* $\mathbf{f}_i = \mathbf{sfp}$, $\mathbf{e}_i = \mathbf{seo}$ *and* $\mathbf{f}_i = \mathbf{fo}$, $\mathbf{e}_i = \mathbf{ep}$ *which are shortly denoted by* $\mathbf{fe}$ *and* $\mathbf{fe}'$*, respectively, in the table.*

	player 2							
A	$\{1,2,5\}$	$\{1,2,3,4,5\}$	$\{2,5\}$	$\{1,2,3,4\}$	$\{2,3,5\}$	$\{1,2,3,4\}$	$\{2\}$	$\{1,2,3\}$
fe	3	3	2	1	1	1	0	0
fe′	5	6	3	3	2	3	0	1
A	$\{1,2,4\}$	$\{1,2,3,4,5\}$	$\{2,5\}$	$\{1,2,3,5\}$	$\{1,2,5\}$	$\{1,2,3,5\}$	$\{2\}$	$\{2,3,5\}$
fe	2	3	2	2	3	2	0	1
fe′	4	6	3	4	5	4	0	2
A	$\{1,2\}$	$\{1,2,3,4,5\}$	$\{1,2\}$	$\{2,3,5\}$	$\{2\}$	$\{1,2,3,5\}$	$\{2,5\}$	$\{1,2,4\}$
fe	1	3	1	1	0	2	2	2
fe′	2	6	2	2	0	4	3	4
	player 1				**player 4**			
A	$\{1,5\}$	$\{1,3,4,5\}$	$\{1,3\}$	$\{1,3,4,5\}$	$\{4\}$	$\{2,4,5\}$	$\{2,4\}$	$\{1,4,5\}$
fe	1	2	2	2	0	0	-1	-1
A	$\{1,3\}$	$\{1,4,5\}$	$\{1\}$	$\{1,4,5\}$				
fe	2	0	0	0				

Table 6.8: Values for coalitions based on f^i_{Borda} for $i \in \{1,2,4\}$ in Example 6.37

Clearly from the definition of f_{Borda} and Proposition 6.34, it follows that $\succeq_i^{FENb}$ is indeed an extension of $\succeq_i^{FEN}$. We state this fact in Proposition 6.38 without proof.

Proposition 6.38 *Let* $(P, \trianglerighteq^{FEN})$ *be an FEN-hedonic game with n players, then it holds that* $\succeq_i^{FENb} \in \text{Ext}(\succeq_i^{FEN})$ *for each fixed choice of* $\mathbf{f}_i$ *and* $\mathbf{e}_i$ *and* $i \in \{1,\dots,n\}$.

Finally, we can define Borda-induced FEN-hedonic games.

Definition 6.39 (Borda-Induced FEN-Hedonic Game) *Let $(P, \trianglerighteq^{FEN})$ be an FEN-hedonic game with n players. For a fixed choice of scoring vectors $\mathbf{f}_i$ and $\mathbf{e}_i$ for $i \in \{1,\dots,n\}$, we define with $\mathscr{G} = (P, \succeq^{FENb})$ the* Borda-induced FEN-hedonic game, *where $\succeq^{FENb}$ is the Borda-induced preference extension of $\succeq^{FEN}$.*

Thus, Borda-induced FEN-hedonic games are a class of FEN-hedonic games with extensions defined by the scoring vectors $\mathbf{f}_i$ and $\mathbf{e}_i$ and each fixed pair of $(\mathbf{f}_i, \mathbf{e}_i)$ defines a subclass. We see from the definition of $\succeq^{FENb}$ that this class of games is a subclass of additively separable hedonic games, where $w_i = f^i_{\text{Borda}}$ for the players $i \in P$.

Observation 6.40 *A Borda-induced FEN-hedonic game is a special additively separable hedonic game.*

While for each Borda-induced FEN-hedonic game there is an additively separable hedonic game with the same values, not every set of values can be derived from given weak preferences with thresholds.

6.3.2 Complexity of Stability

Table 6.9 gives an overview of the results that will be presented in this section. Some of the lower bounds only hold for certain scoring vectors $\mathbf{f}_i$ and $\mathbf{e}_i$, these restrictions are indicated in the footnotes. If no restriction is given, the result holds for all possible combinations of scoring vectors.

Stability	VERIFICATION	Reference	EXISTENCE	Reference
P	P	Corollary 6.41	P	Corollary 6.43
NS	P	Corollary 6.41	NP-complete [1]	Theorem 6.47
IS	P	Corollary 6.41	NP-complete [1]	Theorem 6.48
CIS	P	Corollary 6.41	P	Corollary 6.43
CS	coNP-complete	Theorem 6.42	Σ_2^p-complete [2]	Theorem 6.49
SCS	coNP-complete	Theorem 6.42	coNP-hard, $\in \Sigma_2^p$	Theorem 6.54

[1] for $\mathbf{sfp}, \{\mathbf{seo}, \mathbf{eo}, \mathbf{sep}, \mathbf{ep}\}$ [2] for $\mathbf{sfp}, \{\mathbf{seo}, \mathbf{ep}\}$

Table 6.9: Overview of complexity results regarding stability for Borda-induced FEN-hedonic games

Note that whenever we state the preferences of the players in upcoming proofs, we will, for the sake of readability, omit the indices within the orderings $\trianglerighteq^F$ and $\trianglerighteq^E$. Thus, we write $a \sim b$ or $a \rhd b$ instead of $a \sim_i b$ or $a \rhd_i b$ in the preference of player i.

Verification We start with the verification variants of our stability problems and first state the known tractability results that can be followed directly from known results in additively separable hedonic games (recall Observation 6.40).

Corollary 6.41 *For Borda-induced FEN-hedonic games the problem* α-STABILITY VERIFICATION *is in* P *for each of the stability concepts* $\alpha \in$ {*perfectness, individual stability, contractually individual stability, Nash stability*}.

Verifying whether a given coalition structure in a Borda-induced FEN-hedonic game is stable with respect to the two concepts of group deviation is a far more difficult task. We can show that for both concepts, namely core stability and strict core stability, the verification problems are coNP-complete. The proof is inspired by the corresponding result for games with enemy-based preferences presented by Woeginger [Woe13a].

Theorem 6.42 *For Borda-induced FEN-hedonic games the problems* CORE STABILITY VERIFICATION *and* STRICT CORE STABILITY VERIFICATION *are* coNP*-complete for each choice of* $\mathbf{f}_i$ *and* $\mathbf{e}_i$.

Proof. The upper bound follows from the result for additively separable hedonic games due to Sung and Dimitrov [SD07] and Aziz et al. [ABS13] and Observation 6.40.

To prove coNP-hardness we reduce from the complement of the CLIQUE problem denoted by $\overline{\text{CLIQUE}}$. To do so, let (G,k) be a $\overline{\text{CLIQUE}}$ instance, where $G = (V,H)$ is an undirected graph with vertices $V = \{v_1, v_2, \dots, v_n\}$ and edges $H = \{h_1, h_2, \dots, h_m\}$. We construct the Borda-induced FEN-hedonic game $(P, \succeq^{FENb})$ with $n + n(k-2)$ players in $P = \{v_1, v_2, \dots, v_n\} \cup Q$, where Q is a collection of $n(k-2)$ players $Q = \bigcup_{i=1}^{n} Q_i$ with the sets $Q_i = \{q_{i,1}, q_{i,2}, \dots, q_{i,(k-2)}\}$. Let $\mathcal{N}(v)$ denote the neighborhood of vertex $v \in V$.

The extension $\succeq^{FENb}$ can be derived from the players' weak rankings with double threshold $\trianglerighteq^{FEN}$ as displayed in Table 6.10 (note that when a set of players appears in a preference, the players in the set are unranked).

For each ...	player	$\trianglerighteq^{F}$	N	$\trianglerighteq^{E}$
$i \in \{1,\dots,n\}$	v_i	$\mathcal{N}(v_i) \sim Q_i$	$P - (\mathcal{N}(v_i) \cup \{v_i\} \cup Q_i)$	$\emptyset$
$i \in \{1,\dots,n\}$, $j \in \{1,\dots,k-2\}$	$q_{i,j}$	$\emptyset$	$v_i \sim (Q_i - \{q_{i,j}\})$	$P - (\{v_i\} \cup Q_i)$

Table 6.10: Weak rankings with double threshold of the players in the proof of Theorem 6.42

The players corresponding to the vertices in G are mutual friends if connected by an edge, while every one of these players has $k-1$ friends in Q_i which are no friends of the other v_i-players. For each $i \in \{1,\dots,n\}$ the players in Q_i are indifferent regarding their corresponding player v_i and the players that are in the same Q_i. The remaining players in the game are their enemies, thus these players do not consider anyone to be their friend.

Let $\Gamma = (\Gamma_1, \Gamma_2, \ldots, \Gamma_n)$ with $\Gamma_i = \{v_i\} \cup Q_i$ be the coalition structure. The v_i-players give their coalition in Γ $\ell(k-2)$ points, while ℓ depends on the scoring vector $\mathbf{f}_i$ used for the set of friends and $\ell \geq 1$ holds. Each $q_{i,j} \in Q$ gives their coalition a score of zero (and this is independent from the choice of $\mathbf{f}_i$ and $\mathbf{e}_i$). Note that adding any other player to $\Gamma(q_{i,j})$ turns the score of the coalition from player $q_{i,j}$'s view to a negative value.

We claim that G has a clique of size at most $k-1$ if and only if Γ is (strictly) core-stable.

Only if: Assume that the largest clique in G is of size $k-1$. Since the players in Q do not have friends, they already reach a best possible score with their given coalition. For a weakly blocking coalition $A \subseteq P$ to exist it has to contain at least one player from V preferring A strictly to her original coalition. This can only happen if A consists of a set of players from V forming a clique. Since the largest clique in V is of size at most $k-1$, the players in the clique would assign this coalition a score of $\ell(k-2)$ which is exactly the score each v_i assigns the coalition $\Gamma(v_i)$. Thus, there is no weakly blocking coalition which directly implies that there is neither a blocking one.

If: We show the contraposition. Assume that there were a clique V' of size k in G. Then the players corresponding to the vertices in this clique form a blocking coalition (and thus a weakly blocking one) since every player in the clique gives the coalition V' a score of $\ell(k-1)$ which is bigger than the score of the coalition they are assigned to in Γ. ❑

Existence The first results we present follow directly with Observation 6.40 and known results for additively separable hedonic games. Aziz et al. [ABS13] show that the existence of a perfect coalition structure in a given additively separable hedonic game can be decided in deterministic polynomial time while contractually individually stable coalition structures always exist, leading to the same complexity of the existence problem.

Corollary 6.43 *For Borda-induced FEN-hedonic games the problems* PERFECTNESS EXISTENCE *and* CONTRACTUALLY INDIVIDUAL STABILITY EXISTENCE*, are in* P *for each choice of* $\mathbf{f}_i$ *and* $\mathbf{e}_i$.

For the remaining stability problems, the known upper bounds can be transferred to our games with Observation 6.40, as well. The lower bounds, on the other hand, have to be proven separately. We will show that certain known hardness proofs for additively separable hedonic games can be adapted to also hold for Borda-induced FEN-hedonic games when the scoring vectors $\mathbf{f}_i = \mathbf{sfp}$ and $\mathbf{e}_i = \mathbf{seo}$ are used, assuming that the additively separable hedonic game in the known proof fulfills two properties: the values that the players assign to each other have to be integers and they are not allowed to be symmetric. With these two conditions fulfilled, we show in Construction 6.44 how a Borda-induced FEN-hedonic game can be constructed from a given additively separable hedonic game such that the two games are equivalent in the sense of Lemma 6.46.

Construction 6.44 *Let* $\mathscr{G} = (P, w)$ *be an additively separable hedonic game, where the players* $p_i \in P$ *have values* $w_{p_i} : P - \{p_i\} \to R_{p_i} \subseteq \mathbb{Z}$ *and the values are not symmetric. We construct a Borda-induced FEN-hedonic game* $\mathscr{G}' = (P', \succeq^{FENb})$ *with* $\mathbf{f}_i = \mathbf{sfp}$ *and* $\mathbf{e}_i = \mathbf{seo}$*. Let*

$P' = P \cup D$ be the set of players in $\mathcal{G}'$, where P are the players in the original game $\mathcal{G}$ and we have a set of $(\max\{\bigcup_{p_i \in P} R_{p_i}\} + |\min\{\bigcup_{p_i \in P} R_{p_i}\}| - 2)$ padding players in $D = \{d_1, d_2, \dots\}$.

We first explain how the weak rankings with double threshold have to be constructed for the players in P. To this end let player $p_i \in P$ be a player in the original game, then we define the sets $A^k_{p_i} = \{p_j \in P \mid w_{p_i}(p_j) = k\}$ for $k \in R_{p_i}$ and we know that $\bigcup_{s \in R} A^s_{p_i} = P - \{p_i\}$. We separate the strictly negative values in R_{p_i} from the strictly positive ones and define $R_{p_i} = R^+ \cup R^- \cup \{0\}$ (for the sake of readability we omit the index p_i for R^+ and R^-). We define the sets of friends, enemies, and neutral players of p_i as follows: $N_{p_i} = A^0_{p_i}$, $F_{p_i} = \bigcup_{s \in R^+} A^s_{p_i}$, and $E_{p_i} = \bigcup_{s \in R^-} A^s_{p_i}$.

Assuming that the elements in $R^+ = \{r_1, r_2, \dots, r_{\|R^+\|}\}$ and $R^- = \{r'_1, r'_2, \dots, r'_{\|R^-\|}\}$ are ordered descendingly, we can define $\trianglerighteq^F_{p_i}$ and $\trianglerighteq^E_{p_i}$ as follows (note that for the sake of readability, we omit the index p_i in both $\trianglerighteq^F_{p_i}$ and $\trianglerighteq^E_{p_i}$).

$$\trianglerighteq^F_{p_i} : \underbrace{A^{r_1}}_{\sim} \rhd \underbrace{d_l \rhd \dots \rhd d_k}_{(r_1 - r_2 - 1)\text{-}many} \rhd \underbrace{A^{r_2}}_{\sim} \rhd \underbrace{d_m \rhd \dots \rhd d_n}_{(r_2 - r_3 - 1)\text{-}many} \rhd \dots \rhd \underbrace{A^{r_{\|R^+\|}}}_{\sim} \rhd \underbrace{d_q \rhd \dots \rhd d_t}_{r_{\|R^+\|} - 1\text{-}many}$$

$$\trianglerighteq^E_{p_i} : \underbrace{A^{r'_1}}_{\sim} \rhd \underbrace{d_l \rhd \dots \rhd d_k}_{(-r'_1 + r'_2 - 1)\text{-}many} \rhd \underbrace{A^{r'_2}}_{\sim} \rhd \underbrace{d_m \rhd \dots \rhd d_n}_{(-r'_2 + r'_3 - 1)\text{-}many} \rhd \dots \rhd \underbrace{d_q \rhd \dots \rhd d_t}_{-r'_{\|R^-\|} - 1\text{-}many} \rhd \underbrace{A^{r_{\|R^-\|}}}_{\sim} \rhd D',$$

where D' contains those padding players in D that are not yet positioned in $\trianglerighteq^F_{p_i}$ and $\trianglerighteq^E_{p_i}$.

For the padding players $d_q \in D$ we define $F_{d_q} = \emptyset = N_{d_q}$ and $E = P' - \{d_q\}$, and let them be indifferent between their enemies:

$$\trianglerighteq^{FEN}_{d_q} = (\emptyset \mid \emptyset \mid \underbrace{P' - \{d_k\}}_{\sim}).$$

Note that whenever $\|R^+\| = 1$ or $\|R^-\| = 1$ holds for a player, $\trianglerighteq^F$ and $\trianglerighteq^E$ are defined by the last part of the above description, namely

$$\trianglerighteq^F = \underbrace{A^{r_{\|R^+\|}}}_{\sim} \rhd \underbrace{d_q \rhd \dots \rhd d_t}_{r_{\|R^+\|} - 1\text{-}many}, \quad \trianglerighteq^E = \underbrace{d_q \rhd \dots \rhd d_t}_{-r'_{\|R^-\|} - 1\text{-}many} \rhd \underbrace{A^{r_{\|R^-\|}}}_{\sim} \rhd D'.$$

We illustrate the construction in the following example.

Example 6.45 *Let $\mathcal{G} = (P, w)$ be an additively separable hedonic games with the players $P = \{p_1, p_2, p_3, p_4, p_5\}$ and the values given in Table 6.11, together with the resulting sets A^{-4}, A^1, and A^2.*

We need $2 + 4 - 2 = 4$ padding players d_1, d_2, d_3, and d_4 to construct the weak rankings with double threshold and we present those in Table 6.12.

Lemma 6.46 *Let $\mathcal{G} = (P, w)$ be an additively separable hedonic game, where the values the players assign to each other are integers and the preferences are not symmetric. Let further-*

p_i \ p_j	$w_{p_i}(p_j)$: p_1	p_2	p_3	p_4	p_5	A^2	A^1	A^{-4}
p_1	–	2	2	−4	1	$\{p_2, p_3\}$	$\{p_5\}$	$\{p_4\}$
p_2	2	–	−4	−4	−4	$\{p_1\}$	$\emptyset$	$\{p_3, p_4, p_5\}$
p_3	−4	−4	–	−4	−4	$\emptyset$	$\emptyset$	$\{p_1, p_2, p_4, p_5\}$
p_4	1	1	1	–	1	$\emptyset$	$\{p_1, p_2, p_3, p_5\}$	$\emptyset$
p_5	2	2	2	2	–	$\{p_1, p_2, p_3, p_4\}$	$\emptyset$	$\emptyset$

Table 6.11: Values of the players in $\mathscr{G}$

player	$\trianglerighteq^F$	N	$\trianglerighteq^E$
p_1	$p_2 \sim p_3 \triangleright p_5$	$\emptyset$	$d_1 \triangleright d_2 \triangleright d_3 \triangleright p_4 \triangleright d_4$
p_2	$p_1 \triangleright d_1$	$\emptyset$	$d_2 \triangleright d_3 \triangleright d_4 \triangleright p_3 \sim p_4 \sim p_5$
p_3	$\emptyset$	$\emptyset$	$d_1 \triangleright d_2 \triangleright d_3 \triangleright p_1 \sim p_2 \sim p_4 \sim p_5 \triangleright d_4$
p_4	$p_1 \sim p_2 \sim p_3 \sim p_5$	$\emptyset$	$d_1 \sim d_2 \sim d_3 \sim d_4$
p_5	$p_1 \sim p_2 \sim p_3 \sim p_4 \triangleright d_1$	$\emptyset$	$d_2 \sim d_3 \sim d_4$

Table 6.12: Constructed preferences from Example 6.45

more $\mathscr{G}' = (P', \succeq^{FENb})$ with $P' = P \cup D$ be a Borda-induced FEN-hedonic game with $\mathbf{f}_i = \mathbf{sfp}$ and $\mathbf{e}_i = \mathbf{seo}$ constructed from $\mathscr{G}$ according to Construction 6.44 and let Γ be a coalition structure in $\mathscr{G}$ and $\Gamma' = \Gamma \bigcup_{i=1}^{\|D\|} \{d_i\}$ be a coalition structure in $\mathscr{G}'$. For each stability concept α defined in Section 6.1 it holds that Γ is stable in the sense of α in $\mathscr{G}$ if and only if Γ' is stable in the sense of α in $\mathscr{G}'$.

Proof. Each padding player $d_i \in D$ assigns a negative value to all players in $P' - \{d_i\}$, so there are no acceptable coalitions for $d_i \in D$ except the singleton coalition $\{d_i\}$. Clearly, for each stability concept α defined in Section 6.1 a given coalition structure γ' can only be stable in the sense of α if it assigns each $d_i \in D$ to the coalition $\{d_i\}$. With this, the above equivalence directly follows. ❑

Sung and Dimitrov [SD10, Lemma 2, Theorem 3] show that in additively separable hedonic games the problem NSE is NP-complete.

Theorem 6.47 *In Borda-induced FEN-hedonic games the problem* NASH STABILITY EXISTENCE *is* NP*-complete for the choice of scoring vectors $\mathbf{f}_i = \mathbf{sfp}$ and $\mathbf{e}_i \in \{\mathbf{seo}, \mathbf{eo}, \mathbf{sep}, \mathbf{ep}\}$.*

Proof. With Observation 6.40 and [SD10, Lemma 2] the problem is in NP.

NP-hardness in the setting of additively separable hedonic games is shown by a reduction from X3C and the players in the constructed game assign values from $\{-68, 1, 2, 13, 16\}$ to each other.

For the choice of $\mathbf{f}_i = \mathbf{sfp}$ and $\mathbf{e}_i = \mathbf{seo}$, we can use Construction 6.44 and Lemma 6.46 to apply the argumentation in the proof of [SD10, Theorem 3].

The value -68 is the only negative value that is assigned in the additively separable hedonic game from the original proof and the argumentation remains unchanged if this value was smaller than -68. We show that for the other possible choices of $\mathbf{e}_i$ this negative value, let us call it K, is always at most -68.

Recalling the notation from Construction 6.44, we have that for each player $p_i \in P \subseteq P'$ with $E_{p_i} \neq D$, the ordering of the enemies is

$$\trianglerighteq^E_{p_i} : \underbrace{d_l \triangleright \ldots \triangleright d_k}_{67} \triangleright P'' \triangleright \underbrace{D'}_{\leq 15},$$

where P'' is the set of players, p_i assigns value -68 to in the original game and D' contains up to 16 padding players not contained in F_{p_i}. The set P'' corresponds to the set $E_{p_i,68}$ in the definition of the scoring vectors $\mathbf{e}_i$ in Section 6.3.1 and it is easy to see that for each fixed choice of $\mathbf{e}_i \in \{\mathbf{eo}, \mathbf{sep}, \mathbf{ep}\}$ it holds that $K \leq -68$. ❑

With the exact same approach we adapt the proof by Sung and Dimitrov [SD10, Lemma 2, Theorem 4] showing NP-completeness of the problem ISE in additive separable hedonic games for $\mathbf{f}_i = \mathbf{sfp}$ and $\mathbf{e}_i \in \{\mathbf{seo}, \mathbf{eo}, \mathbf{sep}, \mathbf{ep}\}$.

Theorem 6.48 *In Borda-induced FEN-hedonic games the problem* INDIVIDUAL STABILITY EXISTENCE *is* NP*-complete when* $\mathbf{f}_i = \mathbf{sfp}$ *and* $\mathbf{e}_i \in \{\mathbf{seo}, \mathbf{eo}, \mathbf{sep}, \mathbf{ep}\}$.

Proof. NP membership follows straightforwardly with Observation 6.40 and Lemma 2 in [SD10]. In their NP-hardness proof, Sung and Dimitrov [SD10] construct an additively separable hedonic game from an X3C instance in which the players' values are from $\{-4, 2, 1\}$. We can adapt this proof to our setting by constructing a Borda-induced FEN-hedonic game with Construction 6.44 and applying Lemma 6.46.

For the other choices of $\mathbf{e}_i$ we can argue that assigning a value K that is smaller than -4, the argumentation of the original still holds. For the players $p_i \in P \subseteq P'$ with $E_{p_i} \neq D$ Construction 6.44 defines $\trianglerighteq^E_{p_i}$ to be:

$$\trianglerighteq^E_{p_i} : d_1 \triangleright d_2 \triangleright d_3 \triangleright P'' \triangleright \underbrace{D'}_{\leq 2}.$$

Here we have that P'' corresponds to $E_{p_i,4}$ in the definition of the scoring vectors $\mathbf{e}_i$ in Section 6.3.1 and it is, again, easy to see that for each fixed choice of $\mathbf{e}_i \in \{\mathbf{eo}, \mathbf{sep}, \mathbf{ep}\}$ it holds that $K \leq -4$. ❑

Now we turn to the group deviation stability concepts and we start with analyzing the complexity of the CORE STABILITY EXISTENCE problem. For general additively separable hedonic games, Woeginger [Woe13a] showed that the problem is Σ_2^p-complete. We show that there exists a corresponding Borda-induced FEN-hedonic game with the same properties as the additively separable hedonic game constructed in this proof. We state this result in Theorem 6.49 and, for the sake of comparability, structure the proof analogously to Woeginger [Woe13a].

Theorem 6.49 *In Borda-induced FEN-hedonic games the problem* CORE STABILITY EXISTENCE *is Σ_2^p-complete for the choice of scoring vectors $\mathbf{f}_i = \mathbf{sfp}$ and $\mathbf{e}_i \in \{\mathbf{seo}, \mathbf{ep}\}$.*

Woeginger [Woe13a] shows Σ_2^p-completeness of CORE STABILITY EXISTENCE for additively separable hedonic games with a reduction from 2-QUANTIFIED 3-DNF-SAT defined in Section 2.2. Our approach defined in Construction 6.44 cannot be applied directly, but with careful adaptions we can define a Borda-induced FEN-hedonic game for which Woeginger's argumentation still holds:

Let m be the number of clauses and n the number of variables in a given instance of the problem 2-QUANTIFIED 3-DNF-SAT. The values in the original game are from the set $\{-\infty, -2, 0, \varepsilon, 1, 2, 3, 4, 5, n+2, m+n+1, 4n+m-1\}$, where $-\infty$ denotes a "small enough number" and $\varepsilon = 1/n+1$. To define a Borda-induced FEN-hedonic game we have to define the exact value for $-\infty$ and change ε to a positive integer while preserving the central argumentation. We present the definition of our Borda-induced FEN-hedonic game in Construction 6.50 and show in Lemmas 6.51 through 6.53 where and how Woeginger's argumentation has to be adapted.

Construction 6.50 *Given a* 2-QUANTIFIED 3-DNF-SAT *instance $(X, Y, \phi(X,Y))$ we denote the set of clauses in ϕ by C and we construct the following set of players $P = P_X \cup P_Y \cup P_C \cup \{Q_t, Q_t', Q_t'', Q_f, R, R'\} \cup D$:*

- *For every literal ℓ over X, we construct a corresponding X-player $p(\ell)$ ($2n$ in total). We denote this set with P_X.*
- *For every literal ℓ over Y, we construct a corresponding Y-player $p(\ell)$ ($2n$ in total). We denote this set with P_Y.*
- *For every clause $c \in C$, we construct a corresponding C-player $p(c)$ (m in total). We denote this set with P_C.*
- *We have six structure players Q_t, Q_t', Q_t'', Q_f, R, and R'.*
- *We have a set of padding players D which we will use to generate the preferences providing the needed values.*

The number of padding players is bounded by $\mathcal{O}(n+m) + \mathcal{O}((n+m)(n^2+nm+m^2))$.

The scoring vector for the set of friends is fixed to $\mathbf{f}_i = \mathbf{sfo}$ and we first construct $\trianglerighteq^F$ for the players in P. Note that we change the value of ε from $1/n+1$ to 1 and adjust the score the player Q_t' assigns player Q_t to $n+1$ (instead of 1). Table 6.13 shows $\trianglerighteq^F$ of the players in P and furthermore displays the values that are assigned based on the choice of $\mathbf{f}_i = \mathbf{sfo}$. Whenever set of players are given in a preference, say of player p, we assume that p is indifferent between the players in the set. Furthermore if a single padding player d is given, she can be replaced by an arbitrarily picked player from D. Parts of the preferences that denoted by "$\cdots$" have to be filled with an appropriate number of padding players from D.

The set of neutral players is $N_d = P_C \cup P_X \cup P_Y - \{p(\ell_1), p(\ell_2), p(\ell_3)\}$ for each $d \in D$, $N_{Q_t''} = P_C$, $N_{p(y)} = P_C \cup P_X \cup (P_Y - \{\overline{y}\})$, $N_{p(x)} = P_C \cup P_Y \cup \{Q_t'\}$, and $N_p = \emptyset$ for all remaining players $p \in P$.

For each player $p \in P$ assigning the symbolic value "$-\infty$" to some of her enemies in the original game, we define K_p to be the sum of all positive values p assigns to other players

value:	$m+n+1 \quad \cdots \quad n+2 \quad \cdots \quad 1$
$\trianglerighteq^F_{Q_t}$:	$Q'_t \rhd \cdots \rhd Q''_t \rhd \cdots \rhd P_X \cup P_C$
value:	$4n+2m-1 \quad \cdots \quad 2 \quad 1$
$\trianglerighteq^F_{R}$:	$R' \rhd \cdots \rhd P_C \cup P_X \cup P_Y \rhd d$
value:	$4 \quad 3 \quad 2 \quad 1$
$\trianglerighteq^F_{p(x)}$:	$Q_f \rhd d \rhd R \rhd (P_X - \{\bar{x}\}) \cup \{Q_t\}$
value:	$n+1 \quad \cdots \quad 1$
$\trianglerighteq^F_{Q'_t}$:	$Q_t \rhd \cdots \rhd P_X$
value:	$6 \quad 5$
$\trianglerighteq^F_{p(c)}$:	$Q_t \rhd R \rhd \cdots$
value:	$2 \quad 1$
$\trianglerighteq^F_{p(y)}$:	$R \rhd d$
value:	1
$\trianglerighteq^F_{Q''_t}$:	Q_t
value:	1
$\trianglerighteq^F_{Q_f}$:	P_X
value:	1
$\trianglerighteq^F_{Q_{R'}}$:	R
value:	–
$\trianglerighteq^F_{D}$:	–

Table 6.13: $\trianglerighteq^F$ of the players in the proof of Theorem 6.49

in $P - \{p\}$. Table 6.14 shows the neutral sets and $\trianglerighteq^E$ of the players in P, where D' denotes those padding players not contained in $\trianglerighteq^F$ and not contained in $\trianglerighteq^E$ so far. This completes the construction of the Borda-induced FEN-hedonic game for $\mathbf{f}_i = \mathbf{sfp}$ and $\mathbf{e}_i = \mathbf{seo}$.

For the scoring vectors $\mathbf{f}_i = \mathbf{sfo}$ and $\mathbf{e}_i = \mathbf{ep}$, a similar approach can be used to achieve almost the same values as in the original construction. Only the preferences of the C-players have to be constructed carefully. These players are the only players assigning a different value than $-\infty$ to a subset of their enemies, namely the -2 to those literal-players that are contained in the clause the clause-player corresponds to. With $\mathbf{e}_i = \mathbf{ep}$ we cannot achieve the assignment of value -2, but the assignment of value -3 by adding 12 padding players to the enemy set and due to this change, the players R and Q_t in $\trianglerighteq^F$ have to each gain one point more, thus we have the following adapted preferences:

value:	$-3 \quad \quad -16 \quad -17 \quad -18$
$\trianglerighteq^E_{p(c)}$	$p(\ell_1) \sim p(\ell_2) \sim p(\ell_3) \rhd \cdots \rhd Q'_t \rhd Q_f \rhd R'$
value:	$8 \quad 7$
$\trianglerighteq^F_{p(c)}$:	$Q_t \rhd R \rhd \cdots$

value:	-1						
$\trianglerighteq^E_d$:	$P-\{d\}$						
value:	$\cdots$		$-K_{R'}$		$-K_{R'}-1$		
$\trianglerighteq^E_{R'}$:	$\cdots$	$\triangleright$	$P-\{R\}$	$\triangleright$	D'		
value:	$\cdots$		$-K_{Q_f}$		$-K_{Q_f}-1$		
$\trianglerighteq^E_{Q_f}$:	$\cdots$	$\triangleright$	$P-P_X$	$\triangleright$	D'		
value:	$\cdots$		$-K_{Q''_t}$		$-K_{Q_f}-1$		
$\trianglerighteq^E_{Q''_t}$:	$\cdots$	$\triangleright$	$P_X\cup P_Y\cup\{Q'_t,Q_f,R,R'\}$	$\triangleright$	D'		
value:	$\cdots$		$-K_{p(y)}$		$-K_{p(y)}-1$		
$\trianglerighteq^E_{p(y)}$:	$\cdots$	$\triangleright$	$\{\overline{y},Q_t,Q'_t,Q''_t,Q_f,R'\}$	$\triangleright$	D'		
value:	$\cdots$		$-K_{Q'_t}$		$-K_{Q'_t}-1$		
$\trianglerighteq^E_{Q'_t}$:	$\cdots$	$\triangleright$	$P_C\cup P_Y\cup\{Q''_t,R,R'\}$	$\triangleright$	D'		
value:	$\cdots$		$-K_{p(x)}$		$-K_{p(x)}-1$		
$\trianglerighteq^E_{p(x)}$:	$\cdots$	$\triangleright$	$\{Q''_t,R',\overline{x}\}$	$\triangleright$	D'		
value:	$\cdots$		$-K_R$		$-K_R-1$		
$\trianglerighteq^E_R$:	$\cdots$	$\triangleright$	$\{Q_t,Q'_t,Q''_t,Q_f\}$	$\triangleright$	D'		
value:	$\cdots$		$-K_{Q_t}$		$-K_{Q_t}-1$		
$\trianglerighteq^E_{Q_t}$:	$\cdots$	$\triangleright$	$P_Y\cup\{Q_f,R,R'\}$	$\triangleright$	D'		
value:	-1		-2		$\cdots$		$-K_{p(c)}$
$\trianglerighteq^E_D$:	d	$\triangleright$	$\{p(\ell_1),p(\ell_2),p(\ell_3)\}$	$\triangleright$	$\cdots$	$\triangleright$	$\{Q'_t,Q_f,R'\}$

Table 6.14: $\trianglerighteq^E$ of the players in the proof of Theorem 6.49

The remaining padding players in P that have not been assigned to $\trianglerighteq^F_{p(c)}$ or $\trianglerighteq^E_{p(c)}$ have to be in $N_{p(c)}$. This ensures that Woeginger's argumentation can be adapted straightforwardly.

We will present the argumentation for $\mathbf{f}_i = \mathbf{sfo}$ *and* $\mathbf{e}_i = \mathbf{seo}$ *in detail. Consider the following coalition structure Γ^* that will be used throughout the rest of the argumentation. Let $X = X_1 \cup X_2$ be a partition of X into two sets such that for each $x \in X_1$ we have that $\overline{x} \in X_2$.*

$$\begin{aligned}\Gamma^* =&\{\{Q_f,\{p(x)\mid x\in X_1\}\},\{p(y)\}_{y\in Y},\{R,R'\},\{Q''_t\},\{p(c)\}_{c\in C},\{d\}_{d\in D},\\ &\{Q_t,\{p(x)\mid x\in X_2\},Q'_t\}\}\end{aligned} \tag{6.6}$$

Table 6.15 shows the values each player assigns to her coalition in Γ^*.

Q_f	$\{p(x)\vert x\in X_1\}$	R	R'	$\{p(x)\vert x\in X_2\}$	Q_t	$Q't$	P_Y,P_C,Q''_t,D
n	$n+3$	$4n+2m-1$	1	n	$2n+m+1$	$2n+1$	0

Table 6.15: Values the players assign their coalition in Γ^*

Based on the constructed game, we will show Theorem 6.49 step by step, just as Woeginger did, and we start with the following lemma.

Lemma 6.51 *Let $(P, \succeq^{FEN})$ be a game constructed from a* 2-QUANTIFIED 3-DNF-SAT *instance $(X, Y, \phi(X,Y))$ as in Construction 6.50 and assume that Γ^* is a core-stable coalition structure. Then the following holds for Γ^*.*

1. *Coalition $\Gamma^*(Q_f)$ consists of Q_f and n of the X-Players. For each $x \in X$ either $p(x)$ or $p(\bar{x})$ is in $\Gamma^*(Q_f)$.*
2. *Coalition $\Gamma^*(R)$ cannot consist of R together with n X-players, n Y-players, and all m C-players.*
3. *$\Gamma^*(R) = \{R, R'\}$.*
4. *$Q''_t \notin \Gamma^*(Q_t)$.*
5. *$Q'_t \in \Gamma^*(Q_t)$.*
6. *$\Gamma^*(Q_t) = \{Q_t, Q'_t, \{p(x) | p(x) \notin \Gamma^*(Q_f)\}\}$.*
7. *Γ^* yields a value of 0 for Q''_t, all Y-players, and all C-players.*

Proof. Claim 1 directly follows from [Woe13a, Lemma 4.1] except that for the X-Players all coalitions not containing Q_f yield less than $n+3$ points. The remaining argumentation remains unchanged.

Claims 2 and 3 can be shown with the exact argumentation as in [Woe13a, Lemmas 4.2 and 4.3].

Claim 4 can be shown with a similar argumentation as presented in [Woe13a, Lemma 4.4]: Assume that $Q''_t \in \Gamma^*(Q_t)$. That implies that $\Gamma^*(Q_t) \subseteq \{Q_t, Q''_t\} \cup P_C$ and Q_t assigns a value of at most $m+n+2$, Q'_t assigns a value of at most n (because she is not in a coalition with Q_t), and with Claims 1 and 3 we know that each $p(x)$ assigns a value of at most $n-1$. Now consider the coalition $\{Q_t, \{p(x) | p(x) \notin \Gamma^*(Q_f)\}, Q'_t\}$ that ensures Q_t a value of $m+2n+1$, Q'_t a value of $2n+1$, and the X-players each a value of n and would thus be a blocking coalition.

Claims 5, 6, and 7 can be shown with the exact same argumentation as [Woe13a, Lemmas 4.5, 4.6, and 4.7]. ❑

Lemma 6.52 *Let $(P, \succeq^{FEN})$ be a game constructed from a* 2-QUANTIFIED 3-DNF-SAT *instance $(X, Y, \phi(X,Y))$ as in Construction 6.50. If there exists a core-stable coalition structure Γ^* in this game, then $(X, Y, \phi(X,Y))$ is a yes instance of* 2-QUANTIFIED 3-DNF-SAT.

Proof. This claim can be shown by the exact same argumentation as Woeginger provides in [Woe13a, Section 4]. ❑

Lemma 6.53 *Let $(P, \succeq^{FEN})$ be a game constructed from a* 2-QUANTIFIED 3-DNF-SAT *instance $(X, Y, \phi(X,Y))$ as in Construction 6.50. If $(X, Y, \phi(X,Y))$ is a yes instance of* 2-QUANTIFIED 3-DNF-SAT *then a core-stable coalition structure Γ exists in this game.*

Proof. Assume that $(X,Y,\phi(X,Y))$ is a yes instance of 2-QUANTIFIED 3-DNF-SAT with the truth-assignment τ_X for the variables in X. Define a coalition structure Γ as the one in (6.6) and let $p(x) \in \Gamma(Q_f)$ if and only if x is set to false.

For the sake of contradiction we assume that there is a coalition S^* that blocks the coalition structure Γ. With [Woe13a, Lemmas 5.1, 5.2, and 5.3] and some further argumentation provided by Woeginger, we can show that

1. $\Gamma(Q_f) \not\subseteq S^*$.
2. $R, R' \notin S^*$.
3. $Q_t \notin S^*$.
4. For all $c \in C$, $p(c) \notin S^*$.
5. For all $y \in Y$, $p(y) \notin S^*$.
6. $Q_t'' \notin S^*$.

Furthermore, we have that for all $d \in D$, $p(d) \notin S^*$ which simply follows from the fact that being in a singleton-coalition already maximizes the values of the players in D. Together with Claims 1 to 6, this implies that any possibly blocking coalition S^* is the empty set, thus Γ is a core-stable coalition structure. ❑

Now we can easily conclude the proof of Theorem 6.49.

Proof of Theorem 6.49. The claim follows immediately with Construction 6.50 and Lemmas 6.52 and 6.53. ❑ Theorem 6.49

The complexity of strict core stability existence was settled recently by Peters [Pet15], who established Σ_2^p-completeness. Whether Construction 6.44 is applicable to transfer this proof has to be left open for future work. We establish the following lower bound.

Theorem 6.54 *For Borda-induced FEN-hedonic games the problem* STRICTLY CORE STABILITY EXISTENCE *is* coNP*-hard for each choice of* $\mathbf{f}_i$ *and* $\mathbf{e}_i$.

Proof. We show coNP-hardness by a reduction from $\overline{\text{CLIQUE}}$ with a similar construction as the one used in the proof of Theorem 6.42. To this end let $G = (V,H)$ be an undirected graph with $V = \{v_1, v_2, \ldots, v_n\}$ and $H = \{h_1, h_2, \ldots, h_m\}$ and let $k \geq 2$ be a positive integer. Let $\mathcal{N}(v)$ denote the neighborhood of vertex $v \in V$ and recall that $\mathcal{N}[v] = \mathcal{N}(v) \cup \{v\}$.

Construct the Borda-induced FEN-hedonic game $(P, \succeq^{FENb})$ with the set of players $P = V \cup Q \cup R \cup T$, where the players $v_i \in V$ correspond to the vertices in the graph, $Q = \bigcup_{i=1}^{n} Q_i$ with $Q_i = \{q_{i,1}, q_{i,2}, \ldots, q_{i,(k-2)}\}$, $R = \{r_1, r_2, \ldots, r_n\}$, and $T = \{t_1, t_2, \ldots, t_n\}$. The weak ranking with double threshold of the players are shown in Table 6.16.

So we have that each v_i considers all players in Q_i to be her friends and moreover each other player in V that corresponds to a vertex in G that is connected to the vertex corresponding to player v_i. The players in each Q_i only consider v_i to be a friend, do not care about the other players in Q_i and both r_i, and t_i, while the remaining players are enemies. For the players in R and T we have that for every $i \in \{1, \ldots, n\}$ both players r_i and t_i consider $q_{i,1}$ to be their only

For each ...	player	$\trianglerighteq^F$	N	$\trianglerighteq^E$
$i \in \{1,\dots,n\}$	v_i	$\mathscr{N}(v_i) \sim Q_i$	$P - (\{\mathscr{N}[v_i] \cup Q_i\}))$	$\emptyset$
$i \in \{1,\dots,n\}$, $j \in \{1,\dots,k-2\}$	$q_{i,j}$	v_i	$(Q_i - \{q_{i,j}\}) \sim \{r_i, t_i\}$	$P - (F_{q_{i,j}} \cup N_{q_{i,j}})$
$i \in \{1,\dots,n\}$	r_i	$q_{i,1}$	$Q_i - \{q_{i,1}\}$	$P - (F_{r_i} \cup N_{r_i})$
$i \in \{1,\dots,n\}$	t_i	$q_{i,1}$	$Q_i - \{q_{i,1}\}$	$P - (F_{t_i} \cup N_{t_i})$

Table 6.16: Weak rankings with double threshold of the players in the proof of Theorem 6.54

friend, they both do not care about the other players in Q_i while considering each other to be enemies (and the remaining players are their enemies, as well).

We claim that $(G,k) \notin$ CLIQUE if and only if there exists a strictly core-stable coalition structure for $(P, \succeq^{FENb})$ for each choice of $\mathbf{f}_i$ and $\mathbf{e}_i$.

Only if: Assume there is no clique of size k in G. Then

$$\Gamma = (\Gamma_1^v, \Gamma_2^v, \dots, \Gamma_n^v, \Gamma_1^r, \Gamma_2^r, \dots, \Gamma_n^r, \Gamma_1^t, \Gamma_2^t, \dots, \Gamma_n^t)$$

with $\Gamma_i^v = \{v_i\} \cup Q_i$, $\Gamma_i^r = \{r_i\}$, and $\Gamma_i^t = \{t_i\}$ is a strictly core-stable coalition structure for $(P, \succeq^{FENb})$: The players in the coalitions Γ_i^v are in their best valued coalitions, thus every coalition containing them would not be a weakly blocking coalition. This only leaves the players in R and T which are all enemies, so these cannot form a weakly blocking coalition neither. Thus, the coalition structure is strictly core-stable.

If: We show the contraposition. Assume that there is a clique of size k in G, say V'. To construct a contradiction, let Γ be a strictly core-stable coalition structure. For Γ to be strictly core-stable, the players corresponding to the vertices in the clique V' have to be together in a coalition in Γ and no other players can be contained in this coalition. Let the set $J = \{i \in \{1,\dots,n\} \mid v_i \in V'\}$ denote those indices corresponding to the vertices that are contained in the clique V'. For these $j \in J$ we have that the players in Q_j (and especially $q_{j,1}$) cannot form a coalition with their friend v_j, thus the players r_j and t_j are both interested in forming a coalition with player $q_{j,1}$. The players in each Q_i can be assigned to coalitions in four different ways:

1. $\{r_j, Q_j\}$; then $\{t_j, q_{j,1}\}$ would be a weakly blocking coalition.
2. $\{t_j, Q_j\}$; then $\{r_j, q_{j,1}\}$ would be a weakly blocking coalition.
3. $\{t_j, r_j, Q_j\}$; then both $\{r_j, q_{j,1}\}$ and $\{t_j, q_{j,1}\}$ would be weakly blocking coalitions.
4. $\{Q_j\}$; then $\{r_j, q_{j,1}\}$ and $\{t_j, q_{j,1}\}$ would be weakly blocking coalitions.

We see that in all cases there exists a weakly blocking coalition, thus Γ cannot be strictly core-stable. ❑

Tackling open Problems with Metatheorems In their intriguing work, Peters and Elkind [PE15] establish relations between properties that preferences in hedonic games can fulfill and NP-hardness of certain stability existence problems. They focus on games in which each player $i \in P = \{1, \ldots, n\}$ provides a ranking $\trianglerighteq_i$ over P and based on this ranking, the players in $P - \{i\}$ are divided into a set of friends $F_i = \{j \neq i \mid j \trianglerighteq_i i\}$ and a set of enemies $E = \{j \neq i \mid i \triangleright_i j\}$. They furthermore assume that extensions of these rankings to preferences over coalitions, denoted by $\succeq = (\succeq_1, \ldots, \succeq_n)$, have to allow each player to have arbitrary orderings of coalitions of size 2 and that the game $(P, \succeq)$ can be obtained from $\trianglerighteq$ in deterministic polynomial time.

Recalling Definition 6.26 on page 153, we see that with small redefinitions, our Borda-induced FEN-hedonic games indeed fulfill the above stated requirements: For a fixed player $i \in P$, we can consider the set of neutral players N_i as a part of her set of friends and extend the weak ranking $\trianglerighteq_i$ over $P - \{i\}$ to a ranking $\trianglerighteq'_i$ over P with $\trianglerighteq'_i = \trianglerighteq^F_i \triangleright N_i \sim i \triangleright \trianglerighteq^E_i$, where again i is indifferent between all players in N_i.

Definition 6.55 (Peters and Elkind [PE15]) *A class of hedonic games fulfills the following properties if for each set of n players P and every collection* $\trianglerighteq = (\trianglerighteq_1, \ldots, \trianglerighteq_n)$ *of rankings over players, there is a game* $(P, \succeq)$ *that fulfills the given statement.*

(1) Consistent on pairs*: For all* $i \in P$ *and* $j, k \in F_i \cup \{i\}$ *it holds that* $\{i, j\} \succeq_i \{i, k\}$ *if and only if* $j \trianglerighteq_i k$*.*
(2) Not triangle-hating*: For all* $i \in P$ *and* $j, k \in F_i$ *it holds that* $j \trianglerighteq_i k$ *implies* $\{i, j, k\} \succeq_i \{i, k\}$*.*
(3) {a-b}-toxic*: For all* $i \in P$ *and each* $S \subseteq P$ *it holds that* $\{i\} \succeq_i S$ *if* $\|S \cap F_i\| = a$*, but* $\|S \cap E_i\| \geq b$*.*
(4) Strictly {a-b}-toxic*: For all* $i \in P$ *and each* $S \subseteq P$ *it holds that* $\{i\} \succ_i S$ *if* $\|S \cap F_i\| = a$*, but* $\|S \cap E_i\| \geq b$*.*
(5) Weakly {a-b}-toxic*: For all* $i \in P$ *and each* $S \subseteq P$ *it holds that* $\{i, j\} \succ_i S$ *for all* $j \in F_i$ *if* $\|S \cap F_i\| = a$*, but* $\|S \cap E_i\| \geq b$*.*

We can show that for each choice of scoring vectors, Borda-induced FEN-hedonic games indeed fulfill some of these properties.

Proposition 6.56 *The class of Borda-induced FEN-hedonic games fulfills consistency on pairs, {0-1}-toxicity, as well as strict and weak {0-1}-toxicity, and is not triangle hating.*

Proof. By definition, we have that for each choice of scoring vectors and a fixed player i with $j \in F_i \cup N_i$ and $k \in E_i$ it always holds that

$$f^i_{\text{Borda}}(\{i\}) = 0, \quad f^i_{\text{Borda}}(\{i, j\}) \geq 0, \text{ and } f^i_{\text{Borda}}(\{i, k\}) < 0.$$

From this, the three variants of {0-1}-toxicity follow. The remaining two properties are clearly fulfilled since our scoring vectors provide descending values with respect to $\trianglerighteq'_i$. ❑

Conditions for the NP-hardness of NASH STABILITY EXISTENCE and INDIVIDUAL STABILITY EXISTENCE are given in the next theorem.

Theorem 6.57 (Peters and Elkind [PE15]) *The problems* NASH STABILITY EXISTENCE *and* INDIVIDUAL STABILITY EXISTENCE *are* NP*-hard for a class of hedonic games if the class fulfills consistency on pairs, strict {0-1,1-1,2-2}-toxicity, and is not triangle-hating.*

This theorem, however, is not applicable to our class of Borda-induced FEN-hedonic games when scoring vectors can be chosen from $\{\mathbf{seo},\mathbf{ep}\},\{\mathbf{sfo},\mathbf{fo},\mathbf{sfp},\mathbf{fp}\}$ or $\{\mathbf{sep},\mathbf{eo}\},\{\mathbf{sfo},\mathbf{fp}\}$ as this subclass fails to fulfill the needed property of strict {1-1}-toxicity.

Proposition 6.58 *The subclass of Borda-induced FEN-hedonic games when scoring vectors can be chosen from* $\{\mathbf{seo},\mathbf{ep}\},\{\mathbf{sfo},\mathbf{fo},\mathbf{sfp},\mathbf{fp}\}$ *or* $\{\mathbf{sep},\mathbf{eo}\},\{\mathbf{sfo},\mathbf{fp}\}$ *is not {1-1}-toxic (and thus not strictly {1-1}-toxic).*

Proof. We show the above claim for each combination of the given scoring vectors with the following game as a counter example. Let $P = \{1,2,3,4\}$ be the set of players and we have the weak preferences with thresholds

$$\trianglerighteq_1^{FEN} = (2 \triangleright 4 \mid \emptyset \mid 3),\ \trianglerighteq_2^{FEN} = (1 \mid \emptyset \mid 3 \triangleright 4),\ \trianglerighteq_i^{FEN} = (\emptyset \mid P-\{i\} \mid \emptyset),\ \text{for } i \in \{3,4\}.$$

The values player 1 and player 2 assign to their co-players for different choices of scoring vectors are given in the following table.

	player 1			player 2			
$\mathbf{f}_i$	2	3	4	1	3	4	$\mathbf{e}_i$
sfo	4	-1	3	4	-1	-2	**seo**
fo	2	-4	1	1	-3	-4	**eo**
sfp	2	-4	1	1	-3	-4	**sep**
fp	4	-1	3	4	-1	-2	**ep**

For example, when $\mathbf{f}_i = \mathbf{sfo}$ and $\mathbf{e}_i = \mathbf{sep}$, player 1 assigns player 2 a value of 4, player 3 a value of -4, and player 4 a value of 3. For the coalition $S = \{1,2,3\}$ and an arbitrary choice of $\mathbf{f}_i, \mathbf{e}_i$ from $\{\mathbf{seo},\mathbf{ep}\},\{\mathbf{sfo},\mathbf{fo},\mathbf{sfp},\mathbf{fp}\}$, we have that $f^1_{\text{Borda}}(S) > 0 = f^1_{\text{Borda}}(\{1\})$, which is equivalent to $S \succ_1 \{1\}$. For the same coalition and the scoring vectors from $\{\mathbf{sep},\mathbf{eo}\}\{\mathbf{sfo},\mathbf{fp}\}$ we obtain the same contradiction from player 2's view and we have shown that for these pairs of scoring vectors, (strict) {1-1}-toxicity is not fulfilled. ❑

In Theorems 6.48 and 6.47 we have established NP-hardness of INDIVIDUAL STABILITY EXISTENCE and NASH STABILITY EXISTENCE for scoring vectors from $\{\mathbf{sfp}\},\{\mathbf{sep},\mathbf{eo}\}$, thus it is worth analyzing whether Theorem 6.57 can be applied to the subclass of Borda-induced FEN-games when the choice of scoring vectors is limited to $\{\mathbf{fo}\},\{\mathbf{sep},\mathbf{eo}\}$.

Similar to Theorem 6.57, Peters and Elkind [PE15] provide a result stating conditions for NP-hardness of CORE STABILITY EXISTENCE.

Theorem 6.59 (Peters and Elkind [PE15]) *The problem* CORE STABILITY EXISTENCE *is* NP*-hard for a class of hedonic games if the class fulfills consistency on pairs, {0-1}-toxicity, and weak {1-1,2-2,3-4}-toxicity.*

Unfortunately, for scoring vectors chosen from $\{\mathbf{seo}, \mathbf{ep}\}, \{\mathbf{fp}, \mathbf{fo}, \mathbf{sfo}\}$ this result is not applicable since weak $\{1\text{-}1\}$-toxicity does not hold.

Proposition 6.60 *The subclass of Borda-induced FEN-hedonic games when scoring vectors can be chosen from* $\{\mathbf{seo}, \mathbf{ep}\}, \{\mathbf{fp}, \mathbf{sfp}, \mathbf{fo}, \mathbf{sfo}\}$ *is not weakly* $\{1\text{-}1\}$*-toxic.*

Proof. Recall the game defined in the proof of Proposition 6.58. It holds for each of the above specified choices of scoring vectors that $f^1_{\text{Borda}}(\{1,4\}) = 1 = f^1_{\text{Borda}}(\{1,2,3\})$, which contradicts the condition for weak $\{1\text{-}1\}$-toxicity. ❑

For scoring vectors from $\{\mathbf{sep}, \mathbf{eo}\}, \{\mathbf{fp}, \mathbf{sfp}, \mathbf{fo}, \mathbf{sfo}\}$, however, proving weak $\{1\text{-}1,2\text{-}2,3\text{-}4\}$-toxicity would imply NP-hardness of CORE STABILITY EXISTENCE. We leave this and the question whether Theorem 6.57 can be applied to the given scoring vectors as open question for future work.

6.4 Concluding Remarks and Future Work

In this chapter we have studied the complexity of stability for different representations of hedonic games. We furthermore introduced a new class of hedonic games, namely *FEN-hedonic games* in which the players' preferences are incomplete and suggested a way of extending these preferences using Borda-like comparability functions leading to a new subclass of additively separable hedonic games, which we call *Borda-induced FEN-hedonic games*.

In the context of enemy-based hedonic games we focused on the problems of STRICT CORE STABILITY EXISTENCE and the existence and verification variant of wonderful stability as the complexity of each of these problems was yet unresolved. While for WONDERFUL STABILITY VERIFICATION Rey et al. [RRS⁺15, RRS⁺14] prove NP-completeness and thus pinpoint the exact complexity, for both STRICT CORE STABILITY EXISTENCE and WONDERFUL STABILITY EXISTENCE a lower DP-hardness bound could be shown. These results significantly improve known results. WONDERFUL STABILITY EXISTENCE and STRICT CORE STABILITY EXISTENCE are conjectured to be Θ_2^p- and Σ_2^p-complete, respectively, and we provide a first step for proving Θ_2^p-hardness: We show that coDP-hardness of these problems directly implies their hardness for Θ_2^p.

Since enemy-based hedonic games are a subclass of additively separable hedonic games, the above mentioned results also hold further insights on the until recently unresolved question of the exact complexity of STRICT CORE STABILITY EXISTENCE in additively separable hedonic games. To be precise, the formerly best known lower bound of NP-hardness was improved to DP-hardness. Very recent results by Peters [Pet15], however, settle this glaring open question and show Σ_2^p-completeness.

For our newly introduced class of Borda-induced FEN-hedonic games, we have intensely investigated the complexity of stability with respect to commonly studied stability concepts. There are, however, some pairs of scoring vectors for which the existence problems regarding Nash stability, individual stability, and strict core stability are yet unresolved: For scoring vectors from $\{\mathbf{fp}, \mathbf{fo}, \mathbf{sfo}\}, \{\mathbf{sep}, \mathbf{ep}, \mathbf{seo}, \mathbf{eo}\}$ future research should focus on the complexity of NASH STABILITY EXISTENCE and INDIVIDUAL STABILITY EXISTENCE, while for scoring vectors from **sfp**, {**sep**, **eo**} and {**fp**, **fo**, **sfo**}, {**sep**, **ep**, **seo**, **eo**} the complexity of CORE STABILITY EXISTENCE is yet unsettled. We have seen that for some of these choices, the metatheorems presented in the work of Peters and Elkind [PE15] should be the first approach. Unfortunately for some of the above choices, we could show that the results from Peters and Elkind [PE15] are not applicable. Furthermore, other stability concepts such as strong Nash stability or strong individual stability (see, for instance the work of Karakaya [Kar11]) can be studied and other concepts of breaking incomparabilities compatible with the generalized Bossong-Schweigert extension models are worthwhile to be investigated.

7 Conclusions and Outlook

In this thesis we have studied the computational complexity of standard manipulative attacks on Bucklin and fallback elections and we have defined new variants of the margin of victory problem and analyzed their complexity. We have furthermore introduced the possible winner problem with uncertain weights and have investigated its computational complexity when the weights can be nonnegative rational numbers.

We furthermore have taken a next step to pinpoint the exact complexity of wonderful stability existence and strict core stability existence in enemy-based hedonic games. Moreover, we introduced the class of FEN-hedonic games combining ordinal preferences with the partition of players into friends, enemies, and neutral players. The players' rankings over their co-players are extended to possibly incomplete preferences over coalitions using the generalized Bossong-Schweigert extension principle. As one way of handling such incomparabilities, we defined a subclass of these games, Borda-induced FEN-hedonic games, in which the preferences are additively separable and are derived from Borda-like scoring vectors. For these games we studied the complexity of verification and existence problems of well-known stability concepts.

Our study, however, has left open some unresolved problems, which are summarized in the concluding remarks of the respective chapters.

Being a fast evolving and fruitful line of research, the context of computational social choice yields numerous ways of extending the studies presented in this thesis. In the following we survey some of the possibly most promising directions for general future work.

In the context of voting, much recent research has focused on multiwinner elections in which a set of winners has to be determined, for example to elect a committee or council. For these settings, known voting rules have to be adapted [BKS04] and these new aggregation methods have been intensely studied with respect to their algorithmic properties, see amongst others the work of Meir et al. [MPR+08], Aziz et al. [AGG+15], Amanatidis et al. [ABL+15], Baumeister et al. [BDR15], and Baumeister and Dennisen [BD15]. Most of this work focuses on approval-based elections.

Another interesting line of research is the study of randomized voting rules, which have been of central interest in social choice theory, see the book chapter by Barberà [Bar10]. Recent results are due to Aziz et al. [ABB14] and Brandl et al. [BBH15].

In the context of hedonic game, further research can be taken into the direction of defining new classes of hedonic games or further stability concepts. Also manipulative behavior of players is of high interest, see the work of Rodriguez-Alvarez [Rod09] or Vallée et al. [VBZ+14].

Another interesting approach that can be taken is the definition of *partition correspondences*, which are procedures that determine for a given game a partition of the players into coalitions based on their preferences, but in a centralized manner. For such mechanisms, axiomatic properties can be defined and analyzed. Motivation for such procedures can be found in settings, where the decentralized coalition formation of the players is not feasible, for example when the communication between players is disturbed or not possible at all.

This thesis and much research has focused on the computational complexity of voting and stability problems stated as decision problems. That is, we analyze the complexity of deciding whether a given election can be manipulated, bribed, or controlled, or whether a given game has a stable coalition structure. A natural next step is to consider the complexity of the corresponding search problems, as has been done by Hemaspaandra et al. [HHM13] for voting problems and by Bogomolnaia and Jackson [BJ02], Gairing and Savani [GS10, GS11], and Sung and Dimitrov [SD10] for hedonic games.

Furthermore an interesting approach is to combine aspects from voting theory and game theory in the sense that manipulative activities in elections can be modeled as games, as has been done in the work of Bachrach et al. [BEF11], Elkind et al. [EGR+15], Dutta et al. [DJL01], and Obraztsova et al. [OEP+15].

We conclude by referring to the intriguing work by Bredereck et al. [BCF+14a], who compactly, but nevertheless insightfully survey a collection of research challenges regarding the parameterized complexity of problems in the context of computational social choice.

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Contribution

The results presented in this thesis were published in peer-reviewed conference proceedings and journals. The following list displays my contribution to these publications.

- Prior work regarding electoral control in Bucklin and fallback elections, which is not part of this thesis, has been published in the conference papers [ER10, EPR11, EF10b]. The journal article [EFR+15a] extends and supersedes the above papers. It was jointly and in equal parts written by my coauthors G. Erdélyi, M. Fellows, J. Rothe, and myself.
- The technical report [RS12b] contains a complete survey of my experimental results on control comlpexity in Bucklin, fallback, and plurality elections. A compact summary of these results has been published in the conference paper [RS12a]. Both, the technical report and the conference paper are joint work of my coauthor J. Rothe and myself.
  The journal article [EFR+15b] presents parts of the above mentioned results (for Bucklin and fallback voting) and it was jointly written by my coauthors G. Erdélyi, M. Fellows, J. Rothe, and myself.
- The conference paper [FRR+14] presents an overview of results for manipulation, bribery, and campaign management in Bucklin and fallback elections. Omitted proofs are published in the journal article [FRR+15]. Both works were written jointly and in equal parts with my coauthors P. Faliszewski, Y. Reisch, and J. Rothe.
- The conference paper [RRS14] about the margin of victory and its exact variant has been written jointly with my coauthors Y. Reisch and J. Rothe. The technical contribution is due to Y. Reisch.
- The possible winner problem with uncertain weights and its variants, published in the conference paper [BRR+12], have been developed jointly with my coauthors D. Baumeister, M. Roos, J. Rothe, and L. Xia in equal parts. I contributed the results for the case of rational weights.
- The conference paper [RRS+14], which is extended by the journal article [RRS+15] has been written jointly with my coauthors A. Rey, J. Rothe, and H. Schadrack in equal parts.
- The model of FEN-hedonic games, published in the conference paper [LRR+15] has been developed by my coauthors J. Lang, A. Rey, J. Rothe, H. Schadrack, and myself in equal parts. The definition of Borda-like comparability functions and the complexity results of stability problems for the class of Borda-induced FEN-hedonic games are part of my contribution.
- The survey article [RS13] has been written jointly and in equal parts by J. Rothe and myself.