# Social Welfare in Algorithmic Mechanism Design Without Money

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# Social Welfare in Algorithmic Mechanism Design Without Money

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# Abstract

Social choice theory is concerned with collective decision making under different, possibly contrasting opinions and has been part of the core of society since ancient times. The goal is to implement some socially desired objective while at the same time accounting for the fact that people will act strategically, in order to manipulate the outcomes in their favor.

In this thesis, we consider the well-known objective of social welfare, i.e. the sum of individual utilities as the social objective and following the agenda of algorithmic mechanism design, we study how well our objectives can be approximated by mechanisms that prevent or predict the effects of the agents' strategic nature. We adopt two approaches; on one hand, we study truthful mechanisms and bound their approximation ratios and on the other hand, we study the effect of strategic play on non-truthful mechanisms, by bounding their price of anarchy. Our results provide worst-case guarantees for the performance of mechanisms in voting scenarios and resource allocation problems.

In the first part of the thesis, we consider the general social choice setting, where agents have unrestricted cardinal preferences over a finite set of outcomes and we study the capabilities and limitations of truthful mechanisms for the social welfare objective. We prove upper and lower bounds on the approximation ratio of natural classes of mechanisms, as well as the class of all mechanisms. In the second part, we bound the inefficiency of mechanisms for the one-sided matching problem. We study both truthful and non-truthful mechanisms and prove that some very well-known mechanisms in literature are asymptotically optimal among all mechanisms. Finally, in the last part of the thesis, we study social welfare maximization for the problem of allocating divisible items among agents and bound the price of anarchy of the Fisher market mechanism, a mechanism based on a fundamental market model.

# Resumé


Social choice-teori, som omhandler beslutningsregler for fællesskaber, hvor medlemmerne har divergerende meninger, har haft central betydning for samfundet siden oldtiden. Det generelle mål for teorien er at forstå hvornår og hvordan det er muligt at implementere et socialt ønskværdigt udfald under hensyntagen til at medlemmerne vil agere strategisk og forsøge at påvirke udfaldet i retning af deres egne preferencer.

I denne afhandling studerer vi maximering af social velfærd - summen af individuelle utilities - som er et velkendt mål. Vi forfølger en agenda fra teorien om algoritmisk mekanismedesign idet vi studerer hvor godt mekanismer der tager hensyn til individernes strategiske adfærd kan approksimere den maksimalt mulige sociale velfærd. Mere specifikt forfølger vi to varianter af dette spørgsmål: vi studerer hvor god approksimationen er for sandfærdige mekanismer, når deltagerne taler sandt, og hvor god den er for ikke-sandfærdige mekanismer i en Nash ligevægt (mekanismens såkaldte "price of anarchy").

I den første del af afhandlingen studerer vi den første variant af spørgsmålet for tilfældet af generelle kardinale præferencer over udfald og viser grænser for approksimation både for naturlige restringerede klasser af mekanismer og for alle mekanismer. I den anden del studerer vi begge varianter af spørgsmålet for parringsproblemet med ensidede præferencer. Vi viser at visse velkendte mekanismer er asymptotisk optimale med hensyn til approksimation. I den tredje del studerer vi den anden variant af spørgsmålet for allokering af delelige objekter til agenter med en mekanisme der bygger på den fundamentale Fisher model for markeder.

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*Aris Filos-Ratsikas,*
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# Contents

# Preface

My main area of research is algorithmic mechanism design without money and in particular approximate social welfare maximization in social choice, item assignment and resource allocation settings, when agents have unrestricted preferences over the possible outcomes. My contribution on these topics can be best summarized by the following papers.[1]

1. "Truthful Approximations to Range Voting"
   Aris Filos-Ratsikas and Peter Bro Miltersen. In *Proceedings of the 10th Conference on Web and Internet Economics (WINE '14), LNCS, Springer, pages 175-188, 2014.* (Chapter 2). [77]

2. "Social Welfare in One-sided matchings: Random Priority and beyond"
   Aris Filos-Ratsikas, Søren Kristoffer Stiil Frederiksen and Jie Zhang. In *Proceedings of the 7th Symposium on Algorithmic Game Theory (SAGT '14), LNCS, Springer, pages 1-12, 2014.* (Chapter 4).[79]

3. "Welfare Ratios for One-sided Matching Mechanisms"
   George Christodoulou, Aris Filos-Ratsikas, Søren Kristoffer Stiil Frederiksen, Paul W. Goldberg, Jie Zhang and Jinshan Zhang. *Manuscript*, 2015. (Chapter 5). [57]

4. "The Fisher Market Game: Equilibrium and Welfare"
   Simina Brânzei, Yiling Chen, Xiaotie Deng, Aris Filos-Ratsikas, Søren Kristoffer Stiil Frederiksen, and Jie Zhang. In *Proceedings of the 23rd AAAI Conference on Artificial Intelligence (AAAI 2014), AAAI Press, pages 587-593, 2014.* (Chapter 7). [43]

This thesis is comprised of three parts. The first part (Chapters 1 and 2) is concerned with the fundamental social choice/mechanism design setting and

[1] The order of authors is alphabetical.

how well (randomized) truthful mechanisms can approximate the social welfare of the most preferred outcome, when agents have unrestricted preferences. The second part (Chapters 3, 4 and 5) deals with the problem of assigning (indivisible) items to agents, also known as the one-sided matching problem, and studies the performance of (truthful and non-truthful) mechanisms with respect to the social welfare. Finally, the third part (Chapters 6 and 7), studies the allocation of divisible items under the objective of social welfare maximization.

In addition to the topics studied in this thesis which are mentioned above, during my Ph.D. studies, I worked on several other problems including fair division, kidney exchange and facility location, resulting in the following papers.

5. "The Adjusted Winner Procedure: Characterizations and Equilibria"
   Haris Aziz, Simina Brânzei, Aris Filos-Ratsikas and Søren Kristoffer Stiil Frederiksen. In *Proceedings of the International Joint Conference on Artificial Intelligence (IJCAI '15)*, to appear. [19]

6. "Facility location with double-peaked preferences"
   Aris Filos-Ratsikas, Minming Li, Jie Zhang and Qiang Zhang. In *Proceedings of the 29th AAAI International Conference on Artificial Intelligence (AAAI '15)*, AAAI Press, pages 893-899, 2015. [78]

7. "Randomized assignments for barter exchanges: Fairness vs Efficiency"
   Wenyi Fang, Aris Filos-Ratsikas, Søren Kristoffer Stiil Frederiksen, Pingzhong Tang and Song Zuo. *Manuscript*, 2015. [72]

8. "An improved 2-agent kidney exchange mechanism"
   Ioannis Caragiannis, Aris Filos-Ratsikas and Ariel D. Procaccia. In *Theoretical Computer Science*, To appear, 2015. [49]

9. "On the complexity of the consensus-halving problem"
   Aris Filos-Ratsikas, Søren Kristoffer Stiil Frederiksen, Paul W. Goldberg and Jie Zhang. *Manuscript*, 2015.

# Part I

# The Social Choice Setting

# Chapter 1

# Background

Societies are often faced with problems that must be solved collectively by their members. How should we make a decision that reflects many, often contrasting opinions? How should we aggregate people's choices into a joint decision? As Amartya Sen wrote in his 1988 Nobel Prize lecture [133]:

> *How can we find any rational basis for making such aggregative judgements, as "the society prefers this to that" or "the society should choose this over that" or "this is socially right"?*

This is the subject of *social choice theory*, formulated in its current form and popularized by the pioneering work of Kenneth Arrow [12] in 1951. The fundamental blocks of social choice theory, namely decision making under divergent preferences date back to ancient Greece and the works of Aristotle [11]. Crucial for formalizing and establishing those concepts into a solid theory were the works of J.C. Borda [62] and Marquis De Condorcet [63] in the 18th century, during the French revolution. Condorcet's observation (known as the *Condorcet Paradox*) is still one of the prominent examples used to demonstrate how majority rules, which are perhaps the most well-known voting rules, do not always provide a clear winner. Electing single representatives or comittees are classical examples of social choice problems; in fact most such problems can be interpreted as voting scenarios, where participants express their preferences through votes over a set of possible outcomes.

One answer to the question posed by Amartya Sen above was given (back in the 18th century) by the field of welfare economics and the seminal work of Jeremy Bentham [29], the so-called father of utilitarianism: society should care about "the total utility of a community". Welfare economics (in their original form) have been criticized on the basis of *interpersonal utility comparisons*. Critics, such as Robbins [127] argue that "no common demonitator of feelings is possible" and hence one should not seek to optimize utilitarian objectives, but rather look for efficiency on an individual level, such as Pareto efficiency. However, as we will discuss later, one can think of numerous examples where utilities have natural interpretations and aggregate utility is a reasonable measure of social efficiency. The term "social welfare" is used to denote exactly that; the sum of utilities of the members of the society.

It is conceivable that when faced with collective decisions, people will act *rationally* or *strategically* in order to manipulate the outcomes in their favour. For example, when presented with three possible outcomes and asked to rank them in order of preference, it would make sense for someone to misreport her ranking, if it was clear that her most preferred outcome would not be selected, in order to boost the chances of her second most preferred outcome. The question now becomes, "How should we design a system or an election to prevent or handle such selfish behaviour, given that we do not know the real preferences of people?". This is the topic of the field of *mechanism design* [96, 110, 116].

With some objective in mind (such as social welfare maximization), there are two different approaches to tackling the question above. The first one is to "get rid" of incentives altogether; to design mechanisms (i.e. functions that input preferences and output outcomes) that do not incentivize the participants to report anything but the truth. These mechanisms are called *truthful* or *strategyproof* or *incentive-compatible*, which in game-theoretic terms means that when interacting with such mechanisms, telling the truth is a dominant strategy equilibrium. The other approach would be to let participants strategize "freely" and predict the outcomes using the principles of game theory. In particular, the well-known Nash equilibrium [117] is an established notion of stability of a game. As designers, we could construct mechanisms that are not truthful but perform well (with respect to the objectives) in the equilibrium. We stretch the importance of adopting one of the approaches above: If we simply implement a function of the declared preferences of the participants, the actual outcome might be quite different from the socially desired one.

Of course, managing selfish behaviour does not come without a cost. Imposing the (quite demanding) constraint of truthfulness often renders the social objectives unachievable. The socially optimal outcome, which would be implementable if participants were not strategic, is no longer within reach when restrictions for acting truthfully are in place. What we can hope for, however, is a good *approximation*. In a seminal paper, Procaccia and Tennenholtz [123] coined the term *approximate mechanism design without money* to describe problems of that nature, when some objective is approximately optimized under the strict constraints of truthfulness. In this framework, we can design and compare mechanisms in terms of their *approximation ratio*, a worst-case guarantee for the performance of a mechanism, over all possible inputs of the problem. Similarly, when designing mechanisms that perform well in the equilibrium, the notion of efficiency is the *Price of Anarchy* [103], which bounds the performance of a mechanism in the worst Nash equilibrium, over all instances.

These are the two approaches that are used in this thesis, to evaluate and compare the performance of mechanisms with respect to the social welfare objective.

## 1.1 The setting

In the fundamental social choice/mechanism design setting, there is a finite set of agents $N = \{1, \dots, n\}$ and a finite set of outcomes $A = \{1, \dots, m\}$. Following the standard convention, in the first part of the thesis, we will refer to agents as *voters* and to outcomes as *candidates* or *alternatives*. Each voter $i$ has a private valuation function (or *valuation*) $u_i : A \to \mathbb{R}$, mapping candidates to real values. In the *unrestricted preference setting*, these valuation functions can be arbitrary. The preferences are unrestricted in the sense that there is no reason to a priori assume that some candidates are preferred to others, or that the nature of the problem imposes any restriction on the values voters assign to the candidates.[1] These valuation functions are to be interpreted as von Neumann-Morgenstern utilities, i.e, they are meant to encode orderings on lotteries over outcomes and follow the axioms of the von Neumann-Mongenstern expected utility theory [140]. We will sometimes refer to these valuation functions as *cardinal preferences*, because they express not only the preference orderings of individuals but also the intensity of the preferences, i.e. by *how much* a candidate is preferred to another candidate. Standardly, the function $u_i$ is considered well-defined only up to positive affine transformations. That is, we consider $x \to au_i(x) + b$, for $a > 0$ and any $b$, to be a different representation of $u_i$. Given this, we can fix a canonical representation of the valuation function; we will discuss the usual choices of representation later on in the chapter. Let $V_m$ denote the set of canonically represented valuation functions on $A = \{1, 2, \dots, m\}$. We will call the vector $\mathbf{u} = (u_1, u_2, \dots, u_n) \in {V_m}^n$ a *valuation profile*.

A (direct revelation) *mechanism* (without money) or a *social choice function*[2] is a function $M : V_m \to A$ mapping valuation profiles to candidates. Mechanisms can also be randomized and $M(\mathbf{u})$ is therefore in general a random map. Alternatively, instead of viewing a mechanism as a random map, we can view it as a map from ${V_m}^n$ to $\Delta_m$, the set of probability density functions on $\{1, \dots, m\}$.

[1] Examples of *restricted* or *structured* preference settings are single-peaked or single-crossing preferences and the preferences in the assignment setting that we will discuss in Part 2 of the thesis.

[2] Other names commonly used in literature are *voting rules* or *voting schemes*.

An important distinction that we will make throughout the thesis is that between ordinal mechanisms and general (cardinal) mechanisms. Informally, a mechanism is *ordinal* if the only information it uses is the *rankings* of candidates induced by the valuation functions. Formally,

**Definition 1.1** (Ordinal mechanism)**.** A mechanism $M$ is *ordinal* if it holds that $M(u_i, u_{-i}) = M(u'_i, u_{-i})$, for any voter $i$, any preference profile $\mathbf{u} = (u_i, u_{-i})$, and any valuation function $u'_i$ with the property that for all pairs of candidates $j, j'$, it is the case that $u_i(j) < u_i(j')$ if and only if $u'_i(j) < u'_i(j')$.

where, $u_{-i}$ denotes the valuation functions of all voters besides voter $i$. For example, the mechanism that first selects a voter uniformly at random and then elects her most preferred candidate is ordinal. A mechanism that is not necessarily ordinal is called *cardinal*.

An important class of mechanisms is that of *truthful* (or *strategyproof* or *incentive compatible*) mechanisms, i.e. mechanisms that do not incentivize voters to misreport their valuations. Formally,

**Definition 1.2** (Truthfulness)**.** A mechanism $M$ is *truthful* if for any voter $i \in \{1, \dots, n\}$, and all $\mathbf{u} = (u_i, u_{-i}) \in {V_m}^n$ and $\tilde{u}_i \in V_m$, we have $u_i(M(u_i, u_{-i})) \geq u_i(M(\tilde{u}_i, u_{-i}))$.

For randomized mechanisms, we assume that voters are expected utility maximizers and the corresponding notion is that of *truthfulness-in-expectation.*

**Definition 1.3** (Truthfulness-in-expectation)**.** A mechanism $M$ is *truthful-in-expectation* if for any voter $i \in \{1, \dots, n\}$, and all $\mathbf{u} = (u_i, u_{-i}) \in {V_m}^n$ and $\tilde{u}_i \in V_m$, we have $\mathbb{E}[u_i(M(u_i, u_{-i}))] \geq \mathbb{E}[u_i(M(\tilde{u}_i, u_{-i}))]$.

Throughout this thesis, we will use the term "truthful" to denote both truthful and truthful-in-expectation mechanisms; the distinction will be clear from the context or explicitly stated.

Truthfulness is a quite demanding, but also quite desirable property. A truthful mechanism elicits voters' true private information and does not motivate voters to strategize; they do not need to expend resources to find the best way to act within the rules of the system. Designing truthful mechanisms has been a subject of research for many years. An example of a truthful mechanism is the following: choose a voter and elect her most preferred candidate. This simple mechanism is called a *dictatorship*, since a single, distinguished voter, "the dictator", decides what the outcome should be. It is easily conceivable that such mechanisms are bad; an individual's opinion might not accurately reflect what society wants and hence one should look for more "reasonable" truthful mechanisms. Unfortunately, the celebrated Gibbard-Satterthwaite theorem [86, 131] states that when preferences are unrestricted, dictatorships are pretty much the only truthful mechanisms we can hope for.

**Theorem 1.1** (Gibbard-Satterthwaite [86, 131])**.** *Let $m \geq 3$ and let $M$ be a truthful, onto, deterministic mechanism. Then $M$ is a dictatorship.*

A mechanism $M$ is *onto* if for all candidates $j \in A$, there exists some valuation profile $\mathbf{u}$ such that $M(\mathbf{u}) = j$. In other words, no candidate is excluded from the election for all possible inputs. The fact that there must be at least three candidates is crucial for the theorem; for $m = 2$, performing a majority vote is a truthful mechanism.

*Remark* 1.1. The Gibbard-Satterthwaite theorem is often stated for *ordinal preferences*, i.e. preference orderings over the candidates and not cardinal preferences that we consider here. It is not hard to see that for deterministic mechanisms, this makes no difference, since every truthful mechanism is ordinal. To see this, consider two profiles $(u_i, u_{-i})$ and $(u'_i, u_{-i})$ that induce the same ordering of candidates for voter $i$ (but assign different numerical values) and let $M$ be a deterministic mechanism such that $M(u_i, u_{-i}) = a$ and $M(u'_i, u_{-i}) = b$. If voter $i$ prefers $a$ to $b$, then she would have an incentive to report $u_i$ instead of $u'_i$ in profile $(u'_i, u_{-i})$, violating truthfulness. Similarly, if she prefers $b$ to $a$ then then she would have an incentive to report $u'_i$ instead of $u_i$ in profile $(u_i, u_{-i})$. Therefore, it must be the case then that $a = b$ and the mechanism is ordinal.

Theorem 1.1 seems to eliminate all possibilites for "good" mechanisms in the unrestricted preference setting and for this reason a large body of research has diverted to structured settings, where the impossibility theorem does not hold and more "reasonable" truthful mechanisms can be designed. A prominent example is that of *single-peaked preferences*, where majority outcomes are available and *median voter schemes* achieve them in a truthful fashion. But even in the unrestricted preference setting, not all hope is lost. Crucially, Theorem 1.1 only holds for deterministic mechanisms, but not unlike in classic algorithm design, randomization can be used as a tool to create "better" mechanisms. Indeed, simply selecting the dictator uniformly at random already seems like a better choice; voters are treated fairly and it is much less likely that an individual whose opinions do not reflect those of society will be chosen.

So now the question becomes, what can we do with randomized truthful mechanisms? Is there a characterization similar to that of Theorem 1.1 for the randomized case? The short answer is "no", but there is an important line of work in this direction as well. First, we give a couple of definitions.

**Definition 1.4** (Unilateral mechanism [27, 87])**.** A *unilateral* mechanism $M$ is a mechanism for which there exists a single voter $i^*$ so that for all valuation profiles $(u_{i^*}, u_{-i^*})$ and any alternative valuation profile $u'_{-i^*}$ for the voters except $i^*$, we have $M(u_{i^*}, u_{-i^*}) = M(u_{i^*}, u'_{-i^*})$.

**Definition 1.5** (Duple mechanism [87])**.** A *duple* mechanism $M$ is an ordinal mechanism for which there exist two candidates $j_1^*$ and $j_2^*$ so that for all valuation profiles, $M$ elects all other candidates with probability 0.

Informally, unilateral mechanisms select some voter and then output a candidate at random as a function of that voter's reports only. Dictatorships are unilateral mechanisms; another example is a mechanism that selects a voter and then equiprobably elects one of her $k$ most preferred candidates for some $k \leq m$. Duple mechanisms assign zero election probability to all but two candidates. An example is a mechanism that eliminates all candidates except two, and then performs a majority vote on those two candidates. Note that while duple mechanisms are defined to be ordinal, unilateral mechanisms are not necessarily ordinal. We will provide an example of a non-ordinal unilateral mechanism in the next section.

Gibbard [87] extended the Gibbard-Satterthwaite theorem to the case of randomized ordinal mechanisms.[3]

**Theorem 1.2** (Gibbard [87])**.** *The truthful ordinal mechanisms are exactly the convex combinations of truthful unilateral ordinal mechanisms and truthful duple mechanisms.*

Theorem 1.2 is often interpreted as a negative result, similar in nature to Theorem 1.1. However, there is also an optimistic interpretation of the result that was suggested, e.g., by Barbera [25]: the class of randomized truthful mechanisms is quite rich and contains many arguably "reasonable" mechanisms, in contrast to dictatorships. This is true in particular because since we consider randomized mechanisms, one can choose the distinguished voters or the distuguished candidates at random, resulting in much more desirable mechanisms.

## 1.2 The quest for truthfulness

The characterization of truthful mechanisms of Theorem 1.2 does not apply to cardinal mechanisms. But as we mention before, unilateral mechanisms need not be ordinal; an example is the following mechanism for three candidates, the *quadratic lottery*, which was first introduced and proven to be truthful in [84] and later in [74].

*Mechanism* $Q_n$ (Quadratic Lottery [74, 84]). Select a voter uniformly at random, and let $\alpha$ be the valuation of her second most preferred candidate. Elect

[3] Theorem 1.2 actually holds only when the valuation functions are injective, i.e. voters have distinct values for all candidates. If we allow ties, hierarchical structures come into play [28, 88]. Since whether valuations have ties or not will be important for our results, we will discuss this matter in detail in the next chapters.

her most preferred candidate with probability $(4-\alpha^2)/6$, her second most preferred candidate with probability $(1+2\alpha)/6$ and her least preferred candidate with probability $(1-2\alpha+\alpha^2)/6$.

This could lead one to suspect that a similar characterization to Theorem 1.2 would also apply to cardinal mechanisms. In a followup paper, Gibbard [88], indeed proved a theorem along those lines, but interestingly, his result does *not* apply to truthful *direct* revelation mechanisms,[4] which are the topic of this thesis, but only to *indirect* revelation mechanisms with finite strategy space. Also, the restriction to finite strategy space (which is in direct contradiction to direct revelation) is crucial for the proof.

So, what do we know about cardinal truthful mechanisms in the unrestricted preference domain? Random dictatorships are the only truthful mechanisms that exist if we require unanimity [97], a guarantee that a candidate will be *definitely* elected if all voters rank him on top of their preference lists. Such a property however, restricts the space of truthful mechanisms significantly; removing it allows for many more mechanisms, as indicated also by Theorem 1.2, even in the ordinal case.

Over the years, there have been several attempts of characterizing truthful randomized mechanisms. Freixas [84] provides examples of non-ordinal mechanisms which are better (in the sense of economic efficiency) than all ordinal mechanisms, with respect to the voters' von Neumann-Morgenstern utilities. He also suggests that perhaps focusing on differentiable mechanisms is not too restrictive and advocates the use of the differentiable approach to mechanism design proposed by Laffont and Maskin [106] as a way of designing truthful mechanisms. The idea is that if mechanisms (as functions) are sufficiently differentiable, then incentive constraints can be written as differential equations and truthful mechanisms can be constructed. Barbera et al. [27] dispute this claim by proving that many interesting mechanisms are excluded from this class. Specifically, they prove that if a mechanism is twice continuously differentiable then it is a convex combination of unilaterals.[5] This precludes several natural truthful mechanisms, including duples. On the other hand, Barbera et al. [27] provide a sufficient condition for a mechanism to be truthful; if it is a probability mixture of unilaterals and cardinal "duples", where a duple in their definition means that the mechanism fixes two lotteries and selects a convex combination of the two. These contributions, however, fall short of a characterization. An added difficulty, highlighted in [27], is that when dealing with cardinal mechanisms, one should not expect that these mechanisms are decomposable into convex combinations of unilaterals or duples; integrals of these components might also come into play.

[4] A direct revelation mechanism is a mechanism where voters report a representation of their valuation functions once and the mechanism chooses an output based on those reports, i.e. there is no further interaction between the voters and the mechanism.

[5] Bogomolnaia [34] later relaxed the continuity assumption.

More recently, Feige and Tennenholtz [74] designed truthful cardinal unilateral mechanisms. In fact, for the mechanisms they design, it is the *unique* dominant strategy for a voter to report truthfully; they call those mechanisms *truthful dominant*. The authors provide a geometric characterization of a large class of truthful dominant mechanisms. The mechanisms proposed in [74] will be particularly useful for our purposes in the next chapters. Ehlers et al. [69] prove that every truthful continuous mechanism is ordinal but crucially, they consider a more general domain (referred to as a domain of "cardinal richness"), where affine transformations of utility functions are not simply different representations of the same function. In that sense, non-ordinal truthful mechanisms like those proposed in [84] or [74] are *not* continuous in their setting.

The bottomline of the discussion above is that a complete characterization of truthful randomized mechanisms, similar to Theorem 1.1 is not known. Such a result would be very useful for our investigations in Chapter 2 but perhaps more importantly, it would be a very crucial result in the fundamental social choice setting.

## 1.3 Social welfare and approximate solutions

So far, we have discussed structural properties of truthful mechanisms and the existing results in literature with respect to axiomatic aspects of truthfulness. But the original question of social choice theory is "How should we choose an outcome?" or alternatively using the terminology introduced earlier, "Which mechanism should we choose?". To answer that question, we need to choose an objective function of the voters' valuations to optimize. A natural choice that arguably represents the "socially optimal" solution is to maximize the sum of individual valuations, called the *social welfare*:

**Definition 1.6** (Social welfare)**.** Let $M$ be a mechanism and let $\mathbf{u}$ be a valuation profile. The *social welfare* of $M$ on $\mathbf{u}$ is defined as $\sum_{i=1}^{n} u_i(M(\mathbf{u}))$.

For randomized mechanisms, the objective is *expected* social welfare maximization and the definition is very similar.

If we knew the voters' private information (or if they reported their valuations truthfully), it would be easy to find the solution that maximizes the social welfare; simply elect a candidate that maximizes the quantity above. The challenge comes from the fact that real values are private and voters are strategic; if we maximize the sum of *reported* valuations instead, the outcome might be very different from the social optimum. As mentioned at the beginning of the chapter, there are two possible ways of dealing with this hurdle; either by designing truthful mechanisms, or by estimating the loss in efficiency from the result of the strategic interaction among the participants. In the first part of the thesis, we will only discuss the former; the latter will be considered in the second and third parts of the thesis.

Besides concepts like Pareto dominance and Pareto improvements, there do not seem to be many suggestions in the social choice literature of any well-defined quality measure that would enable us to rigorously compare mechanisms and in particular find the best. In fact, the pure social choice approach would be to evaluate mechanisms based on a set of properties that they satisfy and comparisons between them are indirect and often inconclusive. Fortunately, one of the main conceptual contributions of computer science to mechanism design in general is the suggestion of such a quality measure, namely the notion of the *worst-case approximation ratio* relative to some objective function. Indeed, a large part of the computer science literature on mechanism design (with or without money) is the construction and analysis of approximation mechanisms, following the agenda set by the seminal papers by Nisan and Ronen [119] for the case of mechanisms with money and Procaccia and Tennenholz [123] for the case of mechanisms without money. In particular, with the social welfare objective in mind, the approximation ratio of a (randomized) mechanism is defined as follows.

**Definition 1.7** (Approximation ratio)**.** The *approximation ratio* of a mechanism $M$ is the worst-case ratio (over all valuation profiles) of the expected social welfare of the mechanism on the profile over the maximum social welfare. That is,

$$\text{ratio}(M) = \inf_{\mathbf{u} \in {V_m}^n} \frac{\mathbb{E}[\sum_{i=1}^n u_i(M(\mathbf{u}))]}{\max_{j \in A} \sum_{i=1}^n u_i(j)}.$$

The approximation ratio provides a worst-case guarantee for the performance of a mechanism, very similarly to the notion of the approximation ratio in approximation algorithms or the competitive ratio in online algorithms. We remark here that the notion of the approximation ratio can be applied to any objective function under any constraint; one for instance could aim to approximately maximize the happiness of the least satisfied voter under the constraint that the used mechanism is ordinal. Throughout this thesis (with a noteable exception in Chapter 4), our objective will be to approximately maximize the social welfare under the constraint of truthfulness.

Since the framework descibed above was established, the approximation ratio has been adopted as the standard notion of efficiency of mechanisms in the computer science literature, in several subfields of mechanism design, like *facility location* [7, 81, 82, 108, 109, 123], *assignment problems* [5, 67, 79, 89], *kidney exchange* [15, 49], *peer selection* [8, 37, 80] and *fair division* [48, 59], to name a few.

## 1.4 Canonical representation

As we mentioned earlier, von Neumann-Morgenstern utility functions are well-defined up to positive affine transformations. The choice of transformation does not make a difference when arguing about utilities on an individual basis (for example truthfulness of unilateral mechanisms is independent of the transformation) but it is quite important when considering interpersonal objectives, like social welfare maximization. Applying different transformations to the utility functions would result in different voters having inputs of varying "importance". The standard approach in the literature is to fix some canonical representation, or *normalization* of utilities and there are two popular approaches, *unit-range* and *unit-sum*.

### Unit-range representation

If we use the same transformation for all utility functions, voters' utilities are fixed to lie in an interval with the least preferred and the most preferred candidates mapped to the endpoints of the interval respectively. In the *unit-range* representation, the chosen interval is $[0, 1]$, i.e. $u_i$ is such that $\max_j u_i(j) = 1$ and $\min_j u_i(j) = 0$. This is equivalent to the "zero-one rule" proposed by Hausman [94] for normalizing and comparing von Neumann-Morgenstern utilities. There are two assumptions associated with this normalization.

First, for the (normalized) social welfare objective to be a reasonable criterion of social efficiency, it has to be that the satisfaction that the members of society gain from their extreme choices being elected are (more or less) the same. One can think of scenarios where this is true. If the bus stop is far from A's house, that wouldn't be catastrophic compared to the scenario where the bus stop is far away from B's house, from B's perspective. Of course as Rawls [124] or Sen [134] would possibly argue, A might be quite different from B; maybe A has a walking disability. But such cases could always be true and then different normalizations (incorporating such differences) could be in place. The "uniform" normalizations that we study in this thesis in a sense apply to "the average person" and provide a solid mathematical framework for achieving reasonable results in settings without money. Secondly, unit-range assumes that voters are not indifferent between all the possible outcomes. That's a rather mild restriction, since participants that are indifferent between all choices could potentially be excluded from the election process.

We will assume that utility functions are normalized using the unit-range representation in Chapters 2, 4 and 5. The unit-range representation has been used in the literature in several subfields of mechanism design such as *voting* [26, 27, 84], *information elicitation* [74], *item allocation* [5, 57, 79, 144] or *fair division* [30].

### Unit-sum representation

The other standard normalization in the literature is that of *unit-sum*, where each voter has a total value of 1 for all the outcomes. i.e. $\sum_{j=1}^{m} u_i(j) = 1$. Contrary to unit-range, where voters were assumed to have equal values for the extreme outcomes, the assumption here is that each voter has an overall "satisfaction reserve" that gets distributed between her different choices. The unit-sum normalization has been applied for social welfare maximization in many settings without money including *fair division* and *cake-cutting* [42, 48, 58, 59, 99], *indivisible and divisible item allocation* [43, 57, 60, 75, 79, 89, 91] and *voting* [38] among others.

We will assume the unit-sum representation in Chapters 4, 5 and 7. Finally, let us remark that without any normalization, the bounds obtained from truthful mechanisms or due to strategic interaction are trivial and in fact, no truthful mechanism can outperform the mechanism that elects a candidate at random, without even looking at the valuation functions.

## 1.5 The utilitarian solution and interpersonal utility comparisons

The outcome that (exactly) maximizes the social welfare is often called the *utilitarian solution*[6] in the economics literature. As mentioned at the beginning of the chapter, the utilitarian objective is supported by the field of welfare economics, originating in the 18th century and the works of Jeremy Bentham. As Bentham himself said, "It is the greatest good to the greatest number of people which is the measure of right and wrong". Granted, the concept of utility that the classical welfare economists had in mind is probably different from what von Neumann and Morgenstern coined "utility" in their 1940s revolutionary work, but the principles are there; a decision is beneficial for society if it is beneficial for "most" of its members. A clear point of critisism to this viewpoint, suggested also by social choice and social politics pioneers John Rawls [124] and Amartya Sen [134] is that utilitarianism does not account for individuals that happen to disagree with the consensus and in that sense, such solutions might be quite unfair. On the other side of the spectrum, outcomes that maximize the minimum utility of any member of the society are called *egalitarian solutions*. These outcomes (also known as *maximin solutions*) can also be chosen as objective functions in the approximation ratio framework described above.

The main critisism against utilitarianism however has nothing to do with fairness issues; the objections posed by adversaries in the 1930s were in a sense more deeply philosophical. The main argument, as highlighted in the work of Robbins [127] (who is most often credited as the frontrunner in the race

[6] It i also referred to as *sum-rank*, e.g. see [132].

against utilitarianism) is that comparing utilities of different individuals is not a meaningful operation, since "no common denominator of feelings is possible". In other words, critics argue that there is no sensible unit of measurement that would enable utilities to be compared. This "warfare on welfare" introduced a "note of nihilism" [16, 134] on welfare economics and attention was shifted to Pareto efficiency as a measure of social efficiency. Pareto efficiency (also known as *the Pareto criterion*) states that an outcome is efficient if no other outcome can make someone happier, without making anyone more miserable.

But yet, we do compare utilities in our everyday lives. I would give my Rolling Stones concert ticket to my friend, because "he likes them more than I do", or as J.C. Lester [107] says: "To save a friend from breaking his leg we would usually consider it a small price to sustain a scratch ourselves".[7] More importantly, society seems to make choices based on implicit comparisons: The winner of an election is aimed to be the one that maximizes the overall satisfaction of the community. The location of a bus stop is selected to minimize the average distance from nearby houses. Many times, utilities have natural interpretations as *transportation costs* in facility location problems [7, 123], *compatibility probabilities* in kidney exchange pools [4, 72] or *profits* associated with production plans in a firm [69].

To take the argument one step further, in mechanism design *with* money (e.g. auctions), social welfare maximization is a very standard objective and in fact, the celebrated class of Vickrey-Clarke-Groves (VCG) mechanisms exactly optimizes social welfare (see [120] for details). There, participants are endowed with *quasi-linear utilities* and hence their levels of satisfaction can be expressed in *monetary terms*: a person's value for an outcome is the amount of money that she would be willing to pay to make that outcome come true. The idea is that money serves as a unit of utility and hence comparing and adding up utilities is well-defined. However, settings with money are quite susceptible to the same critisism: a wealthy heir might be willing to pay much more for a Monet than a striving painter, but that does not mean that the former appreciates art more than the latter. As Binmore [33] says: "But, who is to say that apples (or dollars) are the "appropriate" standard of comparison?".

Pareto efficiency on the other hand, the solution proposed by the new welfare economics, is not free of critisism either. As emphatically pointed out in [107], "The Pareto criterion disallows the welfare evaluation of changes from any status quo in an existing society, including one with slavery, if even one person objects to the change". In a more computer science oriented example, allocating all computational resources (such as storage space or memory) to a single person is, in the Pareto sense, an efficient outcome.[8]

[7] These types of examples are related to Harsanyi's theory [92] of *empathetic preferences* [33], see also [134].

[8] Another often quoted example was given in [134], where Sen argues that the burning of Rome by emperor Nero was in fact a Pareto efficient outcome.

The purpose of the discussion above is not so much to participate in the seemingly everlasting debate of whether interpersonal utility comparisons make sense (see [16, 33, 90, 134] for a detailed discussion on the subject), but to point out that there are many examples where such comparisons are meaningful and that statements like “it can never make sense to compare and add up utilities” are, at the very least, overly dismissive.

# Chapter 2

# Truthful approximations to range voting

In this chapter, we design truthful mechanisms that approximately maximize the social welfare, for the fundamental social choice/mechanism design setting defined in Chapter 1. As mentioned earlier, we will assume that valuation functions are canonically represented using the unit-range normalization, throughout the chapter. In light of this representation, a mechanism can be naturally interpreted as a *cardinal voting scheme* in which each voter provides a *ballot* giving each candidate $j \in M$ a numerical score between 0 and 1. A winning candidate is then determined based on the set of ballots. With this interpretation, the well-known *range voting scheme* is simply the determinstic mechanism that elects a socially optimal candidate in $\mathrm{argmax}_{j \in A} \sum_{i=1}^{n} u_i(j)$.[1] In particular, range voting has by construction an approximation ratio of 1 but it is not hard to see that it is not truthful.

## 2.1 Introduction

We obtain results for two different versions of the problem, depending on the number of candidates. First, for *many* candidates, we obtain asymptotic results as functions of the number $m$. Informally, such results bound the

[1] More precisely, range voting elects this candidate *if* ballots are reflecting the true valuation functions $u_i$. The optimal welfare is the benchmark that we compare to, assuming voters where being truthful, to quantify the loss in welfare because of applying truthfulness constraints. Since we only consider truthful mechanisms in this chapter, it is without loss of generality to define the inputs to be the true reports of the voters. In Chapter 5, we will redefine the inputs to account for strategic play.

approximation ratios of mechanisms when the number of candidates becomes large. Then, for $m = 3$, we obtain tighter, constant bounds (independent of then number of voters). Recall that the case of 3 candidates is the first case of interest, because for $m = 2$, Theorem 1.1 does not apply and in fact *majority*, a simple deterministic mechanism is both truthful and optimal (has approximation ratio 1) in our setting.

### Results for many candidates

Before stating our results, we mention for comparison the approximation ratio of some simple truthful mechanisms. Let *random-candidate* be the mechanism that elects a candidate uniformly at random, without looking at the reports. Let *random-favorite* be the mechanism that picks a voter uniformly at random and elects his favorite candidate; i.e., the (unique)[2] candidate to which he assigns valuation 1. Let *random-majority* be the mechanism that picks two candidates uniformly at random and elects one of them by a majority vote. It is not difficult to see that as a function of $m$ and assuming that $n$ is sufficiently large, *random-candidate* as well as *random-favorite* have approximation ratios $\Theta(m^{-1})$, so this is the trivial bound we want to beat. Interestingly, *random-majority* performs even worse, with an approximation ratio of $\Theta(m^{-2})$.

First, we exhibit a randomized truthful mechanism with an approximation ratio of $\Omega(m^{-3/4})$. The mechanism is the following very simple one:

*Mechanism* FRM. With probability 3/4, pick a candidate uniformly at random. With probability 1/4, pick a voter uniformly at random, and pick a candidate uniformly at random from his $\lfloor m^{1/2} \rfloor$ most preferred candidates.

Note that this mechanism is *ordinal*: Its behavior depends only on the *rankings* of the candidates on the ballots, not on their numerical scores. We know no asymptotically better truthful mechanism, even if we allow general (cardinal) mechanisms, i.e., mechanisms that can depend on the numerical scores in other ways.

We also show a negative result: For sufficiently many voters and any truthful ordinal mechanism, there is a valuation profile where the mechanism achieves at most an $O(m^{-2/3})$ fraction of the optimal social welfare in expectation. The negative result also holds for non-ordinal mechanisms that are *mixed-unilateral*, by which we mean mechanisms that elect a candidate based on the ballot of a single randomly chosen voter, i.e. they are convex combinations of unilateral mechanisms.

After the first version of the paper associated with this chapter, we became aware of another mechanism[3], that actually achieves an approximation of

[2]Throughout most of this chapter, we will assume that valuation functions are *injective*, i.e. they assign different values to all candidates. The reason for that choice and how that affects the results will be discussed in detail later in the chapter.

[3]This mechanism is due to Xinye Li, who has kindly agreed to allow me to include his

Table 2.1: Approximation ratios for $m$ candidates. * FRM. ** XL.

| **Approximation ratio** | Ratio | Upper bound |
|---|---|---|
| Ordinal + Mixed unilateral | $\Omega\left(m^{-3/4}\right)^{*}, \Omega\left(m^{-2/3}\right)^{**}$ | $O\left(m^{-2/3}\right)$ |
| Any mechanism | $\Omega\left(m^{-2/3}\right)$ | $O\left(\frac{\log\log m}{\log m}\right)$ |

$\Omega(m^{-2/3})$, closing the gap introduced by our results. The mechanism is the following one, which interestingly, is still mixed-unilateral and ordinal.

*Mechanism* XL. With probability 1/2, pick a voter uniformly at random and elect his most preferred candidate. With probability 1/2, pick a voter uniformly at random and pick a candidate uniformly at random from his $\lfloor m^{1/3} \rfloor$ most preferred candidates.

The important point of this result (in conjuction with our upper bound) is that cardinal information does not help improve the approximation ratio, when considering mixed-unilateral mechanisms.

Regarding general (cardinal) mechanisms, we obtain the following impossibility result. Any mechanism for $m$ alternatives and $n$ agents with $m \geq n^{\lfloor\sqrt{n}\rfloor+2}$, has approximation ratio $O(\log\log m/\log m)$. Interestingly, to obtain this result, we must allow for valuation functions to exhibit ties; i.e. map different alternatives to the same numbers in $[0,1]$. Extending the result to the "no-ties" setting seems like an interesting technical challenge; this is discussed in detail later in the chapter.

The results for many agents are summarized in Table 2.1.

### Results for three candidates

We get tighter bounds for the natural case of $m = 3$ and for this case, we also obtain separation results concerning the approximation ratios achievable by natural restricted classes of truthful mechanisms. Again, we first state the performance of the simple mechanisms defined above for comparison: For

result in my thesis. The mechanism is planned to be included in a journal paper that will extend the results of the conference paper associated with this chapter.

Table 2.2: Approximation ratios for 3 candidates.

| **Approximation ratio** | Ratio | Upper bound |
|---|---|---|
| Ordinal mixed unilateral | 0.610 | 0.611 |
| Ordinal | 0.616 | 0.641 |
| Any mechanism | 0.660 | 0.940 |

the case of $m = 3$, *random-favorite* and *random-majority* both have approximation ratios $1/2 + o(1)$ while *random-candidate* has an approximation ratio of 1/3. We show that for $m = 3$ and large $n$, the best mechanism that is *ordinal* as well as *mixed-unilateral* has an approximation ratio between 0.610 and 0.611. The best *ordinal* mechanism has an approximation ratio between 0.616 and 0.641. Finally, the best *mixed-unilateral* mechanism has an approximation ratio larger than 0.660. In particular, the best mixed-unilateral mechanism strictly outperforms all ordinal ones, even the non-unilateral ordinal ones. The mixed-unilateral mechanism that establishes this is a convex combination of the *quadratic-lottery* [74, 84] and *random-favorite*, that was defined above. For general mechanisms, we prove that no truthful mechanism has an approximation ratio larger than 0.94.

The results for 3 candidates are summarized in Table 2.2.

### Related work

As we mentioned in Chapter 1, the framework introduced by Nisan and Ronen [119] and Procaccia and Tennenholtz [123] allows for evaluating and comparing mechanisms in terms of their approximation ratio. Although such investigations have been very popular in structured settings, somewhat surprisingly, much less work has been done on the unrestricted preference setting, which is the topic of the first part of this thesis. Procaccia [122], in a paper conceptually very closely related to work of this chapter, used the characterization of Theorem 1.2 and proved upper and lower bounds on the approximation ratio achievable by ordinal mechanisms for various objective functions under general preferences. However, he only considered objective functions that can be defined ordinally (such as, e.g. the Borda count), and did in particular not consider approximating the optimal social welfare, which is the objective of this thesis.

Approximate social welfare maximization was considered by Boutilier *et al.* [38] in a very interesting paper closely related to the problem studied here, but crucially, their work did not consider incentives, i.e., they did not require truthfulness of the mechanisms in their investigations. Specifically, they bound the approximation ratio of ordinal mechanisms, where the need for approximate solutions is due to informational limitations and not constraints for truthful behaviour. On the other hand, truthfulness is the pivotal property in our approach and for some of our results, both the need for truthfulness and lack of information factor in the approximation ratio of the mechanisms.

Our investigations are very much helped by the work of Feige and Tennenholtz [74] and Freixas [84] who construct non-ordinal unilateral mechanisms. While the agenda in [74] was mechanisms for which the objective is information elicitation itself rather than mechanisms for approximate optimization of an objective function, the mechanisms suggested there still turn out to be useful for social welfare maximization. In particular, our construction establishing the gap between the approximation ratios for cardinal and ordinal mechanisms for three candidates is based on the *quadratic lottery* [74, 84].

## 2.2 Preliminaries

Recall that $V_m$ denotes the set of canonically represented valuation functions on $A = \{1, 2, \ldots, m\}$. Since valuation function are assumed to be canonically represented as unit-range, $V_m$ is the set of injective functions $u : A \to [0, 1]$ with the property that 0 as well as 1 are contained in the image of $u$.

We let $\mathrm{Mech}_{m,n}$ denote the set of truthful mechanisms for $n$ voters and $m$ candidates. Note that since mechanisms are randomized, they can be interpreted as maps from ${V_m}^n$ to $\Delta_m$. With that in mind, $\mathrm{Mech}_{m,n}$ is a convex subset of the vector space of all maps from ${V_m}^n$ to $\mathbb{R}^m$.

We next define some special classes of mechanisms, based on some standard properties.

**Definition 2.1** (Anonymity [25]). A mechanism $M$ is *anonymous* if the election probabilities do not depend on the names of the voters. Formally, given any permutation $\pi$ on $N$, and any $\mathbf{u} \in (V_m)^n$, we have $M(\mathbf{u}) = M(\pi \cdot \mathbf{u})$, where $\pi \cdot \mathbf{u}$ denotes the vector $(u_{\pi(i)})_{i=1}^n$.

**Definition 2.2** (Neutrality [25]). A mechanism $M$ is *neutral* if the election probabilities do not depend on the names of the candidates. Formally, given any permutation $\sigma$ on $A$, any $\mathbf{u} \in (V_m)^n$, and any candidate $j$, we have $M(\mathbf{u})_{\sigma(j)} = M(u_1 \circ \sigma, u_2 \circ \sigma, \ldots, u_n \circ \sigma)_j$.

We will call a mechanism that is anonymous and neutral, *symmetric*.[4] Note that symmetry implies the following property: if the valuations for two candidates are the same (up to permutations of the voters) then their election probabilities must be the same as well.

Recall the definition of unilateral mechanisms from Chapter 1. In this chapter, we will use the term *mixed-unilateral* for mechanisms that are convex combinations of unilateral truthful mechanisms. Mixed-unilateral mechanisms are quite attractive seen through the "computer science lens": They are mechanisms of *low query complexity*; consulting only a single randomly chosen voter, and therefore deserve special attention in their own right. We define the following classes of mechanisms:

- $\mathrm{Mech}^{\mathbf{U}}_{m,n}$: Mechanisms in $\mathrm{Mech}_{m,n}$ that are mixed-unilateral.
- $\mathrm{Mech}^{\mathbf{OU}}_{m,n}$: Mechanisms in $\mathrm{Mech}_{m,n}$ that are ordinal as well as mixed-unilateral.

[4] Note that *symmetry* here is different from the definition of symmetry in Chapter 3, where symmetry is a property implied by anonymity alone. We abuse the definitions slightly because the two different notions of symmetry are in two separate chapters.

We next give names to some specific important mechanisms.

- $U^q_{m,n} \in \mathrm{Mech}^{\mathbf{OU}}_{m,n}$: Pick a voter uniformly at random and elect uniformly at random a candidate among his $q$ most preferred candidates.
- $D^q_{m,n} \in \mathrm{Mech}^{\mathbf{O}}_{m,n}$, for $\lfloor n/2 \rfloor + 1 \leq q \leq n+1$: Pick two candidates uniformly at random and eliminate all other candidates. Then check for each voter which of the two candidates he prefers and give that candidate a "vote". If a candidate gets at least $q$ votes, she is elected. Otherwise, flip a coin to decide which of the two candidates is elected.

We let *random-favorite* be a nickname for $U^1_{m,n}$ and *random-candidate* be a nickname for $U^m_{m,n}$. We let *random-majority* be a nickname for $D^{\lfloor n/2 \rfloor+1}_{m,n}$. Note also that $D^{n+1}_{m,n}$ is just another name for *random-candidate*. Finally, recall the definition of *quadratic lottery* $Q_n$ from Chapter 1 (repeated here for completeness):

*Mechanism* $Q_n$ (Quadratic Lottery [74, 84]). Select a voter uniformly at random, and let $\alpha$ be the valuation of his second most preferred candidate. Elect his most preferred candidate with probability $(4-\alpha^2)/6$, his second most preferred candidate with probability $(1+2\alpha)/6$ and his least preferred candidate with probability $(1-2\alpha+\alpha^2)/6$.

As mentioned earlier, the mechanism was shown to be in $\mathrm{Mech}^{\mathbf{U}}_{3,n}$ by Freixas [84] and then later by Feige and Tennenholtz [74]. As we will see later, although many cardinal mechanisms were proposed in [74], quadratic lottery is particularly amenable to an approximation ratio analysis due to the fact that the election probabilities are quadratic polynomials.

Recall that ratio($M$) denotes the approximation ratio of a mechanism $M$. We let $r_{m,n}$ denote the best possible approximation ratio (achieved by any truthful mechanism) when there are $n$ voters and $m$ candidates. That is,

$$r_{m,n} = \sup_{M \in \mathrm{Mech}_{m,n}} \mathrm{ratio}(M).$$

Similarly, we let

$$r^{\mathbf{C}}_{m,n} = \sup_{M \in \mathrm{Mech}^{\mathbf{C}}_{m,n}} \mathrm{ratio}(M),$$

for $\mathbf{C}$ being either $\mathbf{O}$, $\mathbf{U}$ or $\mathbf{OU}$. We let $r_m$ denote the asymptotically best possible approximation ratio when the number of voters approaches infinity. That is, $r_m = \liminf_{n\to\infty} r_{m,n}$, and we also extend this notation to the restricted classes of mechanisms with the obvious notation $r^{\mathbf{O}}_m, r^{\mathbf{U}}_m$ and $r^{\mathbf{OU}}_m$.

## Characterization of ordinal symmetric mechanisms

The importance of neutral and anonymous mechanisms is apparent from the following simple lemma. Similar lemmas have been proven before in literature, in different settings [89, 95].

**Lemma 2.1.** *For all $M \in \mathrm{Mech}_{m,n}$, there is a $M' \in \mathrm{Mech}_{m,n}$ such that $M'$ is anonymous and neutral and such that* $\mathrm{ratio}(M') \geq \mathrm{ratio}(M)$*. Similarly, for all $M \in \mathrm{Mech}^{\mathbf{C}}_{m,n}$, there is $M' \in \mathrm{Mech}^{\mathbf{C}}_{m,n}$ so that $M'$ is anonymous and neutral and so that* $\mathrm{ratio}(M') \geq \mathrm{ratio}(M)$*, for $\mathbf{C}$ being either $\mathbf{O}$, $\mathbf{U}$ or $\mathbf{OU}$.*

*Proof.* Given any mechanism $M$, we can "anonymize" and "neutralize" $M$ by applying a uniformly chosen random permutation to the set of candidates and an independent uniformly chosen random permutation to the set of voters before applying $M$. This yields an anonymous and neutral mechanism $M'$ with at least a good an approximation ratio as $M$. Also, if $M$ is ordinal and/or mixed-unilateral, then so is $M'$. If $M$ is truthful, $M'$ will be truthful as well.[5] □

Lemma 2.1 makes the characterizations of the following theorem very useful.

**Theorem 2.1.** *The set of anonymous and neutral mechanisms in $\mathrm{Mech}^{\mathbf{OU}}_{m,n}$ is equal to the set of convex combinations of the mechanisms $U^q_{m,n}$, for $q \in \{1, \ldots, m\}$. Also, the set of anonymous and neutral mechanisms in $\mathrm{Mech}_{m,n}$ that can be obtained as convex combinations of duple mechanisms is equal to the set of convex combinations of the mechanisms $D^q_{m,n}$, for $q \in \{\lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2, \ldots, n, n+1\}$.*

*Proof.* A very closely related statement was shown by Barbera [24]. We sketch how to derive the theorem from that statement.

Barbera (in [24], as summarized in the proof of Theorem 1 in [25]) showed that the anonymous, neutral mechanisms in $\mathrm{Mech}^{\mathbf{OU}}_{m,n}$ are exactly the *point voting schemes* and that the anonymous, neutral mechanisms that are convex combinations of duple mechanisms are exactly *supporting size schemes*. A point voting scheme is given by $m$ real numbers $(a_j)_{j=1}^m$ summing to 1, with $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$. It picks a voter uniformly at random, and elects the candidate he ranks $k$th with probability $a_k$, for $k = 1, \ldots, m$. It is easy to see that the point voting schemes are exactly the convex combinations of $U^q_{m,n}$, for $q \in \{1, \ldots, m\}$. A supporting size scheme is given by $n+1$ real numbers $(b_i)_{i=0}^n$ with $b_n \geq b_{n-1} \cdots \geq b_0 \geq 0$, and $b_i + b_{n-i} = 1$ for $i \leq n/2$. It picks two different candidates $j_1, j_2$ uniformly at random and elects candidate $j_k, k = 1, 2$ with probability $b_{s_k}$ where $s_k$ is the number of voters than rank $j_k$ higher than $j_{3-k}$. It is easy to see that the supporting size schemes are exactly the convex combinations of $D^q_{m,n}$, for $q \in \{\lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2, \ldots, n+1\}$. □

The following corollary is immediate from Theorem 1.2 and Theorem 2.1.

**Corollary 2.1** (Symmetric mechanisms characterization)**.** *The ordinal, anonymous and neutral mechanisms in $\mathrm{Mech}_{m,n}$ are exactly the convex combinations*

[5] However, if $M$ is deterministic and truthful, $M'$ will be randomized and truthful-in-expectation.

*of the mechanisms $U^q_{m,n}$, for $q \in \{1, \ldots, m\}$ and $D^q_{m,n}$, for $q \in \{\lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2, \ldots, n\}$.*

### Scaling with the number of voters

We next present some lemmas that allow us to understand the asymptotic behavior of $r_{m,n}$ and $r^{\mathbf{C}}_{m,n}$ for fixed $m$ and large $n$, for $\mathbf{C}$ being either $\mathbf{O}$, $\mathbf{U}$ or $\mathbf{OU}$.

**Lemma 2.2.** *For any positive integers $n, m, k$, we have $r_{m,kn} \leq r_{m,n}$ and $r^{\mathbf{C}}_{m,kn} \leq r^{\mathbf{C}}_{m,n}$, for $\mathbf{C}$ being either $\mathbf{O}$, $\mathbf{U}$ or $\mathbf{OU}$.*

*Proof.* Suppose we are given any mechanism $M$ in $\mathrm{Mech}_{m,kn}$ with approximation ratio $\alpha$. We will convert it to a mechanism $M'$ in $\mathrm{Mech}_{m,n}$ with the same approximation ratio, hence proving $r_{m,kn} \leq r_{m,n}$. The natural idea is to let $M'$ simulate $M$ on the profile where we simply make $k$ copies of each of the $n$ ballots. More specifically, let $\mathbf{u}' = (u'_1, \ldots, u'_n)$ be a valuation profile with $n$ voters and $\mathbf{u} = (u_1, \ldots, u_{kn})$ be a valuation profile with $kn$ voters, such that $u_{ik+1} = u_{ik+2} = \ldots = u_{(i+1)k} = u'_{i+1}$, for $i = 0, \ldots, n-1$, where "=" denotes component-wise equality. Then let $M'(\mathbf{u}') = M(\mathbf{u})$. To complete the proof, we need to prove that if $M$ is truthful, $M'$ is truthful as well.

Let $\mathbf{u} = (u_1, \ldots, u_{kn})$ be the profile defined above for $kn$ voters and let $\mathbf{u}'$ be the corresponding $n$-voter profile. We will consider deviations of voters with the same valuation functions to the same misreported valuation function $\hat{u}$; without loss of generality, we can assume that these are voters $1, \ldots, k$. For ease of notation, let $\mathbf{u_{i+1}} = (u_{ik+1} = u_{ik+2} = \ldots = u_{(i+1)k})$ be a *block* of valuation functions, for $i = 0, \ldots, n-1$ and note that given this notation, we can write $\mathbf{u} = (\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}) = (u_1, \ldots, u_k, \mathbf{u_2}, \ldots, \mathbf{u_n})$. Let $v^* = \mathbb{E}[u_i(M(\mathbf{u}))]$. Now consider the profile $(\hat{u}, u_2, \ldots, u_k, \mathbf{u_2}, \ldots, \mathbf{u_n})$. By truthfulness, it holds that voter 1's expected utility in the new profile (and with respect to $u_1$) is at most $v^*$. Next, consider the profile $(\hat{u}, \hat{u}, u_3, \ldots, u_k, \mathbf{u_2}, \ldots, \mathbf{u_n})$ and observe that voter 2's utility from misreporting should be at most equal to her utility before misreporting, which is at most $v^*$. Continuing like this, we obtain the valuation profile $(\hat{u}, \hat{u}, \ldots, \hat{u}, \mathbf{u_2}, \ldots, \mathbf{u_n})$ in which the expected utility of voters $1, \ldots, k$ is at most $v^*$ and hence no deviating voter gains from misreporting. Now observe that the new profile $(\hat{u}, \hat{u}, \ldots, \hat{u}, \mathbf{u_2}, \ldots, \mathbf{u_n})$ corresponds to an $n$-voter profile $(\mathbf{\hat{u}_i}', \mathbf{u}'_{-\mathbf{i}}) = (\hat{u}'_1, u'_2, \ldots, u'_n)$ which is obtained from $\mathbf{u}'$ by a single miresport of voter 1. By the discussion above and the way $M'$ was constructed, voter 1 does not benefit from this misreport and since the misreported valuation function was arbitrary, $M'$ is truthful.

The same arguments can be employed for proving the lemma for $r^{\mathbf{C}}_{m,kn} \leq r^{\mathbf{C}}_{m,n}$, for $\mathbf{C}$ being either $\mathbf{O}$, $\mathbf{U}$ or $\mathbf{OU}$. □

**Lemma 2.3.** *For any $n, m$ and $k < n$, we have $r_{m,n} \geq r_{m,n-k} - \frac{km}{n}$. Also, $r^{\mathbf{C}}_{m,n} \geq r^{\mathbf{C}}_{m,n-k} - \frac{km}{n}$, for $\mathbf{C}$ being either $\mathbf{O}$, $\mathbf{U}$, or $\mathbf{OU}$.*

*Proof.* We construct a mechanism $M'$ in $\mathrm{Mech}_{m,n}$ from a mechanism $M$ in $\mathrm{Mech}_{m,n-k}$. The mechanism $M'$ simply simulates $M$ after removing $k$ voters, chosen uniformly at random and randomly mapping the remaining voters to $\{1,\ldots,n\}$. In particular, if $M$ is ordinal (or mixed-unilateral, or both) then so is $M'$. Suppose $M$ has approximation ratio $\alpha$. Consider running $M'$ on any profile where the socially optimal candidate has social welfare $w^*$. Note that $w^* \geq n/m$, since each voter assigns valuation 1 to some candidate. Ignoring $k$ voters reduces the social welfare of any candidate by at most $k$, so $M'$ is guaranteed to return a candidate with expected social welfare at least $\alpha(w^* - k)$. This is at least a $\alpha(1-k/w^*) \geq \alpha - \frac{km}{n}$ fraction of $w^*$. Since the profile was arbitrary, we are done. ☐

**Lemma 2.4** (Scaling lemma)**.** *For any $m, n \geq 2, \epsilon > 0$ and all $n' \geq (n-1)m/\epsilon$, we have $r_{m,n'} \leq r_{m,n} + \epsilon$ and $r^{\mathbf{C}}_{m,n'} \leq r^{\mathbf{C}}_{m,n} + \epsilon$, for $\mathbf{C}$ being either* **O***,* **U***, or* **OU***.*

*Proof.* If $n$ divides $n'$, the statement follows from Lemma 2.2. Otherwise, let $n^*$ be the smallest number larger than $n'$ divisible by $m$; we have $n^* < n' + n$. By Lemma 2.2, we have $r_{m,n^*} = r_{m,n}$. By Lemma 2.3, we have $r_{m,n^*} \geq r_{m,n'} - \frac{(n-1)m}{n^*}$. Therefore, $r_{m,n'} \leq r_{m,n} + \frac{(n-1)m}{n^*} \leq r_{m,n} + \frac{(n-1)m}{n'}$. The same arguments work for proving $r^{\mathbf{C}}_{m,n'} \leq r^{\mathbf{C}}_{m,n} + \epsilon$, for $\mathbf{C}$ being either **O**, **U**, or **OU**. ☐

In particular, Lemma 2.4 implies that $r_{m,n}$ converges to a limit as $n \to \infty$.

### Quasi-combinatorial valuation profiles

It will sometimes be useful to restrict the set of valuation functions to a certain finite domain $R_{m,k}$ for an integer parameter $k \geq m$. Specifically, we define:

$$R_{m,k} = \left\{ u \in V_m | u(A) \subseteq \{0, \frac{1}{k}, \frac{2}{k}, \ldots, \frac{k-1}{k}, 1\} \right\}$$

where $u(A)$ denotes the image of $u$. Given a valuation function $u \in R_{m,k}$, we define its *alternation number* $a(u)$ as

$$a(u) = \#\{j \in \{0,\ldots,k-1\} | [\frac{j}{k} \in u(A)] \oplus [\frac{j+1}{k} \in u(A)]\},$$

where $\oplus$ denotes exclusive-or. That is, the alternation number of $u$ is the number of indices $j$ for which exactly one of $j/k$ and $(j+1)/k$ is in the image of $u$. Since $k \geq m$ and $\{0,1\} \subseteq u(A)$, we have that the alternation number of $u$ is at least 2. We shall be interested in the class of valuation functions $C_{m,k}$ with minimal alternation number. Specifically, we define:

$$C_{m,k} = \{u \in R_{m,k} | a(u) = 2\}$$

and shall refer to such valuation functions as *quasi-combinatorial valuation functions*. Informally, the quasi-combinatorial valuation functions have all valuations as close to 0 or 1 as possible.

Note that a quasi-combinatorial valuation function $u$ is fully described by the value of $k$, together with a partition of $A$ into two sets $A_0$ and $A_1$, with $A_0$ being those candidates close to 0 and $A_1$ being those sets close to 1 together with a ranking of the candidates (i.e., a total ordering $\succ$ on $A$), so that all elements of $A_1$ are greater than all elements of $A_0$ in this ordering. Let the *type* of a quasi-combinatorial valuation function be the partition and the total ordering $(A_0, A_1, \succ)$. Then, a quasi-combinatorial valuation function is given by its type and the value of $k$. For instance, for $m = 3$ and candidates $a, b$ and $c$, one possible type is $(\{b\}, \{a, c\}, \{c \succ a \succ b\})$, and the quasi-combinatorial valuation function $u$ corresponding to this type for $k = 1000$ is $u(a) = 0.999$, $u(b) = 0$, $u(c) = 1$.

The following lemma will be very useful in later sections. It states that in order to analyze the approximation ratio of an ordinal and neutral mechanism, it is sufficient to understand its performance on quasi-combinatorial valuation profiles. The lemma formalizes the intuition that since only ordinal information is available, the worst-case will occur on "extreme" valuation profiles.

**Lemma 2.5** (Quasi-combinatorial Lemma)**.** *Let $M \in \mathrm{Mech}_{m,n}$ be ordinal and neutral. Then*

$$\mathrm{ratio}(M) = \liminf_{k\to\infty} \min_{\mathbf{u}\in(C_{m,k})^n} \frac{\mathbb{E}[\sum_{i=1}^n u_i(M(\mathbf{u}))]}{\sum_{i=1}^n u_i(1)}.$$

*Proof.* For a valuation profile $\mathbf{u} = (u_i)$, define $g(\mathbf{u}) = \frac{\mathbb{E}[\sum_{i=1}^n u_i(M(\mathbf{u}))]}{\sum_{i=1}^n u_i(1)}$. We show the following equations:

$$\begin{aligned}
\mathrm{ratio}(M) &= \inf_{\mathbf{u}\in V_m^n} \frac{\mathbb{E}[\sum_{i=1}^n u_i(M(\mathbf{u}))]}{\max_{j\in A}\sum_{i=1}^n u_i(j)} && (2.1)\\
&= \inf_{\mathbf{u}\in V_m^n} g(\mathbf{u}) && (2.2)\\
&= \liminf_{k\to\infty} \min_{\mathbf{u}\in(R_{m,k})^n} g(\mathbf{u}) && (2.3)\\
&= \liminf_{k\to\infty} \min_{\mathbf{u}\in(C_{m,k})^n} g(\mathbf{u}) && (2.4)
\end{aligned}$$

Equation (2.2) follows from the fact that since $M$ is neutral, it is invariant over permutations of the set of candidates, so there is a worst case instance (with respect to approximation ratio) where the socially optimal candidate is candidate 1. Equation (2.3) follows from the facts that (a) each $\mathbf{u} \in (V_m)^n$ can be written as $\mathbf{u} = \lim_{k\to\infty} \mathbf{v}_k$ where $(\mathbf{v}_k)$ is a sequence so that $\mathbf{v}_k \in (R_{m,k})^n$ and where the limit is with respect to the usual Euclidean topology (with the set of valuation functions being considered as a subset of a finite-dimensional

Euclidean space), and (b) the map $g$ is continuous in this topology (to see this, observe that the denominator in the formula for $g$ is bounded away from 0). Finally, equation (2.4) follows from the following claim:

$$\forall \mathbf{u} \in (R_{m,k})^n \ \exists \mathbf{u'} \in (C_{m,k})^n : g(\mathbf{u'}) \leq g(\mathbf{u}).$$

With $\mathbf{u} = (u_1, \ldots, u_n)$, we shall prove this claim by induction in $\sum_i a(u_i)$ (recall that $a(u_i)$ is the alternation number of $u_i$).

For the induction basis, the smallest possible value of $\sum_i a(u_i)$ is $2n$, corresponding to all $u_i$ being quasi-combinatorial. For this case, we let $\mathbf{u'} = \mathbf{u}$.

For the induction step, consider a valuation profile $\mathbf{u}$ with $\sum_i a(u_i) > 2n$. Then, there must be an $i$ so that the alternation number $a(u_i)$ of $u_i$ is strictly larger than 2 (and therefore at least 4, since alternation numbers are easily seen to be even numbers). Then, there must be $r, s \in \{2, 3, \ldots, k-2\}$, so that $r \leq s$, $\frac{r-1}{k} \notin u_i(A)$, $\{\frac{r}{k}, \frac{r+1}{k}, \ldots, \frac{s-1}{k}, \frac{s}{k}\} \subseteq u_i(A)$ and $\frac{s+1}{k} \notin u_i(A)$. Let $\tilde{r}$ be the largest number strictly smaller than $r$ for which $\frac{\tilde{r}}{k} \in u_i(A)$; this number exists since $0 \in u_i(A)$. Similarly, let $\tilde{s}$ be the smallest number strictly larger than $s$ for which $\frac{\tilde{s}}{k} \in u_i(A)$; this number exists since $1 \in u_i(A)$. We now define a valuation function $u^x \in V_m$ for any $x \in [\tilde{r} - r + 1; \tilde{s} - s - 1]$, as follows: $u^x$ agrees with $u_i$ on all candidates $j$ *not* in $u_i^{-1}(\{\frac{r}{k}, \frac{r+1}{k}, \ldots, \frac{s-1}{k}, \frac{s}{k}\})$, while for candidates $j \in u_i^{-1}(\{\frac{r}{k}, \frac{r+1}{k}, \ldots, \frac{s-1}{k}, \frac{s}{k}\})$ , we let $u^x(j) = u_i(j) + \frac{x}{k}$. Now consider the function $h : x \to g((u^x, u_{-i}))$, where $(u^x, u_{-i})$ denotes the result of replacing $u_i$ with $u^x$ in the profile $\mathbf{u}$. Since $M$ is ordinal, we see by inspection of the definition of the function $g$, that $h$ on the domain $[\tilde{r} - r + 1; \tilde{s} - s - 1]$ is a fractional linear function $x \to (ax + b)/(cx + d)$ for some $a, b, c, d \in \mathbb{R}$. As $h$ is defined on the entire interval $[\tilde{r} - r + 1; \tilde{s} - s - 1]$, we therefore have that $h$ is either monotonically decreasing or monotonically increasing in this interval, or possibly constant. If $h$ is monotonically increasing, we let $\tilde{\mathbf{u}} = (u^{\tilde{r}-r+1}, u_{-i})$, and apply the induction hypothesis on $\tilde{\mathbf{u}}$. If $h$ is monotonically decreasing, we let $\tilde{\mathbf{u}} = (u^{\tilde{s}-s-1}, u_{-i})$, and apply the induction hypothesis on $\tilde{\mathbf{u}}$. If $h$ is constant on the interval, either choice works. This completes the proof.

□

## 2.3 Results for many candidates

Next, we prove our results for the case of many candidates. We start from the approximation guarantees of two trutfhul mechanisms and then prove some upper bounds on the approximation ratio of some general classes of truthful mechanisms, as well as the class of all truthful mechanisms.

### Approximation guarantees

First, we will analyze the approximation ratio of the mechanism $FRM \in \text{Mech}^{\mathbf{OU}}_{m,n}$ that with probability 3/4 elects a uniformly random candidate and

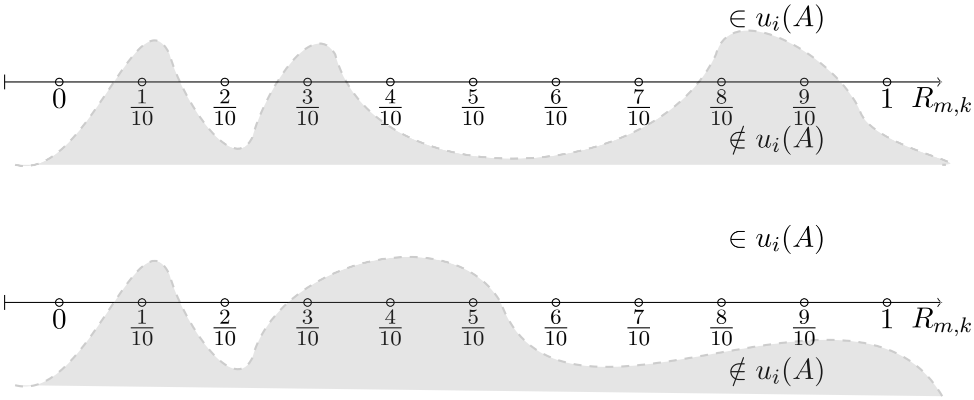


Figure 2.1: Example of the induction step of the proof of Lemma 2.5 for $m = 7$ and $k = 10$. Here, $r = 4$, $s = 7$, $\tilde{r} = 2$ and $\tilde{s} = 10$ and hence $x \in [-1, 2]$. The bottom figure depicts the induced profile when $h(x)$ is monotonely decreasing in $[-1, 2]$.

with probability 1/4 uniformly at random picks a voter and elects a candidate uniformly at random from the set of his $\lfloor m^{1/2} \rfloor$ most preferred candidates.

**Theorem 2.2.** *Let* $n \geq 2, m \geq 3$. *Let* $M = \frac{3}{4}U^m_{m,n} + \frac{1}{4}U^{\lfloor m^{1/2} \rfloor}_{m,n}$. *Then,* $\text{ratio}(M) \geq 0.37m^{-3/4}$.

*Proof.* For a valuation profile $\mathbf{u} = (u_i)$, we define

$$g(\mathbf{u}) = \frac{\mathbb{E}[\sum_{i=1}^n u_i(M(\mathbf{u}))]}{\sum_{i=1}^n u_i(1)}.$$

By Lemma 2.5, since $M$ is ordinal, it is enough to bound from below $g(\mathbf{u})$ for all $\mathbf{u} \in (C_{m,k})^n$ with $k \geq 1000(nm)^2$. Let $\epsilon = 1/k$. Let $\delta = m\epsilon$. Note that all functions of $\mathbf{u}$ map each alternative either to a valuation smaller than $\delta$ or a valuation larger than $1 - \delta$.

Since each voter assigns valuation 1 to at least one candidate, and since $M$ with probability 3/4 picks a candidate uniformly at random from the set of all candidates, we have $E[\sum_{i=1}^n u_i(M(\mathbf{u}))] \geq 3n/(4m)$. Suppose $\sum_{i=1}^n u_i(1) \leq 2m^{-1/4}n$. Then $g(\mathbf{u}) \geq \frac{3}{8}m^{-3/4}$, and we are done. So we shall assume from now on that

$$\sum_{i=1}^n u_i(1) > 2m^{-1/4}n. \tag{2.5}$$

Obviously, $\sum_{i=1}^n u_i(1) \leq n$. Since $M$ with probability 3/4 picks a candidate uniformly at random from the set of all candidates, we have that

$\mathbb{E}[\sum_{i=1}^{n} u_i(M(\mathbf{u}))] \geq \frac{3}{4m} \sum_{i,j} u_i(j)$. So if $\sum_{i,j} u_i(j) \geq \frac{1}{2} nm^{1/4}$, we have $g(\mathbf{u}) \geq \frac{3}{8} m^{-3/4}$, and we are done. So we shall assume from now on that

$$\sum_{i,j} u_i(j) < \frac{1}{2} nm^{1/4}. \tag{2.6}$$

Still looking at the fixed quasi-combinatorial $\mathbf{u}$, let a voter $i$ be called *generous* if his $\lfloor m^{1/2} \rfloor + 1$ most preferred candidates are all assigned valuation greater than $1 - \delta$. Also, let a voter $i$ be called *friendly* if he has candidate 1 among his $\lfloor m^{1/2} \rfloor$ most preferred candidates. Note that if a voter is neither generous nor friendly, he assigns to candidate 1 valuation at most $\delta$. This means that the total contribution to $\sum_{i=1}^{n} u_i(1)$ from such voters is less than $n\delta < 0.001/m$. Therefore, by equation (2.5), the union of friendly and generous voters must be a set of size at least $1.99 m^{-1/4} n$.

If we let $g$ denote the number of generous voters, we have $\sum_{i,j} u_i(j) \geq gm^{1/2}(1-\delta) \geq 0.999 gm^{1/2}$, so by equation (2.6), we have that $0.999 gm^{1/2} < \frac{1}{2} nm^{1/4}$. In particular $g < 0.51 m^{-1/4} n$. So since the union of friendly and generous voters must be a set of size at least a $1.99 m^{-1/4} n$ voters, we conclude that there are at least $1.48 m^{-1/4} n$ friendly voters, i.e. the friendly voters form at least a $1.48 m^{-1/4}$ fraction of the set of all voters. But this ensures that $U_{m,n}^{\lfloor m^{1/2} \rfloor}$ elects candidate 1 with probability at least $1.48 m^{-1/4}/m^{1/2} \geq 1.48 m^{-3/4}$. Then, $M$ elects candidate 1 with probability at least $0.37 m^{-3/4}$ which means that $g(\mathbf{u}) \geq 0.37 m^{-3/4}$, as desired. This completes the proof. □

Next, we will analyze the proof of the performance of the mechanism $XL \in \text{Mech}_{m,n}^{\mathbf{OU}}$ that with probability 1/2 uniformly at random picks a voter and elects his most preferred candidate and with probability 1/2 uniformly at random picks a voter and elects a candidate uniformly at random from the set of his $\lfloor m^{1/3} \rfloor$ most preferred candidates.

In order to prove the approximation ratio of the mechanism, we will restrict the valuation space further, using the nature of the mechanism. Recall that since the mechanism is ordinal, its worst-case approximation ratio is on quasi-combinatorial valuation profiles $C_{m,k}$. Recall the definition of the *type* of a quasi-combinatorial valuation function and the sets $A_0$ and $A_1$ from Section 2.2. We emphasize the association of the sets $A_0$ and $A_1$ with the utility function $u$ by denoting them $A_0^u$ and $A_1^u$ respectively. Now, for a utility function $u \in C_{m,k}$, define:

$$count(u) = |\{j \in M : j \in A_1^u\}| \quad \text{and} \quad rank(u,j) = \left|\{j' \in M : u(j') \geq u(j)\}\right|$$

In other words, $count(u)$ is the number of candidates for which a voter has valuation close to 1 and $rank(u,j)$ is the *rank* of candidate $j$ in a voter's ballot. Now for $k \geq m$, define the sets

$$\begin{aligned}
D^{(a)}_{m,k} &= \{u \in C_k : (count(u) \leq 2) \wedge (1 \in A^u_1)\} \\
D^{(b)}_{m,k} &= \Big\{u \in C_k : (count(u) = 1) \wedge \Big(rank(u,1) > \lfloor m^{1/3} \rfloor\Big)\Big\} \\
D^{(c)}_{m,k} &= \Big\{u \in C_k : count(u) = rank(u,1) = \lfloor m^{1/3} \rfloor + 1\Big\} \\
D_{m,k} &= D^{(a)}_{m,k} \cup D^{(b)}_{m,k} \cup D^{(c)}_{m,k}.
\end{aligned}$$

Intuitively, the set $D^{(a)}_{m,k}$ contains valuation functions for which candidate 1 (who will be used as the social optimum by neutrality of the studied mechanism) is assigned valuation close to 1 but there are at most 2 candidates that are valued highly in total. The set $D^{(b)}_{m,k}$ contains valuations for which only one candidate is assigned a high value and candidate 1 is outranked by at least $\lfloor m^{1/3} \rfloor$ candidates. Finally, the set $D^{(c)}_{m,k}$ contains valuation functions for which candidate 1 is valued highly, but she is outranked by $\lfloor m^{1/3} \rfloor$ candidates, who are thus also valued highly.

We will prove the following lemma about the mechanism $XL$. The lemma states that when bounding the approximation ratio of the mechanism, it suffices to look at a *subset* of quasi-combinatorial profiles, given by the set $D_{m,k}$.

**Lemma 2.6.** *Let $M = \frac{1}{2}U^1_{m,n} + \frac{1}{2}U^{\lfloor m^{1/3} \rfloor}_{m,n}$. Then*

$$\text{ratio}(M) = \liminf_{k \to \infty} \min_{\mathbf{u} \in (D_{m,k})^n} \frac{\mathbb{E}[\sum_{i=1}^n u_i(M(\mathbf{u}))]}{\sum_{i=1}^n u_i(1)}.$$

*Proof.* Similarly to the proof of Lemma 2.5 and since the mechanism $M$ is neutral, for a valuation profile $\mathbf{u} = (u_i)$, we define $g(\mathbf{u}) = \frac{\mathbb{E}[\sum_{i=1}^n u_i(M(\mathbf{u}))]}{\sum_{i=1}^n u_i(1)}$. By the proof of Lemma 2.5, and since $M$ is ordinal, we know that

$$\text{ratio}(M) = \liminf_{k \to \infty} \min_{\mathbf{u} \in (C_{m,k})^n} g(\mathbf{u}),$$

and hence we need to prove that

$$\liminf_{k \to \infty} \min_{\mathbf{u} \in (C_{m,k})^n} g(\mathbf{u}) = \liminf_{k \to \infty} \min_{\mathbf{u} \in (D_{m,k})^n} g(\mathbf{u}).$$

In order to prove that, it suffices to prove

$$\forall \mathbf{u} \in (C_{m,k})^n \; \exists \mathbf{u}' \in (D_{m,k})^n : g(\mathbf{u}') \leq g(\mathbf{u}).$$

The idea is similar to the one used in Lemma 2.5: we will start from an arbitrary quasi-combinatorial profile and inductively "turn" voters' valuation functions into functions in the set $D_{m,k}$, while arguing that the approximation ratio cannot increase. Specifically, consider the profile $\mathbf{u} \in (C_{m,k})^n \backslash (D_{m,k})^n$ and let $i$ be an agent such that $u_i \in C_{m,k} \backslash D_{m,k}$. We will construct a valuation function $v_i \in D_{m,k}$ such that the following conditions are satisfied:

1. $M(v_i, u_{-i}) = M(u_i, u_{-i})$;

2. $1 \in A_1^{u_i} \Rightarrow 1 \in A_1^{v_i}$;

3. $j \in A_0^{u_i} \Rightarrow j \in A_0^{v_i}$ for any $j \neq 1$.

Note that since we are considering the case when $k \to \infty$ and since the argument will be applied inductively, similarly to the proof of Lemma 2.5, satisfying Conditions 1-3 suffices to prove the lemma. Now define

$$S(u) = \Big(\{j \in M : rank(u, j) = 1\}, \{j \in M : 1 < rank(u, j) \leq \lfloor m^{1/3} \rfloor\}\Big)$$

and observe that Condition 1 can alternatively be written as $S(u_i) = S(v_i)$, since the mechanism is invariant to rearranging the ranks of the first $\lfloor m^{1/3} \rfloor$ candidates, while keeping the most preferred candidate fixed. We will consider two cases for the valuation functions $v_i \in D_{m,k}$:

First, if $rank(u_i, 1) \leq \lfloor m^{1/3} \rfloor$ then

- If $rank(u_i, 1) = 1$, let $v_i$ be such that $j \in A_0^{v_i}$ for every $j \neq 1$ and $rank(v_i, j) = rank(u_i, j)$. Then $v_i \in D_{m,k}^{(a)}$ and Conditions 1-3 are satisfied.

- If $rank(u_i, 1) > 1$, let $j'$ be the candidate with $rank(u_i, j') = 1$. Then let $v_i$ be such that $S(v_i) = S(u_i)$ and $j \in A_1^{v_i}$ if and only if $j \in \{1, j'\}$. Then $v_i \in D_{m,k}^{(a)}$ and Conditions 1-3 are satisfied.

Now, if $rank(u_i, 1) > \lfloor m^{1/3} \rfloor$ then

- If $1 \in A_1^{u_i}$, then we have that for any $j$ with $rank(u_i, j) \leq \lfloor m^{1/3} \rfloor$, $j \in A_1$. Therefore, we can let $v_i$ be such that $(v_i) = S(u_i)$, $rank(v_i, 1) = \lfloor m^{1/3} \rfloor + 1$ and $j \in A_1^{v_i}$ if and only if $j = 1$ or $rank(u_i, j) \leq \lfloor m^{1/3} \rfloor$. Then $v_i \in D_{m,k}^{(c)}$ and Conditions 1-3 are satisfied.

- If $1 \in A_0^{u_i}$, let $j'$ be the candidate with $rank(u_i, j') = 1$. Then let $v_i$ be such that $S(v_i) = S(u_i)$ and $j \in A_1^{v_i}$ if and only if $j = j'$. Then $v_i \in D_{m,k}^{(b)}$ and Conditions 1-3 are satisfied.

This completes the proof.

□

We are now ready to bound the approximation ratio of the mechanism.

**Theorem 2.3.** *Let* $M = \frac{1}{2} U_{m,n}^1 + \frac{1}{2} U_{m,n}^{\lfloor m^{1/3} \rfloor}$. *Then,* $\text{ratio}(M) = \Omega(m^{-2/3})$.

*Proof.* Let $\mathbf{u}$ be any valuation profile; by Lemma 2.6, we can assume without loss of generality that $\mathbf{u} \in (D_{m,k})^n$. Similarly to Lemma 2.6 and since the mechanism $M$ is neutral, we can assume that candidate 1 is the social optimum and define

$$g(\mathbf{u}) = \frac{\mathbb{E}[\sum_{i=1}^{n} u_i(M(\mathbf{u}))]}{\sum_{i=1}^{n} u_i(1)}.$$

The minimum ratio will be given by the value of $g(\mathbf{u})$ for $\mathbf{u} \in (D_{m,k})^n$ as $k$ approaches infinity.

Recall the definition of the sets $D_{m,k}^{(a)}, D_{m,k}^{(b)}$ and $D_{m,k}^{(c)}$ and let the $N_1$, $N_2$ and $N_3$ be the sets of voters that have valuations in $D_{m,k}^{(a)}, D_{m,k}^{(b)}$ and $D_{m,k}^{(c)}$ respectively. Let $a = |N_1|$, $b = |N_2|$ and $c = |N_3|$. Furthermore, for ease of notation, let $T$ be the random variable that denotes the voter chosen by the mechanism, let $W$ be the variable denoting the winning candidate and let $F$ be the event that mechanism $U_{m,n}^{1}$ is run. Finally, let

$$X = \lim_{k\to\infty} \frac{\sum_{i=1}^{n} u_i(W)}{\sum_{i=1}^{n} u_i(1)}.$$

Then it holds that:

$$\begin{aligned}
\mathbb{E}[X] \;\; &\geq \;\; \Pr\left[(T \in N_1) \wedge (\neg F) \wedge (W = 1)\right] \\
&+ \;\; \Pr[(T \in N_2) \wedge F] \cdot \mathbb{E}[X|(T \in N_2) \wedge F] \\
&+ \;\; \Pr[(T \in N_3) \wedge \neg F] \cdot \mathbb{E}[X|(T \in N_3) \wedge \neg F]
\end{aligned}$$

We will bound each term of the sum individually. For the first term, since we are calculating the probability that an agent in $N_1$ is selected and candidate 1 is elected under $U_{m,n}^{\lfloor m^{1/3} \rfloor}$, we have

$$\Pr[(T \leq a) \wedge (\neg F) \wedge (W = 1)] = \frac{1}{2} \cdot \frac{a}{n} \cdot \frac{1}{\lfloor m^{1/3} \rfloor} = \frac{a}{2n\lfloor m^{1/3} \rfloor}.$$

For the second term, if we let $Y_1 = \Pr[(T \in N_2) \wedge F] \cdot \mathbb{E}[X|(T \in N_2) \wedge F]$, and $w_i$ denote the (unique) candidate $w$ for which $w \in A_1^{u_i}$, we have

$$\begin{aligned}
Y &= \frac{1}{2}\cdot\frac{b}{n}\cdot\mathbb{E}[X|(T\in N_2\wedge F] \\
&= \frac{1}{2}\cdot\frac{b}{n}\cdot\frac{1}{b}\sum_{i\in N_2}\mathbb{E}[X|(T=i)\wedge F] \\
&= \frac{1}{2}\cdot\frac{1}{n}\cdot\sum_{i\in N_2}\frac{\sum_{j=1}^{n}|w_i\in A_1^{u_j}|}{a+c} \qquad (2.7)\\
&\geq \frac{1}{2}\cdot\frac{1}{n}\sum_{i\in N_2}\frac{\sum_{j\in N_2}|w_i\in A_1^{u_j}|}{a+c} \\
&= \frac{1}{2n(a+c)}\sum_{w=1}^{m-1}|\{i\in N_2 : w\in A_1^{u_i}\}|^2 \qquad (2.8)\\
&\geq \frac{1}{2}\cdot\frac{b}{n}\cdot\frac{1}{b}\cdot\frac{1}{a+c}\cdot\frac{b^2}{m-1} \qquad (2.9)\\
&= \frac{b^2}{2n(m-1)(a+c)}.
\end{aligned}$$

In the above calculation, for Equality 2.7, note that a valuation function in $D_{m,k}^{(b)}$ is such that only one candidate is in set $A_1$ . Since we are calculating the expectation given that mechanism $U_{m,n}^1$ is run, given that agent $i$ is selected, the winner is always the single candidate $w$ with $u_i(w)=1$. Hence, we only need to sum up over these candidates. The equality follows from the fact that voters in $D_{m,k}^{(b)}$ have candidate 1 in $A_0$. Equality 2.8 follows from rearranging the terms and Inequality 2.9 follows from the generalized mean inequality and the fact that

$$\sum_{w=1}^{m-1}|\{i\in N_2 : w\in A_1^{u_i}\}| = b.$$

Using very similar arguments (and substituting $|w_i\in A_1^{u_j}|$ by a term of the form $\sum_{w:w\in A_1^{u_i}}|w\in A_1^{u_j}|$), we can prove that

$$\Pr[(V\in N_3)\wedge\neg F]\ \cdot\mathbb{E}[X|(V\in N_3)\wedge\neg F]\geq\frac{c^2m^{1/3}}{2n(m-1)(a+c)}$$

Therefore, we have that

$$\mathbb{E}[X]\geq\frac{a}{2nm^{1/3}}+\frac{b^2}{2n(m-1)(a+c)}+\frac{c^2m^{1/3}}{2n(m-1)(a+c)},$$

and hence the approximation ratio of $M$ is asymptotically

$$\text{ratio}(M)=\Omega\left(\frac{a}{nm^{1/3}}+\frac{b^2}{nm(a+c)}+\frac{c^2m^{1/3}}{nm(a+c)}\right).$$

Since $a+b+c=n$, it has to be the case that at least one of $a, b$ and $c$ is asymptotically $\Omega(n)$. If $a=\Omega(n)$, then $\text{ratio}(M)=\Omega(m^{-1/3})$ and we are done. Similarly, if $c=\Omega(n)$, it holds that $\text{ratio}(M)=\Omega(m^{-2/3})$ and we are done. Assume from now on that $b=\Omega(n)$ and that $a=o(n)$ and $c=o(n)$. Then, we have

$$\text{ratio}(M)=\Omega\left(\frac{a}{nm^{1/3}}+\frac{n}{m(a+c)}+\frac{c^2m^{1/3}}{nm(a+c)}\right).$$

If $a=\Omega(a+c)$, then it holds that

$$\text{ratio}(M)=\Omega\left(\frac{a}{nm^{1/3}}+\frac{n}{ma}\right)=\Omega(m^{-2/3})$$

and we are done. Similarly, if $c=\Omega(a+c)$ then it holds that

$$\text{ratio}(M)=\Omega\left(\frac{n}{mc}+\frac{c}{nm^{-2/3}}\right)=\Omega(m^{-2/3}).$$

This completes the proof.

□

## Upper bounds

We next show our first negative result. We show that any convex combination of (not necessarily ordinal) unilateral and duple mechanisms performs poorly.

**Theorem 2.4.** *Let $m\geq 20$ and let $n=m-1+g$ where $g=\lfloor m^{2/3}\rfloor$. For any mechanism $M$ that is a convex combination of unilateral and duple mechanisms in $\text{Mech}_{m,n}$, we have $\text{ratio}(M)\leq 5m^{-2/3}$.*

*Proof.* Let $k=\lfloor m^{1/3}\rfloor$. By applying the same proof technique as in the proof of Lemma 2.1, we can assume that $M$ can be decomposed into a convex combination of mechanisms $M_\ell$, with each $M_\ell$ being anonymous as well as neutral, and each $M_\ell$ either being a mechanism of the form $D^q_{m,n}$ for some $q$ (by Theorem 2.1), or a mechanism that applies a truthful one-voter neutral mechanism $U$ to a voter chosen unformly at random.

We now describe a single profile for which any such mechanism $M_\ell$ performs badly. Let $A_1, .., A_g$ be a partition of $\{1,\ldots,kg\}$ with $k$ candidates in each set. The bad profile has the following voters:

- For each $i\in\{1,\ldots,m-1\}$ a voter that assigns 1 to candidate $i$, 0 to candidate $m$ and valuations smaller than $1/m^2$ to the rest.
- For each $j\in\{1,\ldots,g\}$ a vote that assigns valuations strictly bigger than $1-1/m^2$ to members of $A_j$, valuation $1-1/m^2$ to $m$, and valuations smaller than $1/m^2$ to the rest.

Note that the social welfare of candidate $m$ is $(1 - 1/m^2)g$ while the social welfares of the other candidates are all smaller than $2 + 1/m$. Thus, the conditional expected approximation ratio given that the mechanism does not elect $m$ is at most $(2+1/m)/(1-1/m^2)g \leq 3m^{-2/3}$. We therefore only need to estimate the probability that candidate $m$ is elected. For a mechanism of the form $D^q_{m,n}$, candidate $m$ is chosen with probability at most $2/m$, since such a mechanism first eliminates all candidates but two and these two are chosen uniformly at random.

For a mechanism that picks a voter uniformly at random and applies a truthful one-agent neutral mechanism $U$ to the ballot of this voter, we make the following claim: Conditioned on a particular voter $i^*$ being picked, the conditional probability that $m$ is chosen is at most $1/(r+1)$, where $r$ is the number of candidates that outrank $m$ on the ballot of voter $i$. Indeed, if candidate $m$ was chosen with conditional probability strictly bigger than $1/(r+1)$, she would be chosen with strictly higher probability than some other candidate $j^*$ who outranks $m$ on the ballot of voter $i^*$. But if so, since $U$ is neutral, voter $i$ would increase his utility by switching $j^*$ and $m$ on his ballot, as this would switch the election probabilities of $j^*$ and $m$ while leaving all other election probabilities the same. This contradicts that $U$ is truthful. Therefore, our claim is correct. This means that candidate $m$ is chosen with probability at most $1/m+(g/m)\cdot(1/k) \leq 1/m+m^{2/3}/(m(m^{1/3}-1)) \leq 2m^{-2/3}$, since $m \geq 20$.

We conclude that on the bad profile, the expected approximation ratio of any mechanism $M_\ell$ in the decomposition is at most $3m^{-2/3} + 2m^{-2/3} = 5m^{-2/3}$. Therefore, the expected approximation ratio of $M$ on the bad profile is also at most $5m^{-2/3}$. □

**Corollary 2.2.** *For all $m$, and all sufficiently large $n$ compared to $m$, any mechanism $M$ in $\mathrm{Mech}^{\mathbf{O}}_{m,n} \cup \mathrm{Mech}^{\mathbf{U}}_{m,n}$ has approximation ratio $O(m^{-2/3})$.*

*Proof.* Combine Theorem 1.2, Lemma 2.4 and Theorem 2.4. □

Corollary 2.2 provides an upper bound on the approximation ratio of a general class of mechanisms that contains all ordinal mechanisms as well as many cardinal mechanisms. The next question is how well one can do with (general) cardinal mechanisms, which are not mixed-unilateral. We will prove a bound that applies to the class of all truthful mechanisms next; note that the result assumes a departure from our setting and requires that the image of the valuation functions has ties. We will discuss this requirement in detail later in the chapter. The proof of the following theorem is based on Lemma 4.6 that we will state and prove in Chapter 4.

**Theorem 2.5.** *Let $M'$ be any truthful voting mechanism for $n$ voters and $m$ candidates, with $m \geq n^{\lfloor \sqrt{n} \rfloor + 2}$, in the setting with ties. The approximation ratio of $M'$ is $O(\log\log m/\log m)$.*

*Proof.* In Chapter 4, we will prove a related upper bound for the one-sided matching problem.[6] The bound corresponds to an upper bound on the approximation ratio of any truthful mechanism $M$ in the general setting with ties. This is because there is a reduction to the general setting with ties from the setting of the one-sided matching problem.[7]

In the one-sided matching problem, there is a set of $n$ agents and a set of $k$ items and each agent $i$ has a valuation function $v_i : [k] \to [0,1]$ mapping items to real values in the unit interval. Similarly to our definitions, these functions are injective and both 0 and 1 are in their image. A mechanism $M$ on input a valuation profile $\mathbf{v} = (v_1, ..., v_n)$ outputs a *matching* $M(\mathbf{v})$, i.e. an allocation of items to agents such that each agent receives at most one item. Let $M_i(\mathbf{v})$ be the item allocated to agent $i$. For convenience, we will refer to this problem as the *matching setting* and to our problem as the *general setting*.

The reduction works as follows. Let $\mathbf{v} = (v_1, ..., v_n)$ be a valuation profile of the matching setting. We will construct a valuation profile $\mathbf{u} = (u_1, ..., u_n)$ of the general setting that will correspond to $\mathbf{v}$. Let each outcome of the matching setting correspond to a candidate in the general setting. For every agent $i$ and every item $j$ let $u_i(L) = v_i(j)$ for each candidate $L \in A$ that corresponds to a matching in which item $j$ is allocated to agent $i$. Note that the number of candidates is $n^k$ and a bound for the matching setting implies a bound for the general setting. Specifically, the $O(1/\sqrt{n})$ bound proved in Lemma 4.6 translates to a $O(\log\log m/\log m)$ upper bound. □

## 2.4 Results for three candidates

In this section, we consider the special case of three candidates, $m = 3$. To improve readability, we shall denote the three candidates by $a, b$ and $c$, rather than by 1,2 and 3.

When the number of candidates $m$ as well as the number of voters $n$ are small constants, the exact values of $r^{\mathbf{O}}_{m,n}$ and $r^{\mathbf{OU}}_{m,n}$ can be determined. We first give a clean example, and then describe a general method.

**Proposition 2.1.** *For all $M \in \mathrm{Mech}^{\mathbf{O}}_{3,3}$, we have* $\mathrm{ratio}(M) \leq 2/3$.

*Proof.* By Lemma 2.1, we can assume that $M$ is anonymous and neutral. Let $a \succ_i b$ denote the fact that voter $i$ ranks candidate $a$ higher than $b$ in his ballot. Let a *Condorcet* profile be any valuation profile with $a \succ_1 b \succ_1 c$, $b \succ_2 c \succ_2 a$ and $c \succ_3 a \succ_3 b$. Since $M$ is neutral and anonymous, by symmetry, $M$ elects each candidate with probability 1/3. Now, for some small $\epsilon > 0$, consider the Condorcet profile where $u_1(b) = \epsilon$, $u_2(c) = \epsilon$ and $u_3(a) = 1 - \epsilon$. The socially optimal choice is candidate $a$ with social welfare $2 - \epsilon$, while the

[6] The proof will actually be for the setting of $n$ agents and $n$ items but it can be easily adapted to work when the number of items is $\lfloor \sqrt{n} \rfloor + 2$.

[7] We will discuss this reduction explicitly in Chapter 3.

other candidates have social welfare $1+\epsilon$. Since each candidate elected with probability 1/3, the expected social welfare is $(4+\epsilon)/3$. By making $\epsilon$ suffciently small, the approximation ratio on the profile is arbitrarily close to 2/3. □

Note that the same proof works even if the mechanism is not truthful, i.e. the loss in welfare is due to informational limitations. With a case analysis and some pain, it can be proved by hand that *random-majority* achieves an approximation ratio of at least 2/3 on any profile with three voters and three candidates. Together with Proposition 2.1, this implies that $r^{\mathbf{O}}_{3,3} = \frac{2}{3}$.

Rather than presenting the case analysis, we describe a general method for how to exactly and mechanically compute $r^{\mathbf{O}}_{m,n}$ and $r^{\mathbf{OU}}_{m,n}$ and the associated optimal mechanisms for small values of $m$ and $n$. The key is to apply *Yao's principle* [141] and view the construction of a randomized mechanism as devising a strategy for Player I in a two-player zero-sum game $G$ played between Player I, the mechanism designer, who picks a mechanism $M$ and Player II, the adversary, who picks an input profile $\mathbf{u}$ for the mechanism, i.e., an element of $(V_m)^n$. The payoff to Player I is the approximation ratio of $M$ on $\mathbf{u}$. Then, the value of $G$ is exactly the approximation ratio of the best possible randomized mechanism. In order to apply the principle, the computation of the value of $G$ has to be tractable. In our case, Theorem 2.1 allows us to reduce the strategy set of Player I to be finite while Lemma 2.5 allows us to reduce the strategy set of Player II to be finite. This makes the game into a matrix game, which can be solved to optimality using linear programming. The details follow.

For fixed $m, n$ and $k > 2m$, recall that the set of quasi-combinatorial valuation functions $C_{m,k}$ is the set of valuation functions $u$ for which there is a $j$ so that

$$u(A) = \left\{0, \frac{1}{k}, \frac{2}{k}, \ldots, \frac{m-j-1}{k}\right\} \cup \left\{\frac{k-j+1}{k}, \frac{k-j+2}{k}, \ldots, \frac{k-1}{k}, 1\right\}.$$

Recall the definition of the *type* of a quasi-combinatorial valuation function defined in Section 2.2. We see that for any fixed value of $m$, there is a finite set $T_m$ of possible types. In particular, we have $|T_3| = 12$. Let $\eta : T_m \times \mathbb{N} \to C_{m,k}$ be the map that maps a type and an integer $k$ into the corresponding quasi-combinatorial valuation function.

For fixed $m, n$, consider the following matrices $G$ and $H$. The matrix $G$ has a row for each of the mechanisms $U^q_{m,n}$ for $q = 1, \ldots, m$, while the matrix $H$ has a row for each of the mechanisms $U^q_{m,n}$ for $q = 1, \ldots, m$ as well as for each of the mechanisms $D^q_{m,n}$, for $q = \lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2, \ldots, n$. Both matrices have a column for each element of $(T_m)^n$. The entries of the matrices are as follows: Each entry is indexed by a mechanism $M \in \mathrm{Mech}_{m,n}$ (the row index) and by a type profile $\mathbf{t} \in (T_m)^n$ (the column index). We let that entry be

$$c_{M,\mathbf{t}} = \lim_{k\to\infty} \frac{\mathbb{E}[\sum_{i=1}^n u_i(M(\mathbf{u}^k))]}{\max_{j\in A} \sum_{i=1}^n u_i^k(j)},$$

where $u_i^k = \eta(t_i, k)$. Informally, we let the entry be the approximation ratio of the mechanism on the quasi-combinatorial profile of the type profile indicated in the column and with $1/k$ being "infinitisimally small". Note that for the mechanisms at hand, despite the fact that the entries are defined as a limit, it is straightforward to compute the entries symbolically, and they are rational numbers. We now have the following lemma.

**Lemma 2.7.** *The value of $G$, viewed as a matrix game with the row player being the maximizer, is equal to $r^{\mathbf{OU}}_{m,n}$. The value of $H$ is equal to $r^{\mathbf{O}}_{m,n}$. Also, the optimal strategies for the row players in the two matrices, viewed as convex combinations of the mechanisms corresponding to the rows, achieve those ratios.*

*Proof.* We only show the statement for $r^{\mathbf{O}}_{m,n}$, the other proof being analogous. For fixed $k$, consider the matrix $H^k$ defined similarly to $H$, but with entries

$$c_{M,\mathbf{t}} = \frac{\mathbb{E}[\sum_{i=1}^n u_i(M(\mathbf{u}^k))]}{\max_{j\in A}\sum_{i=1}^n u_i^k(j)},$$

where $u_i^k = \eta(t_i, k)$. Viewing $H^k$ as a matrix game, a mixed strategy of the row player can be interpreted as a convex combination of the mechanisms corresponding to the rows, and the expected payoff when the column player responds with a particular column $\mathbf{t}$ is equal to the approximation ratio of $J$ on the valuation profile $(\eta(t_i, k))_i$. Therefore, the value of the game is the worst case approximation ratio of the best convex combination, among profiles of the form $(\eta(t_i, k))_i$ for a type profile $\mathbf{t}$. By Lemma 2.1, $r^{\mathbf{O}}_{m,n}$ is determined by the best available anonymous and neutral ordinal mechanism. By Corollary 2.1, the anonymous and neutral ordinal mechanisms are exactly the convex combinations of the $U^q_{m,n}$ and the $D^q_{m,n}$ mechanisms for various $q$. Given any particular convex combination yielding a mechanism $K$, by Lemma 2.5, its worst case approximation ratio is given by

$$\liminf_{k\to\infty}\min_{\mathbf{u}\in(C_{m,k})^n}\frac{\mathbb{E}[\sum_{i=1}^n u_i(K(\mathbf{u}))]}{\sum_{i=1}^n u_i(a)},$$

which is equal to

$$\liminf_{k\to\infty}\min_{\mathbf{u}\in(C_{m,k})^n}\frac{\mathbb{E}[\sum_{i=1}^n u_i(K(\mathbf{u}^k))]}{\max_{j\in A}\sum_{i=1}^n u_i^k(j)},$$

since $K$ is neutral. This means that no mechanism can have an approximation ratio better than the limit of the values of the games $H^k$ as $k$ approaches infinity. By continuity of the value of a matrix game as a function of its entries, this is equal to the value of $H$. Therefore, $r^{\mathbf{O}}_{m,n}$ is at most the value of $H$. Now consider the mechanism $M$ defined by the optimal strategy for the row player in the matrix game $H$. As the entries of $H_k$ converge to the

Table 2.3: Approximation ratios for $n$ voters.

| **n/Approximation ratio** | $r^{\mathbf{O}}_{3,n}$ | $r^{\mathbf{OU}}_{3,n}$ |
|---|---|---|
| 2 | 2/3 | 2/3 |
| 3 | 2/3 | 105/171 |
| 4 | 2/3 | 5/8 |
| 5 | 6407/9899 | 34/55 |

Table 2.4: Mixed-unilateral ordinal mechanisms for $n$ voters.

| $\mathbf{n}/M$ | $U^1_{3,n}$ | $U^2_{3,n}$ | $U^3_{3,n}$ |
|---|---|---|---|
| 2 | 1/3 | 2/3 | 0 |
| 3 | 9/19 | 10/19 | 0 |
| 4 | 1/2 | 1/2 | 0 |
| 5 | 5/11 | 6/11 | 0 |

entries of $H$ as $k \to \infty$, we have that for any $\epsilon > 0$, and sufficiently large $k$, the strategy is also an $\epsilon$-optimal strategy for $H_k$. Since $\epsilon$ is arbitrary, we have that ratio($M$) is at least the value of $H$, completing the proof. □

When applying Lemma 2.7 for concrete values of $m, n$, one can take advantage of the fact that all mechanisms corresponding to rows are anonymous and neutral. This means that two different columns will have identical entries if they correspond to two type profiles that can be obtained from one another by permuting voters and/or candidates. This makes it possible to reduce the number of columns drastically. After such a reduction, we have applied the theorem to $m = 3$ and $n = 2, 3, 4$ and 5, computing the corresponding optimal approximation ratios and optimal mechanisms. The ratios are given in Table 2.3. The mechanisms achieving the ratios are shown in Table 2.4 and Table 2.5. These mechanisms are in general not unique. Note in particular that a different approximation-optimal mechanism than *random-majority* was found in $\text{Mech}^{\mathbf{O}}_{3,3}$.

Table 2.5: Ordinal mechanisms for $n$ voters.

| $\mathbf{n}/M$ | $U^1_{3,n}$ | $U^2_{3,n}$ | $U^3_{3,n}$ | $D^{\lfloor n/2\rfloor+1}_{n,3}$ | $D^{\lfloor n/2\rfloor+2}_{n,3}$ | $D^{\lfloor n/2\rfloor+3}_{n,3}$ |
|---|---|---|---|---|---|---|
| 2 | 4/100 | 8/100 | 0 | 88/100 | \| | \| |
| 3 | 47/100 | 0 | 0 | 53/100 | 0 | \| |
| 4 | 0 | 0 | 0 | 1 | 0 | \| |
| 5 | 3035/9899 | 0 | 0 | 3552/9899 | 3312/9899 | 0 |

We now turn our attention to the case of three candidates and arbitrarily many voters. In particular, we shall be interested in $r_3^{\mathbf{O}} = \liminf_{n\to\infty} r_{3,n}^{\mathbf{O}}$ and $r_3^{\mathbf{OU}} = \liminf_{n\to\infty} r_{3,n}^{\mathbf{OU}}$. By Lemma 2.4, we in fact have $r_3^{\mathbf{O}} = \lim_{n\to\infty} r_{3,n}^{\mathbf{O}}$ and $r_3^{\mathbf{OU}} = \lim_{n\to\infty} r_{3,n}^{\mathbf{OU}}$.

We present a family of ordinal and mixed-unilateral mechanisms $M_n$ with $\operatorname{ratio}(M_n) > 0.610$. In particular, $r_3^{\mathbf{OU}} > 0.610$. The coefficents $c_1$ and $c_2$ were found by trial-and-error; we present more information about how later.

**Theorem 2.6.** *Let $c_1 = \frac{77066611}{157737759} \approx 0.489$ and $c_2 = \frac{80671148}{157737759} \approx 0.511$. Let $M_n = c_1 \cdot U_{m,n}^1 + c_2 \cdot U_{m,n}^2$. For all $n$, we have* $\operatorname{ratio}(M_n) > 0.610$.

*Proof.* By Lemma 2.5, we have that

$$\operatorname{ratio}(M_n) = \liminf_{k\to\infty} \min_{\mathbf{u}\in(C_{3,k})^n} \frac{\mathbb{E}[\sum_{i=1}^n u_i(M_n(\mathbf{u}))]}{\sum_{i=1}^n u_i(a)}.$$

Recall the definition of the set of *types* $T_3$ of quasi-combinatorial valuation functions on three candidates and the definintion of $\eta$ preceding the proof of Lemma 2.7. From that discussion, we have

$$\liminf_{k\to\infty} \min_{\mathbf{u}\in(C_{m,k})^n} \frac{\mathbb{E}[\sum_{i=1}^n u_i(M_n(\mathbf{u}))]}{\sum_{i=1}^n u_i(a)} = \min_{\mathbf{t}\in(T_3)^n} \liminf_{k\to\infty} \frac{\mathbb{E}[\sum_{i=1}^n u_i(M_n(\mathbf{u}))]}{\sum_{i=1}^n u_i(a)},$$

where $u_i = \eta(t_i, k)$. Also recall that $|T_3| = 12$. Since $M_n$ is anonymous, to determine the approximation ratio of $M_n$ on $\mathbf{u} \in (C_{m,k})^n$, we observe that we only need to know the value of $k$ and the *fraction* of voters of each of the possible 12 types. In particular, fixing a type profile $\mathbf{t} \in (C_{m,k})^n$, for each type $\lambda \in T_3$, let $x_\lambda$ be the fraction of voters in $\mathbf{u}$ of type $\lambda$. For convenience of notation, we identify $T_3$ with $\{1, 2, \ldots, 12\}$ using the scheme depicted in Table 2.6. Let $w_j = \lim_{k\to\infty} \sum_{i=1}^n u_i(i)$, where $u_i = \eta(t_i, k)$, and let $p_j = \lim_{k\to\infty} \Pr[E_j]$, where $E_j$ is the event that candidate $j$ is elected by $M_n$ in an election with valuation profile $\mathbf{u}$ where $u_i = \eta(t_i, k)$. We then have

$$\liminf_{k\to\infty} \frac{\mathbb{E}[\sum_{i=1}^n u_i(M_n(\mathbf{u}))]}{\sum_{i=1}^n u_i(a)} = \frac{p_a \cdot w_a + p_a \cdot w_b + p_c \cdot w_c}{w_a}.$$

Also, from Table 2.6 and the definition of $M_n$, we see:

$$\begin{aligned}
w_a &= n(x_1 + x_2 + x_3 + x_4 + x_5 + x_9) \\
w_b &= n(x_1 + x_5 + x_6 + x_7 + x_8 + x_{11}) \\
w_c &= n(x_4 + x_7 + x_9 + x_{10} + x_{11} + x_{12}) \\
p_a &= (c_1 + c_2/2)(x_1 + x_2 + x_3 + x_4) + (c_2/2)(x_5 + x_6 + x_9 + x_{10}) \\
p_b &= (c_1 + c_2/2)(x_5 + x_6 + x_7 + x_8) + (c_2/2)(x_1 + x_2 + x_{11} + x_{12}) \\
p_c &= (c_1 + c_2/2)(x_9 + x_{10} + x_{11} + x_{12}) + (c_2/2)(x_3 + x_4 + x_7 + x_8)
\end{aligned}$$

Thus we can establish that ratio$(M_n) > 0.610$ for all $n$, by showing that the *quadratic program*

$$\begin{aligned} \text{minimize} \quad & (p_a \cdot w_a + p_b \cdot w_b + p_c \cdot w_c) - 0.610 w_a \\ \text{subject to} \quad & x_1 + x_2 + \cdots + x_{12} = 1, \\ & x_1, x_2, \ldots, x_{12} \geq 0 \end{aligned} \tag{2.10}$$

where $w_a, w_b, w_c, p_a, p_b, p_c$ have been replaced with the above formulae using the variables $x_i$, has a strictly positive minimum (note that the parameter $n$ appears as a multiplicative constant in the objective function and can be removed, so there is only one program, not one for each $n$). This was established rigorously by solving the program symbolically in Maple by a facet enumeration approach[8] (the program being non-convex), which is easily feasible for quadratic programs of this relatively small size. □

We next present a family of ordinal mechanisms $M'_n$ with ratio$(M'_n) >$ 0.616. In particular, $r_3^{\mathbf{O}} > 0.616$. The coefficents $c_1$ and $c_2$ defining the mechanism were again found by trial-and-error; we present more information about how later.

**Theorem 2.7.** *Let* $c'_1 = 0.476, c'_2 = 0.467$ *and* $d = 0.057$ *and let* $M'_n = c'_1 \cdot U^1_{3,n} + c'_2 U^2_{3,n} + d \cdot D^{\lfloor n/2 \rfloor + 1}_{m,n}$. *Then* ratio$(M'_n) > 0.616$ *for all* $n$.

*Proof.* The proof idea is the same as in the proof of Theorem 2.6. In particular, we want to reduce proving the theorem to solving quadratic programs. The fact that we have to deal with the $D^{\lfloor n/2 \rfloor + 1}_{m,n}$, i.e., *random-majority*, makes this task slightly more involved. In particular, we have to solve many programs rather than just one. We only provide a sketch, showing how to modify the proof of Theorem 2.6.

As in the proof of Theorem 2.6, we let $w_j = \lim_{k\to\infty} \sum_{i=1}^n u_i(i)$, where $u_i = \eta(t_i, k)$. The expressions for $w_a, w_b$ and $w_c$ as functions of the variables $x_i$ remain the same as in that proof. Also, we let $p_j = \lim_{k\to\infty} \Pr[E_j]$, where $E_j$ is the event that candidate $j$ is elected by $M'_n$ in an election with valuation profile $\mathbf{u}$ where $u_i = \eta(t_i, k)$. We then have

$$\begin{aligned} p_a &= (c'_1 + c'_2/2)(x_1 + x_2 + x_3 + x_4) + (c'_2/2)(x_5 + x_6 + x_9 + x_{10}) + d \cdot q_a(\mathbf{t}) \\ p_b &= (c'_1 + c'_2/2)(x_5 + x_6 + x_7 + x_8) + (c'_2/2)(x_1 + x_2 + x_{11} + x_{12}) + d \cdot q_b(\mathbf{t}) \\ p_c &= (c'_1 + c'_1/2)(x_9 + x_{10} + x_{11} + x_{12}) + (c'_2/2)(x_3 + x_4 + x_7 + x_8) + d \cdot q_c(\mathbf{t}) \end{aligned}$$

where $q_j(\mathbf{t})$ is the probability that *random-majority* elects candidate $j$ when the type profile is $\mathbf{t}$. Unfortunately, this quantity is not a linear combination of the $x_i$ variables, so we do not immediately arrive at a quadratic program.

However, we can observe that the values of $q_j(\mathbf{t}), j = a, b, c$ depend only on the outcome of the three pairwise majority votes between $a, b$ and $c$, where

[8]The code for the quadratic solver can be found at `http://pastebin.com/j9hQ8EQd`.

Table 2.6: Variables for types of quasi-combinatorial valuation functions with $\epsilon$ denoting $1/k$.

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ | $x_{10}$ | $x_{11}$ | $x_{12}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| A | 1 | 1 | 1 | 1 | $1-\epsilon$ | $\epsilon$ | 0 | 0 | $1-\epsilon$ | $\epsilon$ | 0 | 0 |
| B | $1-\epsilon$ | $\epsilon$ | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | $1-\epsilon$ | $\epsilon$ |
| C | 0 | 0 | $\epsilon$ | $1-\epsilon$ | 0 | 0 | $1-\epsilon$ | $\epsilon$ | 1 | 1 | 1 | 1 |

the majority vote between, say, $a$ and $b$ has three possible outcomes: $a$ wins, $b$ wins, or there is a tie. In particular, there are 27 possible outcomes of the three pairwise majority votes. To show that

$$\min_{\mathbf{t}\in(T_3)^n} \liminf_{k\to\infty} \frac{\mathbb{E}[\sum_{i=1}^n u_i(M'_n(\mathbf{u}))]}{\sum_{i=1}^n u_i(a)} > 0.616,$$

where $u_i = \eta(t_i, k)$, we partition $(T_3)^n$ into 27 sets according to the outcomes of the three majority votes of an election with type profile $\mathbf{t}$ and show that the inequality holds on all 27 sets in the partition. We claim that on each of the 27 sets, the inequality is equivalent to a quadratic program. Indeed, each $q_j(\mathbf{t})$ is now a constant, and the constraint that the outcome is as specified can be expressed as a linear constraint in the $x_i$'s and added to the program. For instance, the condition that $a$ beats $b$ in a majority vote can be expressed as $x_1 + x_2 + x_3 + x_4 + x_9 + x_{10} > 1/2$ while $a$ ties $c$ can be expressed as $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1/2$. Except for the fact that these constraints are added, the program is now constructed exactly as in the proof of Theorem 2.6. Solving[9] the programs confirms the statement of the theorem. □

We next show that $r_3^{\mathbf{OU}} \leq 0.611$ and $r_4^{\mathbf{O}} \leq 0.641$. By Lemma 2.4, it is enough to show that $r_{3,n^*}^{\mathbf{OU}} \leq 0.611$ and $r_{3,n^*}^{\mathbf{O}} \leq 0.641$ for some fixed $n^*$. Therefore, the statements follow from the following theorem.

**Theorem 2.8.** $r_{3,23000}^{\mathbf{OU}} \leq \frac{32093343}{52579253} < 0.611$ *and* $r_{3,23000}^{\mathbf{O}} \leq \frac{41}{64} < 0.641$.

*Proof.* Lemma 2.7 states that the two upper bounds can be proven by showing that the values of two certain matrix games $G$ and $H$ are smaller than the stated figures. While the two games have a reasonable number of rows, the number of columns is astronomical, so we cannot solve the games exactly. However, we can prove upper bounds on the values of the games by restricting the strategy space of the column player. Note that this corresponds to selecting a number of *bad type profiles*. We have constructed a catalogue of just 5 type profiles, each with 23000 voters. Using the "fraction encoding" of profiles suggested in the proof of Theorem 2.6, the profiles are:

[9] To make the program amenable to standard facet enumeration methods of quadratic programming, we changed the sharp inequalities ">" expresssing the majority vote constraints into weak inequalities "≥". Note that this cannot decrease the cost of the optimal solution.

- $x_2 = 14398/23000, x_5 = 2185/23000, x_{11} = 6417/23000$.

- $x_2 = 6000/23000, x_5 = 8000/23000, x_{12} = 9000/23000$.

- $x_1 = 11500/23000, x_{11} = 11500/23000$.

- $x_2 = 9200/23000, x_5 = 4600/23000, x_{12} = 9200/23000$.

- $x_2 = 13800/23000, x_{12} = 9200/23000$.

Solving the corresponding matrix games yields the stated upper bound. □

While the catalogue of bad type profiles of the proof of Theorem 2.8 suffices to prove the theorem, we should discuss how we arrived at this particular "magic" catalogue. This discussion also explains how we arrived at the "magic" coefficients in Theorems 2.6 and 2.7. In fact, we arrived at the catalogue and the coefficients iteratively in a joint local search process (or "co-evolution" process). To get an initial catalogue, we used the fact that we had already solved the matrix games yielding the values of $r^{\mathbf{OU}}_{3,n}$ and $r^{\mathbf{O}}_{3,n}$, for $n = 2, 3, 5$. By the theorem of Shapley and Snow [136], these matrix games have optimal strategies for the column player with support size at most the number of rows of the matrices. One can think of these supports as a small set of bad type profiles for $2, 3$ and $5$ voters. Utilizing that $2, 3$ and $5$ all divide 1000, we scaled all these *up* to 1000 voters. Also, we had solved the quadratic programs of the proofs of Theorem 2.6 and Theorem 2.7, but with inferior coefficients and resulting bounds to the ones stated in this chapter. The quadratic programs obtained their minima at certain type profiles. We added these entries to the catalogue, and scaled all profiles to their least common multiple, i.e. 23000.

Solving the linear programs of the proof of Theorem 2.8 now gave not only an upper bound on the approximation ratio, but the optimal strategy of Player I in the games also suggested reasonable mixtures of the $U^q_{3,n}$ (in the unilateral case) and of the $U^q_{3,n}$ and *random-majority* (all $D^q_{3,n}$ mechanisms except *random-majority* were assigned zero weight) to use for large $n$, making us update the coefficients and bounds of Theorem 2.6 and 2.7, with new bad type profiles being a side product. We also added by hand some bad type profiles along the way, and iterated the procedure until no further improvement was found. In the end we pruned the catalogue into a set of five, giving the same upper bound as we had already obtained.

Next, we prove that $r^{\mathbf{U}}_3$ is between 0.660 and 0.750. The upper bound follows from the following proposition and Lemma 2.4.

**Proposition 2.2.** $r^{\mathbf{U}}_{3,2} \leq 0.75$.

*Proof.* Suppose $M \in \mathrm{Mech}^{\mathbf{U}}_{3,2}$ has $\mathrm{ratio}(M) > 0.75$. By Lemma 2.1, we can assume $M$ is neutral. For some $\epsilon > 0$, consider the valuation profile with

$u_1(a) = u_2(a) = 1 - \epsilon$, $u_1(b) = u_2(c) = 0$, and $u_1(c) = u_2(b) = 1$. As in the proof of Theorem 2.4, by neutrality, we must have that the probability of $a$ being elected is at most $\frac{1}{2}$. The statement follows by considering a sufficiently small $\epsilon$. □

The lower bound follows from an analysis of the *quadratic-lottery* [74, 84]. The main reason that we focus on this particular cardinal mechanism is given by the following lemma. Note that this lemma is equivalent to Lemma 2.5 for the case of three candidates but applies to some cardinal mechanisms as well, not just ordinal ones.

**Lemma 2.8.** *Let $M \in \mathrm{Mech}_{3,n}$ be a convex combination of $Q_n$ and any ordinal and neutral mechanism. Then*

$$\mathrm{ratio}(M) = \liminf_{k\to\infty} \min_{\mathbf{u}\in(C_{m,k})^n} \frac{\mathbb{E}[\sum_{i=1}^n u_i(M(\mathbf{u}))]}{\sum_{i=1}^n u_i(a)}.$$

*Proof.* The proof is a simple modification of the proof of Lemma 2.5. As in that proof, for a valuation profile $\mathbf{u} = (u_i)$, define $g(\mathbf{u}) = \frac{\mathbb{E}[\sum_{i=1}^n u_i(M(\mathbf{u}))]}{\sum_{i=1}^n u_i(a)}$.

We show the following equations:

$$\begin{aligned} \mathrm{ratio}(M) &= \inf_{\mathbf{u}\in V_3^n} \frac{\mathbb{E}[\sum_{i=1}^n u_i(M(\mathbf{u}))]}{\max_{j\in A}\sum_{i=1}^n u_i(j)} && (2.11)\\ &= \inf_{\mathbf{u}\in V_3^n} g(\mathbf{u}) && (2.12)\\ &= \liminf_{k\to\infty} \min_{\mathbf{u}\in(R_{3,k})^n} g(\mathbf{u}) && (2.13)\\ &= \liminf_{k\to\infty} \min_{\mathbf{u}\in(C_{3,k})^n} g(\mathbf{u}) && (2.14) \end{aligned}$$

Equations (2.12) and (2.13) follow as in the proof of Lemma 2.5. Equation (2.14) follows from the following argument. For a profile $\mathbf{u} = (u_i) \in (R_{3,k})^n$, let $c_{\mathbf{u}}$ denote the number of pairs $(i, j)$ with $i$ being a voter and $j$ a candidate, for which $u_i(j) - 1/k$ and $u_i(j) + 1/k$ are both in $[0, 1]$ and both *not* in the image of $u_i$. Then, $C_{3,k}$ consists of exactly those $\mathbf{u}$ in $R_{3,k}$ for which $c_{\mathbf{u}} = 0$. To establish equation (2.14), we merely have to show that for any $\mathbf{u} \in R_{3,k}$ for which $c_{\mathbf{u}} > 0$, there is a $\mathbf{u}' \in R_{3,k}$ for which $g(\mathbf{u}') \le g(\mathbf{u})$ and $c_{\mathbf{u}'} < c_{\mathbf{u}}$. We will now construct such $\mathbf{u}'$. Since $c_{\mathbf{u}} > 0$, there is a pair $(i, j)$ so that $u_i(j) - 1/k$ and $u_i(j) + 1/k$ are both in $[0, 1]$ and both not in the image of $u_i$. Let $\ell_-$ be the smallest integer value so that $u_i(j) - \ell/k$ is not in the image of $u_i$, for any integer $\ell \in \{\ell_-, \ldots, j-1\}$. Let $\ell_+$ be the largest integer value so that $u_i(j) + \ell/k$ is not in the image of $u_i$, for any integer $\ell \in \{j+1, \ldots, \ell_+\}$. We can define a valuation function $u^x \in V_m$ for any $x \in [-\ell_-/k; \ell_+/k]$ as follows: $u^x$ agrees with $u_i$ except on $j$, where we let $u^x(j) = u_i(j) + x$. Let $\mathbf{u}^x = (u^x, u_{-i})$. Now consider the function $h : x \to g(\mathbf{u}^x)$. Since $M$ is a convex combination of *quadratic-lottery* and a neutral ordinal mechanism, we see by inspection of the definition of the function $g$, that $h$ on the domain $[-\ell_-/k; \ell_+/k]$ is the quotient of two quadratic polynomials where the numerator has second derivative being a negative constant and the denominator is postive throughout the interval. This means that $h$ attains its minimum at either $\ell_-/k$ or at $\ell_+/k$. In the first case, we let $\mathbf{u}' = \mathbf{u}^{\ell_-/k}$ and in the second, we let $\mathbf{u}' = \mathbf{u}^{\ell_+/k}$. This completes the proof. $\square$

**Theorem 2.9.** *The limit of the approximation ratio of $Q_n$ as $n$ approaches infinity, is exactly the golden ratio, i.e., $(\sqrt{5}-1)/2 \approx 0.618$. Also, let $M_n$ be the mechanism for $n$ voters that selects* random-favorite *with probability* $29/100$ *and* quadratic-lottery *with probability* $71/100$. *Then,* $\text{ratio}(M_n) > \frac{33}{50} = 0.660$.

*Proof.* (sketch) Lemma 2.8 allows us to proceed completely as in the proof of Theorem 2.6, by constructing and solving appropriate quadratic programs. As the proof is a straightforward adaptation, we leave out the details. $\square$

Mechanism $M_n$ of Theorem 2.9 achieves an approximation ratio strictly better than 0.641. In other words, the best truthful cardinal mechanism for three candidates strictly outperforms all ordinal ones.

We conclude the section with an upper bound on the approximation ratio of any truthful mechanism (under no restrictions).

**Theorem 2.10.** *All mechanisms $M \in \text{Mech}_{m,n}$ for $n \geq 3$ have* $\text{ratio}(M) < 0.94$.

*Proof.* We will prove the theorem for mechanisms in $\text{Mech}_{3,3}$. By applying Lemma 2.4, the theorem holds for any $n \geq 3$.[10]

Assume for contradiction that there exists a mechanism $M \in \text{Mech}_{3,3}$, with $\text{ratio}(M) \geq 0.94$. Consider the valuation profile $\mathbf{u}$ with three voters $\{1, 2, 3\}$, three candidates $\{a, b, c\}$, and valuations $u_1(b) = u_2(b) = u_3(c) = 1$, $u_1(c) = u_2(c) = u_3(b) = 0$, $u_1(a) = 0.7$ and $u_2(a) = u_3(a) = 0.8$. The social optimum on profile $\mathbf{u}$ is candidate $a$, with social welfare $w_a = 2.3$, while $w_b = 2$ and $w_c = 1$. Since $M$'s expected social welfare is at least a 0.94 fraction of $w_a$, i.e. 2.162, the probability of $a$ being elected is at least 0.54, as otherwise the expected social welfare would be smaller than $0.54 \cdot 2.3 + 0.46 \cdot 2 = 2.162$ . The expected utility $\tilde{u}$ of voter 1 in that case is at most $0.54 \cdot 0.7 + 0.46 \cdot 1 = 0.838$.

Next, consider the profile $\mathbf{u}'$ identical to $\mathbf{u}$ except that $u_1'(a) = 0.0001$. Let $p_a, p_b$ and $p_c$ be the probabilities of candidates $a, b$ and $c$ being elected on this profile, respectively. The social optimum is $b$ with social welfare 2. By truthfulness, it must be the case that $0.7p_a + p_b \leq \tilde{u}$, otherwise on profile $\mathbf{u}$, voter 1 would have an incentive to misreport $u_1(a)$ as 0.0001. Also, since $M$

[10] In fact, the theorem holds for any $n, m \geq 3$, by simply adding alternatives for which every voter has valuation almost 0.

has an approximation ratio of at least 0.94, it must be the case that $1.6001 \cdot p_a + 2 \cdot p_b + p_c \geq 1.88$. By those two inequalities, we have:

$$\begin{aligned} 0.9001p_a + p_b + p_c &\geq 1.8800 - \tilde{u} \Rightarrow \\ 0.9001(p_a + p_b + p_c) + 0.0999p_B + 0.0999p_C &\geq 1.8800 - \tilde{u} \Rightarrow \\ 0.0999(p_b + p_c) &\geq 0.10419 \quad \Rightarrow \\ p_b + p_c &\geq 1.42, \end{aligned}$$

which is not possible. Hence, it cannot be that ratio$(M) \geq 0.94$. □

## 2.5 Ties or no ties?

Before we conclude the chaper, we will discuss the issue of whether valuation functions have ties in their image or not and how that affects the results of the chapter. We do not want to declare either the “ties" or the “no ties" model the “right one", so ideally we would like all positive results (approximation guarantees) to be proven for the setting with ties and all negative ones (upper bounds on approximation ratio) to be proven for the setting without ties.

For the positive results, our results are easily adaptable to the “ties” setting. The main issue here is that statements like “elect one of his $k$ most-preferred candidates at random” are not necessarily well-defined in the presence of ties and we need a systematic way to incorporate the fact that mechanisms use some tie-breaking rules to resolve such issues.[11]

For the negative results, with the notable exception of Theorem 2.5, all of the constructed profiles do not exhibit ties and hence they apply to both the “ties” and the “no ties” setting. Note that even though the presence of ties introduces additional ordinal truthful mechanisms that are not captured by Theorem 1.2, when only considering input profiles without ties, the truthful mechanisms are still only convex combinations of unilaterals and duples and the results still hold. The presence of ties is crucial for the proof of Theorem 2.5 however, since otherwise a reduction from the one-sided matching setting to the general setting is not possible. One possible escape route would be the following: Reconstruct the proof of Lemma 4.6 for the general setting, by creating candidates corresponding to the different matchings and following the steps of the proof, *after* slightly perturbing the valuation functions to get rid of ties. This approach seems to fall short however, since it is unclear whether it is possible to apply such a perturbation and still maintain the *symmetry* of the profile that is needed to argue using anonymity and neutrality (which is the equivalent of anonymity used in Lemma 4.6, when applied to the general setting). Another possible way to adapt the proof to the “no-ties” setting could be to use the *probabilistic method* for constructing randomly

[11] In fact, Xinye Li has formalized a way of dealing with such issues; this is also planned to be part of a journal paper associated with the results of this chapter.

generated valuation profiles, instead of a single profile. In general, it looks like an interesting technical problem to prove a theorem equivalent to Theorem 2.5 for the setting without ties.

In this chapter, as well as in Chapter 4, we obtain results that are independent of the presence of ties in the sense described above. For Chapters 5 and 7 however, the presence of ties is important for some of our negative results. While valuation profiles with ties are part of the input in the settings studied there, extending those results to valuation profiles without ties would be interesting, for completeness.

## 2.6 Conclusion and future directions

By the results presented in this chapter, we know that mixed-unilateral mechanisms are asymptotically no better than ordinal mechanisms. Can a cardinal mechanism which is not mixed-unilateral beat this approximation barrier? Getting upper bounds on the performance of general cardinal mechanisms is impaired by the lack of a characterization of cardinal mechanisms a la Gibbard's. Driven by the discussion on the "ties vs no ties" topic, can we adapt the proof of Theorem 2.5 to work in the general setting without ties? For the case of $m = 3$, can we close the gaps for ordinal mechanisms and for mixed-unilateral mechanisms? How well can cardinal mechanisms do for $m = 3$? Can we find a systematic way to analyze the ratio of other cardinal truthful mechanisms proposed in [74], where the worst-case ratio is *not* on quasi-combinatorial profiles?

In a somewhat different approach, perhaps future work could be directed towards analyzing the performance of non-truthful voting mechanisms, with respect to their *price of anarchy*. In fact, we adopt a similar approach for the problem of *one-sided matching* in Chapter 5.[12] The main difference is that while in matching settings, there are examples of well-known non-truthful randomized mechanisms, in voting settings, most popular voting rules are deterministic and could potentially have quite inefficient equilibria. This is not to say that good randomized voting mechanisms could not exist, but the candidate choices in literature seem to be more limited.

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We would like to thank the anonymous reviewer for pointing out that the original proof of Lemma 2.5 was incorrect and for providing a sketch idea on how to fix the error. We would also like to thank the anonymous reviewer for pointing out that Lemma 2.2, which was originally only stated for ordinal mechanisms, holds for all mechanisms as well.

[12] See that chapter for the related definitions.

# Part II

# One-sided matching

# Chapter 3

# Background

In the second part of the thesis, we will consider the *one-sided matching problem*, also known as the *assignment problem* or the *house allocation problem* in the economic literature [98]. Informally, a set of agents have (unrestricted cardinal) preferences over a set of items and the goal is to output a *matching*, i.e. an assignment of items to agents such that each agent receives exactly one item. Instances of matching problems in real life are numerous; assigning students to exams, workers to shifts or clients to time slots are just a few of them [118]. We will start by formally defining the setting.

## 3.1 The setting

Let $N = \{1, \ldots, n\}$ be a finite set of agents and $A = \{1, \ldots, n\}$ be a finite set of indivisible items. An *outcome* is a matching of agents to items, that is, an assignment of items to agents where each agent gets assigned exactly one item. We can view an outcome $\mu$ as a vector $(\mu_1, \mu_2 \ldots, \mu_n)$ where $\mu_i$ is the unique item matched with agent $i$. Let $O$ be the set of all outcomes. Each agent $i$ has a private valuation function mapping outcomes to real numbers that can be arbitrary except for one condition; agents are indifferent between outcomes that match them to the same item. This condition implies that agents only need to specify their valuations for items instead of outcomes and hence the valuation function of an agent $i$ can be instead defined as a map $u_i : A \to \mathbb{R}$ from items to real numbers. Similarly to the setting of Chapter 1, these valuation functions are von Neumann-Morgenstern utilites and are standardly considered to be well-defined up to positive affine transformations, that is, for item $j : j \to \alpha u_i(j) + \beta$ is considered to be a different representation of $u_i$.

Note that the one-sided matching setting is a special case of the basic social choice setting of Chapter 1, where each possible matching corresponds to a candidate. Since agents are indifferent between candidates that correspond to matchings that assign them the same items, the setting of this chapter is *structured*, when compared to the unrestricted preference setting of Chapter 1. On the other hand, the preferences are *unrestricted* with respect to the matching setting, since valuation functions (as maps from items to real values) are arbitrary. Since the natural interpretation of the problem is in terms of matchings, we will use the terms "agents" and "matchings" instead of "voters" and "candidates" for the remainder of the thesis.

Let $V$ be the set of all canonically represented valuation functions of an agent. Call $\mathbf{u} = (u_1, u_2, \dots, u_n)$ a *valuation profile* and let $V^n$ be the set of all valuation profiles with $n$ agents. A *direct revelation mechanism* (without money) is a function $M : V^n \to O$ mapping *reported* valuation profiles to matchings. For a randomized mechanism, we define $M$ to be random map $M : V^n \to O$. For ease of notation, we will let $M_i(\mathbf{u})$ denote the restriction of the outcome of the mechanism to the $i$'th coordinate, which is the item assigned to agent $i$ by the mechanism.

The definition of a *truthful mechanism* is very similar to the one presented in Chapter 1 . We state the formal definition here for completeness:

**Definition 3.1** (Truthfulness)**.** A mechanism $M$ is truthful if for each agent $i$ and all $\mathbf{u} = (u_i, u_{-i}) \in V^n$ and $\tilde{u}_i \in V$ it holds that $u_i(M_i(u_i, u_{-i})) \geq u_i(M_i(\tilde{u}_i, u_{-i}))$, where $u_{-i}$ denotes the valuation profile $\mathbf{u}$ without the $i$'th coordinate.

In other words, if $u_i$ is agent $i$'s true valuation function, then she has no incentive to misreport it. For randomized mechanisms, similarly to Chapter 1, we define *truthfulness-in-expectation*.

**Definition 3.2** (Truthfulness-in-expectation)**.** A mechanism $M$ is *truthful-in-expectation* if for each agent $i$ and all $\mathbf{u} = (u_i, u_{-i}) \in V^n$ and $\tilde{u}_i \in V$ it holds that $\mathbb{E}[u_i(M_i(u_i, u_{-i}))] \geq \mathbb{E}[u_i(M_i(\tilde{u}_i, u_{-i}))]$.

Again, we will use the term "truthful" for both truthful and truthful-in-expectation mechanisms and the distinction will be clear from the context or explicitly stated.

Although we will study both truthful and non-truthful mechanisms in this part of the thesis, we will use the *real* valuation functions $u_i$ as inputs to a mechanism for Chapters 3 and 4; we will redefine the inputs to be *strategies* in Chapter 5. This choice is consistent with the results that we will present in Chapters 3 and 4 ,which will be either on truthful mechanisms or on non-truthful mechanisms *assuming truthful reports*.

Similarly to Chapter 1, we will make the distinction between ordinal and cardinal mechanisms.

**Definition 3.3** (Ordinal mechanism)**.** A mechanism $M$ is *ordinal* if for any $i$, any valuation profile $\mathbf{u} = (u_i, u_{-i})$ and any valuation function $u'_i$ such that for all $j, j' \in M$, $u_i(j) < u_i(j') \Leftrightarrow u'_i(j) < u'_i(j')$, it holds that $M(u_i, u_{-i}) = M(u'_i, u_{-i})$.

A mechanism for which the above does not necessarily hold is cardinal.

## 3.2 Related literature from economics

In the presence of incentives, the one-sided matching problem (often referred to as *the assignment problem* or *house allocation problem*) was originally defined in the seminal paper of Hylland and Zeckhauser [98] and has been studied extensively ever since. There are several surveys discussing the problem (as well as more general matching problems) [3, 137] and we refer the interested reader to those for a in depth exposition of related results that are not necessarily particularly relevant to the results of this thesis.

It is worth noting that while throughout the years research in economics has sometimes considered the variant of the problem where agents' preferences are captured by only ordinal rankings of items, the setting presented in [98] is exactly the same one studied here, where agents have unrestricted cardinal valuations over a set of items. Perhaps even more strikingly, although the matching literature in economics has been dominated by ordinal mechanisms, Hylland and Zeckhauser propose a cardinal mechanism, *the pseudo-market mechanism.* The mechanism first endows agents with artificial budgets of unit capacity and then produces a randomized matching in a market-like fashion: items are treated as divisible commodities, prices are announced and agents purchase their most preferred shares at those prices. The process is repeated until supply meets demand, i.e. all items are entirely allocated and all artificial budgets are exhausted.[1] The pseudo-market mechanism is also sometimes referred to as the *CEEI mechanism* [35], where CEEI stands for "competitive equilibrium from equal incomes", the supply-meets-demand outcome of a market where buyers have equal budgets. The pseudo-market mechanism satisfies the strongest notion of Pareto efficiency for randomized mechanisms, *ex-ante Pareto efficiency*, as well as (ex-ante) *envy-freeness*, a guarantee that no agent would prefer to exchange any other agent's expected allocation with her own. On the other hand, the mechanism is not truthful; truthfulness as a desirable property had already been discussed in [98].

[1] Note that this mechanism is quite similar to the mechanism studied in Chapter 7, but yet also quite different. The main difference comes from the fact that in the pseudo-market mechanism, agents purchase their favorite shares under the additional constraint that ensures that their shares can then be interpreted as probabilities.

### Random Priority

While Hylland and Zeckhauser's mechanism seems like a quite complicated solution, a folklore mechanism that pre-existed their 1979 paper and pressumably dates back to ancient times, is the following simple one, called *random priority*:

*Mechanism* (*Random priority*). On valuation profile $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, pick an ordering of agents $\pi \in S_n$ uniformly at random. For $i = 1$ to $n$ let agent $\pi_i$ pick her favorite item (or one of her favorite items in case of ties) from the set of available items (not already picked by some agent $j$ with $\pi_j < \pi_i$).

The deterministic mechanism that selects some fixed ordering instead of an ordering uniformly at random and then lets agents pick their most preferred available items sequentially is called a *serial dictatorship*. For that reason, and since random priority essentially selects a serial dictatorship uniformly at random, the mechanism is also very often referred to as *random serial dictatorship*.

In terms of desired properties, random priority is very simple to implement and truthful. It also satisfies *anonymity* which implies *symmetry*,[2] a guarantee that agents with the same preferences will be treated equally. On the other hand, it fails to satisfy stronger fairness notions, such as envy-freeness and it is also on the low-end of the economic efficiency spectrum, satisfying only ex-post Pareto efficiency (a property that ensures that every realized outcome will be Pareto efficient).

A large body of literature in economics is centered around the study of random priority. A very interesting result is due to Abdulkadiroğlu and Sönmez [1], who showed that random priority is equivalent to another well-known mechanism; the core for random endowments [135]. Bade [23] further showed that taking any Pareto-optimal, truthful and *non-bossy* deterministic mechanism and uniformly at random assigning agents to roles, results in random priority. Non-bossiness informally means that if an agent does not change her allocation by reporting some different ranking, she does not change any other agent's allocation either.[3] Whether random priority is the only ex-post efficient, truthful and symmetric mechanism is still unknown.

### Probabilistic Serial

Another very imporant mechanism, which was proposed as "a new solution to the random assignment problem" is *probabilistic serial*, introduced by Crès and Moulin [61] and popularized by Bogomolnaia and Moulin [35]:

[2]Recall that the notion of symmetry here is different from the definition in previous chapters. This property is also encountered in literature as *equal treatment of equals*.

[3]An example of a truthful, bossy mechanism is the mechanism that selects an agent, matches her with her most preferred item and then allocates items to agents based only on her ranking [138]. We will revisit this mechanism briefly in Chapter 5.

*Mechanism (Probabilistic Serial).* Each item is interpreted as an infinitely divisible good that all agents can consume at unit speed during the unit time interval $[0, 1]$. Initially each agent consumes her most preferred item (or one of her most preferred items in case of ties) until the item is entirely consumed. Then, the agent moves on to consume the item on top of her preference list, among items that have not yet been entirely consumed. The mechanism terminates when all items have been entirely consumed. The fraction $p_{ij}$ of item $j$ consumed by agent $i$ is then interpreted as the probability that agent $i$ will be matched with item $j$ under the mechanism.

Probabilistic serial is actually one of the mechanisms in the class of *simultaneous eating mechanisms* [35]; these mechanisms are defined similarly to probabilistic serial, for eating speeds that vary with time. This class is actually characterized by *ordinal efficiency*, an efficiency concept that is "between" ex-post and ex-ante Pareto efficiency. Informally, given any set of ordinal preferences, a random assignment is ordinally efficient if there is no other assignment that is better (in the Pareto sense), for *all* von Neumann-Morgenstern utility functions *consistent* with those orderings.

The mechanism also satisfies strong fairness properties, such as anonymity and envy-freeness. On the other hand, probabilistic serial is not truthful. In [35], it is proven that the mechanism is *weakly truthful*, which informally means that given any agent and any true ranking of the items, there is no misreport (no other ranking) that gives the agent a higher utility for *all* valuation functions consistent with the true ranking.

The superior fairness and (economic) efficiency properties of probabilistic serial to random priority has gained the mechanism a lot of popularity throughout the years; one could make the claim that next to random priority, it is the best-studied mechanism in matching literature. Hashimoto et al. [93] give two axiomatic characterizations of probabilistic serial and prove that in the setting where items are not allowed to remain unallocated, the mechanism is characterized by a single property, *ordinal fairness*. Abdulkadiroğlu and Sönmez [2] offer a different characterization of ordinal efficiency, based on the concept of *dominated sets of assignments*. Kesten [102] prove the equivalence of probabilistic serial to two other mechanisms, a variation of the *top trading cycles* algorithm where agents have equal fractional endowments of all items and the limit version of a repeated application of random priority. Katta and Sethuraman [101] extend the results in [35] to allow for indifferences in the preference rankings of the agents.

The incentive properties and the computational aspects of manipulation of probabilistic serial have only been recently studied, with contribution coming mainly from the field of computer science. We will overview the related literature on those topics in Chapter 5.

## 3.3 Truthful mechanisms

In Chapter 1, we talked about the importance of truthful mechanisms and stated the main results characterizing truthful mechanisms in the general domain. Since the one-sided matching setting is restricted, the characterizations of Theorems 1.1 and 1.2 do not necessarily hold. Do we have equivalent results for this setting as well?

Regarding deterministic mechanisms, the most general result that we know of is due to Svensson [138]:

**Theorem 3.1** (Svensson)**.** *[138] Let $M$ be a truthful, neutral and non-bossy mechanism. Then $M$ is a serial dictatorship.*

Here, neutrality means independence on the names of the items, as usual, and non-bossiness was defined earlier. Theorem 3.1 is in a sense analogous to Theorem 1.1. A full characterization of truthful deterministic mechanisms is yet not known.

For randomized truthful mechanisms, recently Mennle and Seuken [113] proved a theorem analogous to Theorem 1.2.

**Theorem 3.2** (Mennle and Seuken [113])**.** *A mechanism $M$ is ordinal and truthful if and only if it satisfies the following three properties:* upper invariance, lower invariance *and* swap monotonicity.

Informally, upper invariance means that a swap in an agent's ranking between two lower-ranked items does not affect the assignment probability of any higher-ranked item; lower invariance is the complementary property. Swap monotonicity requires that when the relative ranking of two items $i, j$ (adjacent in the preference ordering), with $i \succ j$ is swapped, the assignment probabilities are either unaffected, or the probability for $j$ strictly increases and the probability for $i$ strictly decreases.

We remark here that unlike Chapter 2, for the results in Chapter 4, we will not make use of any structural properties of truthful mechanisms, other than truthfulness itself. In that sense, the characterization results are not needed for our proofs. That being said, characterizing general truthful mechanisms for one-sided matching problems is a very interesting open question and perhaps surprisingly, we are not aware of any research done in that direction.

## 3.4 Which mechanism to choose?

Throughout the years and all over the extended literature on one sided-matching problems, comparisons between different mechanisms, mainly random priority and probabilistic serial, were in place and the following question was stated either implicitly or explicitly: "Which mechanism should we choose?".

As we mentioned earlier, comparisons in economics are usually performed through a set of properties satisfied by each mechanism. Random priority is simple and very strong in terms of truthfulness,[4] but lacks in terms of fairness and economic efficiency. Probabilistic serial is weaker in terms of incentive properties, but satisfies better fairness and efficiency criteria. The pseudo-market mechanism is even stronger in terms of efficiency, but it is not truthful and it is not ordinal; eliticing numerical valuations from agents can sometimes be a hard task, not to mention that the mechanism is quite complicated.

What is the best set of properties that we can hope for? A partial answer to this question was provided by Zhou [144], who gave the following impossibility result:

**Theorem 3.3** (Zhou [144])**.** *There is no mechanism M that is ex-ante Pareto efficient, truthful and symmetric.*

Later on, Bogomolnaia and Moulin [35] strengthened the theorem by replacing ex-ante Pareto efficiency with ordinal efficiency:

**Theorem 3.4** (Bogomolnaia and Moulin [35])**.** *There is no mechanism M that is ordinally efficient, truthful and symmetric.*

Featherstone [73] proposed a class of mechanisms that satisfy *rank efficiency*, a concept stronger than ordinal efficiency. However, rank efficiency is incompatible with even weak strategyproofness.

Given the discussion above, qualitative comparisons of different mechanisms do not seem to provide definite answers and the choices are always subject to the very specific goals of the designer. For that reason, a large body of recent literature has quantified and studied tradeoffs between efficiency and truthfulness properties in matching settings but also more general settings [18, 45, 114]. Most related to the discussion here is the work by Mennle and Seuken [114] who quantify truthfulness and propose the use of *hybrid mechanisms*, i.e. convex combinations of different mechanisms to achieve tradeoffs between incentive and efficiency properties.

### Social welfare maximization and overview of results

The second part of this thesis offers an alternative viewpoint to the question phrased earlier. Since agents are endowed with cardinal utilities, the socially optimal solution could be the one that maximizes the social weflare. Note that while the bulk of literature in economics is concerned with ordinal mechanisms, discussions about ex-ante Pareto efficiency indicate that the existence of an underlying cardinal structure is not often in question; the argument is usually

[4]Random priority is actually *universally truthful*, which is a notion stronger than truthfulness-in-expectation. A universally truthful mechanism is a convex combination of deterministic truthful mechanisms.

that it is very hard to ask agents to actually report those numerical values. As we will see in the following chapters, our results imply that with respect to the social welfare objective (and for the case of many items), the best mechanisms are ordinal and hence reporting preference rankings is sufficient.

In Chapter 4, we will study the approximation ratio of mechanisms that are truthful or ordinal (assuming truthful reports). The main result of the chapter is that random priority is asymptotically the best truthful and the best ordinal mechanism for the problem, for the social welfare objective.

In Chapter 5, we will consider *all* mechanisms, even non-truthful ones and we will calculate their inefficiency in the worst Nash equilibrium, using the established notion of the *Price of Anarchy*. Our main result is that both probabilistic serial and random priority are optimal among all mechanisms, even those that are allowed to use the cardinal information of the reports.

Our results indicate that for many items and if social welfare maximization is the goal at hand, both random priority and probabilistic serial are excellent choices.

# Chapter 4

# The approximation ratio of truthful mechanisms

In this chapter, we will (mainly) consider (randomized) truthful mechanisms for the objective of social welfare maximization. The setting that we study is the basic one-sided matching setting introduced in Chapter 3. Similarly to Chapter 2, our measure of efficiency will be the *approximation ratio*. Our main result is the following:

**Theorem 4.1.** *The approximation ratio of random priority is* $\Theta(1/\sqrt{n})$*. Furthermore, random priority is asymptotically the best truthful mechanism and the best ordinal (not necessarily truthful) mechanism for one-sided matching.*

Recall our discussion in Chapter 1 about canonical representations; in this chapter, we will assume both normalizations, *unit-range* and *unit-sum*; Theorem 4.1 holds for both representations.[1]

## 4.1 Related literature

In Chapter 3, we defined random priority and discussed some of the major work in economics on one-sided matchings. Here we will discuss the results that are more relevant to our contributions.

As we mentioned in Chapter 3, a large amount of work in economics has been built around ordinal mechanisms and ordinal measures of efficiency, such

[1]The theorem also holds for an extension to the unit-range representation, when 0 is not required to be in the image of the function; we discuss how in Section 4.5.

as ordinal efficiency. In computer science, a similar line of thought has considered ordinal measures of *aggregate* efficiency. Bhalgat et al. [31] calculate the approximation ratio of random priority and probabilistic serial, when the objective is maximization of *ordinal social welfare*, a notion of efficiency that they define based solely on ordinal information. Other measures of efficiency for one-sided matchings were also studied in Krysta et al. [104], where the authors design truthful mechanisms to approximate the size of a maximum cardinality (or maximal agent weight) Pareto-optimal matching. Chakrabarty and Swamy [50] also consider a purely ordinal setting and proporse *rank approximation* as the measure of efficiency and *lex-truthfulness* as the notion of truthfulness, in the absence of utility functions. However, these measures do not encapsulate the "socially desired" outcome in the way that social welfare does, i.e., they do not necessarily maximize the aggregate happiness of individuals [9], especially since an underlying cardinal valuation structure is, in general, assumed to exist [35, 98, 144].

Social welfare maximization has been studied before in the assignment literature. Anshelevich and Das [9] consider the social welfare objective under unrestricted, *unnormalized* valuations and restrict the space of allowed misreports to obtain reasonable bounds. Guo and Conitzer [89] study assignment problems where there are two agents and multiple items and agents have unrestricted cardinal valuations under the *unit-sum* normalization. The authors provide approximation guarantees and impossibility results for truthful mechanisms in this setting. Independently and at the same time as the results of this chapter were published, Adamzyck et al. [5] studied a setting very similar to the one studied here. In their setting, agents have unrestricted von Neumann-Morgenstern valuations, restricted in the unit interval $[0, 1]$ but not necessarily with 0 and 1 being in the image of the functions. Our lower bound for the unit-range representation can be obtained from their main lemma with some additional arguments, but since the setting that they study is more general, our upper bounds are stronger. In [5], the authors also study the problem under *dichotomous preferences* [36]; in that setting truthful mechanisms that achieve the maximum social welfare exist [67].

## 4.2 Preliminaries

The setting studied in this chapter is the one-sided matching setting presented in Chapter 3. As mentioned earlier, valuation functions will be canonically represented as *unit-range*, i.e., $\max_j u_i(j) = 1$ and $\min_j u_i(j) = 0$ or *unit-sum*, that is $\sum_j u_i(j) = 1$. We will further assume that valuation functions are injective, i.e. agents could assign the same numerical values to different items. This assumption is not crucial for our results and is mainly for convenience, to avoid having to resolve statements such as "the agent's most preferred item". We discuss how our results extend to the "ties" setting in Section 4.5.

A class of mechanisms that turns out to be important for our purposes is that of neutral and anonymous mechanisms. The definitions of anonymity and neutrality are very similar to the ones in Chapter 2, but we will state the formal definitions here as well.

**Definition 4.1** (Anonymity)**.** A mechanism $M$ is *anonymous* if for any valuation profile $(u_1, u_2, \ldots, u_n)$, every agent $i$ and any permutation $\pi : N \to N$ it holds that $M_i(u_1, u_2, \ldots, u_n) = M_{\pi(i)}(u_{\pi(1)}, u_{\pi(2)}, \ldots, u_{\pi(n)})$.

In simple words, an anonymous mechanism is invariant to the names of the agents. Note that in an anonymous mechanism, agents with exactly the same valuation functions must have the same probabilities of receiving each item. This property is called *symmetry.* A neutral mechanism is invariant to the indices of the items, formally:

**Definition 4.2** (Neutrality)**.** A mechanism $M$ is *neutral* if for any valuation profile $(u_1, u_2, \ldots, u_n)$, every item $j$ and any permutation $\sigma : A \to A$ it holds that $M_i(u_1, u_2, \ldots, u_n) = \sigma^{-1}(M_i(u_1 \circ \sigma, u_2 \circ \sigma, \ldots, u_n \circ \sigma))$,

Exactly as we did in Chapter 2, we will measure the performance of a mechanism by its approximation ratio,

$$\text{ratio}(M) = \inf_{\mathbf{u} \in V^n} \frac{\sum_{i=1}^n u_i(M_i(\mathbf{u}))}{\max_{\mu \in O} \sum_{i=1}^n u_i(\mu_i)}$$

where the quantity $\sum_{i=1}^n u_i(M_i(\mathbf{u}))$ is the social welfare of mechanism $M$ on the valuation profile $\mathbf{u}$ and $\max_{\mu \in O} \sum_{i=1}^n u_i(\mu_i)$ is the social welfare of the optimal matching. For ease of notation, let $w^*(\mathbf{u}) = \max_{\mu \in O} \sum_{i=1}^n u_i(\mu_i)$. For the case of randomized mechanisms, we will be interested in the *expected social welfare* $\mathbb{E}\left[\sum_{i=1}^n u_i(M_i(\mathbf{u}))\right]$ of mechanism $M$ and the approximation ratio is defined accordingly.

Next we will state a lemma that will be useful for our proofs. Note that this lemma is very similar to Lemma 2.1 from Chapter 2.

**Lemma 4.1.** *For any mechanism $M$, there exists an anonymous mechanism $M'$ such that* $\text{ratio}(M') \geq \text{ratio}(M)$*. Furthermore, if $M$ is truthful then it holds that $M'$ is truthful.*

*Proof.* Let $M'$ be the mechanism that given any valuation profile $\mathbf{u}$ applies a uniformly random permutation to the set of agents and then applies $M$ on $\mathbf{u}$. The mechanism is clearly anonymous. Furthermore, since $\mathbf{u}$ is a valid input to $M$, the approximation ratio of $M'$ can not be worse than that of $M$, since the approximation ratio is calculated over all possible valuation profiles. For the same reason, if $M$ is truthful and since the permutation is independent of the reports, $M'$ is truthful-in-expectation. □

We conclude the section with the following lemma about random priority. Similar lemmas have been proved in literature (e.g. see Lemma 1 in [35], for a slightly more general statement).

**Lemma 4.2.** *For any valuation profile* $\mathbf{u}$*, the optimal allocation on* $\mathbf{u}$ *is a possible outcome of random priority.*

*Proof.* First, suppose that no agent is matched with her most preferred item in the optimal allocation. Then there must exist agents $i_1, ..., i_k$ such that for each $l$, agent $i_{l+1}$ is matched with agent $i_l$'s most preferred item and agent $i_1$ is matched with agent $i_k$'s most preferred item. By swapping items along this cycle, all agents are better off and the allocation is not optimal.

Now consider any valuation profile $\mathbf{u}$. Since there exists an agent $j$ that is matched with her most preferred item $j$ in the optimal allocation for $\mathbf{u}$, random priority could pick this agent first. If we reduce $\mathbf{u}$ by removing the agent $i$ and item $j$, we obtain a smaller valuation profile $\mathbf{u}'$ where the optimal allocation is the same as in $\mathbf{u}$ but without agent $i$ and item $j$. Then, by inductively applying the same argument, the lemma follows. □

## 4.3 Unit-range valuation functions

In this section, we assume that the representation of the valuation functions is unit-range. It will be useful to consider a special class of valuation functions $C_\epsilon$ that we will refer to as *quasi-combinatorial valuation functions*, a straightforward adaptation of the similar notion in Chapter 2. Recall that informally, a valuation function is quasi-combinatorial if the valuations of each agent for every item are close to 1 or close to 0 (the proximity depends on $\epsilon$). Formally,

$$C_\epsilon = \{u \in V | u(A) \subset [0, \epsilon) \cup (1-\epsilon, 1]\},$$

where $u(A)$ is the image of the valuation function $u$. Let $C_\epsilon^n \subseteq V^n$ be the set of all valuation profiles with $n$ agents whose valuation functions are in $C_\epsilon$. The following lemma implies that when we are trying to prove a lower bound on the approximation ratio of random priority, it suffices to restrict our attention to quasi-combinatorial valuation profiles $C_\epsilon^n \subseteq V^n$ for any value of $\epsilon$. The proof is similar to the proof of Lemma 2.5, but applied to the one-sided matching setting.

**Lemma 4.3.** *Let* $M$ *be an ordinal, anonymous and neutral randomized mechanism for the unit-range representation, and let* $\epsilon > 0$*. Then*

$$\mathrm{ratio}(M) = \inf_{\mathbf{u} \in C_\epsilon^n} \frac{\mathbb{E}[\sum_{i=1}^n u_i(M_i(\mathbf{u}))]}{w^*(\mathbf{u})}.$$

*Proof.* Since $M$ is anonymous and neutral, we can assume that the optimal matching is $\mu^*$ where $\mu^*$ is the matching with $\mu_i^* = i$ for every agent $i \in N$. Given this, then for any valuation profile $\mathbf{u}$, define

$$g(\mathbf{u}) = \frac{\mathbb{E}[\sum_{i=1}^{n} u_i(M_i(\mathbf{u}))]}{\sum_{i=1}^{n} u_i(\mu_i^*)}.$$

Because of this, the approximation ratio can be written as ratio$(M) = \inf_{\mathbf{u}\in V^n} g(\mathbf{u})$. Now since $C_\epsilon^n \subseteq V^n$, the lemma follows from the following claim:

For all $\mathbf{u} \in V^n$ there exists $\mathbf{u}' \in C_\epsilon^n$ such that $g(\mathbf{u}') \le g(\mathbf{u})$

We will prove the claim by induction on $\sum_{i=1}^{n} \#\{u_i(A) \cap [\epsilon, 1-\epsilon]\}$.

*Induction basis:* Since $\sum_{i=1}^{n} \#\{u_i(A) \cap [\epsilon, 1-\epsilon]\} = 0$, one can clearly see that $u_i \in C_\epsilon$ for all $i \in N$. So, for this case, let $\mathbf{u}' = \mathbf{u}$.

*Induction step:* Consider a profile $\mathbf{u} \in V^n$ with $\sum_{i=1}^{n} \#\{u_i(A) \cap [\epsilon, 1-\epsilon]\} > 0$. Clearly, there exists an $i$ such that $\#\{u_i(A) \cap [\epsilon, 1-\epsilon]\} > 0$. By this fact, there exist $l, r \in [\epsilon, 1-\epsilon]$, such that $l \le r$, $u_i(A) \subset [0, \epsilon) \cup [l, r] \cup (1-\epsilon, 1]$ and $\{l, r\} \subseteq u_i(A)$.

Let $\bar{l}$ be the largest number such that $\bar{l} \in [0, \epsilon)$ and $\bar{l} \in u_i(A)$. Similarly, let $\bar{r}$ be the smallest number such that $\bar{r} \in (1-\epsilon, 1]$ and $\bar{r} \in u_i(A)$. Note that both those numbers exist, since $\{0, 1\} \subseteq u_i(A)$. Now let $\tilde{l} = \frac{\bar{l}+\epsilon}{2}$, and $\tilde{r} = \frac{\bar{r}+1-\epsilon}{2}$

For any $x \in [\tilde{l} - l, \tilde{r} - r]$, define a valuation function $u_i^x \in V$ as follows:

$$u_i^x(j) = \begin{cases} u_i(j), & \text{for } j \notin u_i^{-1}([\epsilon, 1-\epsilon]\}) \\ u_i(j) + x, & \text{for } j \in u_i^{-1}([\epsilon, 1-\epsilon]\}). \end{cases}$$

This is still a valid valuation function, since by the choice of the interval $[\tilde{l} - l, \tilde{r} - r]$, there are no ties in the image of the function. Let $(u_i^x, \mathbf{u}_{-i})$ be the valuation profile where all agents have the same valuation functions as in $\mathbf{u}$ except for agent $i$, who has valuation function $u_i^x$. Define the following function $f : x \to g((u_i^x, \mathbf{u}_{-i}))$. Since $M$ is ordinal, by the definition of function $g$, we can see that $f$ on the domain $[\tilde{l} - l, \tilde{r} - r]$ is a fractional linear function $x \to (ax + b)/(cx + d)$ for some $a, b, c, d, \in \mathbb{R}$. Since $f$ is defined on the whole interval $[\tilde{l} - l, \tilde{r} - r]$, it is either monotonically increasing, monotonically decreasing or constant in the interval. If $f$ is monotonically increasing, let $\tilde{\mathbf{u}} = (u^{\tilde{l}-l}, \mathbf{u}_{-i})$, otherwise let $\tilde{\mathbf{u}} = (u^{\tilde{r}-r}, \mathbf{u}_{-i})$. Clearly, $g(\tilde{\mathbf{u}}) \le g(\mathbf{u})$ and

$$\sum_{i=1}^{n} \#\{\tilde{u}_i(A) \cap [\epsilon, 1-\epsilon]\} < \sum_{i=1}^{n} \#\{u_i(A) \cap [\epsilon, 1-\epsilon]\}.$$

Then, apply the induction hypothesis on $\tilde{\mathbf{u}}$. This completes the proof. $\square$

The lemma formalizes the intuition that because the mechanism is ordinal, the worst-case approximation ratio is encountered on extreme valuation profiles.

For the unit-range representation, Theorem 4.1 is given by the following lemmas.

**Lemma 4.4.** *For the unit-range representation,* $\text{ratio}(RP) = \Omega\left(n^{-1/2}\right)$.

*Proof.* Because of Lemma 4.3, for the purpose of computing a lower bound on the approximation ratio of random priority, it is sufficient to only consider quasi-combinatorial valuation profiles. Let $\epsilon \leq 1/n^3$. Then, there exists $k \in \mathbb{N}$ such that

$$|k - w^*(\mathbf{u})| \leq \frac{1}{n^2},$$

where $w^*(\mathbf{u})$ is the social welfare of the maximum weight matching on valuation profile $\mathbf{u}$. Since random priority can trivially achieve an expected welfare of 1 (for any permutation the first agent will be matched to her most preferred item), we can assume that $k \geq \sqrt{n}$, otherwise we are done. Note that the maximum weight matching $\mu^* \in O$ assigns $k$ items to agents with $u_i(\mu_i) \in (1-\epsilon, 1]$. Since random priority is anonymous and neutral, without loss of generality we can assume that these agents are $\{1, \dots, k\}$ and for every agent $j \in N$, it holds that $\mu_j^* = j$. Thus $u_j(j) \in (1-\epsilon, 1]$ for $j = 1, \dots, k$ and $u_j(j) \in [0, \epsilon)$ for $j = k+1, \dots, n$.

Consider any run of random priority; one agent is selected in each round. Let $l \in \{0, \dots, n-1\}$ be any of the $n$ rounds. We will now define the following sets:

$$U_l = \{j \in \{1, \dots, n\} | \text{ agent } j \text{ has not been selected prior to round } l\}$$
$$G_l = \{j \in U_l | u_j(j) \in (1-\epsilon, 1] \text{ and item } j \text{ is still unmatched}\}$$
$$B_l = \{j \in U_l | u_j(j) \in [0, \epsilon) \text{ or item } j \text{ has already been matched to some agent}\}$$

These three families of sets should be interpreted as three sets that change over the course of the execution of random priority. $U_l$ is the set of agents yet to be matched, which is partitioned into $G_l$, the set of "good" agents, that guarantee a welfare of almost 1 when picked, and $B_l$, the set of "bad" agents, that do not guarantee any contribution to the social welfare. For the purpose of calculating a lower bound, we will simply bound the sizes of the sets in these families. Obviously, $G_0 = \{1, \dots, k\}$ and $B_0 = \{k+1, \dots, n\}$.

The probability that an agent $i \in G_l$ is picked in round $l$ of random priority is $|G_l|/(|G_l| + |B_l|)$, whereas the probability that an agent $i \in B_l$ is picked is $|B_l|/(|G_l| + |B_l|)$. By the discussion above, we can assume that whenever an agent from $G_l$ is picked her contribution to the social welfare is at least $1-\epsilon$ whereas the contribution from an agent picked from $B_l$ is less than $\epsilon$. In other words, the expected contribution to the social welfare from round $l$ is at least $|G_l|/(|G_l| + |B_l|) - \epsilon$.

We will now upper bound $|G_l|$ and lower bound $|B_l|$ for each $l$. Consider round $l$ and sizes $|G_l|$ and $|B_l|$. First suppose that some agent $i$ from $G_l$ is picked and the agent is matched with item $j$. If $j \neq i$ and agent $j$ is in $G_l$, then $|G_{l+1}| = |G_l| - 2$ and $|B_{l+1}| = |B_l| + 1$, since agent $j$ no longer has her item from the optimal allocation available and so agent $j$ is in $B_{l+1}$. On the other hand, if $j = i$ or agent $j$ is in $B_l$ then $|G_{l+1}| = |G_l| - 1$ and $|B_{l+1}| = |B_l|$. In either case, $|G_{l+1}| \geq |G_l| - 2$ and $|B_{l+1}| \leq |B_l| + 1$. Intuitively, the picked agent might take away some item from a good agent and turn her into a bad agent.

Now suppose that agent $i$ from $B_l$ is picked and the agent is matched with item $j$. If agent $j$ is in $G_l$ then $|G_{l+1}| = |G_l| - 1$ and $|B_{l+1}| = |B_l|$, since agent $j$ no longer has her item from the optimal allocation available and so agent $j$ is in $B_{l+1}$. On the other hand, if agent $j$ is in $B_l$ then $|G_{l+1}| = |G_l|$ and $|B_{l+1}| = |B_l| - 1$. In either case, $|G_{l+1}| \geq |G_l| - 2$ and $|B_{l+1}| \leq |B_l| + 1$.

To sum up, in each round $l$ of random priority, we can assume the size of $B_l$ increases by at most 1 and the size of $G_l$ decreases by at most 2. Given this and that $|G_0| = k$ and $|B_0| = n - k$ and that $|G_l| > 0$ for $l \leq \lfloor k/2 \rfloor$, we get

$$\mathbb{E}\left[\sum_{i=1}^{n} u_i(RP_i(\mathbf{u}))\right] \geq \sum_{l=0}^{n} \left(\frac{|G_l|}{|G_l| + |B_l|} - \epsilon\right) \geq \sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} \frac{k - 2l}{n - l} - n\epsilon$$

and the ratio is

$$\begin{aligned}\frac{\mathbb{E}\left[\sum_{i=1}^{n} u_i(RP_i(\mathbf{u}))\right]}{w^*(\mathbf{u})} &\geq \frac{\sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} \frac{k-2l}{n-l} - n\epsilon}{k + \frac{1}{n^2}} \geq \frac{\sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} \frac{k-2l}{n-l} - n\epsilon}{2k} \\ &= \sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} \frac{1 - \frac{2l}{k}}{2(n-l)} - \frac{n\epsilon}{2k} > \sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} \frac{1 - \frac{2l}{k}}{2n} - \frac{n\epsilon}{2k} \geq \frac{k - 11}{8n} - \frac{n\epsilon}{2k}.\end{aligned}$$

The bound is clearly minimum when $k$ is minimum, that is, $k = \sqrt{n}$. Since this bound holds for any $\mathbf{u} \in C_\epsilon^n$, we get

$$\text{ratio}(RP) = \inf_{\mathbf{u} \in C_\epsilon^n} \frac{\mathbb{E}[\sum_{i=1}^{n} u_i(M_i(\mathbf{u}))]}{w^*(\mathbf{u})} \geq \frac{\sqrt{n} - 11}{8n} - \frac{n\epsilon}{2\sqrt{n}}.$$

We can choose $\epsilon$ so that the approximation ratio is at least $\frac{1}{20\sqrt{n}}$ for $n \geq 400$ and for $n \leq 400$, the bound holds trivially since random priority matches at least one agent with her most preferred item. □

Next, we state the following lemma about ordinal mechanisms.

**Lemma 4.5.** *Let $M$ be any ordinal mechanism for one-sided matching for the unit-range representation. Then* $\text{ratio}(M) = O\left(n^{-1/2}\right)$.

*Proof.* Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ be the valuation profile where:

$$u_i(j) = \begin{cases} 1 - \frac{j-1}{n} & \text{for } 1 \leq j \leq i \\ \frac{n-j}{n^2} & \text{otherwise} \end{cases} \qquad \forall i \in \{1, \ldots, \lfloor\sqrt{n}\rfloor\}$$

$$u_i(j) = \begin{cases} 1 & \text{for } j = 1 \\ \frac{n-j}{n^2} & \text{otherwise} \end{cases} \qquad \forall i \in \{\lfloor\sqrt{n}\rfloor + 1, \ldots, n\}$$

By Lemma 4.1, we can assume that $M$ is anonymous. Notice that the valuation profile is *ordered*, i.e., $u_i(j) > u_i(j')$ whenever $j < j'$ for all $j, j' \in M$ and all $i \in N$. Thus, any anonymous and ordinal mechanism on input $\mathbf{u}$ must output a uniformly random matching, that is, the probability that agent $i$ is matched with item $j$ is the same for all agents $i$, for every $j \in M$. The expected welfare of the mechanism on valuation profile $\mathbf{u}$ will be

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} u_i(j) &\leq \frac{1}{n}\left[\sum_{i=1}^{\lfloor\sqrt{n}\rfloor}\left(i + \frac{n-i}{n}\right) + \sum_{i=\lfloor\sqrt{n}\rfloor+1}^{n}\left(1 + \frac{n-1}{n}\right)\right] \\ &\leq 4 + \frac{1}{2\sqrt{n}} \leq 5, \end{aligned}$$

where in the above expression, we upper bound each term $\frac{n-j}{n^2}$ by $\frac{1}{n}$ and each term $1 - \frac{j}{n}$ by 1.

On the other hand, the social welfare of the maximum weight matching is

$$\sum_{i=1}^{\lfloor\sqrt{n}\rfloor}\left(1 - \frac{i-1}{n}\right) + \sum_{i=\lfloor\sqrt{n}\rfloor+1}^{n}\frac{n-i}{n^2} \geq \sum_{i=1}^{\lfloor\sqrt{n}\rfloor}\left(1 - \frac{i-1}{n}\right) \geq \lfloor\sqrt{n}\rfloor - 1 \geq \frac{\sqrt{n}}{4},$$

where the final inequality holds for $n \geq 4$, the approximation ratio is at most $\frac{20}{\sqrt{n}}$ for $n \geq 4$, and the bound holds trivially for $n < 4$. $\square$

Note that Lemma 4.5 bounds the performance of all ordinal mechanisms *assuming truthful reporting.* The right way to interpret the result is that even if we assume that agents are honest in their interaction with the mechanism, better social welfare guarantees are not achievable, due to informational limitations. The performance of non-truthful mechanisms in the presence of strategic play will be studied in Chapter 5.

Our final lemma provides a matching upper bound on the approximation ratio of any truthful mechanism.

**Lemma 4.6.** *Let $M$ be a truthful mechanism for one-sided matching for the unit-range representation. Then* $\text{ratio}(M) = O\left(n^{-1/2}\right)$.

*Proof.* By Lemma 4.1, we can assume that Mechanism $M$ is anonymous. Let $k \geq 2$ be a parameter to be chosen later and let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ be the valuation profile where

$$u_i(j) = \begin{cases} 1, & \text{for } j = i \\ \frac{2}{k} - \frac{j}{n}, & \text{for } 1 \leq j \leq k+1, j \neq i \\ \frac{n-j}{n^2}, & \text{otherwise} \end{cases} \qquad \forall i \in \{1, \ldots, k+1\}$$

$$u_i(j) = \begin{cases} 1, & \text{for } j = 1 \\ \frac{2}{k} - \frac{j}{n}, & \text{for } 2 \leq j \leq k+1 \\ \frac{n-j}{n^2}, & \text{otherwise} \end{cases} \qquad \forall i \in \{k+2, \ldots, n\}$$

For $i = 2, \ldots, k+1$, let $\mathbf{u^i} = (u'_i, u_{-i})$ be the valuation profile where all agents besides agent $i$ have the same valuations as in $\mathbf{u}$ and $u'_i = u_{k+2}$. Note that when agent $i$ on valuation profile $\mathbf{u^i}$, reports $u_i$ instead of $u'_i$, the resulting valuation profile is $\mathbf{u}$. Since $M$ is anonymous and $u'_i = u_1 = u_{k+2} = \ldots = u_n$, agent $i$ receives at most a uniform lottery among these agents on valuation profile $\mathbf{u^i}$ and so it holds that

$$\begin{aligned} \mathbb{E}[u'_i(M_i(\mathbf{u^i}))] &\leq \frac{1}{n-k+1} + \sum_{j=2}^{k+1} \frac{1}{n-k+1}\left(\frac{2}{k} - \frac{j}{n}\right) \\ &+ \sum_{j=k+2}^{n} \frac{1}{n-k+1} \cdot \frac{n-j}{n^2} \\ &\leq \frac{4}{n-k+1} \end{aligned}$$

Next observe that since $M$ is truthful, agent $i$ should not increase her expected utility by misreporting $u_i$ instead of $u'_i$ on valuation profile $\mathbf{u^i}$, that is,

$$\mathbb{E}[u'_i(M_i(\mathbf{u^i}))] \geq \mathbb{E}[u'_i(M_i(\mathbf{u}))] \tag{4.1}$$

For all $i = 2, \ldots, k+1$, let $p_i$ be the probability that $M_i(\mathbf{u}) = i$. Then, it holds that

$$\mathbb{E}[u'_i(M_i(\mathbf{u}))] \geq p_i\left(\frac{2}{k} - \frac{i}{n}\right) \geq p_i\left(\frac{2}{k} - \frac{k+1}{n}\right)$$

and by Inequality (4.1) we get

$$\begin{aligned} & p_i\left(\frac{2}{k} - \frac{k+1}{n}\right) \leq \frac{4}{n-k+1} \\ => \quad & p_i \leq \frac{4}{n-k+1} \cdot \frac{kn}{2n-k(k+1)} \leq \frac{4}{n-k} \cdot \frac{kn}{2n-(k+1)^2} \end{aligned}$$

Let

$$p = \frac{4}{n-k} \cdot \frac{kn}{2n-(k+1)^2},$$

i.e. for all $i$, $p_i \leq p$. We will next calculate an upper bound on the expected social welfare achieved by $M$ on valuation profile $\mathbf{u}$.

For item $j = 1$, the contribution to the social welfare is upper bounded by 1. Similarly, for each item $j = k+2, \ldots, n$, her contribution to the social welfare is upper bounded by $1/n$. Overall, the total contribution by item 1 and items $k+2, \ldots, n$ will be upper bounded by 2.

We next consider the contribution to the social welfare from items $j = 2, \ldots, k+1$. Define the random variables

$$X_j = \begin{cases} 1, & \text{if } M_j(\mathbf{u}) = j \\ \frac{2}{k} - \frac{j}{n}, & \text{otherwise} \end{cases}$$

The contribution from items $j = 2, \ldots, k+1$ is then $\sum_{j=2}^{k+1} X_j$ and so we get

$$\mathbb{E}\left[\sum_{j=2}^{k+1} X_j\right] = \sum_{j=2}^{k+1} \mathbb{E}\left[X_j\right] \leq \sum_{j=2}^{k+1} \left(p + \frac{2}{k} - \frac{j}{n}\right) \leq kp + 2$$

Overall, the expected social welfare of mechanism $M$ is at most $4 + pk$ while the social welfare of the optimal matching is

$$k + 1 + \sum_{i=k+2}^{n} \frac{n-i}{n^2},$$

which is at least $k$. Since

$$p = \frac{4}{n-k} \cdot \frac{kn}{2n - (k+1)^2},$$

the approximation ratio of $M$ then is

$$\text{ratio}(M) \leq \frac{4 + pk}{k} = \frac{4}{k} + \frac{4}{n-k} \cdot \frac{kn}{2n - (k+1)^2}$$

Let $k = \lfloor \sqrt{n} \rfloor - 1$ and note that $\sqrt{n} - 2 \leq k \leq \sqrt{n} - 1$. Then,

$$\begin{aligned} \text{ratio}(M) &\leq \frac{4}{k} + \frac{4}{n-k} \cdot \frac{kn}{2n - (k+1)^2} \\ &\leq \frac{4}{\sqrt{n} - 2} + \frac{4}{n - \sqrt{n} + 1} \cdot \frac{(\sqrt{n} - 1)n}{2n - (\sqrt{n})^2} \\ &\leq \frac{4}{\sqrt{n} - 2} + \frac{4}{\sqrt{n}} \leq \frac{12}{\sqrt{n}} + \frac{4}{\sqrt{n}} = \frac{16}{\sqrt{n}}, \end{aligned}$$

The last inequality holds for $n \geq 9$ and for $n < 9$ the bound holds trivially. This completes the proof. □

## 4.4 Unit-sum valuation functions

In this section, we assume that the representation of the valuation functions is unit-sum. We prove Theorem 4.1 using the following three lemmas.

**Lemma 4.7.** *For the unit-sum representation,* $\text{ratio}(RP) = \Omega\left(n^{-1/2}\right)$.

*Proof.* Let $\mathbf{u}$ be any unit-sum valuation profile and let $C$ be the constant in the bound from Lemma 4.4. Suppose first that $w^*(\mathbf{u}) < 4\sqrt{n}/C$; we will show that random priority guarantees an expected social welfare of 1, which proves the lower bound for this case. Consider any agent $i$ and notice that in random priority, the probability that the agent is picked by the $l$'th round is $l/n$, for any $1 \leq l \leq n$ and hence the probability of the agent getting one of her $l$ most preferred items is at least $l/n$. Let $u_i^l$ be agent $i$'s valuation for her $l$'th most preferred item; agent $i$'s expected utility for the first round is then at least $u_i^1/n$. For the second round, in the worst case, agent $i$'s most preferred item has already been matched to a different agent and so the expected utility of the agent for the first two rounds is at least $u_i^1/n + u_i^2/n$. By the same argument, agent $i$'s expected utility after $n$ rounds is at least $\sum_{i=1}^n u_i^l/n = 1/n$. Since this holds for each of the $n$ agents, the expected social welfare is at least 1.

Suppose $w^*(\mathbf{u}) \geq 4\sqrt{n}/C$; we will transform $\mathbf{u}$ to a unit-range valuation profile $\mathbf{u}''$. By Lemma 4.2, the optimal allocation is achieved by a run of random priority, so we know that in the optimal allocation at most one agent will be matched with her least preferred item. Consider the valuation profile $\mathbf{u}'$ where each agent $i$'s valuation for her least preferred item is set to 0 (unless already 0) and the rest are as in $\mathbf{u}$. Since the ordinal rankings of agents are unchanged, random priority performs worse on this valuation profile, and because of Lemma 4.2, $w^*(\mathbf{u}') \geq w^*(\mathbf{u}) - 1/n$. Next consider the profile

$$\mathbf{u}'' = \begin{pmatrix} \mathbf{u}' & \mathbf{1} \\ \mathbf{o}^T & 1 \end{pmatrix}$$

where $\mathbf{o} \in \mathbb{R}^n$ and $\mathbf{o}_j = (j-1)/n^5$. That is, $\mathbf{u}''$ has $n+1$ agents and items, where agents $1, \ldots, n$ have the same valuations for items $1, \ldots, n$ as in $\mathbf{u}'$, every agent has a valuation of 1 for item $n+1$, and agent $n+1$ only has a significant valuation for item $n+1$. Notice that $\mathbf{u}''$ is a unit-range valuation profile, and $w^*(\mathbf{u}'') \geq w^*(\mathbf{u}') + 1$. Furthermore, $\mathbb{E}\left[\sum_{i=1}^n u_i(RP_i(\mathbf{u}'))\right] \geq \mathbb{E}\left[\sum_{i=1}^n u_i(RP_i(\mathbf{u}''))\right] - 2$ and hence

$$\begin{aligned}
\frac{\mathbb{E}\left[\sum_{i=1}^n u_i(RP_i(\mathbf{u}))\right]}{w^*(\mathbf{u})} &\geq \frac{\mathbb{E}\left[\sum_{i=1}^n u_i(RP_i(\mathbf{u}'))\right]}{w^*(\mathbf{u}') + 1/n} \geq \frac{\mathbb{E}\left[\sum_{i=1}^n u_i(RP_i(\mathbf{u}''))\right] - 2}{w^*(\mathbf{u}'') + 1/n - 1} \\
&\geq \frac{\mathbb{E}\left[\sum_{i=1}^n u_i(RP_i(\mathbf{u}''))\right]}{w^*(\mathbf{u}'')} - \frac{2}{w^*(\mathbf{u}'')} \geq \frac{C}{\sqrt{n}} - \frac{2}{w^*(\mathbf{u})} \\
&\geq \frac{C}{\sqrt{n}} - \frac{2}{4\sqrt{n}/C} = \frac{C}{2\sqrt{n}}.
\end{aligned}$$

□

The next lemma bounds the approximation ratio of any ordinal (not necessarily truthful-in-expectation) mechanism.

**Lemma 4.8.** *Let $M$ be an ordinal mechanism for one-sided matching for the unit-sum representation. Then* $\text{ratio}(M) = O\left(n^{-1/2}\right)$.

*Proof.* Assume for ease of notation that $n$ is a square number; the proof can easily be adapted to the general case. By Lemma 4.1, we can assume without loss of generality that $M$ is anonymous. We will use the following valuation profile $\mathbf{u}$ where $\forall i \in \{1, ..., \sqrt{n}\}$:

$$u_i(j) = \begin{cases} 1 - \sum_{j \neq i} u_i(j), & \text{for } j = i, j \leq \sqrt{n} \\ \frac{n-j}{10n^5}, & \text{otherwise} \end{cases}$$

$$u_{i+l\sqrt{n}}(j) = \begin{cases} 1 - \sum_{j \neq i} u_i(j), & \text{for } j = i, j \leq \sqrt{n} \\ \frac{1}{\sqrt{n}} - \frac{j}{10n^2}, & \text{for } j \neq i, j \leq \sqrt{n} \\ \frac{n-j}{10n^5}, & \text{otherwise} \end{cases}, \quad l \in \{1, ..., \sqrt{n}-1\}$$

Intuitively, $\mathbf{u}$ is a valuation profile where for each $1 \leq i \leq \sqrt{n}$, agent $i$'s valuation function induces the same ordering as agent $(i + l \cdot \sqrt{n})$'s valuation function, for $1 \leq l \leq \sqrt{n} - 1$. For agent $i = 1, ..., \sqrt{n}$, because of anonymity, agent $i$ can at most expect to get a uniform lottery over all the items with each of the other $\sqrt{n} - 1$ agents that have the same ordering of valuations. For agents $\sqrt{n} + 1, \ldots, n$, the contribution to the social welfare from items $1, ..., \sqrt{n}$ is at most 2 since their valuations for these items are bounded by $2/\sqrt{n}$, and their contribution to the social welfare from items $\sqrt{n} + 1, ..., n$ is similarly bounded by 1. Thus we can write an upper bound on the expected welfare as:

$$\sum_{i=1}^{\sqrt{n}} \mathbb{E}\left[u_i(M_i(\mathbf{u}))\right] + \sum_{i=\sqrt{n}+1}^{n} \mathbb{E}\left[u_i(M_i(\mathbf{u}))\right] \leq \sum_{i=1}^{\sqrt{n}} \frac{1}{\sqrt{n}} + 3 = 4,$$

while the social welfare of the optimal allocation is at least $\sqrt{n} - 1/10n^3$. From this, we get $\text{ratio}(M) \leq 8/\sqrt{n}$. □

Finally, the upper bound for any truthful mechanism is given by the following lemma.

**Lemma 4.9.** *Let $M$ be a truthful-in-expectation mechanism for one-sided matching for the unit-sum representation. Then* $\text{ratio}(M) = O\left(n^{-1/2}\right)$.

*Proof.* Intuitively, the lemma is true because the valuation profile used in the proof of Lemma 4.6 can be easily modified in a way such that all rows of the matrices of valuations sum up to one. Specifically, consider the following valuation profile:

$$u_i(j) = \begin{cases} 1 - \sum_{j \neq i} u_i(j), & \text{for } j = i \\ \frac{2}{10k} - \frac{j}{10n}, & \text{for } 1 \leq j \leq k+1, j \neq i \\ \frac{n-j}{10n^2}, & \text{otherwise} \end{cases} \qquad \forall i \in \{1, \ldots, k+1\}$$

$$u_i(j) = \begin{cases} 1 - \sum_{j \neq 1} u_i(j), & \text{for } j = 1 \\ \frac{2}{10k} - \frac{j}{10n}, & \text{for } 1 < j \leq k+1 \\ \frac{n-j}{10n^2}, & \text{otherwise} \end{cases} \qquad \forall i \in \{k+2, \ldots, n\}$$

Note that this is exactly the same valuation profile used in the proof of Lemma 4.6 where all entries are divided by ten, except those where the valuation is 1, which are now equal to 1 minus the sum of the valuations for the rest of the items. This modification will only carry a factor of 1/10 through the calculations and hence the proven bound will be the same asymptotically. □

## 4.5 Extensions and special cases

### Allowing ties

Our results extend if we allow ties in the image of the valuation function. All of our upper bounds hold trivially. For the approximation guarantee of random priority, first the mechanism clearly must be equipped with some tie-breaking rule to settle cases where indifferences appear. For all natural (fixed before the execution of the mechanism) tie-breaking rules the lower bounds still hold. To see this, consider any valuation profile with ties and a tie-breaking rule for random priority. We can simply add sufficiently small quantities $\epsilon_{ij}$ to the valuation profile according to the tie-breaking rule and create a new profile without ties. The assignment probabilities of random priority will be exactly the same as for the version with ties, and random priority achieves an $\Omega(1/\sqrt{n})$ approximation ratio on the new profile. Then, since $\epsilon_{ij}$ were sufficiently small, the same bound holds for the original valuation profile.

### [0,1] valuation functions

All of our results apply to the extension of the unit-range representation where 0 is not required to be in the image of the function, that is $\max_j u_i(j) = 1$ and for all $j$, $u_i(j) \in [0, 1]$. This representation captures scenarios where agents are allowed to be (more or less) indifferent between every single item. Since every unit-range valuation profile is also a valid profile for this representation, the upper bounds hold trivially. For the approximation ratio of random priority, we obtain the following corollary.

**Corollary 4.1.** *The approximation ratio of random priority, for the setting with* $[0,1]$ *valuation functions is* $\Omega\left(n^{-1/2}\right)$.

*Proof.* Let $\mathbf{u}$ be any $[0,1]$ valuation profile and let $C$ be the constant in the lower bound of Lemma 4.4. Similarly to the proof of Lemma 4.7, notice that by Lemma 4.2, the optimal matching on $\mathbf{u}$ matches at most one agent with her least-preferred item. So let $\mathbf{u}'$ be the valuation profile in which each agent $i$ has the same valuation for every item as in profile $\mathbf{u}$, except the valuation for her least preferred item is set to 0 (if it is not already 0). Doing this, the expected social welfare of random priority becomes smaller, and $w^*(\mathbf{u}') \geq w^*(\mathbf{u}) - 1$. Notice that $\mathbf{u}'$ is now unit-range and by Lemma 4.4, we get that

$$\begin{aligned}
\frac{\mathbb{E}\left[\sum_{i=1}^n u_i(RP_i(\mathbf{u})\right]}{w^*(\mathbf{u})} &\geq \frac{\mathbb{E}\left[\sum_{i=1}^n u_i(RP_i(\mathbf{u}'))\right]}{w^*(\mathbf{u}')+1} \\
&\geq \frac{\mathbb{E}\left[\sum_{i=1}^n u_i(RP_i(\mathbf{u}'))\right]}{2w^*(\mathbf{u}')} \\
&\geq C/2\sqrt{n}
\end{aligned}$$

□

## 4.6 Improved approximations for small input sizes

Theorem 4.1 implies that random priority is indeed the best truthful mechanism for the problem, when considering the asymptotic behavior of mechanisms. We now consider non-asymptotic behavior by studying the case when $n = 3$ and present a non-ordinal mechanism that achieves better bounds than any ordinal mechanism, when the representation of the valuation functions is unit-range.

Since $n = 3$ and the representation is unit-range, the valuation function of an agent $i$ can be completely specified by a tuple $(a \succ_i b \succ_i c, \alpha_i)$ where $a \succ_i b \succ_i c$ is the ordering of items $1, 2$ and $3$ and $\alpha_i$ is the valuation of agent $i$ for her second to most preferred item. For example, the valuation function $u_i(1) = 0.6, u_i(2) = 1, u_i(3) = 0$ can be written as $(2 \succ_i 1 \succ_i 3, 0.6)$. By this, we can generate all possible valuation profiles with three agents and three items using $\alpha_1, \alpha_2, \alpha_3$ as variables. By anonymity and neutrality, we can compress the search space drastically (by pruning symmetric profiles) and then calculate the ratios on all valuation profiles as functions of $\alpha_1, \alpha_2, \alpha_3$.

For the case of random priority, it is easy to see where those ratios are minimized and the approximation ratio is the worst ratio over all valuation profiles that we consider. It turns out that the approximation ratio of random priority for $n = 3$ is $2/3$. In fact, random priority achieves the optimal approximation ratio among all ordinal mechanisms. To see this, observe that the worst-case ratio of random priority is given by the following ordered valuation profile:

$$\mathbf{u} = \begin{pmatrix} 1 & 1-\epsilon & 0 \\ 1 & \epsilon & 0 \\ 1 & \epsilon & 0 \end{pmatrix}$$

Notice that when $\epsilon$ tends to 0, the ratio of any ordinal mechanism on $\mathbf{u}$ tends to $2/3$. Using a very similar construction, the bound can be extended to any number of agents.

**Theorem 4.2.** *For $n$ agents, the approximation ratio of any ordinal mechanism is at most $\frac{1}{n-1} + \frac{n-2}{2n}$.*

In particular, for $n = 3, 4$ and 5 we obtain bounds of $2/3$, $7/12$ and $11/20$ respectively.

Next, consider the one-agent mechanism that given the reported valuation function matches the agent with her most preferred item with probability $(6-2\alpha^3)/8$, with her second to most preferred item with probability $(1+3\alpha^2)/8$ and with her least preferred item with probability $(1-3\alpha^2+2\alpha^3)/8$. This mechanism, that we will refer to as the *cubic lottery* was presented in [74] and proven by the authors to be truthful. Now consider the following mechanism for the one-sided matching problem:

*Mechanism* (Hybrid mechanism - HM). Uniformly at random fix a permutation $\sigma \in S$ of the agents. Match agent $\sigma(1)$ with item $j \in \{1, 2, 3\}$ with probabilities given by the cubic lottery. Match agent $\sigma(2)$ with her favorite item from the set of still available items. Match agent 3 with the remaining item.

Since the permutation of agents is fixed uniformly at random, this mechanism is truthful. We prove the following theorem.

**Theorem 4.3.** ratio$(HM) = 0.699$.

*Proof.* Observe that the mechanism is anonymous and neutral, hence we can follow the same procedure described above and generate all possible valuation profiles with $n = 3$ and then prune the profile space to obtain a relatively small number of valuation profiles. The ratio on a valuation profile $\mathbf{u}$ will be a function of the form $G(\alpha_1, \alpha_2, \alpha_3) = g_1(\alpha_1, \alpha_2, \alpha_3)/g_2(\alpha_1, \alpha_2, \alpha_3)$ where $g_1 : V_1 \times V_2 \times V_3 \to \mathbb{R}$ is a non-linear function corresponding to the expected social welfare and $g_2 : V_1 \times V_2 \times V_3 \to \mathbb{R}$ is a linear function corresponding to the maximum weight matching on $\mathbf{u}$. Then, to calculate the approximation ratio, we need to solve a non-linear program of the form "minimize $G(\alpha_1, \alpha_2, \alpha_3)$ subject to $\alpha_1, \alpha_2, \alpha_3 \in [0, 1]$" for every valuation profile. The minimum over all valuation profiles is the approximation ratio of the mechanism. We use standard non-linear programming software to obtain the bound; we consider such a computer-assisted proof sufficient. □

Notice that the approximation ratio achieved by the hybrid mechanism is strictly larger than the approximation ratio of any ordinal mechanism. The next question would be whether we can prove similar bounds for other (small) values of $n$. We might be able to extend the technique used above to $n = 4$ by relying heavily on computer-assisted programs to generate the valuation profiles and calculate the ratios but it would be difficult to extend it to any larger number of agents, since the valuation profile space becomes quite large. A different approach for proving approximation guarantees for concrete values of $n$ would be interesting. Finally, it would be interesting to investigate whether random priority obtains the (non-asymptotic) optimal approximation ratio among ordinal mechanisms for all values of $n$.

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# Chapter 5

# The price of anarchy of mechanisms

In this chapter, we consider the efficiency of *all* mechanisms, including non-truthful, cardinal and randomized ones, in terms of their *price of anarchy*, for the social welfare objective. The main result of the chapter is that random priority and probabilistic serial achieve a price of anarchy of $\Omega(1/\sqrt{n})$ which is asymptotically *optimal* among all mechanisms for the problem.

## 5.1 Introduction

In Chapter 4, we (mainly) considered truthful mechanisms in terms of the approximation ratio and proved that random priority is in fact the best (asymptotically) mechanism for one-sided matching settings. The next question that naturally comes to mind is "What about other mechanisms, that are not truthful?". To answer such a question, we need to find a good way to evaluate the performance of non-truthful mechanisms. In Chapter 4, we used the approximation ratio to evaluate the efficiency of non-truthful, ordinal mechanisms, under the assumption that it is only the limited information that results in efficiency loss and not the strategic behaviour of agents.

But what if we wanted to account for such behaviour as well? As we mentioned in the introduction, one way to handle strategic behaviour, other than truthfulness, is to let agents strategize and evaluate the stable outcomes of the *induced game*. In our setting, such a game is played implicitly before the "revelation phase" of the mechanism; agents see the reported valuation functions of others and adjust their reports accordingly, based on the rules of the mechanism used and the associated payoffs. Once an agreement has been reached, the reported valuation profile, which we will call a *strategy profile*, is given as input to a mechanism. By an "agreement", we mean a set of reports such that no agent has any incentive to change her reported valuation. In game theory, this solution is known as the *Nash equilibrium* [117]. Note that a game corresponding to a mechanism might have multiple Nash equilibria but it could also be the case that it does not have any Nash equilibria.

Under the assumption that strategic play will lead to stable outcomes,[1] the standard measure of efficiency in the computer science literature is the *price of anarchy* [103], i.e. the minimum ratio over all valuation profiles of the social welfare achieved by the mechanism in the worst Nash equilibrium[2] over the social welfare of the optimal assignment. Note that similarly to the approximation ratio, the price of anarchy provides worst-case guarantees for the performance of a mechanism.

Since we are now dealing with non-truthful mechanisms, the inputs are not necessarily the agents' true valuations. For that reason, we will redefine the inputs to be *strategies*, i.e. reported valuations which are functions of the true valuations. For general (cardinal) mechanisms, the valuation space and the strategy space will be the same; for ordinal mechanisms, we will define the strategy space to be the set of all permutations of $n$ items, which is sufficient to represent the inputs.

## Our results

In this chapter, we study and compare mechanisms based on their price of anarchy. We will consider both unit-range and unit-sum as the canonical representation of the valuation functions, but since our results for unit-sum are cleaner, we will focus on them for the main exposition of the results and discuss the unit-range normalization as an extension.

As our main contribution, for the unit-sum representation, we bound the inefficiency of the two dominant mechanisms in the literature of one-sided matching problems, *probabilistic serial* and *random priority*. Note that random priority is truthful but it does have other equilibria as well. Our results in Chapter 4 bound the performance of the mechanism in the truthtelling equilibria; here we build upon those results and prove an efficiency guarantee for all equilibria of the mechanism, not just the truthtelling ones. Similar approaches have been done for truthful mechanisms like the second price auction in settings with money [22]. We complement this analysis by showing a *matching* upper bound that applies to *all* cardinal (randomized) mechanisms. As a result, we conclude that those two ordinal mechanisms are optimal.

We start by proving an $\Omega(1/\sqrt{n})$ price of anarchy guarantee for the two mechanisms mentioned above and then we prove that no mechanism can

[1] Assuming of course that Nash equilibria exist for the mechanism in question.

[2] In fact, the price of anarchy can be similarly defined for any solution concept, not just the Nash equilibrium, as we will see thoughout the chapter.

achieve a price of anarchy better than $O(1/\sqrt{n})$. The fact that those mechanisms are ordinal is quite interesting; our results suggest that similarly to the results in Chapter 4, even if we allow mechanisms to use the cardinal nature of the reports, we can not achieve better efficiency guarantees.

We study both the *complete information* and the *incomplete information* settings. In games of complete information, agents know each others' valuation functions and choose their strategies with that information at hand. In settings of incomplete information, agents' valuations are drawn from some known prior distributions; other agents know the distributions but not the actual valuations. We stress that, in analogy to the literature in auctions [32, 56, 64, 76, 139], in the complete information case, our price of anarchy bounds extend from the simplest solutions concepts of pure or mixed Nash equilibrium to the very general concepts of correlated and coarse-correlated equilibria. For the incomplete information case, we show how our results extend for Bayes-Nash equilibria.

We also consider *deterministic* mechanisms and prove that the pure price of anarchy of any mechanism (including cardinal mechanisms) is bounded by $O(1/n^2)$. This result suggests that randomization is essential for non-trivial efficiency guarantees to be achievable.

As an extension to our main results, we consider the price of stability [10], a more optimistic measure of efficiency than the price of anarchy. The price of stability bounds the performance of the mechanism at the *best* Nash equilibrium instead of the worst one. We prove that under a mild "proportionality-like" property, our upper bound of $O(1/\sqrt{n})$ extends to this case as well.

Finally, we prove that our lower bounds of $\Omega(1/\sqrt{n})$ for random priority and probabilistic serial extend to the unit-range representation. For deterministic mechanisms, we prove a price of anarchy upper bound of $O(1/n)$ whereas for general randomized mechanisms, we prove an upper bound of $O(1/n^{1/4})$ with respect to $\epsilon$-approximate Nash equilibria. Proving a $O(1/\sqrt{n})$ matching upper bound for the unit-range representation as well is certainly a very interesting problem.

### Related work

The inefficiency of games has been studied in various contexts in algorithmic game theory [47, 53, 54, 83, 129, 130], with the price of anarchy objective in effect. In mechanism design and settings with money, such as combinatorial auctions, a recent body of literature studies the price of anarchy of item-bidding auctions with second-price or first-price payment rules (see [32, 56, 64, 76, 139] for an non-exclusive list). Similar approaches have been adopted in settings without money [46, 55, 75, 143]; we will also use the same approach for *divisible item allocation* in Chapter 7.

Our lower bounds are established by probabilistic serial; we quantify the efficiency loss of the mechanism due to the strategic behaviour of the agents.

The strategic aspects of the mechanism[3] were, somewhat surprisingly, only recently studied. Aziz et al. [20] prove that although sequences of best responses might cycle, the mechanism is guaranteed to have pure Nash equilibria. This is important for our investigations, because we can use the pure Nash equilibrium as the solution concept[4] for analyzing the price of anarchy of probabilistic serial, before moving to more general solution concepts. The authors in [20] as well as in [21] also study computational aspects of manipulating the mechanism. Ekici and Kesten [71] prove that some of the nice prorties of the mechanism are not guaranteed to exist in the *ordinal equilibria* of the induced game. An ordinal equilibrium is a strategy profile (a set of rankings) such that no agent has any incentive to deviate for *at least one* von Neumann-Morgenstern utility function consistent with her true ranking.

## 5.2 Preliminaries

The setting studied here is the one-sided matching setting presented in Chapter 3 and the valuation functions are defined similary. In constrast to previous chapters, however, we will define the inputs of the mechanisms to account for strategic play, which was not an issue when considering truthful mechanisms. We define $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ to be a pure strategy profile, where $s_i$ is the *reported* valuation vector of agent $i$. For general mechanisms, the set of all valuation profiles $V^n$ is also the set of all pure strategy profiles. A *direct revelation mechanism* without money is a function $M : V^n \to O$ mapping strategy profiles to matchings. For a randomized mechanism, we define $M$ to be a random map $M : V^n \to O$. Similarly to Chapter 3, let $M_i(\mathbf{s})$ denote the restriction of the outcome of the mechanism to the $i$'th coordinate. For randomized mechanisms, we will introduce some new notation that will be used throughout this chapter; let

$$p_{ij}^{M,\mathbf{s}} = \Pr[M_i(\mathbf{s}) = j] \quad \text{and} \quad p_i^{M,\mathbf{s}} = (p_{i1}^{M,\mathbf{s}}, \ldots, p_{in}^{M,\mathbf{s}}).$$

When it is clear from the context, we will drop one or both of the superscripts from the terms $p_{ij}^{M,\mathbf{s}}$. The *utility* of an agent from the outcome of a deterministic mechanism $M$ on input strategy profile $\mathbf{s}$ is simply $u_i(M_i(\mathbf{s}))$. For randomized mechanisms, an agent's utility is $\mathbb{E}[u_i(M_i(\mathbf{s}))] = \sum_{j=1}^{n} p_{ij}^{M,\mathbf{s}} u_{ij}$.

Again, we will be interested in ordinal mechanisms; we have defined ordinal mechanisms in Chapter 3 but we redefine them in terms of strategies here.

**Definition 5.1.** A mechanism $M$ is *ordinal* if for any strategy profiles $\mathbf{s}, \mathbf{s}'$ such that for all agents $i$ and for all items $j, \ell$, $s_{ij} < s_{i\ell} \Leftrightarrow s'_{ij} < s'_{i\ell}$, it holds

[3] In terms of *strategic play*, because the properties of the mechanism in terms of truthfulness were considered already in [35].

[4] Note that the mechanism is ordinal and hence it always has mixed Nash equilibria. This is not true for general mechanisms though, where the strategy space is continuous.

that $M(\mathbf{s}) = M(\mathbf{s}')$. A mechanism for which the above does not necessarily hold is *cardinal*.

For ordinal mechanisms, we define the strategy space to be the set of all permutations of $n$ items instead of the space of valuation functions $V^n$. A strategy $s_i$ of agent $i$ is a *preference ordering* of items $(a_1, a_2, \dots, a_n)$ where $a_\ell \succ a_k$ for $\ell < k$. We will write $j \succ_i j'$ to denote that agent $i$ prefers item $j$ to item $j'$ according to her true valuation function and $j \succ_{s_i} j'$ to denote that she prefers item $j$ to item $j'$ according to her strategy $s_i$. When it is clear from the context, we abuse the notation slightly and let $u_i$ denote the truthtelling strategy of agent $i$, even when the mechanism is ordinal. Note that in the setting of this chapter, agents can be indifferent between items and hence the preference order can be a weak ordering.[5]

Recall the definitions of *anonymity* and *neutrality* from chapter 4. The definitions in terms of strategies are straightforward adaptations.

An *equilibrium* is a strategy profile in which no agent has an incentive to deviate to any different strategy. In this chapter, we will first focus on the concept of *pure Nash equilibrium*, formally,

**Definition 5.2.** A strategy profile $\mathbf{s}$ is a *pure Nash equilibrium* if $u_i(M_i(\mathbf{s})) \geq u_i(M_i(s_i', s_{-i}))$ for all agents $i$, and pure deviating strategies $s_i'$.

In Section 5.6, we extend our results to more general equilibrium notions as well as the setting of incomplete information, where agents' values are drawn from known distributions. Let $S_{\mathbf{u}}^M$ denote the set of all pure Nash equilibria of mechanism $M$ under truthful valuation profile $\mathbf{u}$. The measure of efficiency that we will use is the (pure) *price of anarchy*[6]

$$PoA(M) = \inf_{\mathbf{u} \in V^n} \frac{\min_{\mathbf{s} \in S_{\mathbf{u}}^M} SW_M(\mathbf{u}, \mathbf{s})}{SW_{OPT}(\mathbf{u})}$$

where $SW_M(\mathbf{u}, \mathbf{s}) = \sum_{i=1}^n \mathbb{E}[u_i(M_i(\mathbf{s}))]$ is the expected *social welfare* of mechanism $M$ on strategy profile $\mathbf{s}$ under true valuation profile $\mathbf{u}$, and $SW_{OPT}(\mathbf{u}) = \max_{\mu \in O} \sum_{i=1}^n u_i(\mu_i)$ is the social welfare of the optimal matching. Let $OPT(\mathbf{u})$ be the optimal matching on profile $\mathbf{u}$. In Section 5.5 we will define and use a more optimistic notion of efficiency, the *price of stability*.

## 5.3 Price of anarchy guarantees

We start from our price of anarchy lower bounds, for the two mechanisms we consider.

[5] We will assume that both the valuations and the strategies can exhibit ties, although only the former is important for our results.

[6] The price of anarchy is usually defined as the inverse of this ratio and that is also the definition used in the paper associated with this chapter. Here, we use this ratio to maintain consistency throughout the thesis.

### Probabilistic serial

We first consider *probabilistic serial*, which we will refer to as $PS$ for short. Recall the definition of the mechanism from Chapter 3; it is straightforward to apply the definition to the strategies instead of the true valuations.

We prove that the price of anarchy of $PS$ is $\Omega(1/\sqrt{n})$. We start by proving the following lemma, which states that in a pure Nash equilibrium of the mechanism, an agent's utility cannot be much worse than what her utility would be if she were consuming the item she is matched with in the optimal allocation from the beginning of the mechanism until the item is entirely consumed. Let $t_j(\mathbf{s})$ be the time when item $j$ is entirely consumed on profile $\mathbf{s}$ under $PS(\mathbf{s})$.

**Lemma 5.1.** *Let* $\mathbf{u}$ *be any profile of true agent valuations and let* $\mathbf{s}$ *be a pure Nash equilibrium. Let* $i$ *be any agent and let* $j = OPT_i(\mathbf{u})$. *Then,* $\sum_{\ell=1}^{n} p_{i\ell}^{\mathbf{s}} u_{i\ell} \geq \frac{1}{4} \cdot t_j(\mathbf{s}) \cdot u_{ij}$.

*Proof.* Let $\mathbf{s}' = (s_i', s_{-i})$ be the strategy profile obtained from $\mathbf{s}$ when agent $i$ deviates to the strategy $s_i'$ where $s_i'$ is some strategy such that $j \succ_{s_i} \ell$ for all items $\ell \neq j$. If $s_i' = s_i$, i.e. agent $i$ is already consuming item $j$ from the beginning, her utility $u_i(PS_i(\mathbf{s})) = \sum_{\ell=1}^{n} p_{i\ell}^{\mathbf{s}} u_{i\ell}$ is at least $t_j(\mathbf{s}) \cdot u_{ij}$ and we are done. Hence assume that $s_i \neq s_i'$. Obviously, agent $i$'s utility $u_i(PS_i(\mathbf{s}')) = \sum_{l=1}^{n} p_{il}^{\mathbf{s}'} u_{il}$ is at least $t_j(\mathbf{s}') \cdot u_{ij}$ so since $\mathbf{s}$ is a pure Nash equilibrium, it suffices to prove that $t_j(\mathbf{s}') \geq \frac{1}{4} \cdot t_j(\mathbf{s})$.

First, note that if agent $i$ is the only one consuming item $j$ for the duration of the mechanism, then $t_j(\mathbf{s}') = 1$ and we are done. So assume at least one other agent consumes item $j$ at some point, and let $\tau$ be the time when the first agent besides agent $i$ starts consuming item $j$ in $\mathbf{s}'$. Obviously, $t_j(\mathbf{s}') > \tau$, therefore if $\tau \geq \frac{1}{4} \cdot t_j(\mathbf{s})$ then $t_j(\mathbf{s}') \geq \frac{1}{4} \cdot t_j(\mathbf{s})$ and we are done. So assume that $\tau < \frac{1}{4} \cdot t_j(\mathbf{s})$. Next observe that in the interval $[\tau, t_j(\mathbf{s}')]$, agent $i$ can consume at most half of what remains of item $i$ because there exists at least one other agent consuming the item for the same duration. Overall, agent $i$'s consumption is at most $\frac{1}{2} + \frac{1}{4} t_j(\mathbf{s})$ so at least $\frac{1}{2} - \frac{1}{4} t_j(\mathbf{s})$ of the item will be consumed by the rest of the agents.

Now consider all agents other than $i$ in profile $\mathbf{s}$ and let $\alpha$ be the the amount of item $j$ that they have consumed by time $t_j(s)$. Notice that the total consumption speed of an item is non-decreasing in time which means in particular that for any $0 \leq \beta \leq 1$, agents other than $i$ need at least $\beta t_j(s)$ time to consume $\alpha \cdot \beta$ in profile $s$. Next, notice that since agent $i$ starts consuming item $j$ at time 0 in $\mathbf{s}'$ and all other agents use the same strategies in $\mathbf{s}$ and $\mathbf{s}'$, it holds that every agent $k \neq i$ starts consuming item $j$ in $\mathbf{s}'$ no sooner than she does in $\mathbf{s}$. This means that in profile $\mathbf{s}'$, agents other than $i$ will need more time to consume $\beta \cdot \alpha$; in particular they will need at least $\beta t_j(s)$ time, so $t_j(\mathbf{s}') \geq \beta t_j(s)$. However, from the previous paragraph we know that they will consume at least $\frac{1}{2} - \frac{1}{4}t_j(\mathbf{s})$, so letting $\beta = \frac{1}{\alpha}\left(\frac{1}{2} - \frac{1}{4}t_j(\mathbf{s})\right)$ we get

$$t_j(\mathbf{s}') \geq \beta t_j(s) \geq t_j(\mathbf{s})\left(\frac{1}{2} - \frac{1}{4}t_j(\mathbf{s})\right)\frac{1}{\alpha} \geq t_j(\mathbf{s})\left(\frac{1}{2} - \frac{1}{4}t_j(\mathbf{s})\right) \geq \frac{1}{4} \cdot t_j(\mathbf{s})$$

□

We can now prove the pure price of anarchy guarantee of the mechanism.

**Theorem 5.1.** *The pure price of anarchy of probabilistic serial is* $\Omega(1/\sqrt{n})$.

*Proof.* Let $\mathbf{u}$ be any profile of true agent valuations and let $\mathbf{s}$ be any pure Nash equilibrium. First, note that by reporting truthfully, every agent $i$ can get an allocation that is at least as good as $\left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$, regardless of other agents' strategies. To see this, first consider time $t = 1/n$ and observe that during the interval $[0, 1/n]$, agent $i$ is consuming her favorite item (say $a_1$) and hence $p_{ia_1} \geq \frac{1}{n}$. Next, consider time $\tau = 2/n$ and observe that during the interval $[0, 2/n]$, agent $i$ is consuming one or both of her two favorite items ($a_1$ and $a_2$) and hence $p_{ia_1} + p_{ia_2} \geq \frac{2}{n}$. By a similar argument, for any $k$, it holds that $\sum_{j=1}^{n} p_{ia_j} \geq \frac{k}{n}$. This implies that regardless of other agents' strategies, agent $i$ can achieve a utility of at least $\frac{1}{n}\sum_{j=1}^{n} u_{ij}$. Since $\mathbf{s}$ is a pure Nash equilibrium, it holds that $u_i(PS_i(\mathbf{s})) \geq \frac{1}{n}\sum_{j=1}^{n} u_{ij}$ as well. Summing over all agents, we get that

$$SW_{PS}(\mathbf{u}, \mathbf{s}) \geq \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} u_{ij} = 1.$$

If $SW_{OPT}(\mathbf{u}) \leq \sqrt{n}$ then we are done, so assume $SW_{OPT}(\mathbf{u}) > \sqrt{n}$.

Because $PS$ is neutral, we can assume $t_j(\mathbf{s}) \leq t_{j'}(\mathbf{s})$ for $j < j'$ without loss of generality. Observe that for all $j = 1, \ldots, n$, it holds that $t_j(\mathbf{s}) \geq \frac{j}{n}$. This is true because for any $t \in [0, 1]$, by time $t$, exactly $tn$ mass of items must have been consumed by the agents. Since $j$ is the $j$th item that is entirely consumed, by time $t_j(\mathbf{s})$, the mass of items that must have been consumed is at least $j$. By this, we get that $t_j(\mathbf{s})n \geq j$, which implies $t_j(\mathbf{s}) \geq \frac{j}{n}$.

For each $j$ let $i_j$ be the agent that gets item $j$ in the optimal allocation and for ease of notation, let $w_{i_j}$ be her valuation for the item. Now by Lemma 5.1, it holds that $u_{i_j}(PS(\mathbf{s})) \geq \frac{1}{4} \cdot \frac{j}{n} \cdot w_{i_j}$ and $SW_{PS}(\mathbf{u}, \mathbf{s}) \geq \frac{1}{4}\sum_{j=1}^{n} \frac{j}{n} w_{i_j}$. The price of anarchy is then at least

$$\frac{\frac{1}{4n}\sum_{j=1}^{n} j \cdot w_{i_j}}{\sum_{j=1}^{n} w_{i_j}}.$$

Consider the case when the above ratio is minimized and let $k$ be an integer such that $k \leq \sum_{j=1}^{n} w_{i_j} \leq k+1$. Then it must be that $w_{i_j} = 1$ for $j = 1, \ldots, k$ and $w_{i_j} = 0$, for $k + 2 \leq i_j \leq n$. Hence the minimum ratio is $\frac{aw_{i_{k+1}}+b}{k+w_{i_{k+1}}}$, for some $a$, $b > 0$, which is monotone for $w_{i_{k+1}}$ in $[0, 1]$. Therefore, the minimum value of $\frac{aw_{i_{k+1}}+b}{k+w_{i_{k+1}}}$ is achieved when either $w_{i_{k+1}} = 0$ or $w_{i_{k+1}} = 1$. As a result, the minimum value of the ratio is obtained when $\sum_{i=1^n} w_{i_{k+1}} = k$ for some $k$. By simple calculations, the price of anarchy should be at least

$$\frac{\sum_{j=1}^{k} \frac{j}{n}}{4k} \leq \frac{\frac{k(k-1)}{2n}}{4k} = \frac{k-1}{8n},$$

so the price of anarchy is maximized when $k$ is minimized. By the argument earlier, $k > \sqrt{n}$ and hence the ratio is $\Omega(1/\sqrt{n})$. □

In Section 5.6, we extend Theorem 5.1 to broader solution concepts and the incomplete information setting.

### Random priority

We now turn our attention to another mechanism, *random priority*, or *RP* for short. Recall the definition of the mechanism from Chapter 3. Random priority is truthful, but it does have other equilibria as well. From Chapter 4, we know that the welfare of the mechanism in the truthtelling equilibria and the maximum social welfare differ by a multiple of at most $O(\sqrt{n})$. We prove here that this ratio is guaranteed in all equilibria of the mechanism, for any of the equilibrium notions. We start with an interesting lemma when agents' valuation for items are all distinct.

**Lemma 5.2.** *If valuations are distinct, the social welfare is the same in all mixed Nash equilibria of random priority.*

*Proof.* Let $i$ be an agent, and let $B$ be a subset of the items. Let $\mathbf{s}$ be a mixed Nash equilibrium with the property that with positive probability, $i$ will be chosen to select an item at a point when $B$ is the set of remaining items. In that case (by distinctness of $i$'s values), $i$'s strategy should place agent $i$'s favourite item in $B$ on the top of the preference list among items in $B$. Suppose that for items $j$ and $j'$, there is no set of items $B$ that may be offered to $i$ with positive probability, in which either $j$ or $j'$ is optimal. Then $i$ may rank them either way, i.e. can announce $j \succ_i j'$ or $j' \succ_i j$. However, that choice has no effect on the other agents, in particular it cannot affect their social welfare. □

Given Theorem 4.1, Lemma 5.2 implies the following.

**Corollary 5.1.** *If valuations are distinct, the price of anarchy of random priority is* $\Theta(1/\sqrt{n})$.

The same guarantee on the price of anarchy holds even when the true valuations of agents are not necessarily distinct.

**Theorem 5.2.** *The price of anarchy of random priority is $\Omega(1/\sqrt{n})$, even if valuations are not distinct.*

*Proof.* We know from Chapter 4 that the social welfare of random priority given truthful reports, is within $O(\sqrt{n})$ of the social optimum. The social welfare of a (mixed) Nash equilibrium $\mathbf{q}$ cannot be worse than the worst pure profile from $\mathbf{q}$ that occurs with positive probability, so let $\mathbf{s}$ be such a pure profile. We will say that agent $i$ *misranks* items $j$ and $j'$ if $j \succ_i j'$, but $j' \succ_{s_i} j$.

If an agent misranks two items for which she has distinct values, it is because she has 0 probability in $\mathbf{s}$ to receive either item. So we can change $\mathbf{s}$ so that no items are misranked, without affecting the social welfare or the allocation. For items that the agent values equally (which are then not misranked) we can apply arbitrarily small perturbations to make them distinct. Profile $\mathbf{s}$ is thus consistent with rankings of items according to perturbed values and is truthful with respect to these values, which, being arbitrarily close to the true ones, have optimum social welfare arbitrarily close to the true optimal social welfare. □

Theorem 5.2 can be extended to solution concepts more general than the mixed Nash equilibrium. Again, the details are presented in Section 5.6.

## 5.4 Upper bounds

Here, we prove our main upper bound. Note that the result holds for any mechanism, including randomized and cardinal mechanisms. Since we are interested in mechanisms with good properties, it is natural to consider those mechanisms that have pure Nash equilibria.

**Theorem 5.3.** *The pure price of anarchy of any mechanism for one-sided matching is $O(1/\sqrt{n})$.*

*Proof.* Assume $n = k^2$ for some $k \in \mathbb{N}$. Let $M$ be a mechanism and consider the following valuation profile $\mathbf{u}$. There are $\sqrt{n}$ sets of agents and let $G_j$ denote the $j$-th set. For every $j \in \{1, \ldots, \sqrt{n}\}$ and every agent $i \in G_j$, it holds that $u_{ij} = \frac{1}{n} + \alpha$ and $u_{ik} = \frac{1}{n} - \frac{\alpha}{n-1}$, for $k \neq j$, where $\alpha$ is sufficiently small. Let $\mathbf{s}$ be a pure Nash equilibrium and for every set $G_j$, let $i_j = \arg\min_{i \in G_j} p_{ij}^{M,\mathbf{s}}$ (break ties arbitrarily). Observe that for all $j = 1, \ldots, \sqrt{n}$, it holds that $p_{i_j j}^{M,\mathbf{s}} \leq \frac{1}{\sqrt{n}}$ and let $I = \{i_1, i_2, \ldots, i_{\sqrt{n}}\}$. Now consider the valuation profile $\mathbf{u}'$ where:

- For every agent $i \notin I$, $u'_i = u_i$.
- For every agent $i_j \in I$, let $u'_{i_j j} = 1$ and $u'_{i_j k} = 0$ for all $k \neq j$.

We claim that $\mathbf{s}$ is a pure Nash equilibrium under $\mathbf{u}'$ as well. For agents not in $I$, the valuations have not changed and hence they have no incentive to deviate. Assume now for contradiction that some agent $i \in I$ whose most preferred item is item $j$ could deviate to some beneficial strategy $s'_i$. Since agent $i$ only values item $j$, this would imply that $p_{ij}^{M,(s'_i,s_{-i})} > p_{ij}^{M,\mathbf{s}}$. However, since agent $i$ values all items other than $j$ equally under $u_i$ and her most preferred item is item $j$, such a deviation would also be beneficial under profile $\mathbf{u}$, contradicting the fact that $\mathbf{s}$ is a pure Nash equilibrium.

Now consider the expected social welfare of $M$ under valuation profile $\mathbf{u}'$ at the pure Nash equilibrium $\mathbf{s}$. For agents not in $I$ and taking $\alpha$ to be less than $\frac{1}{n^3}$, the contribution to the social welfare is at most 1. For agents in $I$, the contribution to the welfare is then at most $\frac{1}{\sqrt{n}} \cdot \sqrt{n} + 1$ and hence the expected social welfare of $M$ is at most 3. As the optimal social welfare is at least $\sqrt{n}$, the bound follows. □

Interestingly, if we restrict our attention to *deterministic* mechanisms, then we can prove that only trivial pure price of anarchy guarantees are achievable.

**Theorem 5.4.** *The pure price of anarchy of any deterministic mechanism for one-sided matching is $O(1/n^2)$.*

*Proof.* Let $M$ be a deterministic mechanism that always has a pure Nash equilibrium. Let $\mathbf{u}$ be a valuation profile such that for for all agents $i$ and $i'$, it holds that $u_i = u_{i'}$, $u_{i1} = \frac{1}{n} + \frac{1}{n^3}$ and $u_{ij} > u_{ik}$ for $j < k$. Let $\mathbf{s}$ be a pure Nash equilibrium for this profile and assume without loss of generality that $M_i(\mathbf{s}) = i$.

Now fix another true valuation profile $\mathbf{u}'$ such that $u'_1 = u_1$ and for agents $i = 2, \dots, n$, $u'_{i,i-1} = 1 - \epsilon'_{i,i-1}$ and $u_{ij} = \epsilon'_{ij}$ for $j \neq i-1$, where $0 \leq \epsilon'_{ij} \leq \frac{1}{n^3}$, $\sum_{j \neq i-1} \epsilon'_{ij} = \epsilon'_{i,i-1}$ and $\epsilon'_{ij} > \epsilon'_{ik}$ if $j < k$ when $j, k \neq i-1$. Intuitively, in profile $\mathbf{u}'$, each agent $i \in \{2, \dots, n\}$ has valuation close to 1 for item $i-1$ and small valuations for all other items. Futhermore, she prefers items with smaller indices, except for item $i-1$.

We claim that $\mathbf{s}$ is a pure Nash equilibrium under true valuation profile $\mathbf{u}$ as well. Assume for contradiction that some agent $i$ has a benefiting deviation, which matches her with an item that she prefers more than $i$. But then, since the set of items that she prefers more than $i$ in both $\mathbf{u}$ and $\mathbf{u}'$ is $\{1, \dots, i\}$, the same deviation would match her with a more preferred item under $\mathbf{u}$ as well, contradicting the fact that $\mathbf{s}$ is a pure Nash equilibrium. It holds that $SW_{OPT}(\mathbf{u}') \geq n-2$ whereas the social welfare of $M$ is at most $\frac{2}{n}$ and the theorem follows. □

*Remark* 5.1. The mechanism that naively maximizes the sum of the reported valuations with no regard to incentives, when equipped with a lexicographic tie-breaking rule has pure Nash equilibria and also achieves the above ratio in the worst-case, which means that the bounds are tight.

## 5.5 Price of stability

Theorem 5.3 bounds the price of anarchy of all mechanisms. A more optimistic (and hence stronger when proving lower bounds) measure of efficiency is the *price of stability*, i.e. the worst-case ratio over all valuation profiles of optimal social welfare over the welfare attained at the *best* equilibrium, formally defined as

$$PoS(M) = \inf_{\mathbf{u}\in V^n} \frac{\max_{\mathbf{s}\in S^M_{\mathbf{u}}} SW_M(\mathbf{u},\mathbf{s})}{SW_{OPT}(\mathbf{u})}.$$

In this section we extend Theorem 5.3 to the price of stability of all mechanisms that satisfy a "proportionality-like" property. This class is quite large and contains most well-known mechanisms, including probabilistic serial and random priority. We start with a few definitions that will be needed throughout the section.

**Definition 5.3** (Stochastic Dominance [35])**.** Let $a_1 \succ_i a_2 \succ_i \cdots \succ_i a_n$ be the (possibly weak) preference ordering of agent $i$. A random assignment vector $p_i$ for agent $i$ *stochastically dominates* another random assignment vector $q_i$ if $\sum_{j=1}^{k} p_{ia_j} \geq \sum_{j=1}^{k} q_{ia_j}$, for all $k = 1, 2, \cdots, n$. The notation that we will use for this relation is $p_i \succ_i^{sd} q_i$.

**Definition 5.4** (Safe strategy)**.** Let $M$ be a mechanism. A strategy $s_i$ is a *safe strategy* if for any strategy profile $s_{-i}$ of the other players, it holds that $M_i(s_i, s_{-i}) \succ_i^{sd} \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$.

We will say that a mechanism $M$ has a safe strategy if every agent $i$ has a safe strategy $s_i$ in $M$. We now state our lower bound.

**Theorem 5.5.** *The pure price of stability of any mechanism for one-sided matching that has a safe strategy is $O(1/\sqrt{n})$.*

*Proof.* Let $M$ be a mechanism and let $I = \{k+1, \ldots, n\}$ be a subset of agents. Let $\mathbf{u}$ be the following valuation profile.

- For all agents $i \in I$, let $u_{ij} = \frac{1}{k}$ for $j = 1, \cdots, k$ and $u_{ij} = 0$ otherwise.
- For all agents $i \notin I$, let $u_{ii} = 1$ and $u_{ij} = 0, j \neq i$.

Now let $\mathbf{s}$ be a pure Nash equilibrium on profile $\mathbf{u}$ and let $\mathbf{s_i'}$ be a safe strategy of agent $i$. The expected utility of each agent $i \in I$ in the pure Nash equilibrium $\mathbf{s}$ is

$$\mathbb{E}[u_i(\mathbf{s})] = \sum_{j\in[n]} p_{ij}(s_i, s_{-i})v_{ij} \geq \sum_{j\in[n]} p_{ij}(s_i', s_{-i})v_{ij} \geq \frac{1}{n}\sum_{j\in[n]} v_{ij} = \frac{1}{n},$$

due to the fact that $\mathbf{s}$ is pure Nash equilibrium and $s'_i$ is a safe strategy of agent $i$.
On the other hand, the utility of agent $i \in I$ can be calculated by

$$\mathbb{E}[u_i(\mathbf{s})] = \sum_{j\in[n]} p_{ij}(s_i, s_{-i})v_{ij} = \frac{1}{k}\sum_{j=1}^{k} p_{ij}.$$

Because $\mathbf{s}$ is a pure Nash equilibrium, it holds that $\mathbb{E}[u_i] \geq \frac{1}{n}$, so we get that $\sum_{j=1}^{k} p_{ij} \geq \frac{k}{n}, i \in I$. As for the rest of the agents, it holds that

$$\sum_{i\in N\setminus I}\sum_{j=1}^{k} p_{ij} = k - \sum_{i\in I}\sum_{j=1}^{k} p_{ij} \leq k - (n-k)\frac{k}{n} = \frac{k^2}{n}.$$

This implies that the contribution to the social welfare from agents not in $I$ is at most $\frac{k^2}{n}$ and the expected social welfare of $M$ will be at most $1 + \frac{k^2}{n}$. It holds that $SW_{OPT}(\mathbf{u}) \geq k$ and the bound follows by choosing $k = \sqrt{n}$. $\square$

Due to Theorem 5.5, in order to obtain an $O(1/\sqrt{n})$ bound for a mechanism $M$, it suffices to prove that $M$ has a safe strategy. In fact, most reasonable mechanisms, including random priority and probabilistic serial satisfy this property. We observe that this is indeed the case for a large class of mechanisms in the literature, including for example the well-known class of ordinal, envy-free mechanisms:

**Definition 5.5** (Envy-freeness [35, 98])**.** A mechanism $M$ is (ex-ante) *envy-free* if for all agents $i$ and $r$ and all profiles $\mathbf{s}$, it holds that $\sum_{j=1}^{n} p_{ij}s_{ij} \geq \sum_{j=1}^{n} p_{rj}s_{rj}$. Furthermore, if $M$ is ordinal, then this implies $p_i^{M,\mathbf{s}} \succ_{s_i}^{sd} p_r^{M,\mathbf{s}}$.

Given the interpretation of a truthtelling safe strategy as a "proportionality-like" property, the next lemma is not surprising; it is well-known that in fair division settings, envy-freeness implies proportionality, when no resources are wasted[7].

**Lemma 5.3.** *Let $M$ be an ordinal, envy-free mechanism for one-sided matching. Then for any agent $i$, the truthtelling strategy $u_i$ is a safe strategy.*

*Proof.* Let $\mathbf{s} = (u_i, s_{-i})$ be the strategy profile in which agent $i$ is truthtelling and the rest of the agent are playing some strategies $s_{-i}$. Since $M$ is envy-free and ordinal, it holds that $\sum_{j=1}^{l} p_{ij}^{\mathbf{s}} \geq \sum_{j=1}^{l} p_{rj}^{\mathbf{s}}$ for all agents $r \in \{1,\ldots,n\}$ and all $l \in \{1,\ldots,n\}$. Summing up these inequalities for agents $r = 1, 2, \ldots, n$ we obtain

$$n\sum_{j=1}^{l} p_{ij}^{\mathbf{s}} \quad \geq \quad \sum_{j=1}^{l}\sum_{r=1}^{n} p_{rj}^{\mathbf{s}} = l,$$

[7]We can show that all of our lower bounds still hold even if we allow items to remain unallocated with positive probabilities.

which implies that $\sum_{j=1}^{l} p_{ij}^{\mathbf{s}} \geq \frac{l}{n}$, for all $i \in \{1, \ldots, n\}$, for all $l \in \{1, \ldots, n\}$. $\square$

**Lemma 5.4.** *Random priority has truthtelling as a safe strategy.*

*Proof.* Since random priority first fixes an ordering of agents uniformly at random, every agent $i$ has a probability of $\frac{1}{n}$ to be selected first to choose an item, a probability of $\frac{2}{n}$ to be selected first or second and so on. If the agent ranks her items truthfully, then for every $l = 1, \ldots, n$, it holds that $\sum_{i=1}^{l} p_{ij} \geq \frac{l}{n}$. $\square$

Recently, Mennle and Seuken [114] defined the class of *hybrid mechanisms* to obtain tradeoffs between quantified versions of Pareto efficiency and truthfulness. Hybrid mechanisms are convex combinations of different mechanisms, such as probabilistic serial and random priority.

**Lemma 5.5.** *Let $M$ be a hybrid mechanism which is a convex combination of mechanisms that have truthtelling as a safe strategy. Then $M$ has truthtelling as a safe strategy.*

*Proof.* Mechanism $M$ can be written as a convex combination of mechanisms $M_1$, $M_2$, ..., $M_k$ i.e. for every agent $i$ and and strategy profile $\mathbf{s} = (u_i, s_{-i})$, it holds that $p_i^{M,\mathbf{s}} = \sum_{i=j}^{k} \alpha_j p_i^{M_j,\mathbf{s}}$, with $\sum_{j=1}^{k} \alpha_j = 1$. Since truthtelling is a safe strategy for all mechanisms $M_j$, $j = 1, \ldots, k$, it also holds that for every $j$, $\sum_{m=1}^{l} p_{im}^{M_j,\mathbf{s}} \geq \frac{l}{n}$, for all $l = 1, \ldots, n$. This means that for every $l = 1, \ldots, n$, it holds that $\sum_{j=1}^{l} p_{ij}^{M,\mathbf{s}} \geq \alpha \frac{l}{n} + (1-\alpha)\frac{l}{n} = \frac{l}{n}$ and hence $M_i(\mathbf{s}) \succ_i^{sd} (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$. $\square$

Probabilistic serial is ordinal and envy-free [35] and hence from Lemma 5.3 and Lemma 5.5 we obtain the following corollaries.

**Corollary 5.2.** *The price of stability of any ordinal, envy-free mechanism for one-sided matching (including probabilistic serial) is $O(1/\sqrt{n})$.*

**Corollary 5.3.** *The price of stability of any hybrid mechanism for one-sided matching which is a convex combination of mechanisms that have truthtelling as a safe strategy (including random priority) is $O(1/\sqrt{n})$.*

Note that the safe strategy condition is in a sense a minimal condition required for Theorem 5.5, because we can not hope to prove a strong upper bound on the price of stability of all mechanisms.

To see this, consider the following deterministic, *randomly dictatorial* mechanism [138]: Select an agent $i^*$ uniformly at random and match her with her most preferred item $j^*$. Then, fix an ordering of the rest of the agents and match them sequentially according to this ordering to items

$$j_1 \in \operatorname*{argmax}_{j \in A \setminus \{j^*\}} u_{i^*j} \quad , \quad j_2 \in \operatorname*{argmax}_{j \in A \setminus \{j^*, j_1\}} u_{i^*j}$$

and so on, breaking ties arbitrarily. Note that this mechanism is truthful; once agent $i^*$ is selected, she is matched with her most preferred item and the rest of the agents can not influence the outcome. However, it is easy to see that the mechanism has other equilibria as well; any report such that $j^*$ is on top of agent $i^*$'s preference ranking grants the agent maximum utility. In particular, there is some strategy $s_{i^*}$ of agent $i^*$ that results in an welfare-optimal assignment for the rest of the agents. We can prove the following theorem.

**Theorem 5.6.** *The price of stability of the randomly dictatorial mechanism RD is at least* $1/2$*.*

*Proof.* Consider any valuation profile $\mathbf{u}$ and assume first that $SW_{OPT}(\mathbf{u}) \geq 2$. Given the choice of some agent $i$ and her most preferred item $j$, in the best Nash equilibrium, the mechanism outputs a social welfare optimal matching $OPT_{-i}(\mathbf{u_{-i}})$ for agents in $N\backslash\{i\}$ and items in $A\backslash\{j\}$. Since $OPT_{-i}(\mathbf{u_{-i}})$ is optimal, it is at least as good as the matching that matches every agent $l \in N\backslash\{i, i'\}$ with item $OPT_l(\mathbf{u})$, except agent $i'$, the agent for which $OPT_{i'}(\mathbf{u}) = j$, who is matched with item $OPT_i(\mathbf{u})$. In other words, for every realization of randomness, the mechanism produces a matching that is at least as good as $OPT(\mathbf{u})$, except for the allocation of two agents that is swapped: the agent $i$ chosen by the mechanism and the agent that receives agent $i$'s most preferred item in the optimal matching.

Let $v_i$ be the valuation of agent $i$ for her most preferred item and let $w_i$ be her valuation for item $OPT_i(\mathbf{u})$. Then, from the discussion above, it holds that for every choice of agent $i$ (with most preferred item $j$), the welfare achieved is at least $SW_{OPT}(\mathbf{u}) + v_i - w_i - w_j$, which is at least $SW_{OPT}(\mathbf{u}) - 1$, since $v_i \geq w_i$ and $w_j \leq 1$. The price of stability is then at least $1 - 1/SW_{OPT}(\mathbf{u})$ which is at least $1/2$, since $SW_{OPT}(\mathbf{u}) \geq 2$.

Assume from now on that $SW_{OPT}(\mathbf{u}) \leq 2$. Observe that, in the best Nash equilibrium of the randomly dictatorial mechanism $RD$ on input $\mathbf{u}$, the outcome is at least as good as the outcome of random priority; in particular, there exists some Nash equilibrium such that $RD(\mathbf{u}) = RP(\mathbf{u})$. By the proof of Lemma 4.7, we know that, for the unit-sum representation, $SW_{RP}(\mathbf{u}) \geq 1$ and hence $SW_{RD}(\mathbf{u}) \geq 1$ as well. Since $SW_{OPT}(\mathbf{u}) \leq 2$, this proves the theorem. □

It is not hard to see that the price of anarchy of the randomly dictatorial mechanism is $\Theta(1/n)$. Given that bestowing the rights to the allocation to a single agent is intuitively not a good choice, the above result indicates that one should perhaps be careful when adopting the price of stability as the measure of performance.

## 5.6 More general solution concepts

In the previous sections, we employed the pure Nash equilibrium as the solution concept for bounding the inefficiency of mechanisms, mainly because of its simplicity. Here, we describe how to extend our results to broader well-known equilibrium concepts in literature. For completeness, we include the pure Nash equilibrium (that we have already defined) in the discussion below.

We consider five standard equilibrium notions: pure Nash, mixed Nash, correlated, coarse correlated and Bayes-Nash equilibria. For the first four, the agents have full information. In the Bayesian setting, the valuations are drawn from some distributions and agents know their own valuation and the distributions from which the other valuations are drawn. We formally define the different equilibrium concepts.

**Definition 5.6.** Given a mechanism $M$, let $\mathbf{q}$ be a distribution over strategies. Also, for any distribution $\Delta$ let $\Delta_{-i}$ denote the marginal distribution without the $i$th index. Then a strategy profile $\mathbf{q}$ is called a

*pure Nash equilibrium* if $q = \mathbf{s}$, $u_i(M_i(\mathbf{s})) \geq u_i(M_i(s_i', s_{-i}))$,

*mixed Nash equilibrium* if $\mathbf{q} = \times_i q_i$, $\mathbb{E}_{\mathbf{s}\sim\mathbf{q}}[u_i(M_i(\mathbf{s}))] \geq \mathbb{E}_{s_{-i}\sim q_{-i}}[u_i(M_i((s_i', s_{-i})))]$,

*correlated equilibrium* if $\mathbb{E}_{\mathbf{s}\sim\mathbf{q}}[u_i(M_i(\mathbf{s}))|s_i] \geq \mathbb{E}_{\mathbf{s}\sim\mathbf{q}}[u_i(M_i((s_i', s_{-i})))|s_i]$,

*coarse correlated equilibrium* if $\mathbb{E}_{\mathbf{s}\sim\mathbf{q}}[u_i(M_i(\mathbf{s}))] \geq \mathbb{E}_{\mathbf{s}\sim\mathbf{q}}[u_i(M_i((s_i', s_{-i})))]$,

*Bayes-Nash equilibrium* for a distribution $\Delta_u$ where each $(\Delta_u)_i$ is independent, if when $\mathbf{u} \sim \Delta_u$ then $\mathbf{q}(\mathbf{u}) = \times_i q_i(u_i)$ and for all $u_i$ in the support of $(\Delta_u)_i$
$\mathbb{E}_{u_{-i},\mathbf{s}\sim\mathbf{q}(\mathbf{u})}[u_i(M_i(\mathbf{s}))] \geq \mathbb{E}_{u_{-i},s_{-i}\sim q_{-i}(u_{-i})}[u_i(M_i(s_i', s_{-i}))]$,

where the given inequalities hold for all agents $i$, and (pure) deviating strategies $s_i'$. Also notice that for randomized mechanisms , the definitions are with respect to the expectation over the random choices of the mechanism.

It is well known that for the first four classes each is contained in the next class, i.e., pure $\subset$ mixed $\subset$ correlated $\subset$ coarse correlated. If we regard the full information setting as a special case of Bayesian setting, we also have pure $\subset$ mixed $\subset$ Bayesian. This means that for the complete information setting, when proving efficiency guarantees, it suffices to consider the coarse correlated equilibria of a mechanism and in the incomplete information setting, we only need to consider Bayes-Nash equilibria. The mixed, correlated, coarse correlated and Bayesian price of anarchy is defined similarly to the pure price of anarchy.

### Probabilistic serial

In the following, we extend Theorem 5.1 to the case where the solution concept is the coarse correlated equilibrium. Since the class of coarse correlated equilibria includes the classes of mixed Nash and correlated equilibria and since we are proving a price of anarchy bound, the result covers those solution concepts as well.

**Theorem 5.7.** *The coarse correlated price of anarchy of probabilistic serial is* $\Omega(1/\sqrt{n})$*.*

*Proof.* Let $\mathbf{u}$ be any valuation profile. Let $i$ be any agent and let $j = OPT_i(\mathbf{u})$. The intuition here is that in the proof of Lemma 5.1, the inequality $t_j(\mathbf{s}') \geq \frac{1}{4}t_j(\mathbf{s})$ holds for every strategy profile. In particular, it holds for any pure strategy profile $\mathbf{s}$ where $s_i$ is in the support of the distribution of the mixed strategy $q_i$ of agent $i$, for any coarse correlated equilibrium $\mathbf{q}$. Now let $s_i'$ be the pure strategy that places item $j$ on top of agent $i$'s preference list. This implies that

$$\begin{aligned}
\mathbb{E}_{\mathbf{s}\sim\mathbf{q}}[u_i(PS_i(\mathbf{s}))] &\geq \mathbb{E}_{\mathbf{s}\sim\mathbf{q}}[u_i(PS_i(s_i', s_{-i}))] \\
&\geq \mathbb{E}_{\mathbf{s}\sim\mathbf{q}}[u_{ij}t_j(s_i', s_{-i}))] \\
&\geq \frac{1}{4}u_{ij}t_j(\mathbf{s}).
\end{aligned}$$

where the last inequality holds by the discussion above on Lemma 5.1. Using this, we can use very similar arguments to the arguments of the proof of Theorem 5.1 and obtain the bound. □

For the incomplete information setting, when valuations are drawn from some publically known distributions, we can prove the same upper bound on the Bayesian price of anarchy of the mechanism.

**Theorem 5.8.** *The Bayesian price of anarchy of probabilistic serial is* $\Omega(1/\sqrt{n})$*.*

*Proof.* The proof is again similar to the proof of Theorem 5.1. Let $\mathbf{u}$ be a valuation profile drawn from some distribution satisfying the unit-sum constraint. Let $i$ be any agent and let $j_u = OPT(\mathbf{u})_i$, $i \in [n]$. Note that by a similar argument as the one used in the proof of Theorem 5.1, the expected social welfare of $PS$ is at least 1 and hence we can assume that $\mathbb{E}_{\mathbf{u}}[SW_{OPT}(\mathbf{u})] \geq 2\sqrt{2n}+1$.

Observe that in any Bayes-Nash equilibrium $\mathbf{q}(\mathbf{u})$ it holds that

$$\begin{aligned}
\mathbb{E}_{\substack{\mathbf{u}\\ \mathbf{s}\sim\mathbf{q}(\mathbf{u})}}[u_i(\mathbf{s})] &= \mathbb{E}_{u_i}\left[\mathbb{E}_{\substack{u_{-i}\\ \mathbf{s}\sim\mathbf{q}(\mathbf{u})}}[u_i(\mathbf{s})]\right] \\
&\geq \mathbb{E}_{u_i}\left[\mathbb{E}_{\substack{u_{-i}\\ s_{-i}\sim q_{-i}(u_{-i})}}[u_i(s'_i, s_{-i})]\right] \\
&\geq \mathbb{E}_{u_i}\left[\mathbb{E}_{\substack{u_{-i}\\ s_{-i}\sim q_{-i}(u_{-i})}}[u_{ij_u} t_{j_u}(s'_i, s_i)]\right] \\
&\geq \mathbb{E}_{u_i}\left[\mathbb{E}_{\substack{u_{-i}\\ \mathbf{s}\sim\mathbf{q}(\mathbf{u})}}\left[\frac{1}{4} u_{ij_u} t_{j_u}(\mathbf{s})\right]\right] \\
&= \frac{1}{4}\mathbb{E}_{\substack{\mathbf{u}\\ \mathbf{s}\sim\mathbf{q}(\mathbf{u})}}[u_{ij_u} t_{j_u}(\mathbf{s})]
\end{aligned}$$

where the last inequality holds for the same reason as in the proof of Theorem 5.7 and $s'_i$ denotes the strategy that puts item $j_u$ on top of agent $i$'s preference list. Note that this can be a different strategy for every different $\mathbf{u}$ that we sample. For notational convenience, we use $s'_i$ to denote every such strategy. The expected social welfare at the Bayes-Nash equilibrium is then at least

$$\begin{aligned}
\sum_{i=1}^{n} \mathbb{E}_{\mathbf{u},\mathbf{s}\sim\mathbf{q}(\mathbf{u})}[u_i(\mathbf{s})] &\geq \frac{1}{4}\sum_{i\in[n]} \mathbb{E}_{\substack{\mathbf{u}\\ \mathbf{s}\sim\mathbf{q}(\mathbf{u})}}[u_{ij_u} t_{j_u}(\mathbf{s})] \geq \mathbb{E}_{\substack{\mathbf{u}\\ \mathbf{s}\sim\mathbf{q}(\mathbf{u})}}\left[\sum_{i\in[n]} \frac{i}{4n} u_{ij_u}\right] \\
&\geq \mathbb{E}_{\substack{\mathbf{u}\\ \mathbf{s}\sim\mathbf{q}(\mathbf{u})}}\left[\frac{SW_{OPT}(\mathbf{u})(SW_{OPT}(\mathbf{u})-1)}{8n}\right] \\
&= \mathbb{E}_{\mathbf{u}}\left[\frac{SW_{OPT}(\mathbf{u})(SW_{OPT}(\mathbf{u})-1)}{8n}\right] \\
&\geq \frac{\mathbb{E}_{\mathbf{u}}\left[(SW_{OPT}(\mathbf{u}))^2\right] - \mathbb{E}_{\mathbf{u}}\left[SW_{OPT}(\mathbf{u})\right]}{8n} \\
&\geq \frac{\mathbb{E}_{\mathbf{u}}[SW_{OPT}(\mathbf{u})]}{2\sqrt{2n}},
\end{aligned}$$

and the bound follows. □

## Random priority

Next we extend Theorem 5.2 to the large class of coarse correlated equilibria.

**Theorem 5.9.** *The coarse correlated price of anarchy of random priority is $\Omega(1/\sqrt{n})$.*

*Proof.* The argument is very similar to the one used in the proof of Theorem 5.2. Again, if any strategy in the support of a correlated equilibrium $\mathbf{q}$ misranks two items $j$ and $j'$ for any agent $i$, it can only be because agent $i$ has 0 probability of receiving those items, otherwise agent $i$ would deviate to truthtelling, violating the equilibrium condition. The remaining steps are exactly the same as in the proof of Theorem 5.2. □

Again, for the incomplete information case, we prove the same price of anarchy guarantee in the Bayes-Nash equilibria of the mechanism.

**Theorem 5.10.** *The Bayesian price of anarchy of random priority is* $\Omega(1/\sqrt{n})$.

*Proof.* Consider any Bayes-Nash equilibrium $\mathbf{q}(\mathbf{u})$ and let $\mathbf{u}$ be a any sampled valuation profile. The expected social welfare of the random priority can be written as $\mathbb{E}_{\mathbf{u}}\left[\mathbb{E}_{s\sim\mathbf{q}(\mathbf{u})}\left[u_i(\mathbf{s})\right]\right]$. Using the same argument as the one in the proof of Theorem 5.2, we can lower bound the quantity $\mathbb{E}_{s\sim\mathbf{q}(\mathbf{u})}\left[u_i(\mathbf{s})\right]$ by $\Omega\left(\frac{\sqrt{n}}{SW_{OPT}(\mathbf{u})}\right)$ and the bound follows. □

## 5.7 Unit-range valuations

In this section, we discuss how our results extend to the unit-range representation, i.e., $\max_j u_i(j) = 1$ and $\min_j u_i(j) = 0$. In short, the price of anarchy guarantees from Section 5.3 and 5.3 extend directly to the unit-range case. The upper bound from Theorem 5.4 is replaced by an analogous theorem with a different (but still tight) bound whereas the upper bound from Theorem 5.3 is replaced by an $O(1/n^{1/4})$ upper bound on the Price of Anarchy of any mechanism with respect to $\epsilon$-approximate pure Nash equilibria, for all $\epsilon > 0$.

### Price of anarchy guarantees

We extend the price of anarchy guarantees of probabilistic serial and random priority first.

**Theorem 5.11.** *The price of anarchy of probabilistic serial is* $\Omega(1/\sqrt{n})$ *for the unit-range representation.*

*Proof.* First, observe that Lemma 5.1 holds independently of the representation. Secondly, in the proof of Theorem 5.1, it now holds that

$$SW_{PS}(\mathbf{u},\mathbf{s}) \geq \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} u_{ij} \geq 1,$$

which is sufficient for bounding the price of anarchy when $SW_{OPT}(\mathbf{u}) \leq \sqrt{n}$. Finally, the arguments for the case when $SW_{OPT}(\mathbf{u}) > \sqrt{n}$ hold for both representations. It is easy to see that the extension applies to all the other equilibrium notions as well. □

**Theorem 5.12.** *The price of anarchy of random priority is* $\Omega(1/\sqrt{n})$ *for the unit-range representation.*

*Proof.* First observe that Theorem 5.2 is independent of representation. Secondly, since the main theorem in Chapter 4 also holds for the unit-range representation, the proof of Theorem 5.2 extends to unit-range as well. Again, the result holds for all the other solution concepts. □

### Lower bounds for unit-range

Next, we present a price of anarchy lower bound for deterministic mechanisms. First, we prove the following lemma about the structure of equilibria of deterministic mechanisms. Note that the lemma holds independently of the choice of representation.

**Lemma 5.6.** *The set of pure Nash equilibria of any deterministic mechanism is the same for all valuation profiles that induce the same preference orderings of valuations.*

*Proof.* Let $\mathbf{u}$ and $\mathbf{u}'$ be two different valuation profiles that induce the same preference ordering. Let $\mathbf{s}$ be a pure Nash equilibrium under true valuation profile $\mathbf{u}$ and assume for contradiction that it is not a pure Nash equilibrium under $\mathbf{u}'$. Then, there exists an agent $i$ who by deviating from $\mathbf{s}$ is matched to a more preferred item according to $u_i'$. But that item would also be more preferred according to $u_i$ and hence she would have an incentive to deviate from $\mathbf{s}$ under true valuation profile $\mathbf{u}$, contradicting the fact that $\mathbf{s}$ is a pure Nash equilibrium. □

Using Lemma 5.6, we can then prove the following theorem.

**Theorem 5.13.** *The price of anarchy of any deterministic mechanism for one-sided matching that always has pure Nash equilibria is $O(1/n)$ for the unit-range representation.*

*Proof.* Let $M$ be a deterministic mechanism that always has a pure Nash equilbrium and let $\mathbf{u}$ be a valuation profile such that for all agents $i$ and $i'$, it holds that $u_i = u_{i'}$ and $u_{ij} > u_{ik}$, for all items $i < k$. Let $\mathbf{s}$ be a pure Nash equilibrium for this profile and assume without loss of generality that $M_i(\mathbf{s}) = i$. By Lemma 5.6, $\mathbf{s}$ is a pure Nash equilibrium for any profile $\mathbf{u}$ that induces the above ordering of valuations. In particular, it is a pure Nash equilibrium for a valuation profile satisfying

- For agents $i = 1, \ldots, \frac{n}{2}$, $u_{i1} = 1$ and $u_{ij} < \frac{1}{n^3}$, for $j > 1$.
- For agents $i = \frac{n}{2} + 1, \ldots, n$, $u_{ij} > 1 - \frac{1}{n^3}$ for $j = 1, \ldots, n/2$ and $u_{ij} < \frac{1}{n^3}$ for $j = \frac{n}{2} + 1, \ldots, n$.

It holds that $OPT(\mathbf{u}) \geq \frac{n}{2}$, whereas the social welfare of $M$ is at most 2 and the theorem follows. □

Note that the mechanism that maximizes the sum of the reported valuations achieves the above bound and hence the bound is tight.

We now prove a general lower bound for the class of all mechanisms when the solution concept is the $\epsilon$-approximate pure Nash equilibrium. A strategy profile is an $\epsilon$-approximate pure Nash equilibrium if no agent can deviate to a different strategy and improve her utility by more than $\epsilon$. For $\epsilon$-approximate pure Nash equilibria the measure of efficiency is the *$\epsilon$-approximate pure price of anarchy.*

**Theorem 5.14.** *Let $M$ be a mechanism for one-sided matching and let $\epsilon \in (0,1)$. The $\epsilon$-approximate price of anarchy of $M$ is $O(1/n^{1/4})$ for the unit-range representation.*

*Proof.* Assume $n = k^2$, where $k \in \mathbb{N}$ will be the size of a subset $I$ of "important" agents. We consider valuation profiles where, for some parameter $\delta \in (0,1)$,

- all agents have value 1 for item 1,
- there is a subset $I$ of agents with $|I| = k$ for which any agent $i \in I$ has value $\delta^2$ for any item $j \in \{2,\dots,k+1\}$ and 0 for all other items,
- for agent $i \notin I$, $i$ has value $\delta^3$ for items $j \in \{2,\dots,k+1\}$ and 0 for all other items.

Let $\mathbf{u}$ be such a valuation profile and let $\mathbf{s}$ be a Nash equilibrium. In the optimal allocation, members of $I$ receive items $\{2,\dots,k+1\}$ and such an allocation has social welfare $k\delta^2 + 1$.

First, we claim that there are $k(1-2\delta)$ members of $I$ whose utilities in $\mathbf{s}$ are at most $\delta$; call this set $X$. If that were false, then there would be more than $2k\delta$ members of $I$ whose utilities in $\mathbf{s}$ were more than $\delta$. That would imply that the social welfare of $\mathbf{s}$ was more than $2k\delta^2$, which would contradict the optimal social welfare attainable, for large enough $n$ (specifically, $n > 1/\delta^4$).

Next, we claim that there are at least $k(1-2\delta)$ non-members of $I$ whose probability (in $\mathbf{s}$) to receive any item in $\{1,\dots,k+1\}$ is at most $4(k+1)/n$; call this set $Y$. To see this, observe that there are at least $\frac{3}{4}n$ agents who all have probability at most $4/n$ to receive item 1. Furthermore, there are at least $\frac{3}{4}n$ agents who all have probability $\leq 4k/n$ to receive an item from the set $2,\dots,k+1$. Hence there are at least $\frac{1}{2}n$ agents whose probabilities to obtain these items satisfy both properties.

We now consider the operation of swapping the valuations of the agents in sets $X$ and $Y$ so that the members of $I$ from $X$ become non-members, and vice versa. We will argue that given that they were best-responding beforehand, they are $\delta$-best-responding afterwards. Consequently $\mathbf{s}$ is an $\delta$-approximate Nash equilibrium of the modified set of agents. The optimal social welfare is unchanged by this operation since it only involves exchanging the payoff functions of pairs of agents. We show that the social welfare of $\mathbf{s}$ is some fraction of the optimal social welfare, that goes to 0 as $n$ increases and $\delta$ decreases.

Let $I'$ be the set of agents who, after the swap, have the higher utility of $\delta^2$ for getting items from $\{2,\ldots,k+1\}$. That is, $I'$ is the set of agents in $Y$, together with $I$, minus the agents in $X$.

Following the above valuation swap, the agents in $X$ are $\delta$-best responding. To see this, note that these agents have had a reduction to their utilities for the outcome of receiving items from $\{2,\ldots,k+1\}$. This means that a profitable deviation for such agents should result in them being more likely to obtain item 1, in return for them being less likely to obtain an item from $\{2,\ldots,k+1\}$. However they cannot have probability more than $\delta$ to receive item 1, since that would contradict the property that their expected payoff was at most $\delta$.

After the swap, the agents in $Y$ are also $\delta$-best responding. Again, these agents have had their utilities increased from $\delta^3$ to $\delta^2$ for the outcome of receiving an item from $\{2,\ldots,k+1\}$. Hence any profitable deviation for such an agent would involve a reduction in the probability to get item 1 in return for an increased probability to get an item from $\{2,\ldots,k+1\}$. However, since the payoff for any item from $\{2,\ldots,k+1\}$ is only $\delta^2$, such a deviation pays less than $\delta$.

Finally, observe that the social welfare of $\mathbf{s}$ under the new profile (after the swap) is at most $1+3k\delta^3$. To see this, note that (by an earlier argument and the definition of $I'$) $k(1-2\delta)$ members of $I'$ have probability at most $4(k+1)/n$ to receive any item from $\{1,\ldots,k+1\}$. To upper bound the expected social welfare, note that item 1 contributes 1 to the social welfare. Items in $\{2,\ldots,k+1\}$ contribute in total, $\delta^2$ times the expected number of members of $I'$ who get them, plus $\delta^3$ times the expected number of non-members of $I'$ who get them, which is at most $\delta^2 k 2\delta + \delta^3 k(1-2\delta)$ which is less than $3k\delta^3$.

Overall, the price of anarchy is at least $3k\delta^3/(k\delta^2+1)$, which is more than $\delta$. The statement of the theorem is obtained by choosing $\delta$ to be less than $\epsilon$, $n$ large enough for the arguments to hold for the chosen $\delta$, i.e. $n > 1/\delta^4$. □

## Price of stability

Theorem 5.5 does not directly extend to the unit-range representation. It is an open problem whether we can prove a similar result for the unit-range representation as well. Note that for this representation, a simple version of the randomly dictatorial mechanism that we presented in Section 5.5 that selects agent $i^*$ *deterministically* achieves a constant price of stability. We will call this mechanism *purely dictatorial* [138].

**Theorem 5.15.** *The price of stability of the purely dictatorial mechanism $D$ is at least* $1/2$.

*Proof.* Consider any valuation profile $\mathbf{u}$. Let $i^*$ be the agent selected by the mechanism and let $j^*$ be her most preferred item. First note that if $SW_{OPT}(\mathbf{u}) \leq 2$, agent $i^*$ will be matched with her favorite item (which she values at 1, since the representation is unit-range) and we are done. Hence, assume from now on that $SW_{OPT}(\mathbf{u}) > 2$. Recall that the best Nash equilibrium, the mechanism matches agent $i^*$ with item $j^*$ and outputs a welfare-maximizing matching for agents $i \in N \backslash \{i^*\}$ and items $j \in A \backslash \{j^*\}$; let $OPT_{-i^*}$ be that matching. We will argue that $SW_{OPT_{-i^*}}(\mathbf{u}) \geq SW_{OPT}(\mathbf{u}) - 2$. This is true because the matching $OPT_{-i^*}$ is at least as good as the matching $OPT'$ that matches each agent in $N \backslash \{i^*\}$ to $OPT_i(\mathbf{u})$ except some agent $l$, who was matched with item $j^*$ in the optimal allocation, that is now matched to item $OPT_{i^*}(\mathbf{u})$. In the worst case, agent $l$ values item $OPT_{i^*}(\mathbf{u})$ at 0 (and since we are only considering the matching of agents in $N \backslash \{i^*\}$ and items in $A \backslash \{j^*\}$), it holds that $SW_{OPT'}(\mathbf{u}) \geq SW_{OPT}(\mathbf{u}) - 2$ and hence our claim holds.

Now, since valuation vectors are unit-range, the welfare of the mechanism from item $j^*$ is 1 and hence it holds that $SW_D(\mathbf{u}) \geq SW_{OPT}(\mathbf{u}) - 1$ and the price of stability is at least $1 - 1/SW_{OPT}(\mathbf{u})$. Since $SW_{OPT}(\mathbf{u}) > 2$, we obtain the bound. □

## 5.8 Discussion and future work

In this chapter, we considered all mechanisms for one-sided matching and bounded their performance in terms of the price of anarchy. We showed that probabilistic serial and random priority are optimal among all mechanisms, with that objective at hand. We extended our negative results to the price of stability of mechanisms that satisfy a mild property.

There are several future directions to be considered. First of all, unlike Chapters 2 and 4, where most of the results could easily be adapted to work with and without ties, here the presence of ties is essential for our main upper bounds to hold. While indifferences in valuations are part of the setting and hence the profiles used are perfectly valid, as we discussed in Chapter 2, we would ideally like to obtain impossibility results using profiles without ties. That being said, the absence of ties complicates things further when considering best responses; if the mechanism can be completely arbitrary, then very small indifferences between valuations can have a major impact on incentives. For that reason, the proof of Theorem 5.3 does not seem adaptable to the setting with ties. On the other hand, if we impose some (mild) restrictions on mechanisms, then we can extend the theorem to work in the "no-ties" setting. Specifically, if there is any bound on the minimum difference in utility that an agent can have between two outcomes of the mechanism, then we can generate small enough indifferences in the valuation profile used in the proof of Theorem 5.3, construct a profile without ties and obtain the same bound. This property is in fact satisfied by most well-known mechanisms, including *all ordinal* mechanisms.

We proved that under the mild safe strategy assumption which is satisfied by most reasonable mechanisms, our upper bound extends to the price of stability as well. On the other hand, an arguably "unreasonable" mechanism like the randomly dictatorial mechanism achieves a constant price of stability. That being said, the price of anarchy of the mechanism is quite bad. The reason is that the mechanism basically puts all the power in one agent's hands and the price of stability assumes, perhaps rather optimistically, that this agent will "do the right thing". In a sense, while the price of stability is meant to be an optimistic measure of inefficiency of a system in absence of a central planner, mechanisms like the one above delegate the task of central planning to a single agent. In any setting where satisfying a single person can not significantly harm society as a whole, like the one-sided matching setting, such "trivial" mechanisms can always achieve near-optimal outcomes in terms of their price of stability.

Can we prove stonger impossibility results for the unit-range representation? In Chapter 4, the picture is very clear; the choice of normalization is not important for the (asymptotic) results. Here, the picture is a bit more blurry. Is it possible to obtain tight bound for the unit-range case as well?

Finally, it would be interesting to study scenarios where comparisons between mechanisms are not done asymptotically, or the guarantees are not achieved in the worst-case. Considering concrete numbers of agents and items, similarly to the last section of Chapter 4 could be an example. Another option would be to make some sensible distributional assumption on the input profiles and consider "average versions" of the price of anarchy (or the approximation ratio) of mechanisms in that case. Investigating such settings could potentially lead to more answers regarding the quality and practical usefulness of matching mechanisms and the comparisons between them.

We conclude the chapter with a dicussion on the two measures of efficiency that we have considered so far.

## 5.9 Approximation or anarchy?

In Chapter 4, we used the approximation ratio as the measure of performance whereas in the current chapter, we studied the price of anarchy of (not necessarily truthful) mechanisms. In fact, the asymptotic bounds that we proved in both cases turned out to be very similar. Can we then say that e.g. probabilistic serial and random priority are *equally good*? How does the price of anarchy objective fair when compared to the approximation ratio?

First of all, it is indisputable that for truthful mechanisms like random priority, a price of anarchy guarantee is a stronger result than an approximatio ratio guarantee. This is because as we mentioned earlier, truthtelling is *one* equilibrium and not necessarily the worst equilibrium; in that sense a price of anarchy result includes the guarantee given by the approximation ratio. In fact, Theorem 5.3 *implies*[8] the impossibility part of Theorem 2.5 for truthful mechanisms.

One the other hand, comparisons between the price of anarchy of non-truthful mechanisms like probabilistic serial and the approximation ratio of truthful mechanisms are not as direct; the main reason is the following. In a truthful mechanism, truthtelling is not just a Nash equilibrium, it is a *dominant strategy equilibrium*, i.e. a condition that ensures that agents have no incentives to lie about their preferences, *regardless* of what other people will do. On the other hand, the applicability of the Nash equilbrium relies on the assumption that agents will *best respond* to the set of options laid in front of them. In particular, agents will not look "two steps ahead" to find a better report.

Perhaps even more so than the case of "non-myopic" agents, it is conceivable that strategizing within a mechanism is a hard task; people don't have the time or the resources or even perhaps the mental capabilities of finding the best way to act within such a complicated system. It has been shown, for instance [20, 21] that it might be computationally hard for agents to calculate best responses in probabilistic serial.

In other words, we can not expect agents to play truthfully but can we expect them to always play *rationally*? Game theory is based on the principles of rational play and we are by no means trying to question these assumptions. What we are arguing is that knowing that telling the truth is the best thing that one can do, without having to worry about others, is certainly reassuring and while comparisons between the two measures of efficiency are possible, they shouldn't be made impetuously.

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[8]Minus the fact that Theorem 2.5 holds for both normalizations and without ties.

# Part III

# Divisible item allocation

# Chapter 6

# Background

In Part 2 of the thesis, we considered the problem of one-sided matchings and studied the performance of mechanisms for that problem. Recall that the items in that setting where indivisible and agents had to be matched with exactly one item in expectation. In this chapter, we will introduce a related problem that also has been the topic of a large amount of literature, that of *divisible item allocation.*[1] In many resource allocation problems encountered in real life, the resources are divisible; examples are numerous and range from goods like milk or rice in "traditional" markets to distributed resources like memory or storage space or even intangible resources, like sharing connection costs in an online network.

The general question that has concerned researchers for many years is the very same question of social choice, presented in Chapter 1, rephrased in terms of the specific task at hand: "How should we allocate the resources to the participants?" The extended literature on the topic has proposed a number of different solutions in terms of fairness, efficiency and tradeoffs between the two. Similarly to all the previous chapters, we will be concerned with solutions that (approximately) maximize the social weflare of the participants.

## 6.1 The setting

In the *divisible item allocation* setting there is a set $N = \{1, \ldots, n\}$ of agents and a set $A = \{1, \ldots, m\}$ of *divisible items.* Without loss of generality, the supply of each item is assumed to be one unit. Each agent has a *utility function* $u_i : [0, 1]^m \to \mathbb{R}$ that maps a quantity vector of the $m$ items to a

[1]The problem is also encountered in literature as *allocation of heterogeneous goods.*

real value. The quantity $u_i(\mathbf{x}_i)$ represents the agent's utility when receiving $\mathbf{x}_i$ amount of the items. We will call $\mathbf{x}_i$ an *allocation vector* for agent $i$, where $x_{ij}$ represents the amount of item $j$ received by agent $i$. We will use $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ to denote an *allocation* of the $m$ items.[2] The *social welfare* is the sum of individual utilities $\sum_{i=1}^{n} u_i(\mathbf{x}_i)$.

### Utility functions

The utility functions express the agents' satisfaction from receiving different amounts of items. A natural example is the *linear* or *additive* utility function, where agents' satisfaction increases additively when they aquire larger amounts of items, scaled by their relative intensity of preferences for those items. In this thesis, we will consider a more general class of utility functions that have received considerable interest in economics, that of *constant elasticity of substitution (CES)* utilities [14]:

**Definition 6.1** (CES utility function)**.** A utility function is in the class of constant elasticity of substitution if it satisfies

$$u_i(\mathbf{x}_i) = \left( \sum_{j=1}^{m} a_{ij} \cdot x_{ij}^{\rho} \right)^{\frac{1}{\rho}}$$

where $\rho$ parameterizes the family, and $-\infty < \rho \leq 1,\ \rho \neq 0$.

In the definition above, $a_{ij}$ is a parameter of the utility functions. It quantifies how receiving more from item $j$ affects agent $i$'s utility, while the exact effect depends on the specific class of utility functions. The parameters $a_{ij}$ here play the role of *valuations* and hence we will call $\mathbf{a}_i = (a_{ij})_{j \in A}$ a valuation vector.[3] Note that once the class of utility functions is fixed, a buyer's utility function can be completely described by its valuation vector $\mathbf{a}_i$.

Roughly speaking, the elasticity of substitution measures the degree of *complementarity* of items; how easy it is to substitute one item for another and what effect that has on the utility function. The elasticity is *constant* for CES function in the sense that it does not depend on the parameters $\mathbf{x}_i$, and can be written as $\sigma = 1/(1-\rho)$. For different choices of $\rho$ we obtain different utility functions. In the literature, there are three standard choices that give rise to three fundamental classes of utility functions.

[2] If we wanted to compare the current setting with the social choice setting of Chapter 1, the former is not a special case of the latter. It is however a special case of the setting where there is a continuum of candidates, and each allocation corresponds to a candidate.

[3] Note that in previous chapters, valuations and utilities had the same meaning; here we need to make the distinction.

**Definition 6.2** (Linear utility function)**.** A utility function is *linear* if it satisfies
$$u_i(\mathbf{x}_i) = \sum_{j \in A} a_{ij} x_{ij}.$$

Linear utility functions are obtained from the CES formula when $\rho = 1$. In that case, the elasticity tends to infinity, which means that the items are *perfect substitutes*, i.e. one can be used instead of the other. A commonly used example of items that are perfect substitutes is Pepsi and Coca-cola.

**Definition 6.3** (Leontief utility function)**.** A utility function is *Leontief* if it satisfies
$$u_i(\mathbf{x}_i) = \min_{j \in A^i} \left\{ \frac{x_{ij}}{a_{ij}} \right\},$$

where $A^i$ is the set of items for which agent $i$ has non-zero valuation. The Leontief utility function is a "limit case" of the CES function, when $\rho$ approaches $-\infty$. The elasticity $\sigma$ in that case goes to 0 and hence the Leontief function captures the utility of items that are *perfect complements*, e.g. left and right shoes.

**Definition 6.4** (Cobb-Douglas utility function)**.** A utility function is *Cobb-Douglas* if it satisfies
$$u_i(\mathbf{x}_i) = \prod_{j \in A} x_{ij}^{a_{ij}}.$$

The Cobb-Douglas utility function is obtained from CES functions when $\rho$ approaches 0. Since the elasticity $\sigma$ in this case is 1, it is often said that Cobb-Douglas utilities express a *perfect balance* between complementarity and substitutability.

## 6.2 Related literature

Divisible item allocation has been the subject of research for many years, sometimes also under the umbrella of *fair division* or *cake cutting* [41, 115, 128, 142], with the goal often being to achieve "fair" allocations, for different notions of fairness such as equitability, envy-freeness, or proportionality [59]. The employement of *markets* and the accompanying results from classical market design has been advocated as a way of achieving desired allocations [22]. As we will see in the next section, markets are in fact implict mechanisms for allocating divisible goods among agents.

With the social welfare objective in mind, Guo and Conitzer [89] and Han et al. [91] study truthful mechanisms and their approximation ratios, whereas Feldman et al. [75] and Zhang [143] bound the price of anarchy of a non-truthful mechanism, the *proportional share allocation mechanism*. In a slightly different approach, Caragiannis et al. [48] bound the approximation ratio of mechanisms, where the inefficiency is due to the need for fair solutions and not truthfulness; they call this measure *the price of fairness*.

Linear utility functions are perhaps the most wide-spread CES utilities used in the literature [41, 89, 91, 115]. Note that a randomized mechanism in the setting of Chapter 3 for indivisible items can be interpreted as a deterministic mechanism for divisible items, where probabilities are fractions of allocations. For example, probabilistic serial could be used as a deterministic mechanism for $n$ agents and $n$ indivisible items. While both Cobb-Douglas and Leontief utilities are quite important in economics, Leontief utilities are also quite popular in the computer science literature [66, 85, 121], because of the fact that they express values for items that are complements and are in that sense quite fitting to scenarios where computational jobs are allocated available resources in fixed ratios [59].

In Chapter 7, we will study social welfare maximization by a market-like mechanism, the *Fisher market mechanism*. For that reason, we will next introduce the fundamental economic market model of the Fisher market.

## 6.3 The Fisher market

The concept of the market exists since ancient times. In a traditional market, people come with endowments of goods, or *commodities*, and trade those commodities amongst each other, to satisfy their needs. This market setting is perhaps best described by the celebrated *Arrow-Debreu market* [13], the most fundamental market model in economics. In that model, also known as the Walrasian model, the trade described above is done though the following process. An artificial unit of currency if fixed and *prices* for the commodities are announced, based on this currency. Interested buyers then announce their *demands* for the commodities. Based on the stated demands, prices are adjusted and communicated to the buyers again, who in turn announce their new demand sets. This process, called *the tattonement process* is repeated until all items are "sold", i.e. the market *clears*. It has been proven [14, 112] that the process will always converge to a market clearing solution;[4] this solution is called a *market equilibrium* or a *walrasian equilibrium*.

One fundamental special case of the Arrow-Debreu market was introduced by Irving Fisher [40].

**Definition 6.5** (Fisher market)**.** A Fisher market $\mathcal{M}$ consists of a set of buyers and a set of divisible goods. Every buyer $i$ has:

- an initial budget $B_i > 0$, which can be viewed as some currency that can be used to acquire goods but has no intrinsic value to the buyer, and

[4]Interestingly, [14, 112] proved that such a solution is unique if the utility functions are *strongly concave*. This will be important for our results in Chapter 7.

- a utility function $u_i : [0,1]^m \rightarrow \mathbf{R}$ defined as above.

A market outcome is defined as a tuple $\langle \mathbf{p}, \mathbf{x} \rangle$, where $\mathbf{p}$ is a vector of prices for the $m$ items and $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ is an allocation of the $m$ items, with $p_j$ denoting the price of item $j$ and $x_{ij}$ representing the amount of item $j$ received by buyer $i$.

The Fisher market is a subset of the Arrow-Debreu market, in the following sense. There are two types of participants, one seller and multiple buyers. The endowment of the seller consists of all the goods in the market and the endowments of the buyers are different quantities of a single good that we will call *money*. The seller only experiences positive utility from aquiring money, whereas the buyers only have positive values for goods, but they have no value for money. The market clearing condition requires that no buyers are left with any amount of money and the seller is not left with any amount of goods in her possession. This condition can be rephrased as "all budgets are exhausted" and "all items are sold". We define the notion of a market equilibrium in the Fisher market.

**Definition 6.6.** In a Fisher market $\mathcal{M}$, a *market equilibrium* [13, 120] is a market outcome that maximizes the utility of each buyer subject to her budget constraint and clears the market. Formally, $\langle \mathbf{p}, \mathbf{x} \rangle$ is a market equilibrium if and only if

- For all $i \in N$, $\mathbf{x}_i$ maximizes buyer $i$'s utility given prices $\mathbf{p}$ and budget $B_i$.

- Each item $j$ either is completely sold or has price 0, i.e.

$$\left( \sum_{i=1}^{n} x_{ij} - 1 \right) p_j = 0, \ \forall j \in A.$$

- All budgets are spent, i.e. $\sum_{j=1}^{m} p_j \cdot x_{ij} = B_i$, $\forall i \in N$.

For the Fisher market model, a market equilibrium is guaranteed to exist if each item is desired by at least one buyer and each buyer desires at least one item [111].

### The Fisher market mechanism

One can perhaps see the similarities between a market model and the divisible item allocation setting that we study in this part of the thesis. The allocation part $\mathbf{x}$ of a market outcome is an allocation of divisible items to agents and hence a solution to the divisible item allocation problem. In that sense, one can view the Fisher market as a mechanism for allocating divisible goods, that we will call the *Fisher market mechanism*. The mechanism inputs the valuations of the agents, assigns them artificial budgets $B_i$ and computes a market equilibrium, using $\mathbf{x}$ as the allocation. For this to be possible however, the mechanism designer must have a way of computing the market equilibrium without going through the tattonement process.[5] Luckily, for buyers with utility functions from the same class in the CES family (i.e. for some fixed $\rho$), the equilibrium allocation can be captured by the celebrated Eisenberg-Gale convex program [70], one of the few algorithmic results in general equilibrium theory:

**Definition 6.7** (Market equilibrium computation)**.** The allocation of a market equilibrium of a Fisher market $\mathcal{M}$ is given by the solution to the following convex program (the prices are given by the dual variables).

$$\begin{aligned}
\max \quad & \sum_{i=1}^{n} B_i \cdot \log(u_i) \\
s.t. \quad & u_i = \left(\sum_{j=1}^{m} a_{ij} \cdot x_{ij}^{\rho}\right)^{\frac{1}{\rho}}, \quad \forall\, i \in N \\
& \sum_{i=1}^{n} x_{ij} \leq 1, \quad \forall\, j \in A \\
& x_{ij} \geq 0, \quad \forall\, i \in N, j \in A
\end{aligned}$$

For some values of $\rho$, for example $\rho = 1$, the objective function of this convex program is not strictly concave, which means that there may be multiple market equilibria. In that case, the mechanism should be equipped with some tie-breaking rule; the choice of tie-breaking will be quite important for some of our results in Chapter 7.

## 6.4 Social welfare maximization

In Chapter 7, we will study the Fisher market mechanism for divisible item allocation and bound its price of anarchy with respect to the social welfare objective for different utility functions. We will now discuss the main results on social welfare maximization for divisible items mentioned earlier in more detail.

### Truthful mechanisms

The topic of social welfare maximization for divisible items in setting without money was studied by Guo and Conitzer [89] for the case of two agents and for linear utility functions under the unit-sum representation. The authors propose a class of truthful mechanisms and analyze their approximation ratios.

[5] Otherwise the mechanism would not be direct revelation.

Similarly to the Fisher market mechanism, their mechanisms endow agents with artificial units of currency that they use to produce an allocation. The problem for the case of many agents was studied by Han et al. [91] where the authors bound the performance of any truthful mechanism.[6]

**Theorem 6.1.** *[91] The approximation ratio of any truthful mechanism for divisible item allocation with linear utility functions for the unit-sum representation is* $O(1/\sqrt{n})$.

Note that the theorem only applies to linear utility functions; to the best of our knowledge, an impossibility result of a similar nature for other classes of CES functions (such as Leontief or Cobb-Douglas) does not exist. In fact, it is an interesting open problem to prove such a bound.

We do not know of any truthful mechanisms that achieve this bound. In fact, we don't know of many good truthful candidates that do not come from indivisible item allocation or are not trivial. Cole et al. [59] propose a truthful mechanism, the *partial allocation* mechanism, which achieves good fairness objectives. It would be interesting to evaluate the approximation ratio of the mechanism with respect to the social welfare. It is unclear whether it would provide good approximations though, since crucially, the mechanism "burns" fractions of items in order to ensure truthfulness and achieve fairness guarantees; intuitively however, wastefulness and social welfare maximization do not seem very compatible.

## Non-truthful mechanisms

Our approach in Chapter 7 is to study a non-truthful mechanism and bound its price of anarchy. A very similar approach was taken by Feldman et al. [75] and Zhang [143], who study the *proportional share allocation* mechanism.

**Definition 6.8** (Proportional share allocation mechanism)**.** In the proportional share allocation mechanism, each agent $i$ is given a budget $B_i$ that she can freely distribute over the $m$ items. The report of each agent is an $m$-dimensional vector $\mathbf{s}_i = (s_{i1}, \ldots, s_{im})$, with the property that $\sum_{j=1}^{m} s_{ij} = B_i$. Given any instance of the agents' reports, $\mathbf{s} = (\mathbf{s}_1, \ldots, \mathbf{s}_n)$, the price of each item $j$ is set to $p_j = \sum_{k=1}^{n} s_{kj}$, and agent $i$ receives $x_{ij} = \frac{s_{ij}}{p_j}$ units of item $j$. If all agents report zero for some item, then that item is kept by the center.

Note that the proportional share allocation mechanism is cardinal; the same applies to the Fisher mechanism that we will study in the next chapter. The authors in [75] and [143] consider the unit-sum representation[7] and prove

[6] The authors also prove a $O(1/m)$-upper bound on the approximation ratio of any truthful mechanism. In our investigations, all of the bounds will be functions of $n$, the number of agents.

[7] In fact, Zhang [143] considers a slightly more general condition.

that the price of anarchy of the proportional share allocation mechanism is $\Omega(1/\sqrt{n})$ for any concave, non-decreasing utility function (including CES functions). Feldman et al. [75] provide a profile for linear utilities for which the bound is tight.

**Theorem 6.2.** *[75, 143] The price of anarchy of the proportional share allocation mechanism for divisible item allocation with CES utility functions is $\Omega(1/\sqrt{n})$. If the utility functions are linear, there exists a profile such that the price of anarchy is $O(1/\sqrt{n})$.*

The lower bound established in the theorem above will be very useful for our results in Chapter 7.

## 6.5 Social welfare and divisible items: An agenda

From the discussion above, it is evident that the literature on social welfare maximization in divisible item allocation settings consists of a few key papers and many open questions. In addition, it seems that most of the papers on this topic[8] do not place themselves as parts of a common literature and present the objectives or the contributions in different ways.

We think that an *agenda* on social welfare maximization in divisible item allocation settings is in place: given some general class of utility functions, such as CES, as well as some natural subclasses, what can we achieve in terms of social welfare? What is the approximation ratio of truthful mechanisms? Is there a truthful mechanism that achieves the $O(1/\sqrt{n})$ barrier for linear utility functions? What are the corresponding upper bounds for other utility classes? What about the price of anarchy of non-truthful mechanisms? Is there such a mechanism that outperforms all truthful ones? Can we obtain upper bounds on the price of anarchy of any mechanism?

[8] Including the paper associated with Chapter 7.

# Chapter 7

# The Fisher market mechanism

In this chapter, we study the *Fisher market mechanism* for allocating divisible items among agents. Our goal is to approximately maximize the social welfare, i.e. the sum of agents utilities for the portions of items allocated to them. Similarly to our approach in Chapter 5, we will consider the equilibrium behaviour of the mechanism and bound its *price of anarchy.* We consider the three major subclasses of CES utility functions, *linear*, *Leontief* and *Cobb-Douglas* utilities. First, we prove that regardless of the agents' utilities, the mechanism has pure Nash equilibria, under a mild condition. Then, we bound the inefficiency of the mechanism, by obtaining price of anarchy results for all three cases.

## 7.1 Introduction, setting and contributions

The strategic aspects of Fisher markets were first studied by Adsul et al. [6] for buyers with linear utility functions. The authors showed the existence of pure Nash equilibria under mild assumptions and provided necessary conditions for a strategy profile to be a pure Nash equilibrium. Chen et al. [51] and Chen et al. [52] bounded the extent to which a buyer can improve his utility by deviating from being truthful for CES utility functions; they call this measure *the incentive ratio.* More recently, Babaioff et al. [22] examined strategic behavior in settings where markets are used as auction mechanisms, similarly to what we do in this chapter. Unlike Fisher markets however, where money has no intrinsic value to buyers, the buyers in the markets studied by Babaioff et al. have *quasi-linear* utilities.

Our contributions are as follows. When buyers have Leontief and Cobb-Douglas utility functions, we show that the Fisher market mechanism always has a pure Nash equilibrium under mild conditions. Together with the results of Adsul et al. [6], who identified a particular pure Nash equilibrium of the Fisher market mechanism for buyers with linear utilities, these results prove the existence of a pure Nash equilibrium for all the three typical CES utility functions. We then prove asymptotic price of anarchy bounds for the Fisher market mechanism for linear, Leontief, and Cobb-Douglas utilities. For Leontief and Cobb-Douglas functions, we obtain tight price of anarchy bounds of $\Theta(1/n)$ and $\Theta(1/\sqrt{n})$ respectively, where $n$ is the number of buyers in the game. For linear utility functions, the price of anarchy is upper bounded by $O(1/\sqrt{n})$ and lower bounded by $\Omega(1/n)$. Proving tight price of anarchy bounds for linear utility functions is left as an open question. A summary of the results can be found in Table 7.1.

### The Fisher market game

Recall the definition of the Fisher market mechanism in Chapter 6; the mechanism inputs the valuations of agents and computes a market equilibrium of the corresponding Fisher market. The outcome of the mechanism is the equilibrium allocation. Similarly to Chapter 5 however, we assume that agents are strategic entities that will not truthfully report their valuations, if misreporting gives them a higher utility. The mechanism then induces a game, the *Fisher market game* [6].

In the same spirit as Adsul et al. [6], we define the Fisher market game as a game with complete information among all agents. The definition is for agents with CES utility functions with a fixed $\rho$. Hence, an agent's utility function can be described by her valuation vector $\mathbf{a}_i$.

**Definition 7.1** (Fisher Market Game)**.** Given a set of items $A = \{1, \ldots, m\}$ and a set of agents $N = \{1, \ldots, n\}$, where each agent $i$ has budget $B_i$ and valuation vector $\mathbf{a}_i$, the *Fisher Market Game* is such that:

- The pure strategy space of each agent $i$ is the set of all possible valuation vectors that $i$ may report: $S_i = \{\mathbf{s}_i \mid \mathbf{s}_i \in \mathbb{R}^m_{\geq 0}\}$. We refer to a strategy $\mathbf{s}_i$ as a *report*.

- Given a strategy profile $\mathbf{s} = (\mathbf{s}_i)_{i=1}^n$, the outcome of the game is any fixed market equilibrium of the Fisher market given by $\langle B_i, \mathbf{s}_i \rangle_i$, after removing all items $j$ such that $\sum_{i \in N} s_{ij} = 0$.

- Let $\mathbf{x}(\mathbf{s}) = (\mathbf{x}_1(\mathbf{s}), \ldots, \mathbf{x}_n(\mathbf{s}))$ denote the market allocation for strategy profile $\mathbf{s}$. For all $i \in N$, agent $i$'s utility at $\mathbf{s}$ is $u_i(\mathbf{x}_i(\mathbf{s}))$, written as $u_i(\mathbf{s})$ for short.

Table 7.1: Summary: the social welfare of the Fisher market mechanism. Lower bound (*) is due to Zhang [143]

| | Cobb-Douglas | Leontief | Linear |
|---|---|---|---|
| UB | $O(1/\sqrt{n})$ | $O(1/n)$ | $O(1/\sqrt{n})$ |
| LB | $\Omega(1/\sqrt{n})$ (*) | $\Omega(1/n)$ | $\Omega(1/n)$ |

We will bound the loss in social welfare in the worst pure Nash equilibrium of the mechanism, over all valuation profiles and for different utility functions. Before we do that however, we need to ask ourselves whether the mechanism has Nash equilibria in the first place. Note that since the strategy space is continuous, existence of Nash equilibria is not guaranteed. As we will see in the next section, the mechanism actually has pure Nash equilibria for all three utility functions that we consider.

### Choosing the budgets

In a Fisher market, the budgets have a natural interpretation, as the amount of commodity that each buyers brings to the market. In other words, from the viewpoint of the seller, some buyers are "richer" than others. On the other hand, in the mechanism for allocation of divisible items interpretation, budgets are only artificial and are chosen by the mechanism. The natural choice would be to assign equal budgets to all agents, which means that all agents are treated equally by the mechanism. A market equilibrium when agents are equipped with equal budgets is often called a *competitive equilibrium from equal incomes (CEEI)* and has received considerable attention in the literature[1] as a way of achieving fair allocations [17, 44, 45].

On the other hand, sometimes the choice of budgets might be *exogenous* and not entirely up to the designer. The designer might often be instructed to treat agents *differently* for social or economical reasons; for example, premier customers might have more "claim" on offered services. Such "favoritism" can be implemented through the choice of different budgets.

For the pure Nash equilibrium existence results in Section 7.2 we will impose no restriction on the budgets (other that the sum of budgets is equal to the sum of prices, which is an internal requirement of the market mechanism). Existence is guaranteed for any budgets chosen by the designer. For the price of anarchy results in Section 7.3, we will make the following assumptions. For the upper bounds, we will assume that all budgets are equal; note that it is sufficient to consider this case because, since the budgets are chosen independently of the reported valuations, for any choice of budgets, we can

[1] The equal budget assumption is also referred to as a *balanced game* in [75] and [143] for the same problem but a different mechanism.

create a "bad" valuation profile adjusted to that choice of budgets on which the mechanism can not outperform the stated bounds.

For the price of anarchy guarantees, we will assume that the budgets satisfy a condition that we call *$\delta$-normalization* [143]; this assumption requires that the budgets are proportional to the agent's utility for receiving all the items. Intuitively, this makes sense; agents that would be more satisfied with complete allocations are given more weight within the mechanism. For linear and Cobb-Douglas utility functions and since we are considering the unit-sum representation of valuations, the condition boils down to agents having equal budgets. For Leontief utilities, there is more flexibility on the choice of budgets, that depends on the valuations of the agents; this has to do with the nature of the utility function. We will discuss these assumptions further in Section 7.3.

### Reny's theorem for games with discontinuous payoffs

The Fisher market game that we will analyze has *discontinuous payoffs* and thus standard methods for proving existence of pure Nash equilibria do not apply. The discontinuities arise because when an agent changes her valuation for an item from some very small positive number to 0, the fraction of that item that the agent receives might exhibit a "jump". Our main tool for dealing with such cases is a non-trivial result due to Reny [125] on the existence of pure Nash equilibria in general games with discontinuous payoffs. We set up the required machinery starting with the following definitions.

**Definition 7.2** (Secure payoff)**.** Agent $i$ can *secure* a payoff of $\alpha$ at strategy profile $(\mathbf{s}_i, \mathbf{s}_{-i}) \in S$ if there is $\bar{\mathbf{s}}_i \in S_i$, such that $u_i(\bar{\mathbf{s}}_i, \mathbf{s}'_{-i}) \geq \alpha$ for all $\mathbf{s}'_{-i}$ close enough to $\mathbf{s}_{-i}$, i.e. if there exists $\epsilon > 0$ such that for any $\mathbf{s}'_{-i}$ with $|\mathbf{s}'_{-i} - \mathbf{s}_{-i}| < \epsilon$ then $u_i(\bar{\mathbf{s}}_i, \mathbf{s}'_{-i}) \geq \alpha$.

In other words, agent $i$ can secure the payoff $\alpha$ if the agent has a strategy guaranteeing a utility of at least $\alpha$ not only at the strategy profile $(\bar{\mathbf{s}}_i, \mathbf{s}_{-i})$, but also at all profiles where agent $i$ plays $\bar{\mathbf{s}}_i$ but the other agents slightly deviate from $\mathbf{s}_{-i}$.

**Definition 7.3** (Closure of the graph of the vector payoff function)**.** A pair $(\mathbf{s}, \mathbf{u}) \in S \times \mathbb{R}^n$ is *in the closure of the graph of the vector payoff function* if $\mathbf{u} \in \mathbb{R}^n$ is the limit of the vector of agent payoffs for some sequence of strategies $(\mathbf{s}^k)_{k \geq 1}$ converging to $\mathbf{s}$. That is, if $\mathbf{u} = \lim_k \Big(u_1(\mathbf{s}^k), \ldots, u_n(\mathbf{s}^k)\Big)$ for some $\mathbf{s}^k \to \mathbf{s}$.

**Definition 7.4** (Better-reply security)**.** A game $G = (S_i, u_i)_{i=1}^n$ is *better-reply secure* if whenever $(\mathbf{s}^*, \mathbf{u}^*)$ is in the closure of the graph of its vector payoff function and $\mathbf{s}^*$ is not a Nash equilibrium, some agent $i$ can secure a payoff strictly above $u_i^*$ at $\mathbf{s}^*$.

**Theorem 7.1.** *[Pure nash equilibrium existence [125]] If the strategy space of each agent $i$, $S_i$, is a non-empty, compact, convex subset of a metric space, the utility function of each agent $i$, $u_i(\mathbf{s}_1, \ldots, \mathbf{s}_n)$ is quasi-concave in the agent's own strategy, $\mathbf{s}_i$, and the game $G = (S_i, u_i)_{i=1}^n$ is better-reply secure, then $G$ has at least one pure Nash equilibrium.*

## 7.2 Existence of pure Nash equilibria

In this section, we study the existence of pure Nash equilibria for the three main classes of CES utility functions. According to the standard definition, a strategy profile is a pure Nash equilibrium if no agent can increase her utility by deviating to some other strategy. In the Fisher market game, since the outcome of the game might be one of several market equilibria, we define a *pure Nash equilibrium of the Fisher market game* to be a strategy profile where for any deviation of any agent $i$, agent $i$'s payoff does not increase, for *any* market equilibrium of the resulting strategy profile. Note that this is only an issue when the utility functions are not strictly concave and hence the market equilibrium is not necessarily unique [13]. We start with Cobb-Douglas utility functions.

### Cobb-Douglas utilities

The main result of this section is that the Fisher market game with Cobb-Douglas utilities has pure Nash equilibria for a large class of valuations that captures most scenarios of interest. That is, existence is guaranteed when the game is *strongly competitive* (i.e. for each item $j \in A$, there exists more than one agent with non-zero valuation for it) and the valuations are *unit-sum* (i.e. $\sum_j a_{ij} = 1$, $\forall i \in N$). We assume the unit-sum representation throughout the chapter, which is also consistent with the literature on divisible item allocation [75, 89, 91]. Strong competitiveness is required in order for a pure Nash equilibrium to exist, since if there is an item desired by a single agent, that agent has an incentive to assign less and less value on this item and still be allocated the item entirely. The very same condition is employed by Adsul et al. [6], Feldman et al. [75] and Zhang [143].

As we mentioned earlier, the game that we study has discontinuous payoffs and hence we will use Theorem 7.1 for proving existence of a pure Nash equilibrium.

Recall that given an allocation $\mathbf{x} = (x_{ij})$, where $x_{ij}$ is the amount received by agent $i$ from good $j$, the utilities are:

$$u_i(\mathbf{x}) = \prod_{j \in A : a_{ij} \neq 0} x_{ij}^{a_{ij}}, \forall i \in N.$$

.

Moreover, since utility functions are strictly concave, the market equilibrium and market prices are unique and have the following succinct form [68]:

$$p_j = \sum_{i=1}^{n} a_{ij} B_i \quad \text{and} \quad x_{ij} = \frac{a_{ij} B_i}{\sum_{k=1}^{n} a_{kj} B_k} \quad \text{whenever} \quad p_j \neq 0.$$

We show that the Fisher market game with Cobb-Douglas utilites and unit-sum valuations is better-reply secure.

**Lemma 7.1.** *The Fisher market game with Cobb-Douglas utilities and unit-sum valuations is better-reply secure.*

*Proof.* Since all games with continuous payoffs are better-reply secure, it is sufficient to check the property at the points where the utility functions are discontinuous [126]. In the Fisher market game with Cobb-Douglas utilities, the discontinuity occurs when there exists an item $j$ such that all agents assign a value of zero towards that item. That is, the utility functions are discontinuous at the points in the set

$$\mathcal{D} = \{\mathbf{s} \in S \mid \exists\, j \in A \text{ such that } s_{ij} = 0, \forall i \in N\}$$

Let $(\mathbf{s}^*, \mathbf{u}^*)$ be in the closure of the graph of the vector payoff function, where $\mathbf{s}^* \in \mathcal{D}$. Then $\mathbf{u}^* = \lim_{K\to\infty}(u_1(\mathbf{s}^K), \ldots, u_n(\mathbf{s}^K))$ for some sequence of strategies $\mathbf{s}^K \to \mathbf{s}^*$. For each sequence term $s^K$, let $s_{ij}^K$ be the report of agent $i$ for item $j$ and $S_j^K$ the sum of reported values for item $j$. Let $J$ be the set of items that no agent declares as valuable (i.e. with strictly positive value) in $\mathbf{s}^*$:

$$J = \{j \in A \mid s_{ij}^* = 0, \forall i \in N \text{ and } \exists i \in N \text{ such that } a_{ij} \neq 0\}$$

Using an average argument, there exist an agent $i$, item $l$ and index $N_0 \in \mathbb{N}$ such that

$$\frac{s_{il}^K}{S_l^K} \leq \frac{2}{3}, \quad \text{for all} \quad K \geq N_0.$$

That is, agent $i$ gets at most 50% of item $l$ in every term of the sequence $(\mathbf{s}^K)_{K\geq 1}$ (except possibly for the first $N_0 - 1$ terms). Let agent $i$ and item $l$ be fixed for the remainder of the proof. Let $S_j^* = \sum_{k=1}^{n} s_{kj}^*$ be the sum of values of the agents for item $j$ at the strategy profile $\mathbf{s}^*$, $S_j^K = \sum_{k=1}^{n} s_{kj}^K$ the sum of values for item $j$ at the strategy profile $\mathbf{s}^K$, and $L_i = \{j \in A \mid s_{ij}^* > 0\}$ the set of items that agent $i$ declares as valuable in the limit.

For every item $k \notin J$, we have that

$$\lim_{K\to\infty} \frac{s_{ik}^K}{S_k^K} = \frac{s_{ik}^*}{S_k^*}.$$

Then the utility of agent $i$ in the limit of the sequence of strategies $(\mathbf{s}^K)_{K\geq 1}$, can be rewritten as follows:

$$\begin{aligned} u_i^* &= \lim_{K\to\infty}\left(\prod_{j\in J}\left(\frac{s_{ij}^K}{S_j^K}\right)^{\alpha_{ij}}\cdot\prod_{j\in L_i}\left(\frac{s_{ij}^K}{S_j^K}\right)^{\alpha_{ij}}\right) \\ &= \prod_{j\in J}\lim_{K\to\infty}\left(\frac{s_{ij}^K}{S_j^K}\right)^{a_{ij}}\cdot\prod_{j\in L_i}\left(\frac{s_{ij}^*}{S_j^*}\right)^{a_{ij}} \\ &\leq \left(\frac{2}{3}\right)^{a_{il}}\cdot\prod_{j\in L_i}\left(\frac{s_{ij}^*}{S_j^*}\right)^{a_{ij}} \end{aligned}$$

We illustrate the case $u_i^* > 0$. If $u_i^* = 0$, the analysis is simpler; agent $i$ can easily secure a strictly positive payoff by declaring a small valuation on the items in $J$, for every $\epsilon$-perturbation of the other agents' strategies around $\mathbf{s}_{-i}^*$. Define the constants

$$\alpha = \sum_{j\in L_i : a_{ij}\neq 0} a_{ij} \quad\text{and}\quad \gamma = \left(\frac{3}{2}\right)^{\frac{a_{il}}{\alpha}},$$

where $\gamma > 1$. Let $\delta > 0$ be fixed such that

$$\delta < \frac{(\gamma-1)\cdot S_j^*}{\gamma\cdot S_j^* - s_{ij}^*}, \quad\text{for all}\quad j\in L_i.$$

Consider a new strategy profile, $\mathbf{s}_i'$, for agent $i$, such that

$$s_{ij}' = \begin{cases} (1-\delta)s_{ij}^* & \text{if } j\in L_i \\ \left(\frac{\delta}{|J|}\right)\cdot\left(\sum_{k\in L_i} s_{ik}^*\right) & \text{if } j\in J \\ s_{ij}^*(=0) & \text{otherwise} \end{cases}$$

Agent $i$'s utility when playing $\mathbf{s}_i'$ against strategies $\mathbf{s}_{-i}^*$ is:

$$\begin{aligned} u_i(\mathbf{s}_i', \mathbf{s}_{-i}^*) &= \prod_{j\in J}\left(\frac{s_{ij}'}{s_{ij}'}\right)^{a_{ij}}\cdot\prod_{j\in L_i}\left(\frac{(1-\delta)s_{ij}^*}{S_j^* - \delta\cdot s_{ij}^*}\right)^{a_{ij}} \\ &= \prod_{j\in L_i}\left(\frac{(1-\delta)s_{ij}^*}{S_j^* - \delta\cdot s_{ij}^*}\right)^{a_{ij}} \end{aligned}$$

Then for each $j\in L_i$, the following inequality holds:

$$\gamma^{a_{il}}\cdot\left(\frac{(1-\delta)\cdot s_{ij}^*}{S_j^* - \delta\cdot s_{ij}^*}\right) > \frac{s_{ij}^*}{S_j^*}. \tag{7.1}$$

By taking the product of Inequality (7.1) over all items $j\in L_i$, we obtain that $u_i(\mathbf{s}_i', \mathbf{s}_{-i}^*) > u_i^*$. The utility of agent $i$ is continuous at $(\mathbf{s}_i', \mathbf{s}_{-i}^*)$, and so for small changes in the strategies of the other agents around $\mathbf{s}^*_{-i}$, agent $i$ still gets a better payoff than at $u_i^*$. That is, there exists $\epsilon > 0$ such that for all feasible strategies $\mathbf{s}'_{-i}$ of the other agents, where $||\mathbf{s}'_{-i} - \mathbf{s}^*_{-i}|| < \epsilon$, it is still the case that $u_i(\mathbf{s}'_i, \mathbf{s}'_{-i}) > u_i^*$. It follows that the game is better-reply secure. □

We now state the theorem.

**Theorem 7.2.** *The Fisher market game with Cobb-Douglas utilities has a pure Nash equilibrium under unit-sum valuations whenever the game is strongly competitive.*

*Proof.* The strategy set of each agent in the Fisher market game with unit-sum, Cobb-Douglas utilities is non-empty, compact, and convex. Moreover, the utilities are quasi-concave in the agents' own strategies; by Lemma 7.1, the game is also better-reply secure. Finally, it can be easily seen that the utility function of each agent is quasi-concave in the agent's own strategy, by an application of the vector composition rule for functions [39]. Thus the conditions of Reny's theorem are met, and so a pure Nash equilibrium is guaranteed to exist. □

### Leontief utilities

For the class of Leontief utility functions, we prove the existence of a pure Nash equilibrium by directly constructing a set of equilibrium strategies. Namely, the uniform strategy profile is a Nash equilibrium regardless of the true valuations; moreover, the statement holds even for games that fail to be strongly competitive. The high level explanation is that Leontief utilities exhibit perfect complementarity, thus reporting a smaller valuation for an item that no other agent desires does not result in an increased utility for the deviator (since utility is taken as a minimum over the allocation/valuation ratios).

We start by analyzing two-agent markets and then extend the result to markets with multiple agents.

**Theorem 7.3.** *Given a Fisher market game for two agents with Leontief utilities, the uniform tuple of strategies is a pure Nash equilibrium, and the agents' utilities are $B_1/\max_j\{a_{1j}\}$ and $B_2/\max_j\{a_{2j}\}$, respectively.*

In order to prove Theorem 7.3, we build upon a result of Chen et al. [51], that describes the best response strategies in two-agent markets. First, for each agent $i \in N$ define the following terms:

$$a_i^{max} = \max_{j \in A}\{a_{ij}\}, \quad a_i^{min} = \min_{j \in A}\{a_{ij}\}, \quad s_i^{max} = \max_{j \in A}\{s_{ij}\}, \quad s_i^{min} = \min_{j \in A}\{s_{ij}\}.$$

Then, given any two-agent market and an arbitrary fixed strategy $\mathbf{s}_2$ of agent 2, the best response strategy of agent 1 is [51]

$$\mathbf{s}_1 = (s_{1j})_{j \in A}, \quad \text{where} \quad s_{1j} = 1 - s_{2j} \cdot \frac{B_2}{s_2^{max}}.$$

In addition, given fixed strategies $(\mathbf{s}_1, \mathbf{s}_2)$, the market equilibrium allocation is unique and the utility of agent 1 is:

$$u_1(\mathbf{s}_1, \mathbf{s}_2) = \min_{j \in A} \left\{ \frac{1 - s_{2j} \cdot (B_2/s_2^{max})}{a_{1j}} \right\}.$$

Agent 2's allocation is given by: $x_{2j} = s_{2j} \cdot (B_2/s_2^{\max})$, $\forall j \in A$, and her utility is the minimum possible (as evaluated using strategy $\mathbf{s}_2$); that is, $u_2'(\mathbf{s}_1, \mathbf{s}_2) = B_2/s_2^{max}$.

For ease of notation we will assume that the prices $p_j$ and the budgets $B_i$ satisfy the identity: $\sum_{j \in A} p_j = \sum_{i \in N} B_i = 1$ [51]; the proof can be adapted for a different choice of total prices and total budgets. Then, the utility of agent $i$ (as evaluated using the agent's strategy $\mathbf{s}_i$) is:

$$u_i' = \frac{B_i}{\sum_{j \in A} p_j s_{ij}}, \text{ where } \frac{B_i}{s_i^{max}} \leq u_i' \leq \frac{B_i}{s_i^{min}}.$$

At a high level, by using $\mathbf{s}_1$, agent 1 forces agent 2 to get the minimum possible utility (as evaluated with respect to the reported valuations $\mathbf{s}_i$); this translates to the worst possible allocation for agent 2, while agent 1 gets all the remaining items.

In order for a pair of strategies $(\mathbf{s}_1, \mathbf{s}_2)$ to be a pure Nash equilibrium, the utility of each agent $i$ (evaluated using her report) satisfies $u_i'(\mathbf{s}_1, \mathbf{s}_2) = B_i/s_i^{max}$, otherwise, some agent could increase her allocation by using the above best response strategy (which would decrease the other agent's allocation). Theorem 7.3 follows from Lemmas 7.2 and 7.3.

**Lemma 7.2.** *For every pair of strategies* $(\mathbf{s}_1, \mathbf{s}_2)$ *that is a pure Nash equilibrium of the Fisher market game with two agents and Leontief utilities, the utility of each agent* $i$*, as evaluated using her true valuations, satisfies the inequality:* $u_i(\mathbf{s}_1, \mathbf{s}_2) \leq B_i/a_i^{max}$.

*Proof.* Let $\mathbb{A}_i = \{k \mid a_{ik} = a_i^{max}\}$. The equilibrium utility of agent $i$ (evaluated using her true valuations) is:

$$u_i = \min_{j \in A} \left\{ \frac{s_{ij} \cdot (B_i/s_i^{max})}{a_{ij}} \right\} \leq \frac{s_{ik} \cdot (B_i/s_i^{max})}{a_i^{max}},$$

where $k \in \mathbb{A}_i$. If it was the case that $u_i > B_i/a_i^{max}$, then it would follow that

$$\frac{B_i}{a_i^{max}} < u_i \leq \frac{s_{ik} \cdot (B_i/s_i^{max})}{a_i^{max}},$$

and so $s_i^{max} < s_{ik}$, which is false. Thus $u_i(s_1, s_2) \leq B_i/a_i^{max}$. $\square$

**Lemma 7.3.** *The* uniform strategy $\left(\frac{1}{m}, \cdots \frac{1}{m}\right)$ *guarantees agent* 1 *a payoff of* $B_1/a_1^{max}$*, regardless of agent* 2*'s strategy.*

*Proof.* Let the strategy of agent 1 be $s_1 = \left(\frac{1}{m}, \dots, \frac{1}{m}\right)$. Consider an arbitrary strategy $s_2$ of agent 2 and the resulting market equilibrium prices, $p'$. Then the utility of agent 1 (as evaluated using the strategy) is:

$$u'_1 = \frac{B_1}{\sum_{j \in A} p'_j \cdot s_{1j}} = \frac{B_1}{\sum_{j \in A} p'_j \cdot \left(\frac{1}{m}\right)} = m \cdot B_1.$$

Then the allocation of agent 1 for each item $j$ is given by:

$$x'_{1j} = s_{1j} \cdot u'_1 = \left(\frac{1}{m}\right) \cdot m \cdot B_1 = B_1.$$

Agent 1's utility (evaluated using her true valuations), is:

$$u_1 = min_{j \in A} \left\{ \frac{x'_{1j}}{a_{1j}} \right\} = min_{j \in A} \left\{ \frac{B_1}{a_{1j}} \right\} = \frac{B_1}{a_1^{max}}.$$

This completes the proof of the lemma. □

Note that by reporting truthfully, agent $i$ gets $u_i = B_i / \left(\sum_{j \in A} p_j \cdot a_{ij}\right)$, where $B_i / a_i^{max} \leq u_i \leq B_i / a_i^{min}$. Thus in any pure Nash equilibrium, the agents fare worse compared to truthful play.

Next we generalize Theorem 7.3 to any number of agents. Note that the best response strategy of Chen et al. [51] does not apply to our game directly. However, we observe that the uniform strategy remains a pure Nash equilibrium regardless of the number of agents.

**Theorem 7.4.** *Given a Fisher market game with Leontief utilities, the uniform strategy is a Nash equilibrium for any number of agents, with utilities $u_i = B_i / a_i^{max}$, for all $i \in N$.*

*Proof.* Let $i$ be any agent and $\mathbf{s}_{-i}$ an arbitrary fixed strategy of the other agents. From the objective function of the Eisenberg-Gale convex program, it can be observed that all the other agents can be seen as equivalent to a (combined) single agent. Thus the market equilibrium allocation can be computed by reducing the game to two agents, $i$ and $-i$. By Theorem 7.3, agent $i$ has no incentive to deviate from the uniform strategy; thus the uniform strategy is also a pure Nash equilibrium of the $n$-agent game. It can be verified that the utilities are $u_i = B_i / a_i^{max}$, for each $i \in N$. □

### Linear Utilities

Finally, the existence of pure Nash equilibria for linear utilities was established by Adsul et al. [6], for strongly competitive games in a more restricted model than ours. In their model, they require that the outcome of the game on a given set of reports $\mathbf{s}$, is the market equilibrium that maximizes the product of every agent's utility according to their true valuations, i.e. if there is a market equilibrium $\mathcal{E}$ that every agent prefers to every other market equilibrium, then the outcome of the game is $\mathcal{E}$. We will refer to this condition as *conflict-freeness* [6].

**Theorem 7.5.** *[Adsul et al. [6]] Given any Fisher market game with linear utilities, there exists a (symmetric) pure Nash equilibrium in which the payoffs are identical to those obtained when agents play truthfully.*

The conflict-freeness condition is meaningful in [6] because it is assumed that the outcome of the game is chosen by the agents, who have complete information over the *real* valuations of other agents.[2] When using the Fisher market as a mechanism however, the designer does not have access to the true values and hence she can not choose the conflict-free market equilibrium that would guarantee stable play from the agents' part. One remedy would be for the designer to ask the agents to tie-break between different market equilibria in case of ties, but that would contradict direct revelation and would perhaps introduce extra incentives for the agents to manipulate.

It is an open question if there is a systematic way to choose the market equilibrium that would allow the market to be used as a mechanism for allocating divisible goods. Given that for $\rho < 1$, the utility functions are strictly concave, it is possible that the outcome for $\rho = 1$ could be defined as the limit of some sequence for $\rho \to 1$, in order to obtain a unique outcome. Whether such a choice would result in the induced game having pure Nash equilibria would have to be proved as well.

## 7.3 Price of anarchy bounds

Having examined the existence of pure Nash equilibria in the Fisher market mechanism, we proceed to study its *price of anarchy* and give asymptotic bounds for the three main classes of CES utility functions. The price of anarchy is defined similarly to Chapter 5.

Following [143], we will impose a condition that we call *$\delta$-normalization*: every agent's utility is proportional to her budget if she is allocated all of the items. More formally,

**Definition 7.5** ($\delta$-normalization)**.** For each agent $i \in [n]$, we have that $u_i(\mathbf{0}) = 0$ and $B_i/u_i(\mathbf{1}) = \delta$, where $\mathbf{0}$ and $\mathbf{1}$ are the all 0 and all 1 vectors.

Notice that by the definition of the linear and Cobb-Douglas utility functions, and since valuations are unit-sum, this condition is equivalent to assigning agents equal budgets, *regardless* of their true valuations of their reports. For Leontief utilities, it could be the case that agents with different valuations

[2] This is in fact the same approach that we take in the paper associated with this chapter.

are given different budgets, but since the designer does not know their true valuations (but only their reported valuations), it seems that assigning equal budgets is the only safe choice.

We start our investigations about the price of anarchy of the mechanism from linear utility functions.

### Linear Utilities

For linear utilities, we begin with the following upper bound.

**Theorem 7.6.** *The Fisher market mechanism with linear utilities and unit-sum valuations has a price of anarchy of $O(1/\sqrt{n})$.*

*Proof.* Consider an valuation profile with $n = m^2 + m$ agents and $m$ items, and recall that for all agents $i$, $B_i = 1$. For every agent $i \in \{1, \ldots, m\}$, define her valuation vector as

$$\mathbf{a}_i = (0, \ldots, 0, 1, 0, \ldots, 0),$$

that is, the vector in which the $i^{th}$ coordinate is set to 1 and all other entries are zero. For every agent $i \in \{m+1, \ldots, m^2+m\}$, define her valuation vector as

$$\mathbf{a}_i = \left(\frac{1}{m}, \ldots, \frac{1}{m}\right).$$

By checking the Karush-Kuhn-Tucker (KKT) [100, 105] conditions of the Eisenberg-Gale convex program [65], we obtain the market equilibrium:

- The prices are: $p_j = m + 1, \forall j \in [m]$
- The allocations are: $x_{ii} = \frac{1}{m+1}, \forall i \in \{1, \ldots, m\}$, $x_{ij} = \frac{1}{m(m+1)}, \forall i \in \{m+1, \ldots, m^2+m\}, \forall j \in [m]$, and $x_{ij} = 0$ everywhere else.

Moreover, for any truthful reporting, any market equilibrium gives the same utility to every agent. Thus, regardless of the chosen allocation, the social welfare under truthfulness is $\frac{2m}{m+1}$. By Theorem 7.5, there exists a Nash equilibrium in which the social welfare is the same as that of the truthful strategy profile, and so there exists a Nash equilibrium with a social welfare of $\frac{2m}{m+1}$. The optimal social welfare is at least $m-1$ and the price of anarchy is at most $\frac{2m}{(m-1)(m+1)}$; asymptotically, the bound is $O(1/\sqrt{n})$. □

We also establish the following lower bound.

**Theorem 7.7.** *The Fisher market mechanism with linear utilities and unit-sum valuations has a price of anarchy of $\Omega(1/n)$.*

*Proof.* By the $\delta$-normalization, we have: $B_i/(\sum_{j=1}^m a_{ij}) = \delta$, which for linear utilities implies that $B_i = B_k, \forall i \neq k$. By only assigning value to her most preferred item, each agent $i$ can guarantee utility proportional to her budget and hence bounded as follows:

$$u_i \geq B_i \cdot \frac{a_i^{max}}{\sum_{k=1}^n B_k} = \frac{a_i^{max}}{n}.$$

In the worst case, all agents prefer the same item, so the price of the item is $\sum_{k=1}^n B_k$ and each agent gets a fraction of $B_i/(\sum_{k=1}^n B_k)$. The optimal social welfare is $W^* \leq \sum_{i=1}^n a_i^{max}$, and so the price of anarchy is:

$$\frac{\sum_{i=1}^n u_i}{W^*} \geq \frac{\frac{1}{n}\sum_{i=1}^n a_i^{max}}{\sum_{i=1}^n a_i^{max}} = \frac{1}{n}. \qquad \square$$

### Cobb-Douglas Utilities

Recall that under Cobb-Douglas utilities with unit-sum valuations, the Fisher market mechanism allocates to each agent $i$ exactly a fraction

$$x_{ij} = \frac{a_{ij} B_i}{\sum_{k=1}^n a_{kj} B_k}$$

of every item $j$. This allocation coincides with that of the *proportional-share allocation mechanism*, studied by Feldman et al. [75] and Zhang [143]. This allows us to prove the following theorem.

**Theorem 7.8.** *The Fisher market game with Cobb-Douglas utilities has a Price of Anarchy of* $\Theta(1/\sqrt{n})$*.*

*Proof.* For the lower bound, we can use the arguments employed by Zhang [143] to show that the proportional-share mechanism for linear utilities has a price of anarchy of $\Omega(1/\sqrt{n})$. The key observation is that the technique used in their proof does not require a specific form of the utility functions; thus the lower bound holds more generally for any concave, non-decreasing utility function. As a result, the Fisher market game with Cobb-Douglas utilities has a price of anarchy of $\Omega(1/\sqrt{n})$.

For the upper bound, consider the same valuation profile that we constructed in Theorem 7.6 to prove the upper bound on the price of anarchy for the Fisher market game for linear utilities (with $n = m^2 + m$ agents and $m$ items). With a simple check, it can be seen that reporting truthfully is a Nash equilibrium. Moreover, under truthfulness, the social welfare is $\frac{2m}{m+1}$, while the optimal social welfare is at least $m - 1$. Thus the price of anarchy is at most $\frac{2m}{(m-1)(m+1)}$, and we get the asymptotic bound of $O(1/\sqrt{n})$. $\square$

### Leontief Utilities

Finally, for Leontief utilities, we give the next tight bound.

**Theorem 7.9.** *The Fisher market mechanism with Leontief utilities and unit-sum valuations has a price of anarchy of* $\Theta(1/n)$.

*Proof.* For the upper bound, consider the following valuation profile, with $m = n$. Let the budget of each agent $i$ be $B_i = \frac{1}{n}$ and the valuations $\mathbf{a}_i = (0, \dots, 1, \dots, 0)$, where $\mathbf{a}_i$ is the vector in which the $i$'th coordinate is 1 and all others are 0. Given the fact that the uniform strategy is a pure Nash equilibrium, regardless of the actual valuations and that $\sum_{k=1}^{n} B_k = 1$,[3] the utility of agent $i$ in the equilibrium is $u_i = B_i/a_i^{max} = B_i$; thus the social welfare is $\sum_i B_i = 1$. On the other hand, the social welfare of truthful reporting is $n$ and the price of anarchy is $1/n$.

For the lower bound, by reporting truthfully, agent $i$ can guarantee

$$u_i = \frac{B_i}{(\sum_{j=1}^{m} p_j a_{ij})} \geq \frac{B_i}{(\sum_{j=1}^{m} p_j a_i^{max})} = \frac{B_i}{(\sum_{k=1}^{n} B_k a_i^{max})}$$

The optimal welfare is $W^* \leq \sum_{i=1}^{n} u_i(\mathbf{1}) = \sum_{i=1}^{n} 1/a_i^{max}$, and so the price of anarchy is:

$$\frac{\sum_{i=1}^{n} u_i}{W^*} \geq \frac{\sum_{i=1}^{n} \left(B_i / \left(\sum_{k=1}^{n} B_k a_i^{max}\right)\right)}{\sum_{i=1}^{n} (1/a_i^{max})} = \frac{\sum_{i=1}^{n} (B_i/a_i^{max})}{\sum_{i=1}^{n} (1/a_i^{max})},$$

since for ease of notation, we can assume that $\sum_{k=1}^{n} B_k = 1$.[3] The $\delta$-normalization implies: $a_i^{max} B_i = \delta$, and so the price of anarchy is at least

$$\sum_{i=1}^{n} \frac{B_i^2}{(\sum_{i=1}^{n} B_i)^2}$$

The price of anarchy is then at least

$$\frac{1}{n} \cdot \frac{(\sum_{i=1}^{n} 1^2)(\sum_{i=1}^{n} B_i^2)}{(\sum_{i=1}^{n} B_i)^2} \geq \frac{1}{n} \cdot \frac{(\sum_{i=1}^{n} 1 \cdot B_i)^2}{(\sum_{i=1}^{n} B_i)^2} = \frac{1}{n},$$

where the inequality follows by the Cauchy-Schwarz inequality. □

## 7.4 Conclusion and future work

It remains an interesting open question whether a pure Nash equilibrium of the Fisher market game exists for CES functions with any value of $\rho$. The main

[3] This assumption is done for notational simplicity, see also [51]; the proof can be adapted to work for any choice of total budgets and total prices.

challenge to answering this question is that there is no explicit formula for the allocation; instead, the allocation rule is a part of the feasible solution to a convex program. For the case of Cobb-Douglas utility functions, where there is such an explicit allocation function, we were able to apply Reny's theorem to prove equilibrium existence. The same technique can be used to obtain similar results for other mechanisms, like the proportional share allocation mechanism studied in [75] and [143], where closed formulas for the allocations exist.[4]

The challenge associated with the lack of an explicit expression for the allocation also carries over into proving a tight lower bound on the price of anarchy of the Fisher market mechanism with linear utility functions. For this case, an additional difficulty is that the market equilibrium may not be unique and hence a proof should take all market equilibria into account. We emphasize here that it is possible that the $\Omega(1/n)$ lower bound on the price of anarchy for the linear utilities case could be obtained without the equal budget assumption, but we also conjecture that some more involved analysis (that somehow manages to sidestep the challenges mentioned above) would yield a $\Theta(1/\sqrt{n})$ price of anarchy tight bound.

The applicability of the mechanism in the linear utility case was discussed in Subsection 7.2; removing the conflict-freeness condition seems essential for the market to be used as a mechanism for allocating divisible items. Figuring out a way to uniquely define the output that does not depend on agents' true values that would still guarantee equilibrium existence or proving that such a way does not exist is an important open question.

More generally, our results place the Fisher market mechanism in the literature of mechanisms for social welfare maximization in divisible item allocation settings without money. The research agenda for this problem was laid out at the end of Chapter 6. Answering some of these questions is definitely a topic of future research.

[4] In fact, in a preliminary version of the paper associated with this chapter, we had a proof of pure Nash equilibrium existence in the proportional share allocation mechanism for any CES utility function.

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