Egalitarian allocation principles

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Voor oma Wies

Preface

With the defense of this dissertation, a period of eight years at Tilburg University comes to an end. I am very grateful to all dear friends, colleagues, and family members for experiencing so many memorable moments and making this an enjoyable and unforgettable period in my life. I want to express special gratitude to the following people.

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Contents

1 Introduction

Egalitarianism is a paradigm of economic thought that favors the idea of equality. Economic equality, or equity, refers to the concept of fairness in economics and underlies many theories of distributive justice. Since the seminal work of Rawls (1971), economic equality plays a central role in fundamental principles of justice and is widely applied within several disciplines of social science. Young (1995) provides a rich survey on equity concepts in both theoretical and practical contexts. The interpretation of equality, and which notions should exactly be equated, depends on the model at hand, its characteristics, and its underlying assumptions. The leading thread of this dissertation is constituted by the implementation and analysis of egalitarianism and corresponding principles in models for allocation problems, in particular bankruptcy problems with nontransferable utility and cooperative games. This contributes to a better understanding of fair allocation rules and their properties.

A bankruptcy problem is an elementary allocation problem in which claimants have individual claims on an insufficient estate. The question arises which of the possible estate allocations could or should be selected. For this, bankruptcy theory studies appropriate bankruptcy rules which prescribe for any bankruptcy problem an efficient and feasible allocation, i.e. an estate allocation for which the individual payoffs are bounded by the corresponding claims. Starting from O'Neill (1982), many scientific studies are devoted to bankruptcy problems with transferable utility where the estate and claims are of a monetary nature. We refer to Thomson (2003) for an extensive survey, to Thomson (2013) for recent advances, and to Thomson (2015) for an update. An egalitarian alternative for the well-known proportional rule in this context is the constrained equal awards rule, which divides the monetary estate as equal as possible under the condition that no claimants are allocated more than their corresponding claims.

The first part of this dissertation builds upon the foundations of bankruptcy problems with nontransferable utility as introduced by Orshan, Valenciano, and Zarzuelo (2003). There, individual payoffs are represented in a utility space and the estate is expressed in a set of attainable utility allocations. Throughout this dissertation, we assume that individual utility is normalized in such a way that allocating nothing corresponds to a utility level of zero. Since claimants generally not only differ in their claims, but also in their utility measure, the implementation of egalitarianism in this model cannot simply boil down to equal division. Instead, it makes sense to compare the claims in relation to the estate. Therefore, we take a solid and deliberate approach using the zero vector and the utopia vector as benchmarks. Since allocating zero to all claimants generates the same well-being as the event in which the bankruptcy problem is not solved, claimants are then comparable in terms of minimal satisfaction and the allocation is in that sense egalitarian. Similarly, when allocating to all claimants their corresponding utopia values, defined as the maximal individual payoffs within the estate, claimants are comparable in terms of maximal satisfaction and the allocation is in that sense egalitarian.[1] In this way, we interpret the utopia vector as an egalitarian direction starting from the zero vector and all payoff allocations following this direction are considered to be relatively equal. This approach leads to an adequate definition of *the constrained relative equal awards rule* for bankruptcy problems with nontransferable utility, which allocates payoffs as relatively equal as possible under the condition that no claimants are allocated more than their corresponding claims. On the class of NTU-bankruptcy problems induced by TU-bankruptcy problems, the constrained relative equal awards rule boils down to the standard constrained equal awards rule.

Focusing on fundamental principles and structures, we study the rich model of bankruptcy problems with nontransferable utility from several perspectives. From an axiomatic perspective, we formulate appropriate properties for bankruptcy rules and study their implications. Our interpretation of egalitarianism is reflected in a property called relative symmetry, which imposes that claimants with relatively equal claims are allocated relatively equal payoffs. Another important property is truncation invariance, which imposes invariance of the prescribed allocation under truncation of the claims by the corresponding utopia values. A higher claim than the corresponding utopia value is then not considered as relevant, supported by the fact that claimants are not allocated more than their utopia values in any feasible estate allocation. We derive several axiomatic characterizations of the constrained relative equal awards rule using these and other properties which are generally based on counterparts within the theory on bankruptcy problems with transferable utility.

[1] In the context of bargaining problems (cf. Nash (1950)), the use of utopia values as initiated by Raiffa (1953) is nowadays fully embedded in the literature.

Besides, we explore the relation of the constrained relative equal awards rule with duality, consistency, and the relative adjustment principle. These fundamental concepts are based on duality, consistency, and the contested garment principle for TU-bankruptcy rules which play an essential role in the seminal work of Aumann and Maschler (1985) on Talmudic principles for monetary bankruptcy problems. Two bankruptcy rules are called dual if one rule allocates awards in the same way as the other rule allocates losses. Consistency notions are based on thought experiments in which bankruptcy problems are reevaluated in case some claimants depart with their allocated payoffs. The relative adjustment principle describes a standard solution for bankruptcy problems with nontransferable utility and two claimants, merging the properties relative symmetry and truncation invariance with minimal rights first. The minimal rights first property requires that first allocating minimal rights, defined as the maximal individual payoffs within the estate when all other claimants are allocated their claims, and subsequently applying the bankruptcy rule to the remaining bankruptcy problem leads to the same payoff allocation as direct application of the bankruptcy rule to the original bankruptcy problem.

Surprisingly, while studying bankruptcy problems with nontransferable utility, we not only encounter similarities with the theory on bankruptcy problems with transferable utility, but also common interesting features from the theory on other well-studied domains, e.g. cost sharing theory, bargaining theory, and game theory. In particular, we show that the constrained relative equal awards rule shares characteristics with the serial mechanism for cost sharing problems (cf. Moulin and Shenker (1992)) by deriving a corresponding axiomatic characterization in terms of relative symmetry and independence of larger relative claims. Moreover, we translate several axioms from bargaining theory concerning changes in the estate or the claims to obtain a new characterization of the constrained relative equal awards rule while elaborating on the similarities between bankruptcy problems with nontransferable utility and bargaining problems with claims as introduced by Chun and Thomson (1992). Furthermore, we discuss the game theoretic modeling of NTU-bankruptcy problems along the lines of Curiel, Maschler, and Tijs (1987) for TU-bankruptcy problems by defining an appropriate coalitional bankruptcy game, focusing on the structure of the core, and characterizing the class of game theoretic bankruptcy rules using truncation invariance. Interestingly, the constrained relative equal awards rule is a game theoretic bankruptcy rule, which means that it can be generalized to a solution for the full class of nontransferable utility games. This is exploited in the second part of this dissertation.

The second part of this dissertation focusses on the incorporation of egalitarianism in transferable utility games and nontransferable utility games. A cooperative game models an allocation problem in which players collectively gain revenues while taking into account the possibility to act in coalitions. Following our interpretation of egalitarianism, the interpersonal relations of utopia values form the key ingredient for the determination of egalitarian payoff allocations. However, to allow for a coherent comparison of egalitarian opportunities within coalitions, it is required to consistently apply a fixed interpretation of equality. For that reason, the utopia values within the grand coalition are used as a common benchmark for egalitarian allocations within any subcoalition. We design an egalitarian negotiation procedure in which players iteratively take their coalitional egalitarian opportunities into consideration. This egalitarian procedure converges to a steady state in which each player has acquired a claim attainable in one or more egalitarian admissible coalitions. These egalitarian claims can be interpreted as aspiration levels for a payoff allocation within the grand coalition. The possibly resulting infeasibility is modeled as a bankruptcy problem in which these egalitarian claims are adopted. By solving these bankruptcy problems in an egalitarian way following from the first part of this dissertation, a new and general solution concept for cooperative games arises, which can be considered as a trade-off between egalitarianism and coalitional rationality.

On the domain of transferable utility games (cf. Von Neumann and Morgenstern (1944)), our interpretation of egalitarianism boils down to equal division and the result of the egalitarian procedure is called *the procedural egalitarian solution*. Remarkably, this is the first single-valued solution which exists for any transferable utility game and coincides with the well-known egalitarian solution of Dutta and Ray (1989) on the class of convex games. On the class of bankruptcy games with transferable utility, the procedural egalitarian solution coincides with the constrained equal awards rule for underlying monetary bankruptcy problems.

On the domain of nontransferable utility games (cf. Shapley and Shubik (1953) and Aumann and Peleg (1960)), the result of the egalitarian procedure is called *the constrained egalitarian solution*. Naturally, the constrained egalitarian solution of a nontransferable utility game induced by a transferable utility game corresponds to the procedural egalitarian solution. On the class of bankruptcy games with nontransferable utility, the constrained egalitarian solution coincides with the constrained relative equal awards rule for underlying bankruptcy problems. On the class of bargaining games, the constrained egalitarian solution induces a new and interesting way to solve bargaining problems on the basis of utopia values, as an alternative for the solutions proposed by Kalai and Smorodinsky (1975) and Kalai and Rosenthal (1978).

The third part of this dissertation is devoted to communication situations which arise when the players of a transferable utility game are subject to cooperation restrictions as modeled by an undirected graph. This outlying part contains no reference to an egalitarian allocation principle. Instead, we focus on the decomposition of network communication games into unanimity games and we introduce a general class of network control values based on the Shapley value (cf. Shapley (1953)) for transferable utility games. The well-studied Myerson value (cf. Myerson (1977)) and position value (cf. Borm, Owen, and Tijs (1992)) both belong to this new class.

Overview

This dissertation is organized as follows. Chapter 2 provides an overview of preliminary notions for bankruptcy problems with transferable utility, transferable utility games, and nontransferable utility games.

Chapter 3 analyzes bankruptcy problems with nontransferable utility following the classical axiomatic theory of bankruptcy by formulating some appropriate properties for bankruptcy rules and studying their implications. We explore duality of bankruptcy rules and we derive several characterizations of the proportional rule and the constrained relative equal awards rule.

Chapter 4 continues on this axiomatic approach by examining the relation of the proportional rule and the constrained relative equal awards rule with several consistency notions and the relative adjustment principle.

Chapter 5 takes an axiomatic bargaining approach to bankruptcy problems with nontransferable utility by characterizing bankruptcy rules in terms of properties from bargaining theory. In particular, we derive new axiomatic characterizations of the proportional rule and the constrained relative equal awards rule using properties which concern changes in the estate or the claims.

Chapter 6 analyzes bankruptcy problems with nontransferable utility from a game theoretic perspective by studying the core of corresponding bankruptcy games. Moreover, we derive a necessary and sufficient condition for a bankruptcy rule to be game theoretic.

Chapter 7 introduces and analyzes the procedural egalitarian solution for transferable utility games. This new concept is based on the result of a coalitional negotiation procedure in which egalitarian considerations play a central role.

Chapter 8 generalizes the procedural egalitarian solution to the constrained egalitarian solution for nontransferable utility games. We explore the new solution using the famous examples of Roth (1980) and Shafer (1980) and we formulate conditions under which it leads to a core element.

Using network control structures, Chapter 9 introduces a general class of network communication games and studies their decomposition into unanimity games. Moreover, we introduce a new class of network control values which contains both the Myerson value and the position value. The decomposition results are used to explicitly express these values in terms of dividends.

2 Preliminaries

Let N be a nonempty and finite set. An *order* of N is a bijection $\sigma : \{1, \ldots, |N|\} \to N$. The set of all orders of N is denoted by $\Pi(N)$. The collection of all subsets of N is denoted by $2^N = \{S \mid S \subseteq N\}$. A collection $\mathcal{B} \subseteq 2^N \setminus \{\emptyset\}$ is a *cover* if $\bigcup_{S \in \mathcal{B}} S = N$, is *independent* if $S \not\subset T$ for all $S, T \in \mathcal{B}$, and is *balanced* if there exists a function $\delta : \mathcal{B} \to \mathbb{R}_{++}$ for which $\sum_{S \in \mathcal{B}: i \in S} \delta(S) = 1$ for all $i \in N$.

A vector $x \in \mathbb{R}^N$ denotes $x = (x_i)_{i \in N}$, and $x_S \in \mathbb{R}^S$ denotes $x_S = (x_i)_{i \in S}$ for any $S \in 2^N$. The zero vector $x \in \mathbb{R}^N$ with $x_i = 0$ for all $i \in N$ is denoted by 0_N. For any $x, y \in \mathbb{R}^N$, $x \leq y$ denotes $x_i \leq y_i$ for all $i \in N$, and $x < y$ denotes $x_i < y_i$ for all $i \in N$. A function $f : \mathbb{R}^N \to \mathbb{R}^N$ is *increasing* if $f(x) \leq f(y)$ and $f(x) \neq f(y)$ for all $x, y \in \mathbb{R}^N$ for which $x \leq y$ and $x \neq y$. A *decreasing* function is defined similarly. A function is *monotonic* if it is increasing or decreasing.

Let $A \subseteq \mathbb{R}^N_+$ be a nonempty, closed, and bounded set. Some related notions are

- the vector of *utopia values* $u^A = (\max\{x_i \mid x \in A\})_{i \in N}$;
- the *convex hull*
$$\operatorname{conv}(A) = \left\{ x \in \mathbb{R}^N_+ \;\middle|\; \exists_{A' \subseteq A, |A'| \in \mathbb{N}} \exists_{\theta : A' \to \mathbb{R}_+, \sum_{y \in A'} \theta(y) = 1} : \sum_{y \in A'} \theta(y) y = x \right\};$$
- the *comprehensive hull* $\operatorname{comp}(A) = \{x \in \mathbb{R}^N_+ \mid \exists_{y \in A} : y \geq x\}$;
- the *strong Pareto set* $\mathrm{SP}(A) = \{x \in A \mid \neg \exists_{y \in A, y \neq x} : y \geq x\}$;
- the *weak Pareto set* $\mathrm{WP}(A) = \{x \in A \mid \neg \exists_{y \in A} : y > x\}$;
- the *strong upper contour set* $\mathrm{SUC}(A) = \{x \in \mathbb{R}^N_+ \mid \neg \exists_{y \in A, y \neq x} : y \geq x\}$;
- the *weak upper contour set* $\mathrm{WUC}(A) = \{x \in \mathbb{R}^N_+ \mid \neg \exists_{y \in A} : y > x\}$.

Note that $\mathrm{SP}(A) \subseteq \mathrm{WP}(A) \subseteq \mathrm{WUC}(A)$ and $\mathrm{SP}(A) \subseteq \mathrm{SUC}(A) \subseteq \mathrm{WUC}(A)$. The set $A \subseteq \mathbb{R}^N_+$ is *nontrivial* if $u^A \in \mathbb{R}^N_{++}$, is *convex* if $A = \mathrm{conv}(A)$, is *comprehensive* if $A = \mathrm{comp}(A)$, and is *nonleveled* if $\mathrm{SP}(A) = \mathrm{WP}(A)$, or equivalently, $\mathrm{SUC}(A) = \mathrm{WUC}(A)$.

2.1 Bankruptcy problems with transferable utility

A *bankruptcy problem with transferable utility* (cf. O'Neill (1982)) is a triple (N, e, c) in which N is a nonempty and finite set of *claimants*, $e \in \mathbb{R}_+$ is an *estate*, and $c \in \mathbb{R}^N_+$ is a vector of *claims* of N on e for which $\sum_{i \in N} c_i \geq e$. Let TUBR^N denote the class of bankruptcy problems with transferable utility and claimant set N. For convenience, a TU-bankruptcy problem on N is denoted by $(e, c) \in \mathrm{TUBR}^N$.

A *bankruptcy rule* $f : \mathrm{TUBR}^N \to \mathbb{R}^N_+$ assigns to any $(e, c) \in \mathrm{TUBR}^N$ a payoff allocation $f(e, c) \in \mathbb{R}^N_+$ for which $\sum_{i \in N} f_i(e, c) = e$ and $f(e, c) \leq c$.

The *proportional rule* $\mathrm{Prop} : \mathrm{TUBR}^N \to \mathbb{R}^N_+$ is the bankruptcy rule which assigns to any $(e, c) \in \mathrm{TUBR}^N$ the payoff allocation

$$\mathrm{Prop}(e, c) = \lambda^{e,c} c,$$

where $\lambda^{e,c} = \max\{t \in [0, 1] \mid \sum_{i \in N} t c_i = e\}$.

The *constrained equal awards rule* $\mathrm{CEA} : \mathrm{TUBR}^N \to \mathbb{R}^N_+$ is the bankruptcy rule which assigns to any $(e, c) \in \mathrm{TUBR}^N$ the payoff allocation

$$\mathrm{CEA}(e, c) = \left(\min\{c_i, a^{e,c}\}\right)_{i \in N},$$

where $a^{e,c} = \max\{t \in [0, e] \mid \sum_{i \in N} \min\{c_i, t\} = e\}$.

The *constrained equal losses rule* $\mathrm{CEL} : \mathrm{TUBR}^N \to \mathbb{R}^N_+$ is the bankruptcy rule which assigns to any $(e, c) \in \mathrm{TUBR}^N$ the payoff allocation

$$\mathrm{CEL}(e, c) = \left(\max\{c_i - b^{e,c}, 0\}\right)_{i \in N},$$

where $b^{e,c} = \min\{t \in \mathbb{R}_+ \mid \sum_{i \in N} \max\{c_i - t, 0\} = e\}$.

2.2 Transferable utility games

A *transferable utility game* (cf. Von Neumann and Morgenstern (1944)) is a pair (N, v) in which N is a nonempty and finite set of *players* and $v : 2^N \to \mathbb{R}$ assigns to each *coalition* $S \in 2^N$ its *worth* $v(S) \in \mathbb{R}$ such that $v(\emptyset) = 0$. The number $\frac{v(S)}{|S|}$ is the *average worth* of coalition $S \in 2^N \setminus \{\emptyset\}$. Let TU^N denote the class of transferable utility games with player set N. For convenience, a TU-game on N is denoted by $v \in \mathrm{TU}^N$. The *subgame* $v_S \in \mathrm{TU}^S$ of $v \in \mathrm{TU}^N$ on $S \in 2^N \setminus \{\emptyset\}$ is defined by $v_S(R) = v(R)$ for all $R \in 2^S$.

Let $v \in \mathrm{TU}^N$. The vector $M^\sigma(v) \in \mathbb{R}^N$ corresponding to $\sigma \in \Pi(N)$ is given by

$$M^\sigma_{\sigma(k)}(v) = v(\{\sigma(1), \ldots, \sigma(k)\}) - v(\{\sigma(1), \ldots, \sigma(k-1)\})$$

for all $k \in \{1, \ldots, |N|\}$. The vector $K(v) \in \mathbb{R}^N$ is given by

$$K_i(v) = v(N) - v(N \setminus \{i\})$$

for all $i \in N$, and the vector $k(v) \in \mathbb{R}^N$ is given by

$$k_i(v) = \max_{S \in 2^N : i \in S} \left\{ v(S) - \sum_{j \in S \setminus \{i\}} K_j(v) \right\}$$

for all $i \in N$. The *core* (cf. Gillies (1959)) is given by

$$\mathcal{C}(v) = \left\{ x \in \mathbb{R}^N \;\middle|\; \sum_{i \in N} x_i = v(N), \forall_{S \in 2^N} : \sum_{i \in S} x_i \geq v(S) \right\},$$

the *Weber set* (cf. Weber (1988)) is given by

$$\mathcal{W}(v) = \operatorname{conv}\left(\{M^\sigma(v) \mid \sigma \in \Pi(N)\}\right),$$

the *core cover* (cf. Tijs and Lipperts (1982)) is given by

$$\mathcal{CC}(v) = \left\{ x \in \mathbb{R}^N \;\middle|\; \sum_{i \in N} x_i = v(N), k(v) \leq x \leq K(v) \right\},$$

and the *reasonable set* (cf. Gerard-Varet and Zamir (1987)) is given by

$$\mathcal{R}(v) = \left\{ x \in \mathbb{R}^N \;\middle|\; \sum_{i \in N} x_i = v(N), \forall_{i \in N} : \min_{\sigma \in \Pi(N)} M^\sigma_i(v) \leq x_i \leq \max_{\sigma \in \Pi(N)} M^\sigma_i(v) \right\}.$$

It is known that $\mathcal{C}(v) \subseteq \mathcal{W}(v) \subseteq \mathcal{R}(v)$ and $\mathcal{C}(v) \subseteq \mathcal{CC}(v) \subseteq \mathcal{R}(v)$.

A transferable utility game $v \in \mathrm{TU}^N$ is

- *monotonic* if $v(S) \leq v(T)$ for all $S, T \in 2^N$ for which $S \subseteq T$;
- *superadditive* if $v(S) + v(T) \leq v(S \cup T)$ for all $S, T \in 2^N$ for which $S \cap T = \emptyset$;
- *convex* (cf. Shapley (1971)) if $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for all $S, T \in 2^N$;
- *balanced* (cf. Bondareva (1963) and Shapley (1967)) if $\sum_{S \in \mathcal{B}} \delta(S) v(S) \leq v(N)$ for all balanced collections $\mathcal{B} \subseteq 2^N \setminus \{\emptyset\}$ and any $\delta : \mathcal{B} \to \mathbb{R}_{++}$ for which $\sum_{S \in \mathcal{B} : i \in S} \delta(S) = 1$ for all $i \in N$.

Note that convexity implies superadditivity. For any convex game $v \in \mathrm{TU}^N$, Shapley (1971) showed that $\max_{\sigma \in \Pi(N)} M_i^\sigma(v) = v(N) - v(N \setminus \{i\})$ and $\min_{\sigma \in \Pi(N)} M_i^\sigma(v) = v(\{i\})$ for all $i \in N$, and Tijs and Lipperts (1982) showed that $k_i(v) = v(\{i\})$ for all $i \in N$.

Bondareva (1963) and Shapley (1967) showed that $\mathcal{C}(v) \neq \emptyset$ if and only if $v \in \mathrm{TU}^N$ is balanced. Ichiishi (1981) showed that $\mathcal{C}(v) = \mathcal{W}(v)$ if and only if $v \in \mathrm{TU}^N$ is convex. A transferable utility game $v \in \mathrm{TU}^N$ is *compromise stable* (cf. Quant, Borm, Reijnierse, and Van Velzen (2005)) if $\mathcal{C}(v) = \mathcal{CC}(v)$ and $\mathcal{CC}(v) \neq \emptyset$. This means that both convexity and compromise stability individually imply balancedness.

We introduce the notion of reasonable stability to describe games for which the core and the reasonable set coincide. Moreover, we show that reasonable stability is equivalent to the combination of convexity and compromise stability.

Definition (Reasonable Stability (cf. Dietzenbacher (2018)))
A transferable utility game $v \in \mathrm{TU}^N$ is *reasonable stable* if $\mathcal{C}(v) = \mathcal{R}(v)$.

Theorem 2.2.1
A transferable utility game is reasonable stable if and only if it is convex and compromise stable.

Proof. Assume that $v \in \mathrm{TU}^N$ is reasonable stable. Then $\mathcal{C}(v) = \mathcal{R}(v)$ and $\mathcal{C}(v) \neq \emptyset$. Since $\mathcal{C}(v) \subseteq \mathcal{W}(v) \subseteq \mathcal{R}(v)$ and $\mathcal{C}(v) \subseteq \mathcal{CC}(v) \subseteq \mathcal{R}(v)$, this means that $\mathcal{C}(v) = \mathcal{W}(v)$ and $\mathcal{C}(v) = \mathcal{CC}(v)$. Hence, $v \in \mathrm{TU}^N$ is convex and compromise stable.

Assume that $v \in \mathrm{TU}^N$ is convex and compromise stable. Since $v \in \mathrm{TU}^N$ is convex, $\min_{\sigma \in \Pi(N)} M_i^\sigma(v) = v(\{i\})$ and $\max_{\sigma \in \Pi(N)} M_i^\sigma(v) = v(N) - v(N \setminus \{i\})$ for all $i \in N$. Moreover, $k_i(v) = v(\{i\})$ for all $i \in N$. This means that $\min_{\sigma \in \Pi(N)} M_i^\sigma(v) = k_i(v)$ and $\max_{\sigma \in \Pi(N)} M_i^\sigma(v) = K_i(v)$ for all $i \in N$, so $\mathcal{CC}(v) = \mathcal{R}(v)$. Since $v \in \mathrm{TU}^N$ is compromise stable, this implies that $\mathcal{C}(v) = \mathcal{CC}(v) = \mathcal{R}(v)$. Hence, $v \in \mathrm{TU}^N$ is reasonable stable. $\square$

The *bankruptcy game with transferable utility* (cf. O'Neill (1982)) $v^{e,c} \in \mathrm{TU}^N$ corresponding to the bankruptcy problem $(e, c) \in \mathrm{TUBR}^N$ is given by

$$v^{e,c}(S) = \max\left\{e - \sum_{i \in N \setminus S} c_i, 0\right\}$$

for all $S \in 2^N$. Curiel, Maschler, and Tijs (1987) showed that bankruptcy games are convex and compromise stable. Quant, Borm, Reijnierse, and Van Velzen (2005) showed that convex and compromise stable games are strategically equivalent to bankruptcy games. By Theorem 2.2.1, this means that bankruptcy games are reasonable stable, and reasonable stable games are strategically equivalent to bankruptcy games.

A *solution* for transferable utility games $f : \mathrm{TU}^N \to \mathbb{R}^N$ assigns to any $v \in \mathrm{TU}^N$ a payoff allocation $f(v) \in \mathbb{R}^N$ for which $\sum_{i\in N} f_i(v) = v(N)$.

A solution $f : \mathrm{TU}^N \to \mathbb{R}^N$ satisfies

– *symmetry* if $f_i(v) = f_j(v)$ for all $v \in \mathrm{TU}^N$ and any $i, j \in N$ for which $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$;

– *dummy invariance* if $f_i(v) = v(\{i\})$ for all $v \in \mathrm{TU}^N$ and any $i \in N$ for which $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subseteq N \setminus \{i\}$;

– *strong strategic covariance* if $f(v) = (\alpha f_i(v') + \beta_i)_{i\in N}$ for all $v, v' \in \mathrm{TU}^N$ and any $\alpha \in \mathbb{R}_{++}$ and $\beta \in \mathbb{R}^N$ for which $v(S) = \alpha v'(S) + \sum_{i\in S} \beta_i$ for all $S \in 2^N$;

– *weak strategic covariance* if $f(v) = (\alpha f_i(v') + \beta)_{i\in N}$ for all $v, v' \in \mathrm{TU}^N$ and any $\alpha \in \mathbb{R}_{++}$ and $\beta \in \mathbb{R}$ for which $v(S) = \alpha v'(S) + \beta|S|$ for all $S \in 2^N$;

– *marginal monotonicity* (cf. Young (1985)) if $f_i(v) \leq f_i(v')$ for all $v, v' \in \mathrm{TU}^N$ and any $i \in N$ for which $v(S\cup\{i\})-v(S) \leq v'(S\cup\{i\})-v'(S)$ for all $S \subseteq N\setminus\{i\}$;

– *coalitional monotonicity* (cf. Young (1985)) if $f_S(v) \leq f_S(v')$ for all $v, v' \in \mathrm{TU}^N$ and any $S \in 2^N$ for which $v(S) \leq v'(S)$ and $v(T) = v'(T)$ for all $T \in 2^N \setminus \{S\}$;

– *aggregate monotonicity* (cf. Megiddo (1974)) if $f(v) \leq f(v')$ for all $v, v' \in \mathrm{TU}^N$ for which $v(N) \leq v'(N)$ and $v(S) = v'(S)$ for all $S \subset N$.

Note that strong strategic covariance implies weak strategic covariance, marginal monotonicity implies coalitional monotonicity, and coalitional monotonicity implies aggregate monotonicity.

2.3 Nontransferable utility games

A *nontransferable utility game* (cf. Shapley and Shubik (1953) and Aumann and Peleg (1960)) is a pair (N, V) in which N is a nonempty and finite set of *players* and V assigns to each *coalition* $S \in 2^N \setminus \{\emptyset\}$ a nonempty, closed, bounded, and comprehensive set of payoff allocations $V(S) \subseteq \mathbb{R}^S_+$. Note that $V(S)$ is explicitly restricted to nonnegative payoff allocations. Let NTU^N denote the class of nontransferable utility games with player set N. For convenience, an NTU-game on N is denoted by $V \in \mathrm{NTU}^N$. The *subgame* $V_S \in \mathrm{NTU}^S$ of $V \in \mathrm{NTU}^N$ on $S \in 2^N \setminus \{\emptyset\}$ is defined by $V_S(R) = V(R)$ for all $R \in 2^S \setminus \{\emptyset\}$. Note that any nonnegative game $v \in \mathrm{TU}^N$ induces the game $V \in \mathrm{NTU}^N$ given by $V(S) = \{x \in \mathbb{R}^S_+ \mid \sum_{i\in S} x_i \leq v(S)\}$ for all $S \in 2^N \setminus \{\emptyset\}$.

Let $V \in \mathrm{NTU}^N$. The *strong core* is given by

$$\mathcal{C}^{\mathcal{S}}(V) = \left\{ x \in V(N) \;\middle|\; \forall_{S \in 2^N \setminus \{\emptyset\}} : x_S \in \mathrm{SUC}(V(S)) \right\}$$

and the *weak core* is given by

$$\mathcal{C}^{\mathcal{W}}(V) = \left\{ x \in V(N) \;\middle|\; \forall_{S \in 2^N \setminus \{\emptyset\}} : x_S \in \mathrm{WUC}(V(S)) \right\}.$$

Note that $\mathcal{C}^{\mathcal{S}}(V) \subseteq \mathcal{C}^{\mathcal{W}}(V)$. Moreover, $\mathcal{C}^{\mathcal{S}}(V) = \mathcal{C}^{\mathcal{W}}(V)$ if $V(S)$ is nonleveled for all $S \in 2^N \setminus \{\emptyset\}$, as is the case for NTU-games induced by TU-games.

A nontransferable utility game $V \in \mathrm{NTU}^N$ is

- *monotonic* if $V(S) \times \{0_{T\setminus S}\} \subseteq V(T)$ for all $S, T \in 2^N \setminus \{\emptyset\}$ for which $S \subseteq T$;
- *superadditive* if $V(S) \times V(T) \subseteq V(S \cup T)$ for all $S, T \in 2^N \setminus \{\emptyset\}$ for which $S \cap T = \emptyset$;
- *ordinal convex* (cf. Vilkov (1977)) if V is superadditive, and $x_{S\cup T} \in V(S \cup T)$ or $x_{S\cap T} \in V(S \cap T)$ for all $S, T \in 2^N \setminus \{\emptyset\}$ for which $S \cap T \neq \emptyset$ and any $x \in \mathbb{R}^N_+$ for which $x_S \in V(S)$ and $x_T \in V(T)$;
- *coalitional merge convex* (cf. Hendrickx, Borm, and Timmer (2002)) if V is superadditive, and for all $R \in 2^N \setminus \{\emptyset\}$ and $S, T \in 2^{N\setminus R} \setminus \{\emptyset\}$ for which $S \subset T$, and any $s \in \mathrm{WP}(V(S))$, $t \in \mathrm{WP}(V(T))$, and $x \in V(S \cup R)$ for which $x_S \geq s$, there exists a $y \in V(T \cup R)$ for which $y_T \geq t$ and $y_R \geq x_R$;
- *balanced* (cf. Scarf (1967)) if for all balanced collections $\mathcal{B} \subseteq 2^N \setminus \{\emptyset\}$, $x \in V(N)$ if $x_S \in V(S)$ for all $S \in \mathcal{B}$.

Note that superadditivity implies monotonicity. Greenberg (1985), Hendrickx, Borm, and Timmer (2002), and Scarf (1967) showed that $\mathcal{C}^{\mathcal{W}}(V) \neq \emptyset$ if $V \in \mathrm{NTU}^N$ is ordinal convex, coalitional merge convex, or balanced, respectively.

A *solution* for nontransferable utility games $F : \mathrm{NTU}^N \to \mathbb{R}^N_+$ assigns to any $V \in \mathrm{NTU}^N$ a payoff allocation $F(V) \in \mathrm{WP}(V(N))$.

Part I

Bankruptcy Problems with Nontransferable Utility

3 Proportionality, Equality, and Duality

3.1 Introduction

A bankruptcy problem is an elementary allocation problem in which claimants have individual claims on an insufficient estate. Bankruptcy theory studies allocations of the estate among the claimants, taking into account the corresponding claims. In a bankruptcy problem with transferable utility (cf. O'Neill (1982)), the estate and claims are of a monetary nature. These problems are well-studied, both from an axiomatic perspective and a game theoretic perspective. We refer to Thomson (2003) for an extensive survey, to Thomson (2013) for recent advances, and to Thomson (2015) for an update.

In a bankruptcy problem with nontransferable utility, claimants have incomparable claims and the estate is expressed in a set of attainable utility allocations. These problems arise when claimants have individual utility functions over their monetary payoffs. NTU-bankruptcy problems form a natural generalization of TU-bankruptcy problems. Thomson (2013) states that, although the passage from TU to NTU is in general fraught with difficulties, an NTU generalization is worthwhile in the search for greater generality.

Orshan, Valenciano, and Zarzuelo (2003) analyzed NTU-bankruptcy problems from a game theoretic perspective by showing that the intersection of the bilateral consistent prekernel and the core is nonempty for every smooth bankruptcy game. Estévez-Fernández, Borm, and Fiestras-Janeiro (2014) redefined NTU-bankruptcy games on the basis of convexity and compromise stability. This chapter, based on Dietzenbacher, Estévez-Fernández, Borm, and Hendrickx (2016), analyzes NTU-bankruptcy problems from an axiomatic perspective by formulating appropriate properties for bankruptcy rules and studying their implications.

Bankruptcy problems with nontransferable utility share characteristics with bargaining problems with claims (cf. Chun and Thomson (1992)) and Nash rationing problems (cf. Mariotti and Villar (2005)). These models are studied on the basis of solutions and axioms originating from bargaining theory. Instead, we aim to generalize monetary bankruptcy problems and particularly show that bankruptcy theory can be extended by adequately reformulating the main notions and properties.

The proportional rule for bankruptcy problems prescribes the efficient allocation which is proportional to the vector of claims. We study the proportional rule for NTU-bankruptcy problems and extend the axiomatic characterizations of Young (1988) and Chun (1988) using adequate generalizations of the properties composition down, composition up, self-duality, and path linearity.

The constrained equal awards rule for TU-bankruptcy problems divides the estate equally such that all claimants are not allocated more than their claims. In bankruptcy problems with nontransferable utility, it makes sense to compare the claims in relation to the estate since claimants differ in their measure of utility. For that reason, we introduce the constrained *relative* equal awards rule for NTU-bankruptcy problems which takes into account the *relative* claims of the claimants, i.e. the claims in relation to their utopia values. We extend the axiomatic characterizations of Dagan (1996), Herrero and Villar (2002), Yeh (2004), and Yeh (2006) using generalizations of the properties symmetry, truncation invariance, conditional full compensation, and claim monotonicity. By extending its axiomatic characterization based on symmetry and independence of larger claims, we show that the constrained relative equal awards rule also shares a characteristic feature with the serial mechanism for cost sharing problems (cf. Moulin and Shenker (1992)). In those problems, agents share a production technology and distribute the joint costs among them.

Two bankruptcy rules are called dual (cf. Aumann and Maschler (1985)) if one rule allocates awards in the same way as the other rule allocates losses. Two properties for bankruptcy rules are called dual (cf. Herrero and Villar (2001)) if for any two dual bankruptcy rules it holds that one rule satisfies one property if and only if the other rule satisfies the other property. We generalize the notions of dual bankruptcy rules and dual properties to the context of NTU-bankruptcy problems without explicitly formulating dual bankruptcy problems. In particular, we exploit duality, we show that the proportional rule is self-dual, and we adequately construct the dual of the constrained relative equal awards rule, the constrained relative equal losses rule.

This chapter is organized in the following way. Section 3.2 formally introduces NTU-bankruptcy problems and defines basic notions for NTU-bankruptcy rules. In Section 3.3, we explore duality and analyze dual properties. Section 3.4 studies the proportional rule and Section 3.5 analyzes the constrained relative equal awards rule.

3.2 Bankruptcy problems

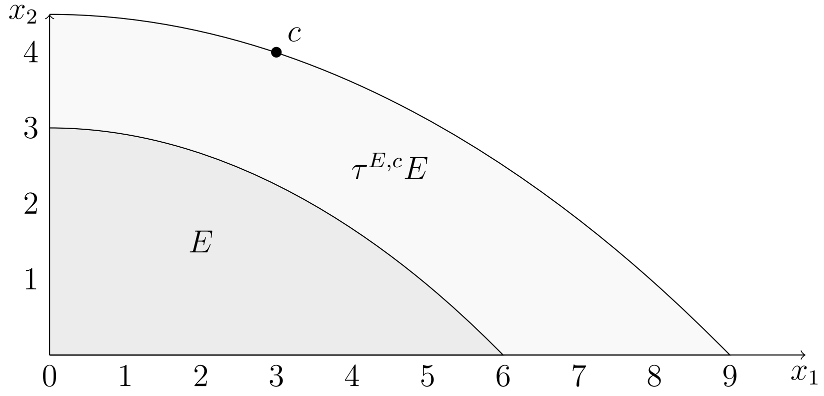
A *bankruptcy problem with nontransferable utility* (cf. Orshan, Valenciano, and Zarzuelo (2003)) is a triple (N, E, c) in which N is a nonempty and finite set of *claimants*, $E \subseteq \mathbb{R}^N_+$ is a nonempty, closed, bounded, comprehensive, and nonleveled *estate*, and $c \in \text{WUC}(E)$ is a vector of *claims* of N on E. Note that $0_N \in E$, and E is nontrivial if and only if $E \neq \{0_N\}$. The estate is expressed in a set of attainable utility allocations which are assumed to be normalized in such a way that allocating nothing corresponds to a utility level of zero. The claim vector represents the individual utility claims on the estate. Let BR^N denote the class of bankruptcy problems with nontransferable utility and claimant set N. For convenience, an NTU-bankruptcy problem on N is denoted by $(E, c) \in \text{BR}^N$. Note that $(E \cup E', c), (E \cap E', c) \in \text{BR}^N$ for all $(E, c), (E', c) \in \text{BR}^N$. Moreover, any bankruptcy problem $(e, c) \in \text{TUBR}^N$ induces the bankruptcy problem $(E, c) \in \text{BR}^N$ in which $E = \{x \in \mathbb{R}^N_+ \mid \sum_{i \in N} x_i \leq e\}$.

Let $(E, c) \in \text{BR}^N$ be such that $E \neq \{0_N\}$. Throughout this chapter, scaling the estate is an essential and fundamental operation which preserves its shape. Applying the scaling operation to the estate allows to analyze the implications for the claimants when their interpersonal relations remain at a constant ratio. For any $t \in \mathbb{R}_+$, the set $tE \subseteq \mathbb{R}^N_+$ is given by $tE = \{tx \mid x \in E\}$. Note that $u^{tE} = tu^E$ for all $t \in \mathbb{R}_+$. Let $x \in \mathbb{R}^N_+$. The scalar $\tau^{E,x} \in \mathbb{R}_+$ is defined in such a way that

$$x \in \text{WP}(\tau^{E,x} E).$$

Note that the conditions on E imply that $\tau^{E,x}$ is well-defined and increasing in x. We have $\tau^{E,x} \leq 1$ if $x \in E$, and $\tau^{E,x} > 1$ if $x \notin E$. Moreover, $\tau^{tE,x} = \frac{\tau^{E,x}}{t}$ for all $t \in \mathbb{R}_{++}$, and $\tau^{E,tx} = t\tau^{E,x}$ for all $t \in \mathbb{R}_+$. Note that $(tE, x) \in \text{BR}^N$ for all $t \in [0, \tau^{E,x}]$.

Example 3.1
Let $N = \{1, 2\}$ and consider the bankruptcy problem $(E, c) \in \text{BR}^N$ given by $E = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 12x_2 \leq 36\}$ and $c = (3, 4)$. Then $\tau^{E,c} = 1\frac{1}{2}$ and $\tau^{E,c}E = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 18x_2 \leq 81\}$. This is illustrated as follows.



△

A *bankruptcy rule* $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ assigns to any $(E, c) \in \mathrm{BR}^N$ a payoff allocation $f(E, c) \in \mathrm{WP}(E)$ for which $f(E, c) \leq c$. Note that $f(E, c) = 0_N$ if and only if $E = \{0_N\}$, and $f(E, c) = c$ if and only if $c \in E$.

Let $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ be a bankruptcy rule, let $(E, c) \in \mathrm{BR}^N$ be such that $E \neq \{0_N\}$, and let $x \in \mathbb{R}^N_+$. The *payoff path* $p_f^{E,x} : [0, \tau^{E,x}] \to \mathbb{R}^N_+$ of f from 0_N to x is defined by

$$p_f^{E,x}(t) = f(tE, x)$$

for all $t \in [0, \tau^{E,x}]$. Note that $p_f^{E,x}$ is injective.

All payoff allocations obtained from scaling the estate are represented by the payoff path of the corresponding bankruptcy rule. The path monotonicity property describes bankruptcy rules for which payoffs increase when the estate is enlarged by a scaling operation.[1] Path monotonicity is a stronger property than path continuity, as is the case for TU-bankruptcy rules.

Definition (Path Monotonicity)
A bankruptcy rule $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ satisfies *path monotonicity* if $p_f^{E,c}$ is increasing for all $(E, c) \in \mathrm{BR}^N$ for which $E \neq \{0_N\}$.

Definition (Path Continuity)
A bankruptcy rule $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ satisfies *path continuity* if $p_f^{E,c}$ is continuous for all $(E, c) \in \mathrm{BR}^N$ for which $E \neq \{0_N\}$.

Lemma 3.2.1
Let $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ be a bankruptcy rule. If f satisfies path monotonicity, then f satisfies path continuity.

Proof. Assume that f satisfies path monotonicity. Suppose that f does not satisfy path continuity. Then there exists an $(E, c) \in \mathrm{BR}^N$ such that $E \neq \{0_N\}$ and $p_f^{E,c}$ is not continuous at a certain $\hat{t} \in [0, \tau^{E,c}]$. Suppose that $\hat{t} \in (0, \tau^{E,c})$. Since $p_f^{E,c}$ is increasing,

$$\lim_{t \uparrow \hat{t}} p_f^{E,c}(t) = \sup_{t \in [0, \hat{t})} p_f^{E,c}(t) \leq p_f^{E,c}(\hat{t}) \leq \inf_{t \in (\hat{t}, \tau^{E,c}]} p_f^{E,c}(t) = \lim_{t \downarrow \hat{t}} p_f^{E,c}(t).$$

Since $p_f^{E,c}$ is not continuous at $\hat{t}$, $\lim_{t \uparrow \hat{t}} p_f^{E,c}(t) \neq p_f^{E,c}(\hat{t})$ or $p_f^{E,c}(\hat{t}) \neq \lim_{t \downarrow \hat{t}} p_f^{E,c}(t)$. This means that $\sup_{t \in [0,\hat{t})} p_f^{E,c}(t) \neq p_f^{E,c}(\hat{t})$ or $p_f^{E,c}(\hat{t}) \neq \inf_{t \in (\hat{t}, \tau^{E,c}]} p_f^{E,c}(t)$. Suppose that $\sup_{t \in [0,\hat{t})} p_f^{E,c}(t) \neq p_f^{E,c}(\hat{t})$. Then there exists an $x \in \mathbb{R}^N_+$ such that $\sup_{t \in [0,\hat{t})} p_f^{E,c}(t) \leq x \leq p_f^{E,c}(\hat{t})$ and $\sup_{t \in [0,\hat{t})} p_f^{E,c}(t) \neq x \neq p_f^{E,c}(\hat{t})$. This means that $t < \tau^{E,x} < \hat{t}$ for all $t \in [0, \hat{t})$. This is not possible. Similarly, $p_f^{E,c}(\hat{t}) \neq \inf_{t \in (\hat{t}, \tau^{E,c}]} p_f^{E,c}(t)$ is not possible. One of these cases also applies if $\hat{t} \in \{0, \tau^{E,c}\}$. Hence, f satisfies path continuity. □

[1] A stronger monotonicity property based on estate inclusion instead of estate scaling appears in Section 5.2 as estate monotonicity.

3.3 Duality

In this section, we explore duality and analyze dual properties for bankruptcy rules. Two rules are called dual (cf. Aumann and Maschler (1985)) if one rule allocates awards in the same way as the other rule allocates losses. We generalize this idea to rules for bankruptcy problems with nontransferable utility.

Definition (Dual Bankruptcy Rules)
Two bankruptcy rules $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ and $g : \mathrm{BR}^N \to \mathbb{R}^N_+$ are *dual* if

$$f(E,c) = c - g(\tau^{E,c-f(E,c)}E,c)^2 \text{ and } g(E,c) = c - f(\tau^{E,c-g(E,c)}E,c)$$

for all $(E,c) \in \mathrm{BR}^N$ for which $E \neq \{0_N\}$.

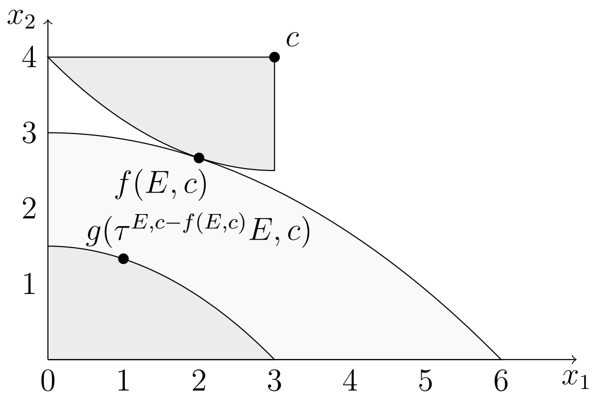
Note that for any two dual bankruptcy rules $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ and $g : \mathrm{BR}^N \to \mathbb{R}^N_+$,

$$f(tE,c) = c - g(\tau^{E,c-f(tE,c)}E,c)$$

for all $(E,c) \in \mathrm{BR}^N$ for which $E \neq \{0_N\}$ and any $t \in [0, \tau^{E,c}]$.

Following our scaling approach, a dual rule assigns the corresponding losses to the bankruptcy problem obtained by scaling the estate in opposite direction from the claims point such that the boundary intersects with the awards assigned by the original rule, as illustrated by the following example.

Example 3.2
Let $N = \{1,2\}$ and consider the bankruptcy problem $(E,c) \in \mathrm{BR}^N$ given by $E = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 12x_2 \leq 36\}$ and $c = (3,4)$ as in Example 3.1. Let $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ and $g : \mathrm{BR}^N \to \mathbb{R}^N_+$ be two dual bankruptcy rules. Then $f(E,c) = c - g(\tau^{E,c-f(E,c)}E,c)$. This is illustrated as follows.

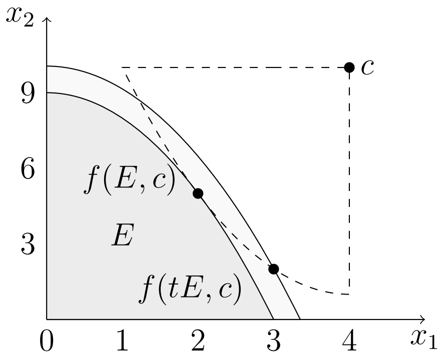


△

The following example shows that, contrary to TU-bankruptcy rules, a dual NTU-bankruptcy rule does not necessarily exist.

[2] Note that $(\tau^{E,c-f(E,c)}E,c) \in \mathrm{BR}^N$ since $\tau^{E,c-f(E,c)} \in [0, \tau^{E,c}]$.

Example 3.3
Let $N = \{1, 2\}$ and consider the bankruptcy problem $(E, c) \in \mathrm{BR}^N$ given by $E = \{x \in \mathbb{R}^N_+ \mid x_1^2 + x_2 \le 9\}$ and $c = (4, 10)$. Then $(2, 5), (1, 8) \in \mathrm{WP}(E)$. Let $t \in [0, \tau^{E,c}]$ be given by $t = \frac{1}{9}(1 + \sqrt{82})$ and let $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ be a bankruptcy rule such that $f(E, c) = (2, 5)$ and $f(tE, c) = (3, 2)$. This is illustrated as follows.



Suppose that there exists a dual bankruptcy rule $g : \mathrm{BR}^N \to \mathbb{R}^N_+$. Then

$$\begin{aligned} &(2,5) = f(E,c) = c - g(\tau^{E,c-f(E,c)}E, c) = c - g(\tau^{E,(2,5)}E, c) = c - g(E,c) \\ \text{and} \quad &(3,2) = f(tE,c) = c - g(\tau^{E,c-f(tE,c)}E, c) = c - g(\tau^{E,(1,8)}E, c) = c - g(E,c). \end{aligned}$$

This is impossible. Hence, f does not have a dual bankruptcy rule. △

To still justify the term duality, we show that a dual rule is unique.

Lemma 3.3.1
Let $f : \mathrm{BR}^N \to \mathbb{R}^N_+$, $g : \mathrm{BR}^N \to \mathbb{R}^N_+$, and $h : \mathrm{BR}^N \to \mathbb{R}^N_+$ be three bankruptcy rules. If f and g are dual, and f and h are dual, then $g = h$.

Proof. Assume that f and g are dual, and that f and h are dual. Let $(E, c) \in \mathrm{BR}^N$ be such that $E \neq \{0_N\}$. Then

$$\begin{aligned} g(E,c) &= c - f(\tau^{E,c-g(E,c)}E, c) = h(\tau^{E,c-f(\tau^{E,c-g(E,c)}E,c)}E, c) \\ &= h(\tau^{E,g(E,c)}E, c) \qquad = h(E,c), \end{aligned}$$

where the first and third equality follow from duality of f and g, the second equality follows from duality of f and h, and the last equality follows from $g(E, c) \in \mathrm{WP}(E)$ implying that $\tau^{E,g(E,c)} = 1$. Hence, $g = h$. □

A rule is self-dual if it coincides with its dual.

Definition (Self-Dual Bankruptcy Rule)
A bankruptcy rule $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ is *self-dual* if $f(E, c) = c - f(\tau^{E,c-f(E,c)}E, c)$ for all $(E, c) \in \mathrm{BR}^N$ for which $E \neq \{0_N\}$.

The remainder of this section studies relations between properties of two dual bankruptcy rules. Two properties for bankruptcy rules are *dual* (cf. Herrero and Villar (2001)) if for any two dual bankruptcy rules, one property is satisfied by one rule if and only if the other property is satisfied by the other rule. A property for bankruptcy rules is *self-dual* if any two dual bankruptcy rules either both satisfy the property, or neither. Note that self-duality is a self-dual property. We show that path monotonicity is a self-dual property as well.

Lemma 3.3.2
Path monotonicity is self-dual.

Proof. Let $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ and $g : \mathrm{BR}^N \to \mathbb{R}^N_+$ be two dual bankruptcy rules. Assume that f satisfies path monotonicity. Let $(E, c) \in \mathrm{BR}^N$ be such that $E \neq \{0_N\}$. Then

$$p_f^{E,c}(t) = f(tE, c) = c - g(\tau^{E,c-f(tE,c)}E, c) = c - p_g^{E,c}(\tau^{E,c-p_f^{E,c}(t)})$$

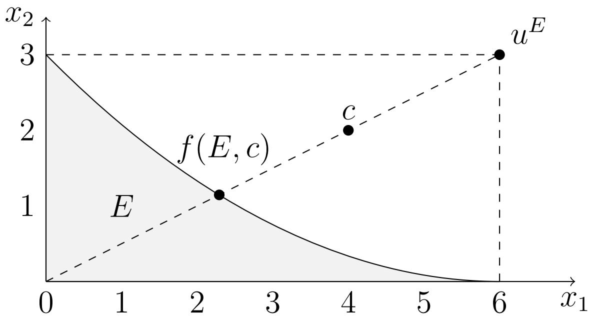
for all $t \in [0, \tau^{E,c}]$, where the second equality follows from duality. Since $p_f^{E,c}$ is increasing, this means that $\tau^{E,c-p_f^{E,c}(t)}$ is decreasing in t. This implies that $p_g^{E,c}$ is increasing, so g satisfies path monotonicity. Hence, path monotonicity is self-dual. □

Next, we study a self-dual symmetry property. The idea of equality, equity, or symmetry underlies many theories of economic justice (cf. Rawls (1971) and Young (1995)). The interpretation of symmetry depends on the underlying model. In a bankruptcy problem with nontransferable utility, claimants not only differ in their claims, but also differ in their measure of utility. It makes sense to compare their claims in relation to the estate. Preserving the most important characteristics, the maximal individual payoffs within the estate, or utopia values, form a natural benchmark for a symmetry property.

Definition (Relative Symmetry)
A bankruptcy rule $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ satisfies *relative symmetry* if $f_i(E, c)u_j^E = f_j(E, c)u_i^E$ for all $(E, c) \in \mathrm{BR}^N$ and any $i, j \in N$ for which $c_i u_j^E = c_j u_i^E$.

Note that relative symmetry is an interpretation of equality which is covariant under individual rescaling of utility. Moreover, for any bankruptcy problem $(E, c) \in \mathrm{BR}^N$ in which $E = \{x \in \mathbb{R}^N_+ \mid \sum_{i \in N} x_i \leq e\}$, induced by a bankruptcy problem $(e, c) \in \mathrm{TUBR}^N$, $u_i^E = e$ for all $i \in N$ and relative symmetry boils down to the standard property equal treatment of claimants with equal claims.

Example 3.4
Let $N = \{1, 2\}$ and consider the bankruptcy problem $(E, c) \in \mathrm{BR}^N$ given by $E = \{x \in \mathbb{R}^N_+ \mid \sqrt{6x_1} + 2x_2 \leq 6\}$ and $c = (4, 2)$. Then $u^E = (6, 3)$ and $f(E, c) = (9 - 3\sqrt{5}, \frac{9}{2} - \frac{3}{2}\sqrt{5})$ for any bankruptcy rule $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ satisfying relative symmetry. This is illustrated as follows.



△

Lemma 3.3.3
Relative symmetry is self-dual.

Proof. Let $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ and $g : \mathrm{BR}^N \to \mathbb{R}^N_+$ be two dual bankruptcy rules. Assume that f satisfies relative symmetry. Let $(E, c) \in \mathrm{BR}^N$ be such that $E \neq \{0_N\}$ and let $i, j \in N$ be such that $c_i u^E_j = c_j u^E_i$. Denote $d = \tau^{E,c-g(E,c)}$. Then $f_i(dE, c)u^{dE}_j = f_j(dE, c)u^{dE}_i$ since $c_i u^{dE}_j = c_j u^{dE}_i$. This means that

$$\begin{aligned} g_i(E, c)u^E_j &= (c_i - f_i(dE, c))u^E_j \quad = c_i u^E_j - f_i(dE, c)u^E_j \\ &= c_j u^E_i - f_j(dE, c)u^E_i = (c_j - f_j(dE, c))u^E_i \ = g_j(E, c)u^E_i, \end{aligned}$$

where the first and last equality follow from duality, and the third equality follows from relative symmetry. This means that g satisfies relative symmetry. Hence, relative symmetry is self-dual. □

Two other interesting properties from TU-bankruptcy theory are composition down and composition up. Composition down implies that solutions on the payoff path can replace the claim vector when the estate is scaled down. Composition up implies that solutions on the payoff path can act as a new origin from which the estate is scaled again.[3]

Definition (Composition Down)
A bankruptcy rule $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ satisfies *composition down* if

$$f(tE, c) = f(tE, f(E, c))$$

for all $(E, c) \in \mathrm{BR}^N$ for which $E \neq \{0_N\}$ and any $t \in [0, 1]$.

[3]Another composition property based on estate inclusion instead of estate scaling appears in Section 5.2 as step-by-step negotiations.

Definition (Composition Up)
A bankruptcy rule $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ satisfies *composition up* if

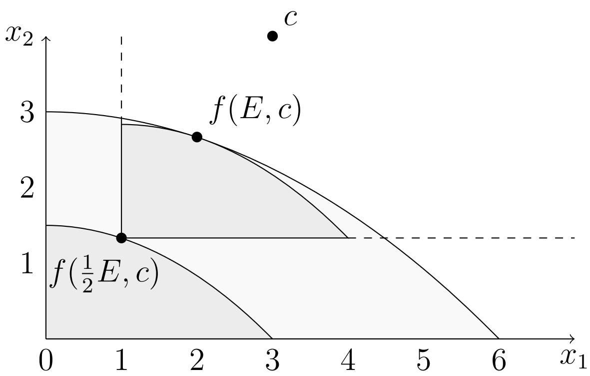
$$f(E,c) = f(tE,c) + f(\tau^{E,f(E,c)-f(tE,c)}E, c - f(tE,c))$$

for all $(E,c) \in \mathrm{BR}^N$ for which $E \neq \{0_N\}$ and any $t \in [0,1]$.

Example 3.5
Let $N = \{1,2\}$ and consider the bankruptcy problem $(E,c) \in \mathrm{BR}^N$ given by $E = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 12x_2 \leq 36\}$ and $c = (3,4)$ as in Example 3.1. Then

$$f(E,c) = f\left(\frac{1}{2}E, c\right) + f\left(\tau^{E,f(E,c)-f(\frac{1}{2}E,c)}E, c - f\left(\frac{1}{2}E, c\right)\right)$$

for any bankruptcy rule $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ satisfying composition up. This is illustrated as follows.



△

Both composition properties are stronger than path monotonicity.

Lemma 3.3.4
Let $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ be a bankruptcy rule.

(i) If f satisfies composition down, then f satisfies path monotonicity.

(ii) If f satisfies composition up, then f satisfies path monotonicity.

Proof. (i) Assume that f satisfies composition down. Let $(E,c) \in \mathrm{BR}^N$ be such that $E \neq \{0_N\}$ and let $t_1, t_2 \in [0, \tau^{E,c}]$ be such that $t_1 < t_2$. Then $\frac{t_1}{t_2} \in [0,1)$ and

$$\begin{aligned} p_f^{E,c}(t_1) = f(t_1E,c) \qquad &= f\left(\frac{t_1}{t_2}t_2E, c\right) = f\left(\frac{t_1}{t_2}t_2E, f(t_2E,c)\right) \\ = f(t_1E, f(t_2E,c)) \leq f(t_2E,c) \qquad &= p_f^{E,c}(t_2), \end{aligned}$$

where the third equality follows from composition down and the inequality follows from the definition of a bankruptcy rule. Moreover, $p_f^{E,c}(t_1) \neq p_f^{E,c}(t_2)$ since $p_f^{E,c}$ is injective. Hence, f satisfies path monotonicity.

(*ii*) Assume that f satisfies composition up. Let $(E,c) \in \mathrm{BR}^N$ be such that $E \neq \{0_N\}$ and let $t_1, t_2 \in [0, \tau^{E,c}]$ be such that $t_1 < t_2$. Then $\frac{t_1}{t_2} \in [0,1)$ and

$$\begin{aligned} p_f^{E,c}(t_2) &= f(t_2E,c) \\ &= f\left(\frac{t_1}{t_2}t_2E,c\right) + f\left(\tau^{t_2E,f(t_2E,c)-f\left(\frac{t_1}{t_2}t_2E,c\right)}t_2E, c - f\left(\frac{t_1}{t_2}t_2E,c\right)\right) \\ &= f(t_1E,c) + f(\tau^{E,f(t_2E,c)-f(t_1E,c)}E, c - f(t_1E,c)) \\ &\geq f(t_1E,c) \\ &= p_f^{E,c}(t_1), \end{aligned}$$

where the second equality follows from composition up and the inequality follows from the definition of a bankruptcy rule. Moreover, $p_f^{E,c}(t_2) \neq p_f^{E,c}(t_1)$ since $p_f^{E,c}$ is injective. Hence, f satisfies path monotonicity. □

Finally, we show that composition down and composition up are dual properties.

Lemma 3.3.5

Composition down and composition up are dual.

Proof. Let $f : \mathrm{BR}^N \to \mathbb{R}_+^N$ and $g : \mathrm{BR}^N \to \mathbb{R}_+^N$ be two dual bankruptcy rules.

First, assume that f satisfies composition down. Then f satisfies path monotonicity by Lemma 3.3.4. Then g satisfies path monotonicity by Lemma 3.3.2. Let $(E,c) \in \mathrm{BR}^N$ be such that $E \neq \{0_N\}$ and let $t \in [0,1]$. If $t \in \{0,1\}$, then $g(E,c) = g(tE,c) + g(\tau^{E,g(E,c)-g(tE,c)}E, c - g(tE,c))$. Suppose that $t \in (0,1)$. Denote $d = \tau^{E,c-g(E,c)}$ and denote $d' = \tau^{E,c-g(tE,c)}$. Then $d < d'$ since $g(tE,c) \leq g(E,c)$ and $g(tE,c) \neq g(E,c)$. This means that $\frac{d}{d'} \in [0,1)$ and

$$\begin{aligned} g(E,c) - g(tE,c) &= (c - f(dE,c)) - (c - f(d'E,c)) \\ &= f(d'E,c) - f(dE,c) \\ &= f(d'E,c) - f(dE, f(d'E,c)) \\ &= f(d'E,c) - \left(f(d'E,c) - g(\tau^{E,f(d'E,c)-f(dE,f(d'E,c))}E, f(d'E,c))\right) \\ &= g(\tau^{E,g(E,c)-g(tE,c)}E, c - g(tE,c)), \end{aligned}$$

where the first and fourth equality follow from duality, and the third equality follows from composition down. Hence, g satisfies composition up.

Next, assume that g satisfies composition up. Then g satisfies path monotonicity by Lemma 3.3.4. Then f satisfies path monotonicity by Lemma 3.3.2. Let $(E,c) \in \mathrm{BR}^N$ be such that $E \neq \{0_N\}$ and let $t \in [0,1]$. If $t \in \{0,1\}$, then $f(tE,c) = f(tE, f(E,c))$. Suppose that $t \in (0,1)$. Denote $d = \tau^{E,c-f(E,c)}$ and denote $d' = \tau^{E,c-f(tE,c)}$. Then $d < d'$ since $f(tE,c) \leq f(E,c)$ and $f(tE,c) \neq f(E,c)$. This means that $\frac{d}{d'} \in [0,1)$ and

$$\begin{aligned}
f(tE,c) &= c - g(d'E,c) \\
&= c - \Big(g(dE,c) + g(\tau^{E,g(d'E,c)-g(dE,c)}E, c - g(dE,c))\Big) \\
&= f(E,c) - g(\tau^{E,f(E,c)-f(tE,c)}E, f(E,c)) \\
&= f(E,c) - \Big(f(E,c) - f(\tau^{E,f(E,c)-g(\tau^{E,f(E,c)-f(tE,c)}E,f(E,c))}E, f(E,c))\Big) \\
&= f(\tau^{E,f(tE,c)}E, f(E,c)) \\
&= f(tE, f(E,c)),
\end{aligned}$$

where the first, third, fourth, and fifth equality follow from duality, the second equality follows from composition up, and the last equality follows from $f(tE,c) \in \mathrm{WP}(tE)$ implying that $\tau^{E,f(tE,c)} = t$. Hence, f satisfies composition down. $\square$

3.4 The proportional rule

This section introduces the proportional rule for bankruptcy problems with nontransferable utility and provides three axiomatic characterizations.

Definition (Proportional Rule)
The *proportional rule* $\mathrm{Prop} : \mathrm{BR}^N \to \mathbb{R}^N_+$ is the bankruptcy rule which assigns to any $(E,c) \in \mathrm{BR}^N$ the payoff allocation

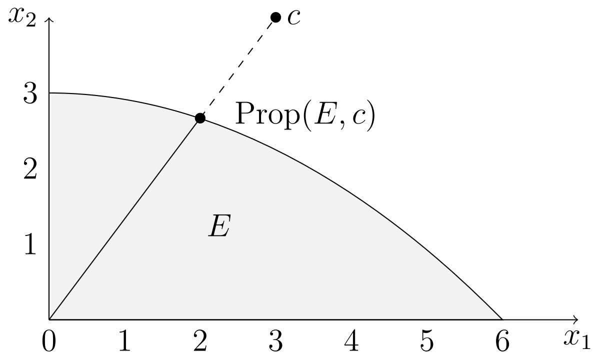
$$\mathrm{Prop}(E,c) = \lambda^{E,c}c,$$

where $\lambda^{E,c} = \max\{t \in [0,1] \mid tc \in \mathrm{WP}(E)\}$.

Note that if $E \neq \{0_N\}$, then $\lambda^{E,c} = \frac{1}{\tau^{E,c}}$,

$$\begin{aligned}
\lambda^{tE,c} &= t\lambda^{E,c} \text{ for all } t \in [0, \tau^{E,c}], \\
\text{and } \lambda^{E,tc} &= \frac{\lambda^{E,c}}{t} \text{ for all } t \geq \lambda^{E,c}.
\end{aligned}$$

Example 3.6
Let $N = \{1, 2\}$ and consider the bankruptcy problem $(E, c) \in \mathrm{BR}^N$ given by $E = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 12x_2 \leq 36\}$ and $c = (3, 4)$ as in Example 3.1. Then $\lambda^{E,c} = \frac{1}{\tau^{E,c}} = \frac{2}{3}$ and $\mathrm{Prop}(E, c) = \lambda^{E,c} c = (2, 2\frac{2}{3})$. This is illustrated as follows.



△

The characterization of the proportional rule for TU-bankruptcy problems in terms of composition down and self-duality (cf. Thomson (2016)), or composition up and self-duality (cf. Young (1988)), can be extended to NTU-bankruptcy problems.

Theorem 3.4.1

(i) The proportional rule is the unique self-dual bankruptcy rule satisfying composition down.

(ii) The proportional rule is the unique self-dual bankruptcy rule satisfying composition up.

Proof. Since (ii) follows from (i) and Lemma 3.3.5, it suffices to prove only (i).

First, let $(E, c) \in \mathrm{BR}^N$ be such that $E \neq \{0_N\}$. Then

$$\begin{aligned}\mathrm{Prop}(\tau^{E,c-\mathrm{Prop}(E,c)}E, c) &= \lambda^{\tau^{E,(1-\lambda^{E,c})c}E,c}c = \tau^{E,(1-\lambda^{E,c})c}\lambda^{E,c}c = (1-\lambda^{E,c})\tau^{E,c}\lambda^{E,c}c \\ &= (1-\lambda^{E,c})c \qquad = c - \lambda^{E,c}c \qquad\quad = c - \mathrm{Prop}(E, c).\end{aligned}$$

Hence, the proportional rule is self-dual. Let $t \in [0, 1]$. Then

$$\begin{aligned}\mathrm{Prop}(tE, \mathrm{Prop}(E, c)) &= \lambda^{tE,\mathrm{Prop}(E,c)}\mathrm{Prop}(E, c) = \lambda^{tE,\lambda^{E,c}c}\lambda^{E,c}c \\ &= \lambda^{tE,c}c \qquad\qquad\qquad\qquad = \mathrm{Prop}(tE, c).\end{aligned}$$

Hence, the proportional rule satisfies composition down.

Second, let $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ be a self-dual bankruptcy rule satisfying composition down. Then f satisfies path monotonicity by Lemma 3.3.4. Then f satisfies path continuity by Lemma 3.2.1. Let $(E, c) \in \mathrm{BR}^N$. If $E = \{0_N\}$, then $f(E, c) = 0_N = \mathrm{Prop}(E, c)$. Suppose that $E \neq \{0_N\}$. Let $x \in \mathbb{R}^N_+$. For any $s \in [0, \sum_{i\in N} x_i]$, there exists a unique $t \in [0, \tau^{E,x}]$ for which $\sum_{i\in N} f_i(tE, x) = s$. Let $t \in [0, \tau^{E,x}]$. Then

$$\begin{aligned} p_f^{E,x}(\tau^{E,x-p_f^{E,x}(t)}) &= f(\tau^{E,x-p_f^{E,x}(t)}E, x) = f(\tau^{E,x-f(tE,x)}E, x) \\ &= x - f(tE, x) \qquad = x - p_f^{E,x}(t), \end{aligned}$$

where the third equality follows from self-duality. This means that for any vector $y \in \mathbb{R}^N_+$ on the payoff path of f from 0_N to x, $x - y$ is also on the payoff path of f from 0_N to x. Let $t' \in [0, t]$. Then

$$p_f^{E,p_f^{E,x}(t)}(t') = f(t'E, p_f^{E,x}(t)) = f(t'E, f(tE, x)) = f(t'E, x) = p_f^{E,x}(t'),$$

where the third equality follows from composition down. This means that for any vector $y \in \mathbb{R}^N_+$ on the payoff path of f from 0_N to x, any vector on the payoff path of f from 0_N to y is also on the payoff path of f from 0_N to x.

Now, let $t \in [0, \tau^{E,x}]$ be such that $\sum_{i\in N} f_i(tE, x) = \frac{1}{2}\sum_{i\in N} x_i$. Then $f(tE, x)$ and $x - f(tE, x)$ are both on the payoff path of f from 0_N to x. Moreover,

$$\sum_{i\in N}(x_i - f_i(tE, x)) = \sum_{i\in N} x_i - \sum_{i\in N} f_i(tE, x) = \sum_{i\in N} x_i - \frac{1}{2}\sum_{i\in N} x_i = \frac{1}{2}\sum_{i\in N} x_i.$$

This means that $f(tE, x) = \frac{1}{2}x$, so $\frac{1}{2}x$ is on the payoff path of f from 0_N to x.

In particular, $\frac{1}{2}c$ is on the payoff path of f from 0_N to c, and $\frac{1}{4}c$ is on the payoff path of f from 0_N to $\frac{1}{2}c$, which means that $\frac{1}{4}c$ and $\frac{3}{4}c$ are on the payoff path of f from 0_N to c. Continuing this reasoning, $\frac{m}{2^n}c$ is on the payoff path of f from 0_N to c for any $m, n \in \mathbb{N}$ for which $m \leq 2^n$. Since f satisfies path continuity, this means that tc is on the payoff path of f from 0_N to c for any $t \in [0, 1]$. In other words, $f(E, c) = \lambda^{E,c}c = \mathrm{Prop}(E, c)$. Hence, $f = \mathrm{Prop}$. □

The bankruptcy rule $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ which assigns to any $(E, c) \in \mathrm{BR}^N$ the payoff allocation

$$f(E, c) = \begin{cases} \left(\min\{\frac{1}{2}c_i, \eta\}\right)_{i\in N} & \text{if } \frac{1}{2}c \notin E; \\ \left(\max\{\frac{1}{2}c_i, c_i - \eta\}\right)_{i\in N} & \text{if } \frac{1}{2}c \in E, \end{cases}$$

where $\eta \in \mathbb{R}_+$ is such that $f(E, c) \in \mathrm{WP}(E)$, is also self-dual. This means that the proportional rule is not the unique self-dual bankruptcy rule.

Chun (1988) characterized the proportional rule in terms of a linearity axiom. We extend this characterization by showing that the proportional rule is the only rule with a linear payoff path for any bankruptcy problem.

Definition (Path Linearity)
A bankruptcy rule $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ satisfies *path linearity* if

$$f(\theta E + (1-\theta)tE, c) = \theta f(E, c) + (1-\theta) f(tE, c)$$

for all $(E, c) \in \mathrm{BR}^N$ for which $E \neq \{0_N\}$, any $t \in [0, \tau^{E,c}]$, and any $\theta \in [0, 1]$.

Theorem 3.4.2
The proportional rule is the unique bankruptcy rule satisfying path linearity.

Proof. First, let $(E, c) \in \mathrm{BR}^N$ be such that $E \neq \{0_N\}$, let $t \in [0, \tau^{E,c}]$, and let $\theta \in [0, 1]$. Then

$$\begin{aligned}
\mathrm{Prop}(\theta E + (1-\theta)tE, c) &= \lambda^{\theta E + (1-\theta)tE, c} c \\
&= \lambda^{(\theta + (1-\theta)t)E, c} c \\
&= (\theta + (1-\theta)t)\lambda^{E,c} c \\
&= \theta \lambda^{E,c} c + (1-\theta) t \lambda^{E,c} c \\
&= \theta \lambda^{E,c} c + (1-\theta) \lambda^{tE,c} c \\
&= \theta \mathrm{Prop}(E, c) + (1-\theta)\mathrm{Prop}(tE, c).
\end{aligned}$$

Hence, the proportional rule satisfies path linearity.

Second, let $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ be a bankruptcy rule satisfying path linearity. Let $(E, c) \in \mathrm{BR}^N$. If $E = \{0_N\}$, then $f(E, c) = 0_N = \mathrm{Prop}(E, c)$. Suppose that $E \neq \{0_N\}$. Then

$$\begin{aligned}
f(E, c) &= f(\lambda^{E,c}\tau^{E,c}E + (1-\lambda^{E,c})0\tau^{E,c}E, c) \\
&= \lambda^{E,c} f(\tau^{E,c}E, c) + (1-\lambda^{E,c}) f(0\tau^{E,c}E, c) \\
&= \lambda^{E,c} f(\tau^{E,c}E, c) + (1-\lambda^{E,c}) f(\{0_N\}, c) \\
&= \lambda^{E,c} c + (1-\lambda^{E,c}) 0_N \\
&= \lambda^{E,c} c \\
&= \mathrm{Prop}(E, c),
\end{aligned}$$

where the second equality follows from path linearity. Hence, $f = \mathrm{Prop}$. □

3.5 The constrained relative equal awards rule

This section introduces the constrained relative equal awards rule for bankruptcy problems with nontransferable utility and provides four axiomatic characterizations. The constrained relative equal awards rule generalizes the constrained equal awards rule for bankruptcy problems with transferable utility which divides the estate equally such that all claimants are not allocated more than their claims. Following our interpretation of equality and symmetry in bankruptcy problems with nontransferable utility, it makes sense to define a rule which allocates payoffs *relatively* equal such that all claimants are not allocated more than their claims.

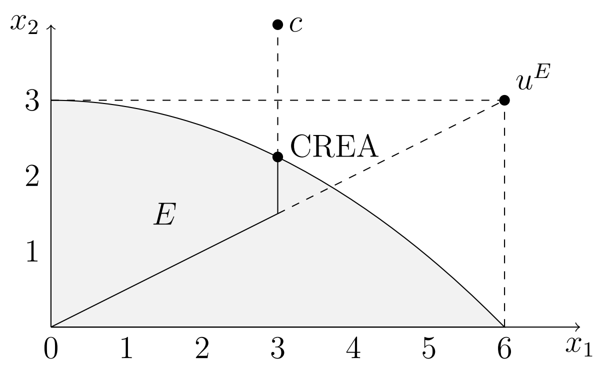
Definition (Constrained Relative Equal Awards Rule)
The *constrained relative equal awards rule* $\mathrm{CREA} : \mathrm{BR}^N \to \mathbb{R}^N_+$ is the bankruptcy rule which assigns to any $(E, c) \in \mathrm{BR}^N$ the payoff allocation

$$\mathrm{CREA}(E,c) = \left(\min\{c_i, \alpha^{E,c} u_i^E\}\right)_{i \in N},$$

where $\alpha^{E,c} = \max\{t \in [0,1] \mid (\min\{c_i, t u_i^E\})_{i \in N} \in \mathrm{WP}(E)\}$.

Note that for any bankruptcy problem $(E, c) \in \mathrm{BR}^N$ in which $E = \{x \in \mathbb{R}^N_+ \mid \sum_{i \in N} x_i \leq e\}$, induced by a bankruptcy problem $(e, c) \in \mathrm{TUBR}^N$, $u_i^E = e$ for all $i \in N$ and the constrained relative equal awards rule coincides with the standard constrained equal awards rule.

Example 3.7
Let $N = \{1, 2\}$ and consider the bankruptcy problem $(E, c) \in \mathrm{BR}^N$ given by $E = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 12x_2 \leq 36\}$ and $c = (3, 4)$ as in Example 3.1. Then $u^E = (6, 3)$, $\alpha^{E,c} = \frac{3}{4}$, and $\mathrm{CREA}(E, c) = (3, 2\frac{1}{4})$. This is illustrated as follows.



△

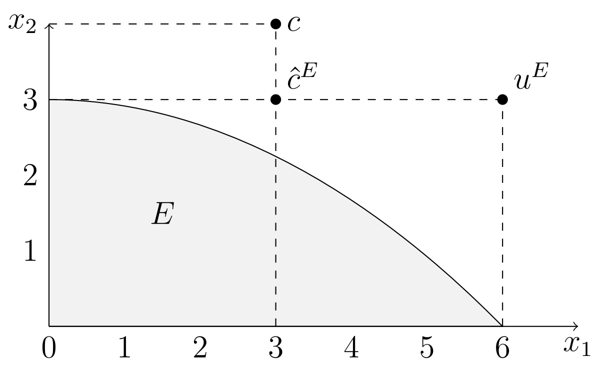
Throughout this section, we refer to the appendix of this chapter for derivations of the specific properties stated for the constrained relative equal awards rule.

Let $(E, c) \in \mathrm{BR}^N$. The vector of *truncated claims* $\hat{c}^E \in \mathbb{R}^N_+$ is defined by

$$\hat{c}^E = \left(\min\{c_i, u_i^E\}\right)_{i \in N}.$$

Note that $\hat{c}^E \in \mathrm{WUC}(E)$ and $f(E, c) \leq \hat{c}^E$ for any bankruptcy rule $f : \mathrm{BR}^N \to \mathbb{R}^N_+$.

Example 3.8
Let $N = \{1, 2\}$ and consider the bankruptcy problem $(E, c) \in \mathrm{BR}^N$ given by $E = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 12x_2 \leq 36\}$ and $c = (3, 4)$ as in Example 3.1 and Example 3.7. Then $\hat{c}^E = (3, 3)$. This is illustrated as follows.



△

The truncation invariance property requires that bankruptcy rules only take the truncated claims of the claimants into account.

Definition (Truncation Invariance)
A bankruptcy rule $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ satisfies *truncation invariance* if $f(E, c) = f(E, \hat{c}^E)$ for all $(E, c) \in \mathrm{BR}^N$.

Inspired by Dagan (1996), we axiomatically characterize the constrained relative equal awards rule using the properties relative symmetry, composition up, and truncation invariance. Note that the proportional rule also satisfies relative symmetry and composition up, but does not satisfy truncation invariance.

Theorem 3.5.1
The constrained relative equal awards rule is the unique bankruptcy rule satisfying relative symmetry, truncation invariance, and composition up.

Proof. By Lemma 3.A.1, Lemma 3.A.2, and Lemma 3.A.3, the constrained relative equal awards rule satisfies relative symmetry, truncation invariance, and composition up. Let $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ be a bankruptcy rule satisfying relative symmetry, truncation invariance, and composition up. Then f satisfies path monotonicity by Lemma 3.3.4. Then f satisfies path continuity by Lemma 3.2.1.

Let $(E, c) \in \text{BR}^N$ be such that $E \neq \{0_N\}$. Suppose that $f(tE, c) \neq \text{CREA}(tE, c)$ for some $t \in [0, \tau^{E,c}]$. Let $\hat{t} = \inf\{t \in [0, \tau^{E,c}] \mid f(tE, c) \neq \text{CREA}(tE, c)\}$. Since f and CREA satisfy path continuity, $\hat{t} \in [0, \tau^{E,c})$ and $f(\hat{t}E, c) = \text{CREA}(\hat{t}E, c)$. Denote $N = \{1, \ldots, n\}$ such that $\frac{c_1}{u_1^E} \leq \cdots \leq \frac{c_n}{u_n^E}$. Let $k \in N$ be such that $f_i(\hat{t}E, c) = c_i$ for all $i < k$, and $f_i(\hat{t}E, c) = \hat{t}\alpha^{\hat{t}E,c} u_k^E < c_i$ for all $i \geq k$.

Let $m = \min\{\|x\| \mid x \in \text{WP}(E)\}$. Note that the conditions on E imply that m exists. Take $\varepsilon \in (0, m(\frac{c_k - f_k(\hat{t}E,c)}{u_k^E}))$. Since f satisfies path continuity, there exists a $\delta > 0$ such that $\|f(tE, c) - f(\hat{t}E, c)\| < \varepsilon$ for all $t \in (\hat{t}, \min\{\hat{t} + \delta, \tau^{E,c}\})$. Let $t \in (\hat{t}, \min\{\hat{t} + \delta, \tau^{E,c}\})$. Denote $d = \tau^{E, f(tE,c) - f(\hat{t}E,c)}$. Then

$$m\left(\frac{c_k - f_k(\hat{t}E, c)}{u_k^E}\right) > \varepsilon > \|f(tE, c) - f(\hat{t}E, c)\| = \|f(dE, c - f(\hat{t}E, c))\| \geq dm,$$

where the equality follows from composition up. This means that $d < \frac{c_k - f_k(\hat{t}E,c)}{u_k^E}$. Let $x \in \mathbb{R}_+^N$ be defined by

$$x_i = \begin{cases} 0 & \text{for all } i \in N \text{ for which } i < k; \\ u_i^{dE} & \text{for all } i \in N \text{ for which } i \geq k. \end{cases}$$

Then

$$x_i = 0 = c_i - c_i = c_i - f_i(\hat{t}E, c) = c_i - \text{CREA}_i(\hat{t}E, c)$$

for all $i \in N$ for which $i < k$. Moreover,

$$\begin{aligned} x_i = u_i^{dE} = du_i^E \quad &< \left(\frac{c_k - f_k(\hat{t}E, c)}{u_k^E}\right) u_i^E \leq \left(\frac{c_i}{u_i^E} - \frac{\hat{t}\alpha^{\hat{t}E,c} u_k^E}{u_k^E}\right) u_i^E \\ = c_i - \hat{t}\alpha^{\hat{t}E,c} u_i^E = c_i - f_i(\hat{t}E, c) \quad &= c_i - \text{CREA}_i(\hat{t}E, c) \end{aligned}$$

for all $i \in N$ for which $i \geq k$. Then

$$\begin{aligned} f(dE, c - f(\hat{t}E, c)) = f(dE, x) \quad &= \lambda^{dE,x} x \\ &= \text{CREA}(dE, x) = \text{CREA}(dE, c - \text{CREA}(\hat{t}E, c)), \end{aligned}$$

where the first and last equality follow from truncation invariance, and the second and third equality follow from relative symmetry. Moreover,

$$\begin{aligned} f(tE, c) &= f(\hat{t}E, c) + f(dE, c - f(\hat{t}E, c)) \\ &= \text{CREA}(\hat{t}E, c) + \text{CREA}(dE, c - \text{CREA}(\hat{t}E, c)) \\ &= \text{CREA}(tE, c), \end{aligned}$$

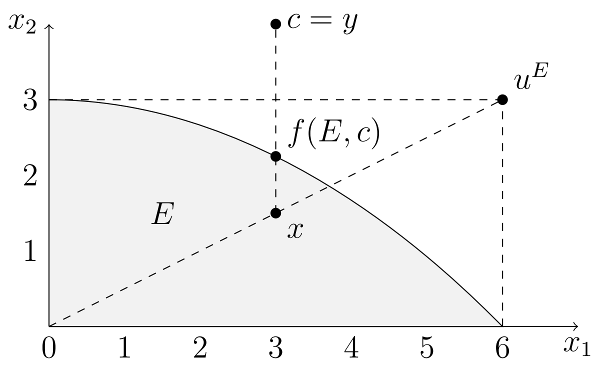
where the first and the last equality follow from composition up. This contradicts the definition of $\hat{t}$. Hence, $f(tE, c) = \text{CREA}(tE, c)$ for all $t \in [0, \tau^{E,c}]$. □

The second axiomatic characterization is in the spirit of Yeh (2006), who showed that the constrained equal awards rule for TU-bankruptcy problems is the only rule that satisfies claim monotonicity and a property which requires that the claimants with small enough claims are fully compensated. We generalize this idea to a conditional full compensation property for NTU-bankruptcy rules based on the relative claims and characterize the constrained relative equal awards rule in terms of claim monotonicity and conditional full compensation.

Definition (Conditional Full Compensation)
A bankruptcy rule $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ satisfies *conditional full compensation* if $f_i(E, c) = c_i$ for all $(E, c) \in \mathrm{BR}^N$ for which $E \neq \{0_N\}$ and any $i \in N$ for which

$$\left(\min\left\{\frac{c_i}{u_i^E}, \frac{c_j}{u_j^E}\right\} u_j^E\right)_{j\in N} \in E.$$

Example 3.9
Let $N = \{1, 2\}$ and consider the bankruptcy problem $(E, c) \in \mathrm{BR}^N$ given by $E = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 12x_2 \leq 36\}$ and $c = (3, 4)$ as in Example 3.1 and Example 3.7. Let $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ be a bankruptcy rule satisfying conditional full compensation, let $x \in \mathbb{R}^N_+$ be given by $x = (c_1, \min\{\frac{c_1}{u_1^E}, \frac{c_2}{u_2^E}\}u_2^E) = (3, 1\frac{1}{2})$, and let $y \in \mathbb{R}^N_+$ be given by $y = (\min\{\frac{c_1}{u_1^E}, \frac{c_2}{u_2^E}\}u_1^E, c_2) = (3, 4)$. Then $x \in E$ and $y \notin E$. This means that $f(E, c) = (3, 2\frac{1}{4})$. This is illustrated as follows.



△

Definition (Claim Monotonicity)
A bankruptcy rule $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ satisfies *claim monotonicity* if $f_i(E, c) \leq f_i(E, c')$ for all $(E, c) \in \mathrm{BR}^N$, any $i \in N$, and any $c' \in \mathbb{R}^N_+$ for which $c'_i \geq c_i$ and $c'_{N\setminus\{i\}} = c_{N\setminus\{i\}}$.

Theorem 3.5.2

The constrained relative equal awards rule is the unique bankruptcy rule satisfying conditional full compensation and claim monotonicity.

Proof. By Lemma 3.A.4 and Lemma 3.A.5, the constrained relative equal awards rule satisfies conditional full compensation and claim monotonicity. Let $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ be a bankruptcy rule satisfying conditional full compensation and claim monotonicity.

Let $(E, c) \in \mathrm{BR}^N$ be such that $E \neq \{0_N\}$. Let $i \in N$ be such that $\mathrm{CREA}_i(E, c) = c_i$. Then

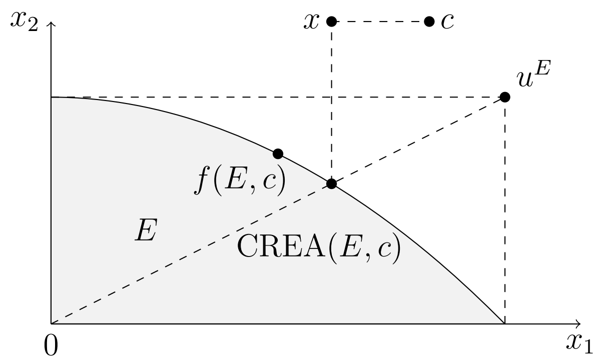
$$\min\left\{\frac{c_i}{u_i^E}, \frac{c_j}{u_j^E}\right\} u_j^E \leq \min\left\{\alpha^{E,c}, \frac{c_j}{u_j^E}\right\} u_j^E = \min\{c_j, \alpha^{E,c} u_j^E\} = \mathrm{CREA}_j(E, c)$$

for all $j \in N$. Since E is comprehensive, this means that

$$\left(\min\left\{\frac{c_i}{u_i^E}, \frac{c_j}{u_j^E}\right\} u_j^E\right)_{j \in N} \in E.$$

Since f satisfies conditional full compensation, this implies that $f_i(E, c) = c_i$.

Suppose that $f(E, c) \neq \mathrm{CREA}(E, c)$. Since E is nonleveled, there exists a $k \in N$ such that $f_k(E, c) < \mathrm{CREA}_k(E, c) = \alpha^{E,c} u_k^E < c_k$. Let $x \in \mathbb{R}^N_+$ be defined by $x_k = \alpha^{E,c} u_k^E$ and $x_{N \backslash \{k\}} = c_{N \backslash \{k\}}$.



Then $f(E, c) \leq x \leq c$ and

$$\min\left\{\frac{x_k}{u_k^E}, \frac{x_j}{u_j^E}\right\} u_j^E = \min\left\{\alpha^{E,c}, \frac{c_j}{u_j^E}\right\} u_j^E = \min\{c_j, \alpha^{E,c} u_j^E\} = \mathrm{CREA}_j(E, c)$$

for all $j \in N$. This means that

$$\left(\min\left\{\frac{x_k}{u_k^E}, \frac{x_j}{u_j^E}\right\} u_j^E\right)_{j \in N} \in E.$$

Since f satisfies conditional full compensation, this implies that $f_k(E, x) = x_k$. Then $f_k(E, x) > f_k(E, c)$, which contradicts that f satisfies claim monotonicity. Hence, $f(E, c) = \mathrm{CREA}(E, c)$. □

Next, we generalize the characterization of Herrero and Villar (2002) and Yeh (2004) by showing that the constrained relative equal awards rule is the only rule satisfying conditional full compensation and composition down.

Theorem 3.5.3
The constrained relative equal awards rule is the unique bankruptcy rule satisfying conditional full compensation and composition down.

Proof. By Lemma 3.A.4 and Lemma 3.A.6, the constrained relative equal awards rule satisfies conditional full compensation and composition down. Let $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ be a bankruptcy rule satisfying conditional full compensation and composition down. Then f satisfies path monotonicity by Lemma 3.3.4. Then f satisfies path continuity by Lemma 3.2.1.

Let $(E, c) \in \mathrm{BR}^N$ be such that $E \neq \{0_N\}$. Let $i \in N$ be such that $\mathrm{CREA}_i(E, c) = c_i$. Then

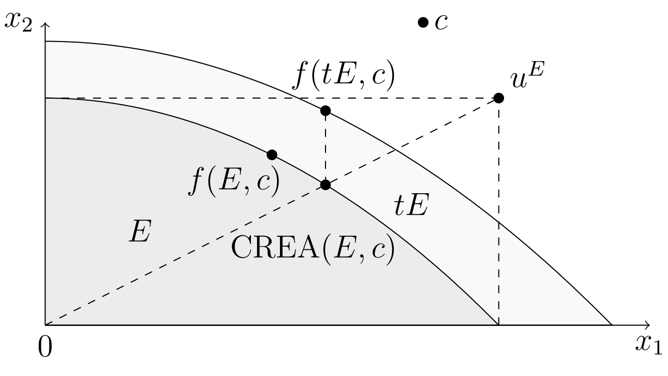
$$\min\left\{\frac{c_i}{u_i^E}, \frac{c_j}{u_j^E}\right\} u_j^E \leq \min\left\{\alpha^{E,c}, \frac{c_j}{u_j^E}\right\} u_j^E = \min\{c_j, \alpha^{E,c} u_j^E\} = \mathrm{CREA}_j(E, c)$$

for all $j \in N$. Since E is comprehensive, this means that

$$\left(\min\left\{\frac{c_i}{u_i^E}, \frac{c_j}{u_j^E}\right\} u_j^E\right)_{j \in N} \in E.$$

Since f satisfies conditional full compensation, this implies that $f_i(E, c) = c_i$.

Suppose that $f(E, c) \neq \mathrm{CREA}(E, c)$. Since E is nonleveled, there exists a $k \in N$ such that $f_k(E, c) < \mathrm{CREA}_k(E, c) = \alpha^{E,c} u_k^E < c_k$. Since f satisfies path monotonicity and path continuity, there exists a $t \in (1, \tau^{E,c})$ such that $f_k(tE, c) = \alpha^{E,c} u_k^E$.



Then

$$\begin{aligned}\min\left\{\frac{f_k(tE, c)}{u_k^E}, \frac{f_j(tE, c)}{u_j^E}\right\} u_j^E &\leq \min\left\{\alpha^{E,c}, \frac{c_j}{u_j^E}\right\} u_j^E \\ &= \min\{c_j, \alpha^{E,c} u_j^E\} \\ &= \mathrm{CREA}_j(E, c)\end{aligned}$$

for all $j \in N$.

Since E is comprehensive,

$$\left(\min\left\{\frac{f_k(tE,c)}{u_k^E},\frac{f_j(tE,c)}{u_j^E}\right\}u_j^E\right)_{j\in N}\in E.$$

Since f satisfies conditional full compensation, this means that $f_k(E, f(tE, c)) = f_k(tE, c)$. Since f satisfies composition down, this implies that $f_k(E, c) = f_k(tE, c)$. This is a contradiction. Hence, $f(E, c) = \text{CREA}(E, c)$. □

The serial mechanism for cost sharing problems is characterized by symmetry and a property which requires that individual cost shares are independent of the cost shares of agents with higher demands (cf. Moulin and Shenker (1992)). Interestingly, we can also formulate a fourth characterization of the constrained relative equal awards rule based on relative symmetry and independence of larger relative claims.

Definition (Independence of Larger Relative Claims)
A bankruptcy rule $f : \text{BR}^N \to \mathbb{R}^N_+$ satisfies *independence of larger relative claims* if $f_i(E, c) = f_i(E, c')$ for all $(E, c) \in \text{BR}^N$, any $i \in N$, and any $c' \in \mathbb{R}^N_+$ for which $c'_j \geq c_j$ and $c'_{N\setminus\{j\}} = c_{N\setminus\{j\}}$ for some $j \in N \setminus \{i\}$ for which $c_j u_i^E \geq c_i u_j^E$.

Theorem 3.5.4
The constrained relative equal awards rule is the unique bankruptcy rule satisfying relative symmetry and independence of larger relative claims.

Proof. By Lemma 3.A.1 and Lemma 3.A.7, the constrained relative equal awards rule satisfies relative symmetry and independence of larger relative claims. Let $f : \text{BR}^N \to \mathbb{R}^N_+$ be a bankruptcy rule satisfying relative symmetry and independence of larger relative claims.

Let $(E, c) \in \text{BR}^N$ be such that $E \neq \{0_N\}$. If $c \in E$, then $f(E, c) = c = \text{CREA}(E, c)$. Suppose that $c \notin E$. Denote $N = \{1, \ldots, n\}$ such that $\frac{c_1}{u_1^E} \leq \cdots \leq \frac{c_n}{u_n^E}$. Let $k \in N$ be such that $\text{CREA}_i(E, c) = c_i$ for all $i \in N$ for which $i < k$, and $\text{CREA}_i(E, c) = \alpha^{E,c} u_i^E < c_i$ for all $i \in N$ for which $i \geq k$. Then

$$f_i(E, c) = f_i(E, \text{CREA}(E, c)) = \text{CREA}_i(E, c)$$

for all $i \in N$ for which $i < k$, where the first equality follows from independence of larger relative claims. For any $i \in N$ for which $i \geq k$, let $x^i \in \mathbb{R}^N_+$ be defined by

$$x_j^i = \begin{cases} c_j & \text{for all } j \leq i; \\ \frac{c_i}{u_i^E} u_j^E & \text{for all } j > i. \end{cases}$$

Then

$$f_k(E, c) = f_k(E, x^k) = \alpha^{E,x^k} u_k^E = \text{CREA}_k(E, x^k) = \text{CREA}_k(E, c),$$

where the first and last equality follow from independence of larger relative claims, and the second and third equality follow from relative symmetry. Next, this argument can be applied to claimant $k+1$, and so on. Following this reasoning, $f_i(E, c) = \text{CREA}_i(E, c)$ for all $i \in N$ for which $i \geq k$. Hence, $f(E, c) = \text{CREA}(E, c)$. □

Finally, we explore duality in the context of the constrained relative equal awards rule. We introduce the constrained relative equal losses rule for NTU-bankruptcy problems which allocates losses relatively equal such that all claimants are allocated a nonnegative payoff and show that the constrained relative equal awards rule and the constrained relative equal losses rule are dual. By Lemma 3.3.3, Lemma 3.3.5, Lemma 3.A.1, Lemma 3.A.3, and Lemma 3.A.6, this means that the constrained relative equal losses rule also satisfies relative symmetry, composition down, and composition up.

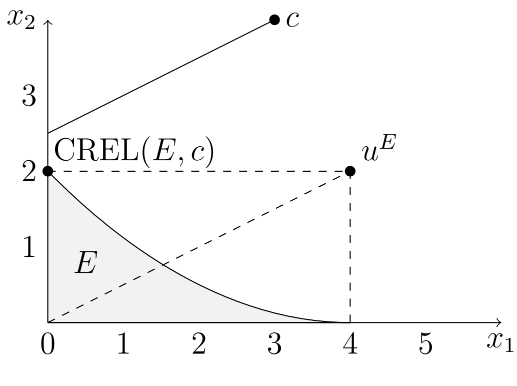
Definition (Constrained Relative Equal Losses Rule)
The *constrained relative equal losses rule* $\text{CREL} : \text{BR}^N \to \mathbb{R}_+^N$ is the bankruptcy rule which assigns to any $(E, c) \in \text{BR}^N$ for which $E \neq \{0_N\}$ the payoff allocation

$$\text{CREL}(E, c) = \left(\max\{0, c_i - \beta^{E,c} u_i^E\}\right)_{i \in N},$$

where $\beta^{E,c} = \min\{t \in \mathbb{R}_+ \mid (\max\{0, c_i - t u_i^E\})_{i \in N} \in \text{WP}(E)\}$.

Note that for any bankruptcy problem $(E, c) \in \text{BR}^N$ in which $E = \{x \in \mathbb{R}_+^N \mid \sum_{i \in N} x_i \leq e\}$, induced by a bankruptcy problem $(e, c) \in \text{TUBR}^N$, $u_i^E = e$ for all $i \in N$ and the constrained relative equal losses rule coincides with the standard constrained equal losses rule.

Example 3.10
Let $N = \{1, 2\}$ and consider the bankruptcy problem $(E, c) \in \text{BR}^N$ given by $E = \{x \in \mathbb{R}_+^N \mid \sqrt{x_1} + x_2 \leq 2\}$ and $c = (3, 4)$. Then $u^E = (4, 2)$, $\beta^{E,c} = 1$, and $\text{CREL}(E, c) = (0, 2)$. This is illustrated as follows.



△

Proposition 3.5.5
The constrained relative equal awards rule and the constrained relative equal losses rule are dual.

Proof. Let $(E, c) \in \mathrm{BR}^N$ be such that $E \neq \{0_N\}$. Denote $d = \tau^{E,c-\mathrm{CREL}(E,c)}$. Suppose that $d\alpha^{dE,c} \leq \beta^{E,c}$. Then

$$\begin{aligned}\mathrm{CREA}_i(dE, c) = \min\{c_i, \alpha^{dE,c} u_i^{dE}\} &= c_i - \max\{0, c_i - d\alpha^{dE,c} u_i^E\} \\ \leq c_i - \max\{0, c_i - \beta^{E,c} u_i^E\} &= c_i - \mathrm{CREL}_i(E, c)\end{aligned}$$

for all $i \in N$. Since E is nonleveled, this means that $\mathrm{CREA}(dE, c) = c - \mathrm{CREL}(E, c)$. Similarly, $d\alpha^{dE,c} \geq \beta^{E,c}$ leads to $\mathrm{CREA}(dE, c) = c - \mathrm{CREL}(E, c)$.

Denote $d' = \tau^{E,c-\mathrm{CREA}(E,c)}$. Suppose that $d'\beta^{d'E,c} \leq \alpha^{E,c}$. Then

$$\begin{aligned}\mathrm{CREL}_i(d'E, c) = \max\{0, c_i - \beta^{d'E,c} u_i^{d'E}\} &= c_i - \min\{c_i, d'\beta^{d'E,c} u_i^E\} \\ \geq c_i - \min\{c_i, \alpha^{E,c} u_i^E\} &= c_i - \mathrm{CREA}_i(E, c)\end{aligned}$$

for all $i \in N$. Since E is nonleveled, this means that $\mathrm{CREL}(d'E, c) = c - \mathrm{CREA}(E, c)$. Similarly, $d'\beta^{d'E,c} \geq \alpha^{E,c}$ leads to $\mathrm{CREL}(d'E, c) = c - \mathrm{CREA}(E, c)$. Hence, the constrained relative equal awards rule and the constrained relative equal losses rule are dual. □

Future research could search for axiomatic characterizations of the constrained relative equal losses rule for bankruptcy problems with nontransferable utility.

3.A Appendix

Lemma 3.A.1
The constrained relative equal awards rule satisfies relative symmetry.

Proof. Let $(E, c) \in \mathrm{BR}^N$ and let $i, j \in N$ be such that $c_i u_j^E = c_j u_i^E$. Then

$$\begin{aligned}\mathrm{CREA}_i(E, c) u_j^E &= \min\{c_i, \alpha^{E,c} u_i^E\} u_j^E \\ &= \min\{c_i u_j^E, \alpha^{E,c} u_i^E u_j^E\} \\ &= \min\{c_j u_i^E, \alpha^{E,c} u_j^E u_i^E\} \\ &= \min\{c_j, \alpha^{E,c} u_j^E\} u_i^E \\ &= \mathrm{CREA}_j(E, c) u_i^E.\end{aligned}$$

Hence, the constrained relative equal awards rule satisfies relative symmetry. □

Lemma 3.A.2
The constrained relative equal awards rule satisfies truncation invariance.

Proof. Let $(E, c) \in \mathrm{BR}^N$. Then

$$\begin{aligned}\mathrm{CREA}_i(E, \hat{c}^E) = \min\{\hat{c}_i^E, \alpha^{E,\hat{c}^E} u_i^E\} \quad &= \min\{\min\{c_i, u_i^E\}, \alpha^{E,\hat{c}^E} u_i^E\} \\ = \min\{c_i, u_i^E, \alpha^{E,\hat{c}^E} u_i^E\} &= \min\{c_i, \alpha^{E,\hat{c}^E} u_i^E\}\end{aligned}$$

for all $i \in N$. Since E is nonleveled, this means that $\mathrm{CREA}(E, c) = \mathrm{CREA}(E, \hat{c}^E)$. Hence, the constrained relative equal awards rule satisfies truncation invariance. □

Lemma 3.A.3
The constrained relative equal awards rule satisfies composition up.

Proof. Let $(E, c) \in \mathrm{BR}^N$ be such that $E \neq \{0_N\}$ and let $t \in [0, 1]$. If $t \in \{0, 1\}$, then

$$\mathrm{CREA}(E, c) = \mathrm{CREA}(tE, c) + \mathrm{CREA}(\tau^{E,\mathrm{CREA}(E,c)-\mathrm{CREA}(tE,c)} E, c - \mathrm{CREA}(tE, c)).$$

Suppose that $t \in (0, 1)$. Since $\mathrm{CREA}(tE, c) \in tE$, we have $(\min\{\frac{c_i}{t}, \alpha^{tE,c} u_i^E\})_{i \in N} \in E$. Since E is comprehensive, this means that $(\min\{c_i, t\alpha^{tE,c} u_i^E\})_{i \in N} \in E$. This implies that $\mathrm{CREA}(tE, c) \leq \mathrm{CREA}(E, c) \leq c$. Denote $d = \tau^{E,\mathrm{CREA}(E,c)-\mathrm{CREA}(tE,c)}$. Suppose that $d\alpha^{dE,c-\mathrm{CREA}(tE,c)} \leq \alpha^{E,c} - t\alpha^{tE,c}$. Then

$$\mathrm{CREA}_i(dE, c - \mathrm{CREA}(tE, c)) = 0 = c_i - c_i = \mathrm{CREA}_i(E, c) - \mathrm{CREA}_i(tE, c)$$

for all $i \in N$ for which $\mathrm{CREA}_i(tE, c) = c_i$. Moreover,

$$\begin{aligned}\mathrm{CREA}_i(dE, c - \mathrm{CREA}(tE, c)) &= \min\{c_i - \mathrm{CREA}_i(tE, c), \alpha^{dE,c-\mathrm{CREA}(tE,c)} u_i^{dE}\} \\ &= \min\{c_i - \alpha^{tE,c} u_i^{tE}, d\alpha^{dE,c-\mathrm{CREA}(tE,c)} u_i^E\} \\ &\leq \min\{c_i - \alpha^{tE,c} u_i^{tE}, \alpha^{E,c} u_i^E - t\alpha^{tE,c} u_i^E\} \\ &= \min\{c_i, \alpha^{E,c} u_i^E\} - \alpha^{tE,c} u_i^{tE} \\ &= \mathrm{CREA}_i(E, c) - \mathrm{CREA}_i(tE, c)\end{aligned}$$

for all $i \in N$ for which $\mathrm{CREA}_i(tE, c) = \alpha^{tE,c} u_i^{tE}$. Since E is nonleveled, this means that

$$\mathrm{CREA}(dE, c - \mathrm{CREA}(tE, c)) = \mathrm{CREA}(E, c) - \mathrm{CREA}(tE, c).$$

Similarly, $d\alpha^{dE,c-\mathrm{CREA}(tE,c)} \geq \alpha^{E,c} - t\alpha^{tE,c}$ leads to

$$\mathrm{CREA}(dE, c - \mathrm{CREA}(tE, c)) = \mathrm{CREA}(E, c) - \mathrm{CREA}(tE, c).$$

Hence, the constrained relative equal awards rule satisfies composition up. □

Lemma 3.A.4

The constrained relative equal awards rule satisfies conditional full compensation.

Proof. Let $(E, c) \in \mathrm{BR}^N$ be such that $E \neq \{0_N\}$ and let $i \in N$ be such that

$$\left(\min\left\{\frac{c_i}{u_i^E}, \frac{c_j}{u_j^E}\right\} u_j^E\right)_{j\in N} \in E.$$

Suppose that $\mathrm{CREA}_i(E, c) = \alpha^{E,c} u_i^E$. Then

$$\mathrm{CREA}_j(E, c) = \min\{c_j, \alpha^{E,c} u_j^E\} = \min\left\{\alpha^{E,c}, \frac{c_j}{u_j^E}\right\} u_j^E \leq \min\left\{\frac{c_i}{u_i^E}, \frac{c_j}{u_j^E}\right\} u_j^E$$

for all $j \in N$. Since E is nonleveled, this means that $\mathrm{CREA}_i(E, c) = c_i$. Hence, the constrained relative equal awards rule satisfies conditional full compensation. □

Lemma 3.A.5

The constrained relative equal awards rule satisfies claim monotonicity.

Proof. Let $(E, c) \in \mathrm{BR}^N$, let $i \in N$, and let $c' \in \mathbb{R}_+^N$ be such that $c'_i \geq c_i$ and $c'_{N\setminus\{i\}} = c_{N\setminus\{i\}}$. If $\alpha^{E,c'} \geq \alpha^{E,c}$, then

$$\mathrm{CREA}_i(E, c') = \min\{c'_i, \alpha^{E,c'} u_i^E\} \geq \min\{c_i, \alpha^{E,c} u_i^E\} = \mathrm{CREA}_i(E, c).$$

Suppose that $\alpha^{E,c'} \leq \alpha^{E,c}$. For all $j \in N \setminus \{i\}$,

$$\mathrm{CREA}_j(E, c') = \min\{c'_j, \alpha^{E,c'} u_j^E\} \leq \min\{c_j, \alpha^{E,c} u_j^E\} = \mathrm{CREA}_j(E, c).$$

Since E is nonleveled, this means that $\mathrm{CREA}_i(E, c') \geq \mathrm{CREA}_i(E, c)$. Hence, the constrained relative equal awards rule satisfies claim monotonicity. □

Lemma 3.A.6

The constrained relative equal awards rule satisfies composition down.

Proof. Let $(E, c) \in \mathrm{BR}^N$ be such that $E \neq \{0_N\}$ and let $t \in [0, 1]$. If $t \in \{0, 1\}$, then $\mathrm{CREA}(tE, c) = \mathrm{CREA}(tE, \mathrm{CREA}(E, c))$. Suppose that $t \in (0, 1)$. Since $\mathrm{CREA}(tE, c) \in tE$, we have $(\min\{\frac{c_i}{t}, \alpha^{tE,c} u_i^E\})_{i\in N} \in E$. Since E is comprehensive, this means that $(\min\{c_i, t\alpha^{tE,c} u_i^E\})_{i\in N} \in E$. This implies that $\mathrm{CREA}(tE, c) \leq \mathrm{CREA}(E, c) \leq c$. Suppose that $\alpha^{tE,\mathrm{CREA}(E,c)} \leq \alpha^{tE,c}$. Then

$$\begin{aligned}
\mathrm{CREA}_i(tE, \mathrm{CREA}(E, c)) &= \min\{\mathrm{CREA}_i(E, c), \alpha^{tE,\mathrm{CREA}(E,c)} u_i^{tE}\} \\
&\leq \min\{\min\{c_i, \alpha^{E,c} u_i^E\}, \alpha^{tE,c} u_i^{tE}\} \\
&= \min\{c_i, \alpha^{E,c} u_i^E, \alpha^{tE,c} u_i^{tE}\} \\
&= \min\{\mathrm{CREA}_i(E, c), \mathrm{CREA}_i(tE, c)\} \\
&= \mathrm{CREA}_i(tE, c)
\end{aligned}$$

for all $i \in N$.

This means that $\text{CREA}(tE, \text{CREA}(E, c)) \leq \text{CREA}(tE, c)$. Since E is nonleveled, this implies that $\text{CREA}(tE, \text{CREA}(E, c)) = \text{CREA}(tE, c)$. Similarly, $\alpha^{tE,\text{CREA}(E,c)} \geq \alpha^{tE,c}$ leads to $\text{CREA}(tE, \text{CREA}(E, c)) = \text{CREA}(tE, c)$. Hence, the constrained relative equal awards rule satisfies composition down. □

Lemma 3.A.7
The constrained relative equal awards rule satisfies independence of larger relative claims.

Proof. Let $(E, c) \in \text{BR}^N$, let $i \in N$, and let $c' \in \mathbb{R}^N_+$ be such that $c'_j \geq c_j$ and $c'_{N\setminus\{j\}} = c_{N\setminus\{j\}}$ for some $j \in N \setminus \{i\}$ for which $c_j u^E_i \geq c_i u^E_j$. If $\alpha^{E,c'} \geq \alpha^{E,c}$, then

$$\text{CREA}_k(E, c') = \min\{c'_k, \alpha^{E,c'} u^E_k\} \geq \min\{c_k, \alpha^{E,c} u^E_k\} = \text{CREA}_k(E, c)$$

for all $k \in N$, which means that $\text{CREA}(E, c') = \text{CREA}(E, c)$ since E is nonleveled. Suppose that $\alpha^{E,c'} < \alpha^{E,c}$. Then

$$\text{CREA}_k(E, c') = \min\{c'_k, \alpha^{E,c'} u^E_k\} \leq \min\{c_k, \alpha^{E,c} u^E_k\} = \text{CREA}_k(E, c)$$

for all $k \in N \setminus \{j\}$. Since E is nonleveled, this means that $\text{CREA}_j(E, c') \geq \text{CREA}_j(E, c)$. Moreover, if $\text{CREA}_j(E, c') = \text{CREA}_j(E, c)$, then $\text{CREA}(E, c') = \text{CREA}(E, c)$. Suppose that $\text{CREA}_j(E, c') > \text{CREA}_j(E, c)$. Then $\min\{c'_j, \alpha^{E,c'} u^E_j\} > \min\{c_j, \alpha^{E,c} u^E_j\}$. Since $\alpha^{E,c'} < \alpha^{E,c}$, this means that $\frac{c_i}{u^E_i} \leq \frac{c_j}{u^E_j} < \alpha^{E,c'} < \alpha^{E,c}$. Then

$$\begin{aligned} \text{CREA}_i(E, c') = \min\{c'_i, \alpha^{E,c'} u^E_i\} \quad &= \min\left\{\frac{c_i}{u^E_i}, \alpha^{E,c'}\right\} u^E_i \\ = \min\left\{\frac{c_i}{u^E_i}, \alpha^{E,c}\right\} u^E_i = \min\{c_i, \alpha^{E,c} u^E_i\} \quad &= \text{CREA}_i(E, c). \end{aligned}$$

Hence, the constrained relative equal awards rule satisfies independence of larger relative claims. □

4 Consistency and the Relative Adjustment Principle

4.1 Introduction

The proportional rule, the constrained equal awards rule, and the constrained equal losses rule can be considered as the three basic bankruptcy rules for bankruptcy problems with transferable utility. Herrero and Villar (2001) called these bankruptcy rules the three musketeers. Similarly, we interpret the proportional rule, the constrained relative equal awards rule, and the constrained relative equal losses rule as the three basic bankruptcy rules for bankruptcy problems with nontransferable utility. Another well-studied rule for bankruptcy problems with transferable utility, which according to Herrero and Villar (2001) plays the role of D'Artagnan, is the so-called Talmud rule. Aumann and Maschler (1985) showed that the Talmud rule is the unique TU-bankruptcy rule satisfying consistency and the contested garment principle. This chapter, based on Dietzenbacher, Borm, and Estévez-Fernández (2017), studies generalizations of these two concepts to bankruptcy problems with nontransferable utility on which a generalized Talmud rule can be based in future research.

Following Thomson (2011), the consistency principle can be stated as follows. Consider a bankruptcy problem and the corresponding payoff allocation assigned by a particular bankruptcy rule. Suppose that some claimants depart with their allocated payoffs and that the remaining claimants reevaluate their allocated payoffs. The bankruptcy rule is called consistent if it prescribes for this reduced problem the same payoffs for the involved claimants. The design of these reduced problems for NTU-bankruptcy problems is however not straightforward, and different modeling choices have different consequences.

We examine the relation of the proportional rule, the constrained relative equal awards rule, and the constrained relative equal losses rule with several consistency notions. The proportional rule satisfies a multilateral consistency notion which converts reduced problems into new bankruptcy problems for the remaining claimants. This result is used to derive new axiomatic characterizations from those in the previous chapter using an elevator lemma. The constrained relative equal awards rule and the constrained relative equal losses rule do not satisfy multilateral consistency, but they do satisfy consistency on a restricted domain which includes NTU-bankruptcy problems induced by TU-bankruptcy problems. Inspired by Young (1987), we also introduce a class of parametric bankruptcy rules which contains the three basic bankruptcy rules, and we show that all parametric bankruptcy rules satisfy a consistency notion which interprets the reduced problem as the original bankruptcy problem where the departing claimants leave a footprint behind.

The contested garment principle for TU-bankruptcy rules describes a standard solution for bankruptcy problems with two claimants where they first concede the minimal rights to each other and subsequently divide the remaining estate equally. To adequately generalize this two-claimant solution to the relative adjustment principle for NTU-bankruptcy rules, we study minimal rights in NTU-bankruptcy problems. The minimal rights first property requires that first allocating minimal rights, the maximal individual payoffs within the estate when all other claimants are allocated their claims, and subsequently applying the bankruptcy rule to the remaining bankruptcy problem, leads to the same payoff allocation as application of the bankruptcy rule to the original bankruptcy problem.

The three basic bankruptcy rules do not satisfy minimal rights first. Inspired by Thomson and Yeh (2008), we introduce the truncation operator and minimal rights operator which 'force' bankruptcy rules to satisfy truncation invariance and minimal rights first, respectively. The new bankruptcy rules obtained by applying both operators to existing ones form the class of adjusted bankruptcy rules. All adjusted counterparts of bankruptcy rules which satisfy relative symmetry coincide on the class of bankruptcy problems with two claimants. This is called the relative adjustment principle for NTU-bankruptcy rules which generalizes the contested garment principle for TU-bankruptcy rules. The new principle merges the properties truncation invariance, minimal rights first, and a restricted form of relative symmetry.

This chapter is organized in the following way. Section 4.2 discusses several consistency notions and introduces the class of parametric bankruptcy rules. Section 4.3 introduces the class of adjusted bankruptcy rules and studies the relative adjustment principle.

4.2 Consistency

Consistency requires that application of a bankruptcy rule to a reduced problem leads to the same payoffs for the involved claimants as within the original bankruptcy problem. For TU-bankruptcy problems, the estate of such a reduced problem can simply be defined as the original estate subtracted with the allocated payoffs to the departing claimants (cf. Aumann and Maschler (1985)). For NTU-bankruptcy problems, the design of such a reduced problem is not straightforward. We discuss several ways to generalize the consistency property for TU-bankruptcy rules.

A natural option is to convert the reduced problem into a new problem for the remaining claimants in which the estate is defined as the part of the original estate where all departing claimants are allocated their corresponding payoffs. For this, we need to extend the domain of bankruptcy rules to bankruptcy problems for any nonempty subset of claimants. Formally, let $\overline{\text{BR}}^N$ denote $\bigcup_{S \in 2^N \setminus \{\emptyset\}} \text{BR}^S$. A bankruptcy rule f on $\overline{\text{BR}}^N$ assigns to any $(E, c) \in \text{BR}^S$ for which $S \in 2^N \setminus \{\emptyset\}$ a payoff allocation $f(E, c) \in \text{WP}(E)$ for which $f(E, c) \leq c$.

Let $(E, c) \in \text{BR}^N$, let $x \in \mathbb{R}^N_+$, and let $S \in 2^N \setminus \{\emptyset\}$. The set of payoff allocations $E^x_S \subseteq \mathbb{R}^S_+$ is defined by

$$E^x_S = \left\{ y \in \mathbb{R}^S_+ \;\middle|\; (y, x_{N \setminus S}) \in E \right\}.$$

Note that $(E^{f(E,c)}_S, c_S) \in \text{BR}^S$ for any bankruptcy rule f on $\overline{\text{BR}}^N$.

A rule is multilaterally consistent if it assigns to all reduced problems the same payoffs for the remaining claimants as within the original problem.

Definition (Multilateral Consistency)
A bankruptcy rule f on $\overline{\text{BR}}^N$ satisfies *multilateral consistency* if

$$f_S(E, c) = f(E^{f(E,c)}_S, c_S)$$

for all $(E, c) \in \text{BR}^N$ and any $S \in 2^N \setminus \{\emptyset\}$.

The corresponding weaker property which only considers reduced problems for two remaining claimants is called bilateral consistency.

Definition (Bilateral Consistency)
A bankruptcy rule f on $\overline{\text{BR}}^N$ satisfies *bilateral consistency* if

$$f_S(E, c) = f(E^{f(E,c)}_S, c_S)$$

for all $(E, c) \in \text{BR}^N$ and any $S \in 2^N$ for which $|S| = 2$.

In other words, a rule is bilaterally consistent if it assigns to all reduced problems with two claimants the same payoffs for the remaining claimants as within the original problem. This principle can also be applied in reverse direction. Consider a problem and a corresponding feasible payoff allocation. Suppose that for all reduced problems with two claimants a rule prescribes the corresponding payoffs within this allocation. Then the rule is called conversely consistent (cf. Thomson (2011)) if it assigns this payoff allocation to the original problem.

Definition (Converse Consistency)
A bankruptcy rule f on $\overline{\text{BR}}^N$ satisfies *converse consistency* if $f(E,c) = x$ for all $(E,c) \in \text{BR}^N$ and any $x \in \text{WP}(E)$ for which $x \leq c$ and $x_S = f(E_S^x, c_S)$ for all $S \in 2^N$ for which $|S| = 2$.

If a bilateral consistent rule coincides with a conversely consistent rule on the class of two-claimant problems, then the rules coincide for any problem. This type of result is known as an elevator lemma (cf. Thomson (2011)).

Lemma 4.2.1 (Elevator Lemma)
Let f and g be two bankruptcy rules on $\overline{\text{BR}}^N$. If f satisfies bilateral consistency, g satisfies converse consistency, and $f(E,c) = g(E,c)$ for all $(E,c) \in \text{BR}^S$ for which $S \in 2^N$ and $|S| = 2$, then $f = g$.

Proof. Let $(E,c) \in \text{BR}^N$ and let $x = f(E,c)$. Since f satisfies bilateral consistency, $x_S = f(E_S^x, c_S)$ for all $S \in 2^N$ for which $|S| = 2$. This means that $x_S = g(E_S^x, c_S)$ for all $S \in 2^N$ for which $|S| = 2$. Since g satisfies converse consistency, this implies that $g(E,c) = x$. Hence, $f(E,c) = g(E,c)$. □

For a rule which satisfies both bilateral consistency and converse consistency, the Elevator Lemma can be used to extend axiomatic characterizations from problems with two claimants to problems with any number of claimants.

Theorem 4.2.2
Consider a bankruptcy rule satisfying bilateral consistency and converse consistency. Any axiomatic characterization for bankruptcy problems with two claimants yields an axiomatic characterization for bankruptcy problems with any number of claimants if bilateral consistency or converse consistency is required in addition.

Proof. Let f be a bankruptcy rule on $\overline{\text{BR}}^N$ satisfying bilateral consistency and converse consistency. Let g be a bankruptcy rule on $\overline{\text{BR}}^N$ satisfying the properties in the axiomatic characterization of f on the class of two-claimant bankruptcy problems and either bilateral consistency or converse consistency. Then $g(E,c) = f(E,c)$ for all $(E,c) \in \text{BR}^S$ for which $S \in 2^N$ and $|S| = 2$. Since f satisfies bilateral consistency and converse consistency, $g = f$ by Lemma 4.2.1. □

An example of a rule satisfying both bilateral consistency and converse consistency is the proportional rule.

Lemma 4.2.3
The proportional rule satisfies multilateral consistency.

Proof. Let $(E, c) \in \mathrm{BR}^N$ and let $S \in 2^N \setminus \{\emptyset\}$. Then $\mathrm{Prop}_S(E, c) = \lambda^{E,c} c_S$ and

$$\mathrm{Prop}(E_S^{\mathrm{Prop}(E,c)}, c_S) = \lambda^{E_S^{\mathrm{Prop}(E,c)}, c_S} c_S,$$

where $\lambda^{E,c} \in [0, 1]$ is such that $\lambda^{E,c} c \in \mathrm{WP}(E)$ and $\lambda^{E_S^{\mathrm{Prop}(E,c)}, c_S} \in [0, 1]$ is such that

$$\lambda^{E_S^{\mathrm{Prop}(E,c)}, c_S} c_S \in \mathrm{WP}(E_S^{\mathrm{Prop}(E,c)}).$$

Since E is comprehensive, this means that

$$\left(\lambda^{E_S^{\mathrm{Prop}(E,c)}, c_S} c_S, \lambda^{E,c} c_{N \setminus S}\right) \in \mathrm{WP}(E).$$

Since E is nonleveled, this implies that $\mathrm{Prop}_S(E, c) = \mathrm{Prop}(E_S^{\mathrm{Prop}(E,c)}, c_S)$. Hence, the proportional rule satisfies multilateral consistency. □

Lemma 4.2.4
The proportional rule satisfies converse consistency.

Proof. Let $(E, c) \in \mathrm{BR}^N$ and let $x \in \mathrm{WP}(E)$ be such that $x \leq c$ and $x_S = \mathrm{Prop}(E_S^x, c_S)$ for all $S \in 2^N$ for which $|S| = 2$. Then $\mathrm{Prop}(E, c) = \lambda^{E,c} c$ and $x_S = \lambda^{E_S^x, c_S} c_S$ for all $S \in 2^N$ for which $|S| = 2$. This means that $x = tc$ for some $t \in [0, 1]$. Since E is nonleveled, this implies that $\mathrm{Prop}(E, c) = x$. Hence, the proportional rule satisfies converse consistency. □

From Theorem 3.4.1, Theorem 3.4.2, Theorem 4.2.2, Lemma 4.2.3, and Lemma 4.2.4, we derive the following corollary.

Corollary 4.2.5

(i) The proportional rule is the unique bankruptcy rule satisfying self-duality and composition down on the class of bankruptcy problems with two claimants, and bilateral consistency or converse consistency.

(ii) The proportional rule is the unique bankruptcy rule satisfying self-duality and composition up on the class of bankruptcy problems with two claimants, and bilateral consistency or converse consistency.

(iii) The proportional rule is the unique bankruptcy rule satisfying path linearity on the class of bankruptcy problems with two claimants, and bilateral consistency or converse consistency.

To show that the proportional rule is not the only rule satisfying multilateral consistency, we introduce the constrained equal awards rule and the constrained equal losses rule for bankruptcy problems with nontransferable utility. The *constrained equal awards rule* $\text{CEA} : \text{BR}^N \to \mathbb{R}^N_+$ is the bankruptcy rule which assigns to any $(E, c) \in \text{BR}^N$ the payoff allocation

$$\text{CEA}(E, c) = (\min\{c_i, a\})_{i \in N} ,$$

where $a \in \mathbb{R}_+$ is such that $\text{CEA}(E, c) \in \text{WP}(E)$. The *constrained equal losses rule* $\text{CEL} : \text{BR}^N \to \mathbb{R}^N_+$ is the bankruptcy rule which assigns to any $(E, c) \in \text{BR}^N$ the payoff allocation

$$\text{CEL}(E, c) = (\max\{c_i - b, 0\})_{i \in N} ,$$

where $b \in \mathbb{R}_+$ is such that $\text{CEL}(E, c) \in \text{WP}(E)$. Where the constrained *relative* equal awards rule and the constrained *relative* equal losses rule aim to allocate payoffs and losses relatively equal among the claimants, respectively, the constrained equal awards rule and the constrained equal losses rule aim to allocate payoffs and losses absolutely equal among the claimants, respectively.

The constrained equal awards rule and the constrained equal losses rule satisfy multilateral consistency. However, the following example shows that the constrained relative equal awards rule and the constrained relative equal losses rule do not satisfy multilateral consistency.

Example 4.1
Let $N = \{1, 2, 3\}$ and consider the bankruptcy problem $(E, c) \in \text{BR}^N$ given by $E = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 2x_2 + x_3^2 \leq 4\}$ and $c = (2, 2, 2)$. Then $u^E = (2, 2, 2)$ and

$$\text{Prop}(E, c) = \text{CREA}(E, c) = \text{CREL}(E, c) = (1, 1, 1).$$

This means that

$$E_{\{1,2\}}^{\text{Prop}(E,c)} = E_{\{1,2\}}^{\text{CREA}(E,c)} = E_{\{1,2\}}^{\text{CREL}(E,c)} = \left\{x \in \mathbb{R}^{\{1,2\}}_+ \,\middle|\, x_1^2 + 2x_2 \leq 3\right\},$$

which implies that

$$\begin{aligned} \text{Prop}(E_{\{1,2\}}^{\text{Prop}(E,c)}, c_{\{1,2\}}) &= (1, 1, \cdot) , \\ \text{CREA}(E_{\{1,2\}}^{\text{CREA}(E,c)}, c_{\{1,2\}}) &= \left(\tfrac{1}{2}\sqrt{15} - \tfrac{1}{2}\sqrt{3}, \tfrac{3}{4}\sqrt{5} - \tfrac{3}{4}, \cdot\right) , \\ \text{and } \text{CREL}(E_{\{1,2\}}^{\text{CREL}(E,c)}, c_{\{1,2\}}) &= \left(\tfrac{1}{6}\sqrt{72\sqrt{3} - 9} - \tfrac{1}{2}\sqrt{3}, \tfrac{1}{4}\sqrt{24\sqrt{3} - 3} - \sqrt{3} + \tfrac{5}{4}, \cdot\right) . \end{aligned}$$

Hence, the constrained relative equal awards rule and the constrained relative equal losses rule do not satisfy multilateral consistency.

However,

$$E^{\text{Prop}(E,c)}_{\{1,3\}} = E^{\text{CREA}(E,c)}_{\{1,3\}} = E^{\text{CREL}(E,c)}_{\{1,3\}} = \left\{x \in \mathbb{R}^{\{1,3\}}_{+} \;\middle|\; x_1^2 + x_3^2 \leq 2\right\}$$

and

$$\begin{aligned} \text{Prop}(E^{\text{Prop}(E,c)}_{\{1,3\}}, c_{\{1,3\}}) &= \text{CREA}(E^{\text{CREA}(E,c)}_{\{1,3\}}, c_{\{1,3\}}) \\ &= \text{CREL}(E^{\text{CREL}(E,c)}_{\{1,3\}}, c_{\{1,3\}}) \\ &= (1, \cdot, 1)\,. \end{aligned}$$

△

As illustrated in Example 4.1, the constrained relative equal awards rule and the constrained relative equal losses rule do satisfy consistency on the restricted domain of reduced problems for which the ratio of utopia values is equal to the ratio of utopia values in the original problem. We introduce the restricted consistency property to describe these type of bankruptcy rules. Peters, Tijs, and Zarzuelo (1994) introduced a similar property for bargaining solutions.

Definition (Restricted Consistency)
A bankruptcy rule f on $\overline{\text{BR}}^N$ satisfies *restricted consistency* if

$$f_S(E,c) = f(E_S^{f(E,c)}, c_S)$$

for all $(E,c) \in \text{BR}^N$ and any $S \in 2^N \setminus \{\emptyset\}$ for which $u^{E_S^{f(E,c)}} = tu_S^E$ for some $t \in [0,1]$.

Note that both multilateral consistency and restricted consistency generalize the consistency notion for TU-bankruptcy rules.

Proposition 4.2.6
The constrained relative equal awards rule satisfies restricted consistency.

Proof. Let $(E,c) \in \text{BR}^N$ and let $S \in 2^N \setminus \{\emptyset\}$ be such that $u^{E_S^{\text{CREA}(E,c)}} = tu_S^E$ for some $t \in [0,1]$. Then $\text{CREA}_i(E,c) = \min\{c_i, \alpha^{E,c}u_i^E\}$ for all $i \in S$ and

$$\begin{aligned} \text{CREA}(E_S^{\text{CREA}(E,c)}, c_S) &= (\min\{c_i, \alpha^{E_S^{\text{CREA}(E,c)},c_S} u_i^{E_S^{\text{CREA}(E,c)}}\})_{i\in S} \\ &= (\min\{c_i, t\alpha^{E_S^{\text{CREA}(E,c)},c_S} u_i^E\})_{i\in S}, \end{aligned}$$

where $\alpha^{E,c} \in [0,1]$ is such that $\text{CREA}(E,c) \in \text{WP}(E)$ and $\alpha^{E_S^{\text{CREA}(E,c)},c_S} \in [0,1]$ is such that

$$\text{CREA}(E_S^{\text{CREA}(E,c)}, c_S) \in \text{WP}(E_S^{\text{CREA}(E,c)}).$$

Since E is comprehensive, this means that

$$\left(\text{CREA}(E_S^{\text{CREA}(E,c)}, c_S), \text{CREA}_{N\setminus S}(E,c)\right) \in \text{WP}(E).$$

Since E is nonleveled, this implies that $\text{CREA}_S(E,c) = \text{CREA}(E_S^{\text{CREA}(E,c)}, c_S)$.
Hence, the constrained relative equal awards rule satisfies restricted consistency. □

Proposition 4.2.7
The constrained relative equal losses rule satisfies restricted consistency.

Proof. Let $(E,c) \in \mathrm{BR}^N$ and let $S \in 2^N \setminus \{\emptyset\}$ be such that $u^{E_S^{\mathrm{CREL}(E,c)}} = tu_S^E$ for some $t \in [0,1]$. If $E = \{0_N\}$, then $\mathrm{CREL}_S(E,c) = \mathrm{CREL}(E_S^{\mathrm{CREL}(E,c)}, c_S)$. Suppose that $E \neq \{0_N\}$. Then $\mathrm{CREL}_i(E,c) = \max\{0, c_i - \beta^{E,c} u_i^E\}$ for all $i \in S$ and

$$\begin{aligned}\mathrm{CREL}(E_S^{\mathrm{CREL}(E,c)}, c_S) &= (\max\{0, c_i - \beta^{E_S^{\mathrm{CREL}(E,c)}, c_S} u_i^{E_S^{\mathrm{CREL}(E,c)}}\})_{i \in S} \\ &= (\max\{0, c_i - t\beta^{E_S^{\mathrm{CREL}(E,c)}, c_S} u_i^E\})_{i \in S},\end{aligned}$$

where $\beta^{E,c} \in \mathbb{R}_+$ is such that $\mathrm{CREL}(E,c) \in \mathrm{WP}(E)$ and $\beta^{E_S^{\mathrm{CREL}(E,c)}, c_S} \in \mathbb{R}_+$ is such that

$$\mathrm{CREL}(E_S^{\mathrm{CREL}(E,c)}, c_S) \in \mathrm{WP}(E_S^{\mathrm{CREL}(E,c)}).$$

Since E is comprehensive, this means that

$$\Big(\mathrm{CREL}(E_S^{\mathrm{CREL}(E,c)}, c_S), \mathrm{CREL}_{N \setminus S}(E,c)\Big) \in \mathrm{WP}(E).$$

Since E is nonleveled, this implies that $\mathrm{CREL}_S(E,c) = \mathrm{CREL}(E_S^{\mathrm{CREL}(E,c)}, c_S)$. Hence, the constrained relative equal losses rule satisfies restricted consistency. □

Converting reduced problems into new problems for the remaining claimants by projecting on the corresponding lower dimensional space tends to lose characteristics of the original problems. Instead, one could also interpret the reduced problem as the original problem where the payoffs of the departing claimants are fixed. In a sense, the original problem has already been solved for the departing claimants and they leave a footprint behind. To formalize this approach, we first need to redefine bankruptcy rules on the domain of footprint bankruptcy problems.

A *footprint bankruptcy problem* is a quintuple (N, E, c, x, S) in which $(E,c) \in \mathrm{BR}^N$ is a bankruptcy problem, $x \in \mathbb{R}_+^N$ is a vector of footprints, and $S \in 2^N \setminus \{\emptyset\}$ is the set of remaining claimants such that $(E_S^x, c_S) \in \mathrm{BR}^S$. Let FBR^N denote the class of footprint bankruptcy problems with claimant set N. For convenience, a footprint bankruptcy problem on N is denoted by $(E,c,x,S) \in \mathrm{FBR}^N$, and $(E,c,x,N) \in \mathrm{FBR}^N$ is abbreviated to $(E,c) \in \mathrm{FBR}^N$.

A bankruptcy rule f on FBR^N assigns to any footprint bankruptcy problem $(E,c,x,S) \in \mathrm{FBR}^N$ a payoff allocation $f(E,c,x,S) \in \mathrm{WP}(E)$ for which

$$f_S(E,c,x,S) \leq c_S \text{ and } f_{N \setminus S}(E,c,x,S) = x_{N \setminus S}.$$

Note that $(E,c,f(E,c),S) \in \mathrm{FBR}^N$ for all $(E,c) \in \mathrm{BR}^N$, any $S \in 2^N \setminus \{\emptyset\}$, and any bankruptcy rule f on FBR^N.

The proportional rule Prop on FBR^N is the bankruptcy rule which assigns to any footprint bankruptcy problem $(E, c, x, S) \in \mathrm{FBR}^N$ the payoff allocation for which

$$\mathrm{Prop}_S(E, c, x, S) = \lambda^{E,c,x,S} c_S,$$

where $\lambda^{E,c,x,S} = \max\{t \in [0, 1] \mid (tc_S, x_{N\setminus S}) \in \mathrm{WP}(E)\}$.

The constrained relative equal awards rule CREA on FBR^N is the bankruptcy rule which assigns to any footprint bankruptcy problem $(E, c, x, S) \in \mathrm{FBR}^N$ the payoff allocation for which

$$\mathrm{CREA}_S(E, c, x, S) = \left(\min\{c_i, \alpha^{E,c,x,S} u_i^E\}\right)_{i\in S},$$

where $\alpha^{E,c,x,S} = \max\{t \in [0, 1] \mid ((\min\{c_i, tu_i^E\})_{i\in S}, x_{N\setminus S}) \in \mathrm{WP}(E)\}$.

The constrained relative equal losses rule CREL on FBR^N is the bankruptcy rule which assigns to any footprint bankruptcy problem $(E, c, x, S) \in \mathrm{FBR}^N$ for which $E \neq \{0_N\}$ the payoff allocation for which

$$\mathrm{CREL}_S(E, c, x, S) = \left(\max\{0, c_i - \beta^{E,c,x,S} u_i^E\}\right)_{i\in S},$$

where $\beta^{E,c,x,S} = \min\{t \in \mathbb{R}_+ \mid ((\max\{0, c_i - tu_i^E\})_{i\in S}, x_{N\setminus S}) \in \mathrm{WP}(E)\}$.

We now introduce the footprint consistency property to describe rules which prescribe the same payoff allocation for the original problem as for any footprint bankruptcy problem in which the departing claimants fix their allocated payoffs.

Definition (Footprint Consistency)
A bankruptcy rule f on FBR^N satisfies *footprint consistency* if

$$f(E, c) = f(E, c, f(E, c), S)$$

for all $(E, c) \in \mathrm{BR}^N$ and any $S \in 2^N \setminus \{\emptyset\}$.

Inspired by Young (1987), we introduce the class of parametric rules for which the payoff allocated to a claimant only depends on individual characteristics within the bankruptcy problem and a common parameter.

Definition (Parametric Bankruptcy Rule)
A bankruptcy rule f on FBR^N is *parametric* if there exists a function $r^f : \mathbb{R}^3_+ \to \mathbb{R}_+$, monotonic in its third argument, such that $f_S(E, c, x, S) = (r^f(c_i, u_i^E, \theta^{E,c,x,S}))_{i\in S}$ for all $(E, c, x, S) \in \mathrm{FBR}^N$ and some parameter $\theta^{E,c,x,S} \in \mathbb{R}_+$.

Theorem 4.2.8
All parametric bankruptcy rules satisfy footprint consistency.

Proof. Let f be a parametric bankruptcy rule on FBR^N, let $(E, c) \in \mathrm{BR}^N$, and let $S \in 2^N \setminus \{\emptyset\}$. Then $f_{N\setminus S}(E, c) = f_{N\setminus S}(E, c, f(E, c), S)$. Moreover, $f_i(E, c) = r^f(c_i, u_i^E, \theta^{E,c})$ and $f_i(E, c, f(E, c), S) = r^f(c_i, u_i^E, \theta^{E,c,f(E,c),S})$ for all $i \in S$. Since r^f is monotonic in its third argument, this means that $f_S(E, c) \leq f_S(E, c, f(E, c), S)$ or $f_S(E, c) \geq f_S(E, c, f(E, c), S)$. Since E is nonleveled, this implies that $f(E, c) = f(E, c, f(E, c), S)$. Hence, f satisfies footprint consistency. □

Specific examples of parametric rules are the proportional rule, the constrained relative equal awards rule, and the constrained relative equal losses rule. We have

$$\begin{aligned}
r^{\mathrm{Prop}}(c_i, u_i^E, \theta^{E,c,x,S}) &= \theta^{E,c,\mathrm{Prop}(E,c),S} c_i, & \theta^{E,c,\mathrm{Prop}(E,c),S} &= \lambda^{E,c,x,S},\\
r^{\mathrm{CREA}}(c_i, u_i^E, \theta^{E,c,x,S}) &= \min\{c_i, \theta^{E,c,\mathrm{CREA}(E,c),S} u_i^E\}, & \theta^{E,c,\mathrm{CREA}(E,c),S} &= \alpha^{E,c,x,S},\\
r^{\mathrm{CREL}}(c_i, u_i^E, \theta^{E,c,x,S}) &= \max\{0, c_i - \theta^{E,c,\mathrm{CREL}(E,c),S} u_i^E\} \text{ and } & \theta^{E,c,\mathrm{CREL}(E,c),S} &= \beta^{E,c,x,S}
\end{aligned}$$

for all $(E, c, x, S) \in \mathrm{FBR}^N$ and any $i \in S$.

Corollary 4.2.9
The proportional rule, the constrained relative equal awards rule, and the constrained relative equal losses rule satisfy footprint consistency.

Example 4.2
Let $N = \{1, 2, 3\}$ and consider the bankruptcy problem $(E, c) \in \mathrm{BR}^N$ given by $E = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 2x_2 + x_3^2 \leq 4\}$ and $c = (2, 2, 2)$ as in Example 4.1. Then

$$\begin{aligned}
\mathrm{Prop}_{\{1,2\}}(E, c, \mathrm{Prop}(E, c), \{1, 2\}) &= \mathrm{CREA}_{\{1,2\}}(E, c, \mathrm{CREA}(E, c), \{1, 2\})\\
&= \mathrm{CREL}_{\{1,2\}}(E, c, \mathrm{CREL}(E, c), \{1, 2\})\\
&= (1, 1, \cdot).
\end{aligned}$$

△

4.3 The relative adjustment principle

The contested garment principle for TU-bankruptcy rules (cf. Aumann and Maschler (1985)) describes a standard solution for bankruptcy problems with two claimants where they first concede the minimal rights to each other and subsequently divide the remaining estate equally. To adequately generalize this two-claimant solution to the relative adjustment principle for NTU-bankruptcy rules, we first study minimal rights in NTU-bankruptcy problems.

The minimal right of a claimant in a TU-bankruptcy problem is defined as the remaining part of the estate when all other claimants are allocated their claims (cf. Curiel, Maschler, and Tijs (1987)). Following Estévez-Fernández, Borm, and Fiestras-Janeiro (2014), we define the minimal right of a claimant in an NTU-bankruptcy problem as the maximal individual payoff within the estate when all other claimants are allocated their claims.

Let $(E, c) \in \mathrm{BR}^N$. The vector of *minimal rights* $m(E, c) \in \mathbb{R}^N_+$ is defined by

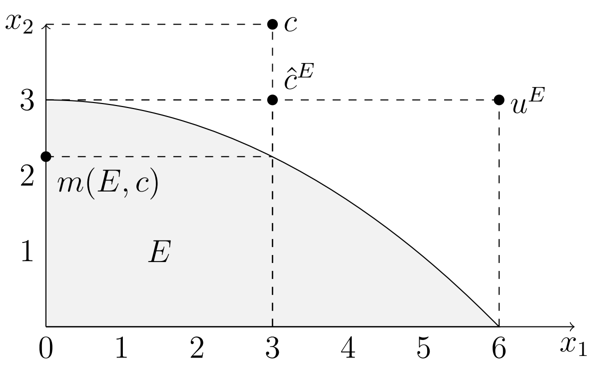
$$m_i(E, c) = \begin{cases} \max\{x \mid (x, c_{N\setminus\{i\}}) \in E\} & \text{if } (0, c_{N\setminus\{i\}}) \in E; \\ 0 & \text{if } (0, c_{N\setminus\{i\}}) \notin E \end{cases}$$

for all $i \in N$. We have $m(E, c) \in E$ and $m(E, c) \leq \hat{c}^E \leq c$, which means that

$$((E - \{m(E, c)\})_+, c - m(E, c)) \in \mathrm{BR}^N.$$

Moreover, $m(E, c) \leq f(E, c) \leq \hat{c}^E$ for any bankruptcy rule $f : \mathrm{BR}^N \to \mathbb{R}^N_+$.

Example 4.3
Let $N = \{1, 2\}$ and consider the bankruptcy problem $(E, c) \in \mathrm{BR}^N$ given by $E = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 12x_2 \leq 36\}$ and $c = (3, 4)$. Then $u^E = (6, 3)$, $\hat{c}^E = (3, 3)$, and $m(E, c) = (0, 2\frac{1}{4})$. This is illustrated as follows.



△

The following lemma derives some elementary relations between truncated claims and minimal rights.

Lemma 4.3.1
Let $(E, c) \in \mathrm{BR}^N$. *Then*

(i) $\widehat{\hat{c}^E}^E = \hat{c}^E$;

(ii) $m((E - \{m(E,c)\})_+, c - m(E,c)) = 0_N$;

(iii) $m(E, \hat{c}^E) = m(E,c)$;

(iv) $\widehat{c - m(E,c)}^{(E-\{m(E,c)\})_+} = \hat{c}^E - m(E,c)$.

Proof. (i) Let $i \in N$. Then

$$\widehat{\hat{c}^E}^E_i = \min\{\hat{c}^E_i, u^E_i\} = \min\{\min\{c_i, u^E_i\}, u^E_i\} = \min\{c_i, u^E_i\} = \hat{c}^E_i.$$

(ii) Let $i \in N$. Suppose that $m_i((E - \{m(E,c)\})_+, c - m(E,c)) > 0$. Then

$$\Big(m_i((E - \{m(E,c)\})_+, c - m(E,c)), (c - m(E,c))_{N\setminus\{i\}}\Big) \in (E - \{m(E,c)\})_+.$$

This means that

$$\Big(m_i((E - \{m(E,c)\})_+, c - m(E,c)) + m_i(E,c), c_{N\setminus\{i\}}\Big) \in E.$$

This contradicts the definition of $m_i(E,c)$.

(iii) Let $i \in N$. If $\hat{c}^E_{N\setminus\{i\}} = c_{N\setminus\{i\}}$, then $m_i(E, \hat{c}^E) = m_i(E,c)$. If $\hat{c}^E_{N\setminus\{i\}} \neq c_{N\setminus\{i\}}$, then $(0, c_{N\setminus\{i\}}) \notin E$, so $m_i(E, \hat{c}^E) = 0 = m_i(E,c)$.

(iv) Let $i \in N$. If $m_{N\setminus\{i\}}(E,c) = 0_{N\setminus\{i\}}$, then $u_i^{(E-\{m(E,c)\})_+} = u^E_i - m_i(E,c)$ and

$$\begin{aligned}
\widehat{(c - m(E,c))}_i^{(E-\{m(E,c)\})_+} &= \min\Big\{c_i - m_i(E,c), u_i^{(E-\{m(E,c)\})_+}\Big\} \\
&= \min\{c_i - m_i(E,c), u^E_i - m_i(E,c)\} \\
&= \min\{c_i, u^E_i\} - m_i(E,c) \\
&= \hat{c}^E_i - m_i(E,c).
\end{aligned}$$

Suppose that there exists a $j \in N \setminus \{i\}$ for which $m_j(E,c) > 0$. Then $\hat{c}^E_i = c_i$ and $(m_j(E,c), c_{N\setminus\{j\}}) \in E$. Since E is comprehensive and $m(E,c) \leq c$, this means that $(c_i, m_{N\setminus\{i\}}(E,c)) \in E$, so $(c_i - m_i(E,c), 0_{N\setminus\{i\}}) \in (E - \{m(E,c)\})_+$. This implies that $u_i^{(E-\{m(E,c)\})_+} \geq c_i - m_i(E,c)$. Then

$$\begin{aligned}
\widehat{(c - m(E,c))}_i^{(E-\{m(E,c)\})_+} &= \min\Big\{c_i - m_i(E,c), u_i^{(E-\{m(E,c)\})_+}\Big\} \\
&= c_i - m_i(E,c) \\
&= \hat{c}^E_i - m_i(E,c).
\end{aligned}$$

□

The minimal rights first property requires that first allocating minimal rights and subsequently applying the rule to the remaining problem leads to the same payoff allocation as application of the rule to the original problem.

Definition (Minimal Rights First)
A bankruptcy rule $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ satisfies *minimal rights first* if

$$f(E, c) = m(E, c) + f((E - \{m(E, c)\})_+, c - m(E, c))$$

for all $(E, c) \in \mathrm{BR}^N$.

The following example shows that the proportional rule, the constrained relative equal awards rule, and the constrained relative equal losses rule do not satisfy minimal rights first.

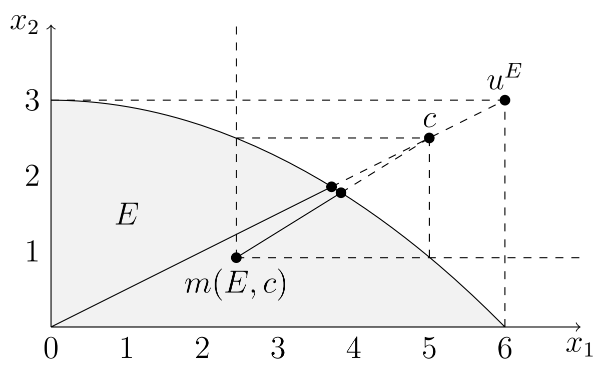
Example 4.4
Let $N = \{1, 2\}$ and consider the bankruptcy problem $(E, c) \in \mathrm{BR}^N$ given by $E = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 12x_2 \leq 36\}$ and $c = (5, 2\frac{1}{2})$. Then $m(E, c) = (\sqrt{6}, \frac{11}{12})$. Let $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ be a bankruptcy rule satisfying relative symmetry, e.g. the proportional rule, the constrained relative equal awards rule, or the constrained relative equal losses rule. Then $f(E, c) = (3\sqrt{5} - 3, \frac{3}{2}\sqrt{5} - \frac{3}{2})$. However,

$$m_1(E, c) + f_1((E - \{m(E, c)\})_+, c - m(E, c)) = \sqrt{\tfrac{18745\sqrt{6} - 36630}{860\sqrt{6} - 1944}} + \tfrac{19}{2\sqrt{6} - 10}$$

and

$$m_2(E, c) + f_2((E - \{m(E, c)\})_+, c - m(E, c)) = \sqrt{\tfrac{6766945\sqrt{6} - 13223430}{6638400\sqrt{6} - 16108416}} + \tfrac{1735\sqrt{6} - 5198}{1720\sqrt{6} - 3888}.$$

This is illustrated as follows.



△

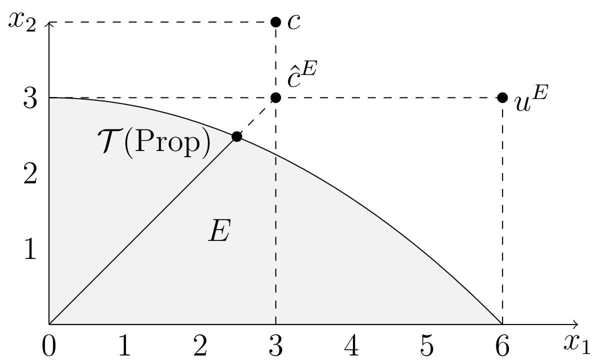
By Proposition 3.5.5, the constrained relative equal awards rule and the constrained relative equal losses rule are dual rules, and by Lemma 3.A.2, the constrained relative equal awards rule satisfies truncation invariance. This means that minimal rights first and truncation invariance are not dual properties, in contrast to the TU-bankruptcy context (cf. Herrero and Villar (2001)).

Inspired by Thomson and Yeh (2008), we introduce two operators which 'force' rules to satisfy truncation invariance and minimal rights first. Let BR denote the class of bankruptcy problems with an arbitrary set of claimants. A bankruptcy rule f on BR assigns to any $(E, c) \in \mathrm{BR}^N$ with arbitrary N a payoff allocation $f(E, c) \in \mathrm{WP}(E)$ for which $f(E, c) \leq c$. Let $\mathcal{F}$ denote the space of all bankruptcy rules on BR. The *truncation operator* $\mathcal{T} : \mathcal{F} \to \mathcal{F}$ assigns to any bankruptcy rule $f \in \mathcal{F}$ the bankruptcy rule $\mathcal{T}(f) \in \mathcal{F}$ which assigns to any $(E, c) \in \mathrm{BR}^N$ with arbitrary N the payoff allocation

$$\mathcal{T}(f)(E, c) = f(E, \hat{c}^E).$$

In particular, the *truncated proportional rule* $\mathcal{T}(\mathrm{Prop}) \in \mathcal{F}$ assigns to any $(E, c) \in \mathrm{BR}^N$ with arbitrary N the payoff allocation $\mathcal{T}(\mathrm{Prop})(E, c) = \mathrm{Prop}(E, \hat{c}^E)$.

Example 4.5
Let $N = \{1, 2\}$ and consider the bankruptcy problem $(E, c) \in \mathrm{BR}^N$ given by $E = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 12x_2 \leq 36\}$ and $c = (3, 4)$ as in Example 4.3. Then $\lambda^{E,\hat{c}^E} = 2\sqrt{2} - 2$ and $\mathcal{T}(\mathrm{Prop})(E, c) = (6\sqrt{2} - 6, 6\sqrt{2} - 6)$. This is illustrated as follows.



△

The *minimal rights operator* $\mathcal{M} : \mathcal{F} \to \mathcal{F}$ assigns to any bankruptcy rule $f \in \mathcal{F}$ the bankruptcy rule $\mathcal{M}(f) \in \mathcal{F}$ which assigns to any $(E, c) \in \mathrm{BR}^N$ with arbitrary N the payoff allocation

$$\mathcal{M}(f)(E, c) = m(E, c) + f((E - \{m(E, c)\})_+, c - m(E, c)).$$

A bankruptcy rule on BR satisfies a property if it satisfies that property on BR^N for all arbitrary N. Note that $f \in \mathcal{F}$ satisfies truncation invariance if and only if $f = \mathcal{T}(f)$, and $f \in \mathcal{F}$ satisfies minimal rights first if and only if $f = \mathcal{M}(f)$. In particular, this means that CREA $= \mathcal{T}$(CREA) by Lemma 3.A.2.

The next theorem studies some consequences of the truncation operator and the minimal rights operator for the properties of the rules to which they are applied.

Theorem 4.3.2
Let $f \in \mathcal{F}$ be a bankruptcy rule.

(i) Then $\mathcal{T}(f)$ satisfies truncation invariance.

(ii) Then $\mathcal{M}(f)$ satisfies minimal rights first.

(iii) If f satisfies relative symmetry, then $\mathcal{T}(f)$ satisfies relative symmetry.

(iv) If f satisfies truncation invariance, then $\mathcal{M}(f)$ satisfies truncation invariance.

(v) If f satisfies minimal rights first, then $\mathcal{T}(f)$ satisfies minimal rights first.

Proof. (i) Let $(E, c) \in \mathrm{BR}^N$ with arbitrary N. Then

$$\mathcal{T}(f)(E, \hat{c}^E) = f(E, \widehat{\hat{c}^E}^E) = f(E, \hat{c}^E) = \mathcal{T}(f)(E, c),$$

where the second equality follows from Lemma 4.3.1(i).

(ii) Let $(E, c) \in \mathrm{BR}^N$ with arbitrary N. Then

$$\begin{aligned} &m(E, c) + \mathcal{M}(f)((E - \{m(E, c)\})_+, c - m(E, c)) \\ &= m(E, c) + f((E - \{m(E, c)\})_+, c - m(E, c)) \\ &= \mathcal{M}(f)(E, c), \end{aligned}$$

where the first equality follows from Lemma 4.3.1(ii).

(iii) Assume that f satisfies relative symmetry. Let $(E, c) \in \mathrm{BR}^N$ with arbitrary N and let $i, j \in N$ be such that $c_i u_j^E = c_j u_i^E$. Then

$$\begin{aligned} \hat{c}_i^E u_j^E &= \min\{c_i, u_i^E\} u_j^E &&= \min\{c_i u_j^E, u_i^E u_j^E\} \\ &= \min\{c_j u_i^E, u_j^E u_i^E\} &&= \min\{c_j, u_j^E\} u_i^E = \hat{c}_j^E u_i^E. \end{aligned}$$

Since f satisfies relative symmetry, this means that

$$\mathcal{T}(f)_i(E, c) u_j^E = f_i(E, \hat{c}^E) u_j^E = f_j(E, \hat{c}^E) u_i^E = \mathcal{T}(f)_j(E, c) u_i^E.$$

(*iv*) Assume that f satisfies truncation invariance. Let $(E,c) \in \mathrm{BR}^N$ with arbitrary N. Then

$$\begin{aligned}
\mathcal{M}(f)(E,\hat{c}^E) &= m(E,\hat{c}^E) + f((E - \{m(E,\hat{c}^E)\})_+, \hat{c}^E - m(E,\hat{c}^E)) \\
&= m(E,c) + f((E - \{m(E,c)\})_+, \hat{c}^E - m(E,c)) \\
&= m(E,c) + f((E - \{m(E,c)\})_+, \widehat{c - m(E,c)}^{(E-\{m(E,c)\})_+}) \\
&= m(E,c) + f((E - \{m(E,c)\})_+, c - m(E,c)) \\
&= \mathcal{M}(f)(E,c),
\end{aligned}$$

where the second equality follows from Lemma 4.3.1(*iii*), the third equality follows from Lemma 4.3.1(*iv*), and the fourth equality follows from f satisfying truncation invariance.

(*v*) Assume that f satisfies minimal rights first. Let $(E,c) \in \mathrm{BR}^N$ with arbitrary N. Then

$$\begin{aligned}
&m(E,c) + \mathcal{T}(f)((E - \{m(E,c)\})_+, c - m(E,c)) \\
&= m(E,c) + f((E - \{m(E,c)\})_+, \widehat{c - m(E,c)}^{(E-\{m(E,c)\})_+}) \\
&= m(E,c) + f((E - \{m(E,c)\})_+, \hat{c}^E - m(E,c)) \\
&= m(E,\hat{c}^E) + f((E - \{m(E,\hat{c}^E)\})_+, \hat{c}^E - m(E,\hat{c}^E)) \\
&= f(E,\hat{c}^E) \\
&= \mathcal{T}(f)(E,c),
\end{aligned}$$

where the second equality follows from Lemma 4.3.1(*iv*), the third equality follows from Lemma 4.3.1(*iii*), and the fourth equality follows from f satisfying minimal rights first. □

The purpose of Theorem 4.3.2 is twofold. First, it shows that the truncation operator and the minimal rights operator indeed 'force' rules to satisfy truncation invariance and minimal rights first, respectively. Second, it studies the preservation of properties under the truncation operator and the minimal rights operator. Both operators preserve truncation invariance and minimal rights first, and relative symmetry is preserved under the truncation operator, as is the case for TU-bankruptcy rules. However, as illustrated by Example 4.4, relative symmetry is not preserved under the minimal rights operator, in contrast to TU-bankruptcy rules.

Let $f \in \mathcal{F}$. By Theorem 4.3.2, $\mathcal{T}(f)$ satisfies truncation invariance and $\mathcal{M}(f)$ satisfies minimal rights first, which means that $\mathcal{T}(\mathcal{T}(f)) = \mathcal{T}(f)$ and $\mathcal{M}(\mathcal{M}(f)) = \mathcal{M}(f)$. By Theorem 4.3.2, $\mathcal{T}(\mathcal{M}(f))$ and $\mathcal{M}(\mathcal{T}(f))$ both satisfy truncation invariance and minimal rights first, which means that $\mathcal{T}(\mathcal{M}(f)) = \mathcal{T}(\mathcal{M}(\mathcal{T}(f))) = \mathcal{T}(\mathcal{M}(\mathcal{M}(f)))$ and $\mathcal{M}(\mathcal{T}(f)) = \mathcal{M}(\mathcal{T}(\mathcal{T}(f))) = \mathcal{M}(\mathcal{T}(\mathcal{M}(f)))$. Hence, nothing changes when one of the operators is applied more than once. However, the two operators can be combined to obtain a rule which satisfies both truncation invariance and minimal rights first. The following proposition shows that the order in which the operators are applied does not matter, as is the case for TU-bankruptcy rules.

Proposition 4.3.3
Let $f \in \mathcal{F}$. *Then* $\mathcal{T}(\mathcal{M}(f)) = \mathcal{M}(\mathcal{T}(f))$.

Proof. Let $(E, c) \in \mathrm{BR}^N$ with arbitrary N. Then

$$\begin{aligned}
\mathcal{T}(\mathcal{M}(f))(E,c) &= \mathcal{M}(f)(E, \hat{c}^E) \\
&= m(E, \hat{c}^E) + f((E - \{m(E, \hat{c}^E)\})_+, \hat{c}^E - m(E, \hat{c}^E)) \\
&= m(E, c) + f((E - \{m(E, c)\})_+, \hat{c}^E - m(E, c)) \\
&= m(E, c) + f((E - \{m(E, c)\})_+, \widehat{c - m(E, c)}^{(E-\{m(E,c)\})_+}) \\
&= m(E, c) + \mathcal{T}(f)((E - \{m(E, c)\})_+, c - m(E, c)) \\
&= \mathcal{M}(\mathcal{T}(f))(E, c),
\end{aligned}$$

where the third equality follows from Lemma 4.3.1(iii) and the fourth equality follows from Lemma 4.3.1(iv). □

The bankruptcy rule $\mathcal{T}(\mathcal{M}(f))$ is the *adjusted counterpart* of the rule $f \in \mathcal{F}$. Three examples of adjusted rules are the adjusted proportional rule[1] $\mathcal{T}(\mathcal{M}(\mathrm{Prop}))$, the adjusted constrained relative equal awards rule $\mathcal{T}(\mathcal{M}(\mathrm{CREA}))$, and the adjusted constrained relative equal losses rule $\mathcal{T}(\mathcal{M}(\mathrm{CREL}))$. On the class of bankruptcy problems with two claimants, these three adjusted rules coincide. This standard solution is called the relative adjustment principle.[2]

Definition (Relative Adjustment Principle)
A bankruptcy rule $f \in \mathcal{F}$ satisfies the *relative adjustment principle* if it assigns to any $(E, c) \in \mathrm{BR}^N$ with arbitrary N for which $|N| = 2$ the payoff allocation

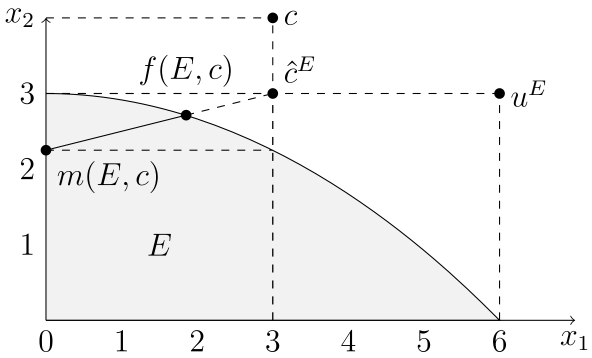
$$f(E, c) = m(E, c) + \kappa^{E,c} \left(\hat{c}^E - m(E, c) \right),$$

where $\kappa^{E,c} = \max\{t \in [0, 1] \mid m(E, c) + t(\hat{c}^E - m(E, c)) \in \mathrm{WP}(E)\}$.

[1] The adjusted proportional rule for TU-bankruptcy problems was introduced by Curiel, Maschler, and Tijs (1987).

[2] For TU-bankruptcy problems, Aumann and Maschler (1985) called this standard solution the contested garment principle. Later, Thomson (2003) named it the concede-and-divide principle.

Example 4.6
Let $N = \{1, 2\}$ and consider the bankruptcy problem $(E, c) \in \mathrm{BR}^N$ given by $E = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 12x_2 \leq 36\}$ and $c = (3, 4)$ as in Example 4.3. Then $f(E, c) = (\frac{3}{2}\sqrt{5} - \frac{3}{2}, \frac{3}{8}\sqrt{5} + \frac{15}{8})$ for any bankruptcy rule $f \in \mathcal{F}$ satisfying the relative adjustment principle. This is illustrated as follows.



△

In order to axiomatically study the relative adjustment principle, we introduce the class of simple bankruptcy problems.

Definition (Simple Bankruptcy Problem)
A bankruptcy problem $(E, c) \in \mathrm{BR}^N$ is *simple* if $\hat{c}^E = c$ and $m(E, c) = 0_N$.

Let SBR^N denote the class of simple bankruptcy problems with claimant set N.

Lemma 4.3.4
Let $(E, c) \in \mathrm{BR}^N$. *Then* $((E - \{m(E, c)\})_+, \hat{c}^E - m(E, c)) \in \mathrm{SBR}^N$.

Proof. We have

$$\begin{aligned}\widehat{(\hat{c}^E - m(E,c))}^{(E-\{m(E,c)\})_+} &= \widehat{(\hat{c}^E - m(E,\hat{c}^E))}^{(E-\{m(E,\hat{c}^E)\})_+} \\ &= \widehat{\hat{c}^E}^E - m(E, \hat{c}^E) \\ &= \hat{c}^E - m(E, c),\end{aligned}$$

where the first equality follows from Lemma 4.3.1(*iii*), the second equality follows from Lemma 4.3.1(*iv*), and the third equality follows from Lemma 4.3.1(*i*) and Lemma 4.3.1(*iii*). Moreover,

$$m((E - \{m(E,c)\})_+, \hat{c}^E - m(E,c)) = m((E - \{m(E,\hat{c}^E)\})_+, \hat{c}^E - m(E,\hat{c}^E)) = 0_N,$$

where the first equality follows from Lemma 4.3.1(*iii*) and the second equality follows from Lemma 4.3.1(*ii*). □

A rule satisfies the *simple counterpart* of a property if it satisfies that property on the class of simple problems. For example, a rule $f \in \mathcal{F}$ satisfies *simple relative symmetry* if $f_i(E,c)u_j^E = f_j(E,c)u_i^E$ for all $(E,c) \in \text{SBR}^N$ with arbitrary N and any $i,j \in N$ for which $c_i u_j^E = c_j u_i^E$. Note that all bankruptcy rules satisfy *simple truncation invariance* and *simple minimal rights first.*

If a rule satisfies a property, then Lemma 4.3.4 implies that its adjusted counterpart satisfies the simple counterpart of that property. For example, the adjusted counterpart of any relatively symmetric rule satisfies simple relative symmetry. Inspired by Dagan (1996), we show that the relative adjustment principle is equivalent to the combination of simple relative symmetry, truncation invariance, and minimal rights first on the class of bankruptcy problems with two claimants. In particular, since the adjusted counterpart of any rule satisfies truncation invariance and minimal rights first, this means that the adjusted counterpart of any relatively symmetric bankruptcy rule satisfies the relative adjustment principle.

Theorem 4.3.5
A bankruptcy rule on BR *satisfies the relative adjustment principle if and only if it satisfies simple relative symmetry, truncation invariance, and minimal rights first on* BR^N *for all arbitrary* N *for which* $|N| = 2$.

Proof. Let $f \in \mathcal{F}$ be a bankruptcy rule satisfying the relative adjustment principle. Let $(E,c) \in \text{SBR}^N$ with arbitrary N be such that $|N| = 2$ and let $i,j \in N$ be such that $c_i u_j^E = c_j u_i^E$. Then

$$\begin{aligned} f_i(E,c)u_j^E &= \left(m_i(E,c) + \kappa^{E,c}\left(\hat{c}_i^E - m_i(E,c)\right)\right)u_j^E \\ &= \kappa^{E,c} c_i u_j^E \\ &= \kappa^{E,c} c_j u_i^E \\ &= \left(m_j(E,c) + \kappa^{E,c}\left(\hat{c}_j^E - m_j(E,c)\right)\right)u_i^E \\ &= f_j(E,c)u_i^E. \end{aligned}$$

Hence, f satisfies simple relative symmetry. Now, let $(E,c) \in \text{BR}^N$ with arbitrary N be such that $|N| = 2$. Then

$$\begin{aligned} f(E,\hat{c}^E) &= m(E,\hat{c}^E) + \kappa^{E,\hat{c}^E}\left(\widehat{\hat{c}^E}^E - m(E,\hat{c}^E)\right) \\ &= m(E,c) + \kappa^{E,c}\left(\hat{c}^E - m(E,c)\right) \\ &= f(E,c), \end{aligned}$$

where the second equality follows from Lemma 4.3.1(i) and Lemma 4.3.1(iii). Hence, f satisfies truncation invariance.

Moreover,

$$\begin{aligned}
& m(E,c) + f((E - \{m(E,c)\})_+, c - m(E,c)) \\
&= m(E,c) + \kappa^{(E-\{m(E,c)\})_+, c-m(E,c)} \left(\widehat{c - m(E,c)}^{(E-\{m(E,c)\})_+} \right) \\
&= m(E,c) + \kappa^{E,c} \left(\hat{c}^E - m(E,c) \right) \\
&= f(E,c),
\end{aligned}$$

where the first equality follows from Lemma 4.3.1(*ii*) and the second equality follows from Lemma 4.3.1(*iv*). Hence, f satisfies minimal rights first.

Let $f \in \mathcal{F}$ be a bankruptcy rule satisfying simple relative symmetry, truncation invariance, and minimal rights first on BR^N for all arbitrary N for which $|N| = 2$. Let $(E,c) \in \mathrm{BR}^N$ with arbitrary N be such that $|N| = 2$. Since f satisfies truncation invariance and minimal rights first,

$$f(E,c) = m(E,c) + f((E - \{m(E,c)\})_+, \hat{c}^E - m(E,c)).$$

By Lemma 4.3.4, $((E - \{m(E,c)\})_+, \hat{c}^E - m(E,c)) \in \mathrm{SBR}^N$. Let $i \in N$ and let $j \in N \setminus \{i\}$. Then

$$\begin{aligned}
u_i^{(E-\{m(E,c)\})_+} &= \max\{x_i \mid x \in (E - \{m(E,c)\})_+\} \\
&= \max\{x_i \mid (x_i + m_i(E,c), m_j(E,c)) \in E\} \\
&= \begin{cases} u_i^E - m_i(E,c) & \text{if } m_j(E,c) = 0; \\ c_i - m_i(E,c) & \text{if } m_j(E,c) > 0 \end{cases} \\
&= \begin{cases} u_i^E - m_i(E,c) & \text{if } \hat{c}_i^E = u_i^E; \\ c_i - m_i(E,c) & \text{if } \hat{c}_i^E = c_i \end{cases} \\
&= \hat{c}_i^E - m_i(E,c).
\end{aligned}$$

This means that

$$\left(\hat{c}_i^E - m_i(E,c) \right) u_j^{(E-\{m(E,c)\})_+} = \left(\hat{c}_j^E - m_j(E,c) \right) u_i^{(E-\{m(E,c)\})_+}.$$

Since f satisfies simple relative symmetry, this implies that

$$f((E - \{m(E,c)\})_+, \hat{c}^E - m(E,c)) = \kappa^{E,c} \left(\hat{c}^E - m(E,c) \right).$$

Then

$$\begin{aligned}
f(E,c) &= m(E,c) + f((E - \{m(E,c)\})_+, \hat{c}^E - m(E,c)) \\
&= m(E,c) + \kappa^{E,c} \left(\hat{c}^E - m(E,c) \right).
\end{aligned}$$

Hence, f satisfies the relative adjustment principle. $\square$

By Theorem 4.3.5, all adjusted counterparts of rules which satisfy relative symmetry coincide. Since the proportional rule, the constrained relative equal awards rule, and the constrained relative equal losses rule satisfy relative symmetry, we derive the following corollary.

Corollary 4.3.6
The adjusted proportional rule, the adjusted constrained relative equal awards rule, and the adjusted constrained relative equal losses rule satisfy the relative adjustment principle.

Future research could study generalizations of other bankruptcy rules which satisfy the relative adjustment principle on the class of TU-bankruptcy problems, such as the random arrival rule (cf. O'Neill (1982)), the minimal overlap rule (cf. O'Neill (1982)), and the Talmud rule (cf. Aumann and Maschler (1985)).

5 Bargaining Axioms

5.1 Introduction

Bankruptcy problems with nontransferable utility share some similarities with bargaining problems with claims as introduced by Chun and Thomson (1992). In a bargaining problem (cf. Nash (1950)), agents need to agree upon a surplus allocation within a feasible set while taking into account their individual disagreement payoffs. Chun and Thomson (1992) enriched these bargaining problems with a vector of claims. In a bankruptcy problem with nontransferable utility, the estate is of a similar nature as the feasible set in a bargaining problem. However, in a bankruptcy problem, it is assumed that individual utility is normalized in such a way that allocating nothing corresponds to a utility level of zero. Therefore, it is convenient to consider the zero vector as a natural benchmark for allocations within bankruptcy problems instead of an exogenous disagreement point as within bargaining problems.

This chapter, based on Dietzenbacher and Peters (2018), takes an axiomatic bargaining approach to bankruptcy problems with nontransferable utility by characterizing bankruptcy rules in terms of properties from bargaining theory. We consider the role of the claims vector within bankruptcy problems as being 'dual' to the role of the disagreement point within bargaining problems. Where the disagreement point serves as a lower bound for rational payoff allocations within a bargaining problem, the claims vector serves as an upper bound for feasible payoff allocations within a bankruptcy problem.[1]

[1] Although not addressed in this chapter, one could also consider the role of the minimal rights vector within bankruptcy problems as being analogous to the role of the disagreement point within bargaining problems. Following this approach, Herrero (1997) interpreted the minimal rights vector of a bankruptcy problem as an endogenous disagreement point of a bargaining problem with claims.

Following the classical axiomatic theory of bargaining, we formulate several properties which concern changes in the estate or the claims, where the latter ones are based on axioms concerning changes in the disagreement point, and study their implications. In particular, we translate several axioms from bargaining theory to the domain of bankruptcy problems with nontransferable utility, study their relations, and combine them with the properties relative symmetry and truncation invariance from bankruptcy theory to derive new axiomatic characterizations of the proportional rule, the truncated proportional rule, and the constrained relative equal awards rule.

Alternatively, one could also interpret solutions for bargaining problems as new rules for bankruptcy problems, in line with the work of Dagan and Volij (1993) for bankruptcy problems with transferable utility. Future research allows to formalize this reverse approach in order to further connect bankruptcy problems with bargaining problems.

This chapter is organized in the following way. In Section 5.2, we introduce and study the implications of axioms concerning changes in the estate. In Section 5.3, we introduce and study the implications of axioms concerning changes in the claims.

5.2 Estate Axioms

In this section, we introduce and study the implications of axioms concerning changes in the estate. Starting from the well-known independence of irrelevant alternatives axiom introduced by Nash (1950), several axioms concerning changes in the feasible set of bargaining problems have been proposed in the literature. As exploited by Roth (1977) for the independence of irrelevant alternatives axiom, in the formulation of these properties the disagreement point is required to be fixed. We translate these properties to the domain of bankruptcy problems with nontransferable utility in such a way that the vector of claims is required to be fixed.

Let $(E, c) \in \mathrm{BR}^N$. Throughout this chapter, the set of *positive claimants* is defined by

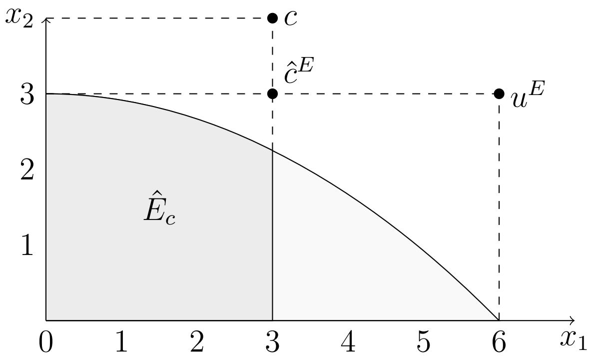
$$N_+^c = \{i \in N \mid c_i > 0\}$$

and the *truncated estate* $\hat{E}_c \subseteq \mathbb{R}_+^N$ is defined by

$$\hat{E}_c = \{x \in E \mid x \leq c\}.$$

Note that $u^{\hat{E}_c} = \hat{c}^E$ and $\hat{E}_c = \hat{E}_{\hat{c}^E}$.

Example 5.1
Let $N = N_+^c = \{1, 2\}$ and consider the bankruptcy problem $(E, c) \in \mathrm{BR}^N$ given by $E = \{x \in \mathbb{R}_+^N \mid x_1^2 + 12x_2 \leq 36\}$ and $c = (3, 4)$. Then $\hat{E}_c = \hat{E}_{\hat{c}^E} = \{x \in \mathbb{R}_+^N \mid x_1^2 + 12x_2 \leq 36, x_1 \leq 3\}$, $u^E = (6, 3)$, and $\hat{c}^E = u^{\hat{E}_c} = (3, 3)$. This is illustrated as follows.



△

Definition (Estate Axioms)
A bankruptcy rule $f : \mathrm{BR}^N \to \mathbb{R}_+^N$ satisfies

- *step-by-step negotiations* if $f(E, c) = f(E', c) + f((E - \{f(E', c)\})_+, c - f(E', c))$ for all $(E, c), (E', c) \in \mathrm{BR}^N$ for which $E' \subseteq E$;
- *estate monotonicity* if $f(E, c) \geq f(E', c)$ for all $(E, c), (E', c) \in \mathrm{BR}^N$ for which $E' \subseteq E$;
- *domination* if $f(E, c) \leq f(E', c)$ or $f(E, c) \geq f(E', c)$ for all $(E, c), (E', c) \in \mathrm{BR}^N$;
- *independence of irrelevant alternatives* if $f(E, c) = f(E', c)$ for all $(E, c) \in \mathrm{BR}^N$ and $(E', c) \in \mathrm{BR}^N$ for which $E' \subseteq E$ and $f(E, c) \in \mathrm{WP}(E')$;
- *independence of undominating alternatives* if $f(E, c) = f(E', c)$ for all $(E, c) \in \mathrm{BR}^N$ and $(E', c) \in \mathrm{BR}^N$ for which $E' \subseteq E$ and $f(E', c) \in \mathrm{WP}(E)$;
- *independence of unclaimed alternatives* if $f(E, c) = f(E', c)$ for all $(E, c) \in \mathrm{BR}^N$ and $(E', c) \in \mathrm{BR}^N$ for which $\hat{E}_c = \hat{E}'_c$.

The axioms step-by-step negotiations, estate monotonicity, domination, and independence of undominating alternatives are based on bargaining axioms of Kalai (1977), Roth (1979), and Thomson and Myerson (1980). The independence of unclaimed alternatives axiom, describing bankruptcy rules which only take the truncated estate into account, is used in a similar form by Chun and Thomson (1992) and originates from the bargaining axiom of Peters (2010) describing bargaining solutions which only take the rational payoff allocations within the feasible set into account.

The following lemma studies the relations between the estate axioms. Some of these relations bear similarities with the results of Thomson and Myerson (1980) on the domain of bargaining problems. Throughout this chapter, we refer to the appendix for the derivations of properties satisfied by specific bankruptcy rules.

Lemma 5.2.1
Let $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ be a bankruptcy rule.

(i) If f satisfies step-by-step negotiations, then f satisfies estate monotonicity.

(ii) Then f satisfies estate monotonicity if and only if f satisfies domination.

(iii) If f satisfies estate monotonicity, then f satisfies independence of irrelevant alternatives.

(iv) If f satisfies estate monotonicity, then f satisfies independence of undominating alternatives.

(v) If f satisfies independence of irrelevant alternatives, then f satisfies independence of unclaimed alternatives.

(vi) If f satisfies independence of undominating alternatives, then f satisfies independence of unclaimed alternatives.

Proof. (i) Assume that f satisfies step-by-step negotiations. Let $(E, c), (E', c) \in \mathrm{BR}^N$ be such that $E' \subseteq E$. Then

$$f(E, c) = f(E', c) + f((E - \{f(E', c)\})_+, c - f(E', c)) \geq f(E', c).$$

Hence, f satisfies estate monotonicity.

(ii) Assume that f satisfies estate monotonicity. Let $(E, c), (E', c) \in \mathrm{BR}^N$. Suppose that $f(E, c) \in E'$. Then $f(E, c) \in \mathrm{WP}(E \cap E')$, $f(E \cap E', c) \leq f(E, c)$, and $f(E \cap E', c) \leq f(E', c)$. Since E is nonleveled, this implies that $f(E, c) = f(E \cap E', c) \leq f(E', c)$.

Now suppose that $f(E, c) \notin E'$. Then $f(E, c) \in \mathrm{WP}(E \cup E')$, $f(E, c) \leq f(E \cup E', c)$, and $f(E', c) \leq f(E \cup E', c)$. Since E is nonleveled, this implies that $f(E, c) = f(E \cup E', c) \geq f(E', c)$. Hence, f satisfies domination.

Assume that f satisfies domination. Let $(E, c), (E', c) \in \mathrm{BR}^N$ be such that $E' \subseteq E$. Then $f(E, c) \leq f(E', c)$ or $f(E, c) \geq f(E', c)$. Since E is nonleveled, this implies that $f(E', c) \leq f(E, c)$. Hence, f satisfies estate monotonicity.

(iii) Assume that f satisfies estate monotonicity. Let $(E,c),(E',c) \in \mathrm{BR}^N$ be such that $E' \subseteq E$ and $f(E,c) \in \mathrm{WP}(E')$. Then $f(E,c) \geq f(E',c)$. Since E is nonleveled, this implies that $f(E,c) = f(E',c)$. Hence, f satisfies independence of irrelevant alternatives.

(iv) Assume that f satisfies estate monotonicity. Let $(E,c),(E',c) \in \mathrm{BR}^N$ be such that $E' \subseteq E$ and $f(E',c) \in \mathrm{WP}(E)$. Then $f(E,c) \geq f(E',c)$. Since E is nonleveled, this implies that $f(E,c) = f(E',c)$. Hence, f satisfies independence of undominating alternatives.

(v) Assume that f satisfies independence of irrelevant alternatives. Let $(E,c) \in \mathrm{BR}^N$ and $(E',c) \in \mathrm{BR}^N$ be such that $\hat{E}_c = \hat{E}'_c$. Then $f(E,c), f(E',c) \in \mathrm{WP}(E \cap E')$. This implies that $f(E,c) = f(E \cap E', c) = f(E',c)$. Hence, f satisfies independence of unclaimed alternatives.

(vi) Assume that f satisfies independence of undominating alternatives. Let $(E,c),(E',c) \in \mathrm{BR}^N$ be such that $\hat{E}_c = \hat{E}'_c$. Then $f(E,c), f(E',c) \in \mathrm{WP}(E \cup E')$. This implies that $f(E,c) = f(E \cup E', c) = f(E',c)$. Hence, f satisfies independence of unclaimed alternatives. □

As shown by the following two rules, the axioms independence of irrelevant alternatives and independence of undominating alternatives are independent.

The bankruptcy rule $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ which assigns to any $(E,c) \in \mathrm{BR}^N$ the payoff allocation

$$f(E,c) = \begin{cases} (\hat{c}_1^E, \max\{x \mid (\hat{c}_1^E, x) \in E\}) & \text{if } N = \{1,2\} \text{ and } \hat{c}_1^E \geq \hat{c}_2^E; \\ (\max\{x \mid (x, \hat{c}_2^E) \in E\}, \hat{c}_2^E) & \text{if } N = \{1,2\} \text{ and } \hat{c}_1^E < \hat{c}_2^E; \\ \mathrm{Prop}(E,c) & \text{otherwise} \end{cases}$$

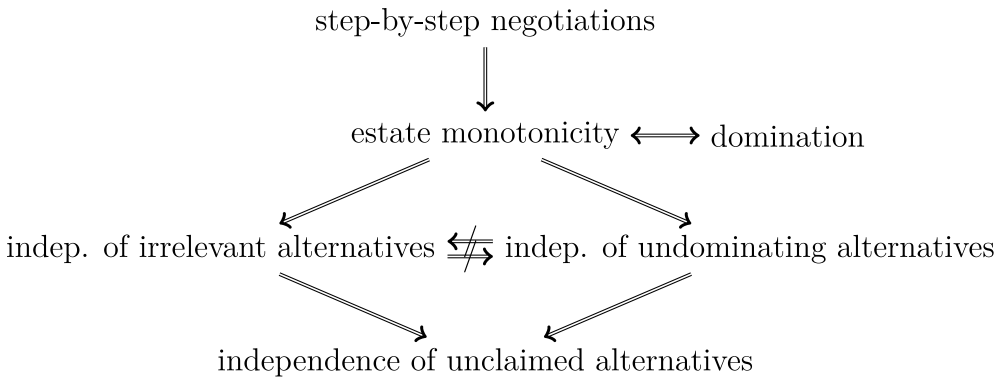
satisfies independence of irrelevant alternatives, but does not satisfy independence of undominating alternatives.

The bankruptcy rule $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ which assigns to any $(E,c) \in \mathrm{BR}^N$ the payoff allocation

$$f(E,c) = \begin{cases} (\hat{c}_1^E, \max\{x \mid (\hat{c}_1^E, x) \in E\}) & \text{if } N = \{1,2\} \text{ and } \hat{c}_1^E \leq \hat{c}_2^E; \\ (\max\{x \mid (x, \hat{c}_2^E) \in E\}, \hat{c}_2^E) & \text{if } N = \{1,2\} \text{ and } \hat{c}_1^E > \hat{c}_2^E; \\ \mathrm{Prop}(E,c) & \text{otherwise} \end{cases}$$

satisfies independence of undominating alternatives, but does not satisfy independence of irrelevant alternatives.

The relations of all estate axioms can be summarized by the following diagram.



The axioms independence of irrelevant alternatives and independence of undominating alternatives are independent. However, if relative symmetry is required, then independence of irrelevant alternatives and independence of undominating alternatives become equivalent and are only satisfied by the proportional rule.

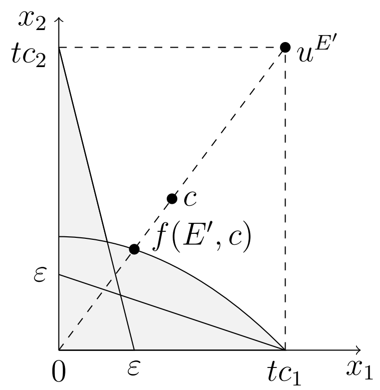
Theorem 5.2.2
The proportional rule is the unique bankruptcy rule satisfying relative symmetry and independence of irrelevant alternatives.

Proof. By Lemma 5.2.1 and Lemma 5.A.1, the proportional rule satisfies independence of irrelevant alternatives. Let $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ be a bankruptcy rule satisfying relative symmetry and independence of irrelevant alternatives. Let $(E, c) \in \mathrm{BR}^N$. If $E = \{0_N\}$, then $f(E, c) = 0_N = \mathrm{Prop}(E, c)$. Suppose that $E \neq \{0_N\}$. Then $u^E \in \mathbb{R}^N_{++}$ and $N^c_+ \neq \emptyset$. Denote

$$t = \max_{i \in N^c_+} \left\{ \frac{u^E_i}{c_i} \right\} \text{ and } \varepsilon = \min_{i \in N^c_+} \{\mathrm{Prop}_i(E, c)\}.$$

Define

$$E' = \bigcup_{i \in N^c_+} \mathrm{comp}\left(\mathrm{conv}\left(\left\{\left(tc_i, 0_{N\setminus\{i\}}\right)\right\} \cup \left\{\left(\varepsilon, 0_{N\setminus\{j\}}\right) \;\middle|\; j \in N \setminus \{i\}\right\}\right)\right) \cup E.$$



Then $(E',c) \in \mathrm{BR}^N$ and $E \subseteq E'$. Moreover, $u^{E'}_{N^c_+} = tc_{N^c_+}$ and $\lambda^{E',c} = \lambda^{E,c}$. We have $c_i u^{E'}_j = tc_ic_j = c_j u^{E'}_i$ for all $i,j \in N^c_+$. Since f satisfies relative symmetry, this means that $f(E',c) = \lambda^{E',c}c = \lambda^{E,c}c = \mathrm{Prop}(E,c)$. Since f satisfies independence of irrelevant alternatives, this implies that $f(E,c) = f(E',c) = \mathrm{Prop}(E,c)$. □

Theorem 5.2.3
The proportional rule is the unique bankruptcy rule satisfying relative symmetry and independence of undominating alternatives.

Proof. By Lemma 5.2.1 and Lemma 5.A.1, the proportional rule satisfies independence of undominating alternatives. Let $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ be a bankruptcy rule satisfying relative symmetry and independence of undominating alternatives. Let $(E,c) \in \mathrm{BR}^N$. If $E = \{0_N\}$, then $f(E,c) = 0_N = \mathrm{Prop}(E,c)$. Suppose that $E \neq \{0_N\}$. Then $u^E \in \mathbb{R}^N_{++}$ and $N^c_+ \neq \emptyset$. If $|N^c_+| = 1$, then $f(E,c) = (u^E_{N^c_+}, 0_{N\setminus N^c_+}) = \mathrm{Prop}(E,c)$. Suppose that $|N^c_+| \geq 2$. Denote

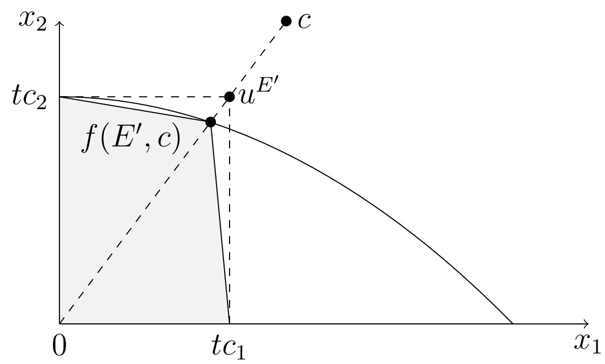
$$t = \min_{i \in N^c_+} \left\{ \frac{u^E_i}{c_i} \right\}.$$

Let $\varepsilon \in \mathbb{R}^N_{++}$ be defined by

$$\varepsilon_i = \begin{cases} \frac{1}{|N^c_+|-1} \left(tc_i - \mathrm{Prop}_i(E,c) \right) & \text{for all } i \in N^c_+; \\ u^E_i & \text{for all } i \in N \setminus N^c_+. \end{cases}$$

Define $E' = \mathrm{comp}(\mathrm{conv}(A_1 \cup A_2)) \cap E$, where

$$A_1 = \left\{ \left(\left(\mathrm{Prop}_i(E,c) + |N^c_+ \setminus S| \varepsilon_i \right)_{i \in S}, 0_{N\setminus S} \right) \,\middle|\, S \in 2^{N^c_+} \setminus \{\emptyset\} \right\}$$
$$\text{and } A_2 = \left\{ \left(\varepsilon_i, 0_{N\setminus\{i\}} \right) \,\middle|\, i \in N \setminus N^c_+ \right\}.$$



Then $(E',c) \in \mathrm{BR}^N$ and $E' \subseteq E$. Moreover, $u^{E'}_{N^c_+} = tc_{N^c_+}$ and $\lambda^{E',c} = \lambda^{E,c}$. We have $c_i u^{E'}_j = tc_ic_j = c_j u^{E'}_i$ for all $i,j \in N^c_+$. Since f satisfies relative symmetry, this means that $f(E',c) = \lambda^{E',c}c = \lambda^{E,c}c = \mathrm{Prop}(E,c)$. Since f satisfies independence of undominating alternatives, this implies that $f(E,c) = f(E',c) = \mathrm{Prop}(E,c)$. □

The constrained relative equal awards rule satisfies relative symmetry, but does not satisfy independence of unclaimed alternatives. The constrained equal awards rule satisfies step-by-step negotiations, but does not satisfy relative symmetry. This is summarized in the following table.

	Prop	CREA	CEA
relative symmetry	+	+	−
step-by-step negotiations	+	−	+
estate monotonicity	+	−	+
domination	+	−	+
independence of irrelevant alternatives	+	−	+
independence of undominating alternatives	+	−	+
independence of unclaimed alternatives	+	−	+

This means that relative symmetry is independent of any estate axiom. This implies that the properties in an axiomatic characterization of the proportional rule remain independent if independence of irrelevant alternatives in Theorem 5.2.2 or independence of undominating alternatives in Theorem 5.2.3 is strengthened to domination, estate monotonicity, or step-by-step negotiations.

The proportional rule is not the unique rule satisfying relative symmetry and independence of unclaimed alternatives, since the truncated proportional rule also satisfies these two properties. Nevertheless, these two properties lead to the proportional rule for a large class of problems.

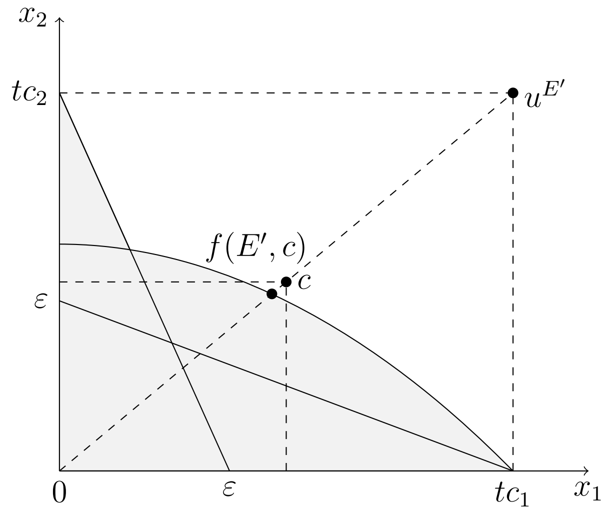
Lemma 5.2.4
Let $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ be a bankruptcy rule. If f satisfies relative symmetry and independence of unclaimed alternatives, then $f(E,c) = \mathrm{Prop}(E,c)$ for all $(E,c) \in \mathrm{BR}^N$ for which $c < u^E$.

Proof. Assume that f satisfies relative symmetry and independence of unclaimed alternatives. Let $(E,c) \in \mathrm{BR}^N$ be such that $c < u^E$. Then $u^E \in \mathbb{R}^N_{++}$ and $|N^c_+| \geq 2$. Denote

$$t = \max_{i \in N^c_+} \left\{ \frac{u^E_i}{c_i} \right\} \text{ and } \varepsilon = \min_{i,j \in N^c_+} \left\{ x_i \mid (x_i, c_j, 0_{N\setminus\{i,j\}}) \in \mathrm{WP}(E) \right\}.$$

Define

$$E' = \bigcup_{i \in N^c_+} \mathrm{comp}\left(\mathrm{conv}\left(\left\{\left(tc_i, 0_{N\setminus\{i\}}\right)\right\} \cup \left\{\left(\varepsilon, 0_{N\setminus\{j\}}\right) \;\middle|\; j \in N \setminus \{i\}\right\}\right)\right) \cup E.$$



Then $(E',c) \in \mathrm{BR}^N$ and $\hat{E}_c = \hat{E}'_c$. Moreover, $u^{E'}_{N^c_+} = tc_{N^c_+}$ and $\lambda^{E',c} = \lambda^{E,c}$. We have $c_i u^{E'}_j = tc_i c_j = c_j u^{E'}_i$ for all $i,j \in N^c_+$. Since f satisfies relative symmetry, this means that $f(E',c) = \lambda^{E',c} c = \lambda^{E,c} c = \mathrm{Prop}(E,c)$. Since f satisfies independence of unclaimed alternatives, this implies that $f(E,c) = f(E',c) = \mathrm{Prop}(E,c)$. $\square$

If we combine independence of unclaimed alternatives with the bankruptcy axioms relative symmetry and truncation invariance, and the weak technical requirement claims continuity, we derive an axiomatic characterization of the truncated proportional rule by using Lemma 5.2.4.

Definition (Claims Continuity)
A bankruptcy rule $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ satisfies *claims continuity* if $f(E,c)$ is continuous in c for all $(E,c) \in \mathrm{BR}^N$.

Theorem 5.2.5
The truncated proportional rule is the unique bankruptcy rule satisfying relative symmetry, truncation invariance, independence of unclaimed alternatives, and claims continuity.

Proof. By Theorem 4.3.2, Lemma 5.A.2, and Lemma 5.A.3, the truncated proportional rule satisfies relative symmetry, truncation invariance, independence of unclaimed alternatives, and claims continuity. Let $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ be a bankruptcy rule satisfying relative symmetry, truncation invariance, independence of unclaimed alternatives, and claims continuity. Let $(E,c) \in \mathrm{BR}^N$. If $E = \{0_N\}$, then $f(E,c) = 0_N = \mathcal{T}(\mathrm{Prop})(E,c)$. Suppose that $E \neq \{0_N\}$. Then $u^E \in \mathbb{R}^N_{++}$ and $N^c_+ \neq \emptyset$. If $|N^c_+| = 1$, then $f(E,c) = (u^E_{N^c_+}, 0_{N \setminus N^c_+}) = \mathcal{T}(\mathrm{Prop})(E,c)$.

Suppose that $|N_+^c| \geq 2$. Let $\{x_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathrm{WUC}(E)$ defined by $x_k = \frac{1}{k}\mathrm{Prop}(E, \hat{c}^E) + (1 - \frac{1}{k})\hat{c}^E$ for all $k \in \mathbb{N}$. Then $x_k < u^E$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} x_k = \hat{c}^E$. Since f satisfies relative symmetry and independence of unclaimed alternatives, Lemma 5.2.4 implies that $f(E, x_k) = \mathrm{Prop}(E, x_k) = \mathrm{Prop}(E, \hat{c}^E) = \mathcal{T}(\mathrm{Prop})(E, c)$ for all $k \in \mathbb{N}$. Since f satisfies claims continuity, this means that $f(E, \hat{c}^E) = \lim_{k\to\infty} f(E, x_k) = \mathcal{T}(\mathrm{Prop})(E, c)$. Since f satisfies truncation invariance, this implies that $f(E, c) = f(E, \hat{c}^E) = \mathcal{T}(\mathrm{Prop})(E, c)$. □

To show that the properties in Theorem 5.2.5 are independent, we introduce the restricted truncated proportional rule. The *restricted truncated proportional rule* $\mathrm{RTProp} : \mathrm{BR}^N \to \mathbb{R}_+^N$ is the bankruptcy rule which assigns to any $(E, c) \in \mathrm{BR}^N$ the payoff allocation

$$\mathrm{RTProp}(E, c) = \begin{cases} \mathrm{Prop}(E, c) & \text{if } c < u^E; \\ (tu_S^E, 0_{N\setminus S}) & \text{otherwise,} \end{cases}$$

where $S = \{i \in N \mid c_i \geq u_i^E\}$ and $t \in [0, 1]$ is such that $\mathrm{RTProp}(E, c) \in \mathrm{WP}(E)$.

The restricted truncated proportional rule satisfies relative symmetry, truncation invariance, and independence of unclaimed alternatives, but does not satisfy claims continuity. The constrained relative equal awards rule satisfies relative symmetry, truncation invariance, and claims continuity, but does not satisfy independence of unclaimed alternatives. The proportional rule satisfies relative symmetry, independence of unclaimed alternatives, and claims continuity, but does not satisfy truncation invariance. The constrained equal awards rule satisfies truncation invariance, independence of unclaimed alternatives, and claims continuity, but does not satisfy relative symmetry. This is summarized in the following table.

	$\mathcal{T}$(Prop)	RTProp	CREA	Prop	CEA
relative symmetry	+	+	+	+	−
truncation invariance	+	+	+	−	+
indep. of unclaimed alternatives	+	+	−	+	+
claims continuity	+	−	+	+	+

This means that the properties in Theorem 5.2.5 are independent.

5.3 Claims Axioms

In this section, we introduce and study the implications of axioms concerning changes in the claims. Several axioms concerning changes in the disagreement point of bargaining problems have been proposed in the literature. We translate these properties to the domain of bankruptcy problems in such a way that they concern similar changes in the vector of claims while the estate is required to be fixed.

Definition (Claims Axioms)
A bankruptcy rule $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ satisfies

- *claims linearity* if $f(E,c) = f(E, \theta c + (1-\theta)c')$ for all $(E,c),(E,c') \in \mathrm{BR}^N$ for which $f(E,c) = f(E,c')$ and any $\theta \in \mathbb{R}$ for which $(E, \theta c + (1-\theta)c') \in \mathrm{BR}^N$;
- *weak claims linearity* if $f(E,c) = f(E, \theta c + (1-\theta)f(E,c))$ for all $(E,c) \in \mathrm{BR}^N$ and any $\theta \in \mathbb{R}_+$;
- *claims convexity* if $f(E,c) = f(E, \theta c + (1-\theta)c')$[2] for all $(E,c),(E,c') \in \mathrm{BR}^N$ for which $f(E,c) = f(E,c')$ and any $\theta \in [0,1]$;
- *weak claims convexity* if $f(E,c) = f(E, \theta c + (1-\theta)f(E,c))$ for all $(E,c) \in \mathrm{BR}^N$ and any $\theta \in [0,1]$.

The claims linearity axiom describes bankruptcy rules for which all claim vectors on the line connecting two claim vectors with equal outcomes lead to the same payoff allocation. The claims convexity axiom is based on a bargaining axiom of Livne (1988) and Chun and Thomson (1990). If there is uncertainty about which of the two claim vectors with equal outcomes applies, then any expected value leads to the same payoff allocation. The corresponding weaker axioms of claims linearity and claims convexity, which only require that linear or convex combinations of the claim vector and its outcome lead to the same payoff allocation, are based on bargaining axioms of Peters and Van Damme (1991) and Peters (2010).

The following lemma studies the relations between the claims axioms.

Lemma 5.3.1
Let $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ be a bankruptcy rule.

(i) If f satisfies claims linearity, then f satisfies claims convexity.

(ii) If f satisfies claims linearity, then f satisfies weak claims linearity.

(iii) If f satisfies claims convexity, then f satisfies weak claims convexity.

(iv) If f satisfies weak claims linearity, then f satisfies weak claims convexity.

[2] Note that $(E, \theta c + (1-\theta)c') \in \mathrm{BR}^N$ for all $(E,c),(E,c') \in \mathrm{BR}^N$ for which $f(E,c) = f(E,c')$ and any $\theta \in [0,1]$.

Proof. (i) Assume that f satisfies claims linearity. Let $(E, c), (E, c') \in \mathrm{BR}^N$ be such that $f(E, c) = f(E, c')$ and let $\theta \in [0, 1]$. Then $(E, \theta c + (1 - \theta)c') \in \mathrm{BR}^N$. By claims linearity, $f(E, c) = f(E, \theta c + (1 - \theta)c')$. Hence, f satisfies claims convexity.

(ii) Assume that f satisfies claims linearity. Let $(E, c) \in \mathrm{BR}^N$ and let $\theta \in \mathbb{R}_+$. Then $f(E, c) = f(E, f(E, c))$ and $(E, \theta c + (1-\theta) f(E, c)) \in \mathrm{BR}^N$. By claims linearity, $f(E, c) = f(E, \theta c + (1 - \theta) f(E, c))$. Hence, f satisfies weak claims linearity.

(iii) Assume that f satisfies claims convexity. Let $(E, c) \in \mathrm{BR}^N$ and let $\theta \in [0, 1]$. Then $f(E, c) = f(E, f(E, c))$. By claims convexity, $f(E, c) = f(E, \theta c + (1 - \theta) f(E, c))$. Hence, f satisfies weak claims convexity.

(iv) Assume that f satisfies weak claims linearity. Let $(E, c) \in \mathrm{BR}^N$ and let $\theta \in [0, 1]$. By weak claims linearity, $f(E, c) = f(E, \theta c + (1 - \theta) f(E, c))$. Hence, f satisfies weak claims convexity. □

As shown by the following two rules, the axioms weak claims linearity and claims convexity are independent.

The *restricted constrained relative equal awards rule* $\mathrm{RCREA} : \mathrm{BR}^N \to \mathbb{R}^N_+$, the bankruptcy rule which assigns to any $(E, c) \in \mathrm{BR}^N$ the payoff allocation

$$\mathrm{RCREA}(E, c) = \begin{cases} \mathrm{CREA}(E, c) & \text{if } c < u^E \text{ or } c \geq \lambda^{E,u^E} u^E; \\ (t u^E_S, 0_{N \setminus S}) & \text{otherwise,} \end{cases}$$

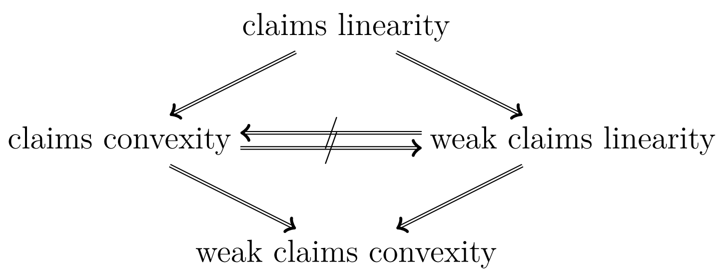
where $S = \{i \in N \mid c_i \geq u^E_i\}$ and $t \in [0, 1]$ is such that $\mathrm{RCREA}(E, c) \in \mathrm{WP}(E)$, satisfies claims convexity, but does not satisfy weak claims linearity.

The bankruptcy rule $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ which assigns to any $(E, c) \in \mathrm{BR}^N$ the payoff allocation

$$f(E, c) = \begin{cases} (\hat{c}^E_1, \min\{c_2, t u^E_2\}, \min\{c_3, t u^E_3\}) & \text{if } N = \{1, 2, 3\} \text{ and } c_2 u^E_3 = c_3 u^E_2; \\ \mathrm{CREA}(E, c) & \text{otherwise,} \end{cases}$$

where $t \in [0, 1]$ is such that $f(E, c) \in \mathrm{WP}(E)$, satisfies weak claims linearity, but does not satisfy claims convexity.

The relations of all claims axioms can be summarized by the following diagram.



The constrained relative equal awards rule is not the unique rule satisfying relative symmetry, truncation invariance, and claims convexity, since the restricted constrained relative equal awards rule also satisfies these three properties. However, the constrained relative equal awards rule is the only bankruptcy rule satisfying relative symmetry, truncation invariance, and weak claims linearity.

Theorem 5.3.2
The constrained relative equal awards rule is the unique bankruptcy rule satisfying relative symmetry, truncation invariance, and weak claims linearity.

Proof. By Lemma 3.A.1, Lemma 3.A.2, and Lemma 5.A.4, the constrained relative equal awards rule satisfies relative symmetry, truncation invariance, and weak claims linearity. Let $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ be a bankruptcy rule satisfying relative symmetry, truncation invariance, and weak claims linearity. Let $(E, c) \in \mathrm{BR}^N$. If $c \in E$, then $f(E, c) = c = \mathrm{CREA}(E, c)$. Suppose that $c \notin E$. Denote $S = \{i \in N \mid f_i(E, c) < c_i\}$. Then $S \neq \emptyset$. Let $x = \theta c + (1 - \theta)f(E, c)$ for some $\theta \in \mathbb{R}_+$ be such that $x_S \geq u^E_S$. Then $\hat{x}^E_S = u^E_S$ and $\hat{x}^E_i u^E_j = u^E_i u^E_j = \hat{x}^E_j u^E_i$ for all $i, j \in S$. Since f satisfies relative symmetry, this means that $f_S(E, \hat{x}^E) = tu^E_S$ for some $t \in [0, 1]$. Since f satisfies truncation invariance, this implies that $f_S(E, x) = f_S(E, \hat{x}^E) = tu^E_S$. Since f satisfies weak claims linearity, $f_S(E, c) = f_S(E, x) = tu^E_S$. Then $f_S(E, c) \leq \alpha^{E,c} u^E_S$, since otherwise $f(E, c) \geq \mathrm{CREA}(E, c)$ and $f(E, c) \neq \mathrm{CREA}(E, c)$, which contradicts that E is nonleveled.

Suppose that there exists an $i \in N \setminus S$ such that $f_i(E, c) > \alpha^{E,c} u^E_i$. Then $f_j(E, c)u^E_i \leq \alpha^{E,c} u^E_j u^E_i < f_i(E, c)u^E_j$ for all $j \in S$. Let $y = \theta c + (1-\theta)f(E, c)$ for some $\theta \in \mathbb{R}_{++}$ be such that $y_j u^E_i = f_i(E, c)u^E_j$ for some $j \in S$. Then $y_i u^E_j = f_i(E, c)u^E_j = y_j u^E_i$. Since f satisfies relative symmetry, this means that $f_i(E, y)u^E_j = f_j(E, y)u^E_i$. Since f satisfies weak claims linearity, this implies that $f_i(E, c)u^E_j = f_j(E, c)u^E_i$. This is a contradiction. Hence, $f_i(E, c) \leq \min\{c_i, \alpha^{E,c} u^E_i\} = \mathrm{CREA}_i(E, c)$ for all $i \in N$. Since E is nonleveled, this implies that $f(E, c) = \mathrm{CREA}(E, c)$. □

The truncated proportional rule satisfies relative symmetry and truncation invariance, but does not satisfy weak claims linearity. The proportional rule satisfies relative symmetry and weak claims linearity, but does not satisfy truncation invariance. The constrained equal awards rule satisfies truncation invariance and weak claims linearity, but does not satisfy relative symmetry. This is summarized in the following table.

	CREA	$\mathcal{T}$(Prop)	Prop	CEA
relative symmetry	+	+	+	−
truncation invariance	+	+	−	+
weak claims linearity	+	−	+	+

This means that the properties in Theorem 5.3.2 are independent.

The axioms concerning changes in the claims can also be combined with the independence of unclaimed alternatives axiom. The proportional rule is not the unique rule satisfying relative symmetry and independence of unclaimed alternatives, since the truncated proportional rule also satisfies these two properties. However, if weak claims linearity is required in addition, then these properties are only satisfied by the proportional rule.

Theorem 5.3.3
The proportional rule is the unique bankruptcy rule satisfying relative symmetry, independence of unclaimed alternatives, and weak claims linearity.

Proof. By Lemma 5.2.1, Lemma 5.3.1, Lemma 5.A.1, and Lemma 5.A.5, the proportional rule satisfies independence of unclaimed alternatives and weak claims linearity. Let $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ be a bankruptcy rule satisfying relative symmetry, independence of unclaimed alternatives, and weak claims linearity. Let $(E, c) \in \mathrm{BR}^N$. If $E = \{0_N\}$, then $f(E, c) = 0_N = \mathrm{Prop}(E, c)$. Suppose that $E \neq \{0_N\}$. Then $u^E \in \mathbb{R}^N_{++}$ and $N^c_+ \neq \emptyset$. If $|N^c_+| = 1$, then $f(E, c) = (u^E_{N^c_+}, 0_{N \setminus N^c_+}) = \mathrm{Prop}(E, c)$. Suppose that $|N^c_+| \geq 2$. Let $x = \theta c + (1-\theta)\mathrm{Prop}(E, c)$ for some $\theta \in (0, 1]$ be such that $x < u^E$. Since f satisfies relative symmetry and independence of unclaimed alternatives, Lemma 5.2.4 implies that $f(E, x) = \mathrm{Prop}(E, x) = \mathrm{Prop}(E, c)$. Since f satisfies weak claims linearity, this implies that $f(E, c) = f(E, \frac{1}{\theta}x + (1 - \frac{1}{\theta})f(E, x)) = f(E, x) = \mathrm{Prop}(E, c)$. □

To show that relative symmetry and independence of unclaimed alternatives are independent of any claims axiom, we introduce two other rules. For any $(E, c) \in \mathrm{BR}^N$ for which $N = N^c_+ = \{1, 2\}$, $E \neq \{0_N\}$, and $c \notin E$, let $\xi^{E,c} \in \mathrm{WP}(E)$ be defined such that $\sqrt{\frac{\xi^{E,c}_2}{\xi^{E,c}_1}} = \frac{c_2 - \xi^{E,c}_2}{c_1 - \xi^{E,c}_1}$. Note that $\xi^{E,c}$ exists and is uniquely defined.

The bankruptcy rule $\psi^1 : \mathrm{BR}^N \to \mathbb{R}^N_+$ assigns to any $(E, c) \in \mathrm{BR}^N$ the payoff allocation

$$\psi^1(E, c) = \begin{cases} \xi^{E,c} & \text{if } N = N^c_+ = \{1, 2\},\ E = \{x \in \mathbb{R}^N_+ \mid x_1 + x_2 \leq 1\}, \\ & \text{and } c \notin E; \\ \mathrm{Prop}(E, c) & \text{otherwise.} \end{cases}$$

The bankruptcy rule $\psi^2 : \mathrm{BR}^N \to \mathbb{R}^N_+$ assigns to any $(E, c) \in \mathrm{BR}^N$ the payoff allocation

$$\psi^2(E, c) = \begin{cases} \xi^{E,c} & \text{if } N = N^c_+ = \{1, 2\},\ E \neq \{0_N\}, \text{ and } c \notin E; \\ \mathrm{Prop}(E, c) & \text{otherwise.} \end{cases}$$

The truncated proportional rule satisfies relative symmetry and independence of unclaimed alternatives, but does not satisfy weak claims convexity. The bankruptcy rule ψ^1 satisfies relative symmetry and claims linearity, but does not satisfy independence of unclaimed alternatives. The bankruptcy rule ψ^2 satisfies independence of unclaimed alternatives and claims linearity, but does not satisfy relative symmetry. This is summarized in the following table.

	Prop	$\mathcal{T}$(Prop)	ψ^1	ψ^2
relative symmetry	+	+	+	−
independence of unclaimed alternatives	+	+	−	+
claims linearity	+	−	+	+
weak claims linearity	+	−	+	+
claims convexity	+	−	+	+
weak claims convexity	+	−	+	+

This means that relative symmetry and independence of unclaimed alternatives are independent of any claims axiom. In particular, the properties in Theorem 5.3.3 are independent. Moreover, the properties in the axiomatic characterization of the proportional rule remain independent if weak claims linearity in Theorem 5.3.3 is strengthened to claims linearity.

The proportional rule is not the unique rule satisfying relative symmetry, independence of unclaimed alternatives, and claims convexity. The *restricted proportional rule* RProp : $\mathrm{BR}^N \to \mathbb{R}^N_+$, the bankruptcy rule which assigns to any $(E, c) \in \mathrm{BR}^N$ the payoff allocation

$$\mathrm{RProp}(E, c) = \begin{cases} \mathrm{Prop}(E, c) & \text{if } c < u^E; \\ (tc_S, 0_{N \setminus S}) & \text{otherwise,} \end{cases}$$

where $S = \{i \in N \mid \forall_{j \in N} : c_j u_i^E \leq c_i u_j^E\}$ and $t \in [0, 1]$ is such that $\mathrm{RProp}(E, c) \in \mathrm{WP}(E)$, also satisfies relative symmetry, independence of unclaimed alternatives, and claims convexity. However, if positive claimants are required to get positive awards, then these properties are only satisfied by the proportional rule.

Definition (Positive Awards)
A bankruptcy rule $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ satisfies *positive awards* if $f_{N_+^c}(E, c) > 0_{N_+^c}$ for all $(E, c) \in \mathrm{BR}^N$ for which $E \neq \{0_N\}$.

Theorem 5.3.4
The proportional rule is the unique bankruptcy rule satisfying relative symmetry, independence of unclaimed alternatives, weak claims convexity, and positive awards.

Proof. By Lemma 5.2.1, Lemma 5.3.1, Lemma 5.A.1, Lemma 5.A.5, and Lemma 5.A.6, the proportional rule satisfies independence of unclaimed alternatives, weak claims convexity, and positive awards. Let $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ be a bankruptcy rule satisfying relative symmetry, independence of unclaimed alternatives, weak claims convexity, and positive awards. Let $(E, c) \in \mathrm{BR}^N$. If $E = \{0_N\}$, then $f(E, c) = 0_N = \mathrm{Prop}(E, c)$. Suppose that $E \neq \{0_N\}$. Then $u^E \in \mathbb{R}^N_{++}$ and $N^c_+ \neq \emptyset$. If $|N^c_+| = 1$, then $f(E, c) = (u^E_{N^c_+}, 0_{N \setminus N^c_+}) = \mathrm{Prop}(E, c)$. Suppose that $|N^c_+| \geq 2$. Since f satisfies positive awards, there exists an $x \in \mathbb{R}^N_+$ with $x = \theta c + (1-\theta) f(E, c)$ for some $\theta \in (0, 1]$ such that $x < u^E$. Since f satisfies relative symmetry and independence of unclaimed alternatives, Lemma 5.2.4 implies that $f(E, x) = \mathrm{Prop}(E, x)$. Since f satisfies weak claims convexity, this implies that $f(E, c) = f(E, x) = \mathrm{Prop}(E, x) = \mathrm{Prop}(E, c)$. □

The restricted proportional rule satisfies relative symmetry, independence of unclaimed alternatives, and claims convexity, but does not satisfy positive awards. The truncated proportional rule satisfies relative symmetry, independence of unclaimed alternatives, and positive awards, but does not satisfy weak claims convexity. The constrained relative equal awards rule satisfies relative symmetry, claims convexity, and positive awards, but does not satisfy independence of unclaimed alternatives. The constrained equal awards rule satisfies independence of unclaimed alternatives, claims convexity, and positive awards, but does not satisfy relative symmetry. This is summarized in the following table.

	Prop	RProp	$\mathcal{T}$(Prop)	CREA	CEA
relative symmetry	+	+	+	+	−
indep. of unclaimed alternatives	+	+	+	−	+
claims convexity	+	+	−	+	+
weak claims convexity	+	+	−	+	+
positive awards	+	−	+	+	+

This means that the properties in Theorem 5.3.4 are independent. Moreover, the properties in the axiomatic characterization of the proportional rule remain independent if weak claims convexity in Theorem 5.3.4 is strengthened to claims convexity.

In this chapter, we derived new axiomatic characterizations of the proportional rule, the truncated proportional rule, and the constrained relative equal awards rule for bankruptcy problems with nontransferable utility using axioms from bargaining theory. An overview of the corresponding properties, including the bankruptcy axioms relative symmetry and truncation invariance, the axioms concerning changes in the estate, the axioms concerning changes in the claims, and the weak technical requirements claims continuity and positive awards, is provided in the following table. The constrained equal awards rule is included for illustrative purposes.

	Prop	$\mathcal{T}$(Prop)	CREA	CEA
relative symmetry	+	+	+	−
truncation invariance	−	+	+	+
step-by-step negotiations	+	−	−	+
estate monotonicity	+	−	−	+
domination	+	−	−	+
independence of irrelevant alternatives	+	−	−	+
independence of undominating alternatives	+	−	−	+
independence of unclaimed alternatives	+	+	−	+
claims continuity	+	+	+	+
claims linearity	+	−	−	−
weak claims linearity	+	−	+	+
claims convexity	+	−	+	+
weak claims convexity	+	−	+	+
positive awards	+	+	+	+

The following table provides an overview of the axiomatic characterizations derived in this chapter.

	Prop	Prop	Prop	Prop	$\mathcal{T}$(Prop)	CREA
relative symmetry	⋆	⋆	⋆	⋆	⋆	⋆
truncation invariance					⋆	⋆
indep. of irrelevant alt.	⋆					
indep. of undominating alt.		⋆				
indep. of unclaimed alt.			⋆	⋆	⋆	
claims continuity					⋆	
weak claims linearity			⋆			⋆
weak claims convexity				⋆		
positive awards				⋆		

5.A Appendix

Lemma 5.A.1
The proportional rule satisfies step-by-step negotiations.

Proof. Let $(E, c), (E', c) \in \text{BR}^N$ be such that $E' \subseteq E$. Then $\text{Prop}(E, c) = \lambda^{E,c} c$ and

$$\begin{aligned} & \text{Prop}(E', c) + \text{Prop}((E - \{\text{Prop}(E', c)\})_+, c - \text{Prop}(E', c)) \\ = \; & \lambda^{E',c} c + \text{Prop}((E - \{\lambda^{E',c} c\})_+, c - \lambda^{E',c} c) \\ = \; & \lambda^{E',c} c + \lambda^{(E - \{\lambda^{E',c} c\})_+, (1-\lambda^{E',c})c} (1 - \lambda^{E',c}) c \\ = \; & \left(\lambda^{E',c} + \lambda^{(E - \{\lambda^{E',c} c\})_+, c} \right) c. \end{aligned}$$

Since E is nonleveled, this implies that

$$\text{Prop}(E, c) = \text{Prop}(E', c) + \text{Prop}((E - \{\text{Prop}(E', c)\})_+, c - \text{Prop}(E', c)).$$

Hence, the proportional rule satisfies step-by-step negotiations. □

Lemma 5.A.2
The truncated proportional rule satisfies independence of unclaimed alternatives.

Proof. Let $(E, c), (E', c) \in \text{BR}^N$ be such that $\hat{E}_c = \hat{E}'_c$. Then $\hat{E}_{\hat{c}^E} = \hat{E}_c = \hat{E}'_c = \hat{E}'_{\hat{c}^{E'}}$ and $\hat{c}^E = u^{\hat{E}_c} = u^{\hat{E}'_c} = \hat{c}^{E'}$. By Lemma 5.2.1 and Lemma 5.A.1, the proportional rule satisfies independence of unclaimed alternatives. Then

$$\mathcal{T}(\text{Prop})(E, c) = \text{Prop}(E, \hat{c}^E) = \text{Prop}(E', \hat{c}^{E'}) = \mathcal{T}(\text{Prop})(E', c).$$

Hence, the truncated proportional rule satisfies independence of unclaimed alternatives. □

Lemma 5.A.3
The truncated proportional rule satisfies claims continuity.

Proof. Let $(E, c) \in \text{BR}^N$. Then $\lim_{x \to c} \hat{x}^E = \hat{c}^E$, $\lim_{x \to c} \lambda^{E, \hat{x}^E} = \lambda^{E, \hat{c}^E}$, and

$$\begin{aligned} \lim_{x \to c} \mathcal{T}(\text{Prop})(E, x) &= \lim_{x \to c} \text{Prop}(E, \hat{x}^E) = \lim_{x \to c} \lambda^{E, \hat{x}^E} \hat{x}^E = \lim_{x \to c} \lambda^{E, \hat{x}^E} \lim_{x \to c} \hat{x}^E \\ &= \lambda^{E, \hat{c}^E} \hat{c}^E \qquad = \text{Prop}(E, \hat{c}^E) = \mathcal{T}(\text{Prop})(E, c). \end{aligned}$$

Hence, the truncated proportional rule satisfies claims continuity. □

Lemma 5.A.4
The constrained relative equal awards rule satisfies weak claims linearity.

Proof. Let $(E, c) \in \mathrm{BR}^N$ and let $\theta \in \mathbb{R}_+$. Suppose that $\alpha^{E,\theta c+(1-\theta)\mathrm{CREA}(E,c)} \geq \alpha^{E,c}$. Then

$$\begin{aligned} &\mathrm{CREA}_i(E, \theta c + (1-\theta)\mathrm{CREA}(E,c)) \\ =\ &\min\{\theta c_i + (1-\theta)\mathrm{CREA}_i(E,c), \alpha^{E,\theta c+(1-\theta)\mathrm{CREA}(E,c)} u_i^E\} \\ =\ &\min\{\mathrm{CREA}_i(E,c) + \theta(c_i - \mathrm{CREA}_i(E,c)), \alpha^{E,\theta c+(1-\theta)\mathrm{CREA}(E,c)} u_i^E\} \\ \geq\ &\min\{\mathrm{CREA}_i(E,c), \alpha^{E,c} u_i^E\} \\ =\ &\mathrm{CREA}_i(E,c) \end{aligned}$$

for all $i \in N$. Since E is nonleveled, this implies that

$$\mathrm{CREA}(E,c) = \mathrm{CREA}(E, \theta c + (1-\theta)\mathrm{CREA}(E,c)).$$

Now, suppose that $\alpha^{E,\theta c+(1-\theta)\mathrm{CREA}(E,c)} \leq \alpha^{E,c}$. Then

$$\begin{aligned} &\mathrm{CREA}_i(E, \theta c + (1-\theta)\mathrm{CREA}(E,c)) \\ =\ &\min\{\theta c_i + (1-\theta)\mathrm{CREA}_i(E,c), \alpha^{E,\theta c+(1-\theta)\mathrm{CREA}(E,c)} u_i^E\} \\ =\ &\min\{\mathrm{CREA}_i(E,c) + \theta(c_i - \mathrm{CREA}_i(E,c)), \alpha^{E,\theta c+(1-\theta)\mathrm{CREA}(E,c)} u_i^E\} \\ \leq\ &\min\{\mathrm{CREA}_i(E,c) + \theta(c_i - \mathrm{CREA}_i(E,c)), \alpha^{E,c} u_i^E\} \\ =\ &\mathrm{CREA}_i(E,c) \end{aligned}$$

for all $i \in N$. Since E is nonleveled, this implies that

$$\mathrm{CREA}(E,c) = \mathrm{CREA}(E, \theta c + (1-\theta)\mathrm{CREA}(E,c)).$$

Hence, the constrained relative equal awards rule satisfies weak claims linearity. □

Lemma 5.A.5
The proportional rule satisfies claims linearity.

Proof. Let $(E, c), (E, c') \in \mathrm{BR}^N$ be such that $\mathrm{Prop}(E,c) = \mathrm{Prop}(E,c')$, and let $\theta \in \mathbb{R}$ be such that $(E, \theta c + (1-\theta)c') \in \mathrm{BR}^N$. If $E = \{0_N\}$, then $\mathrm{Prop}(E,c) = 0_N = \mathrm{Prop}(E, \theta c + (1-\theta)c')$. Suppose that $E \neq \{0_N\}$. Then $\lambda^{E,c} c = \lambda^{E,c'} c'$ and

$$\mathrm{Prop}(E, \theta c + (1-\theta)c') = \lambda^{E,\theta c+(1-\theta)c'}(\theta c + (1-\theta)c') = \lambda^{E,\theta c+(1-\theta)c'}\left(\theta + (1-\theta)\frac{\lambda^{E,c}}{\lambda^{E,c'}}\right) c.$$

Since E is nonleveled, this implies that $\mathrm{Prop}(E,c) = \mathrm{Prop}(E, \theta c + (1-\theta)c')$. Hence, the proportional rule satisfies claims linearity. □

Lemma 5.A.6
The proportional rule satisfies positive awards.

Proof. Let $(E, c) \in \mathrm{BR}^N$ be such that $E \neq \{0_N\}$ and let $i \in N_+^c$. Then $\mathrm{Prop}_i(E,c) = \lambda^{E,c} c_i > 0$. Hence, the proportional rule satisfies positive awards. □

6 Bankruptcy Games

6.1 Introduction

Already in the work of O'Neill (1982), bankruptcy problems with transferable utility are analyzed from a game theoretic perspective by studying a corresponding bankruptcy game. One model for bankruptcy games with nontransferable utility was introduced by Orshan, Valenciano, and Zarzuelo (2003). However, Estévez-Fernández, Borm, and Fiestras-Janeiro (2014) pointed out that coalitions can attain payoff allocations outside the estate in this game, which contradicts the original idea of O'Neill (1982). They redefined NTU-bankruptcy games to stay in line with this original idea, while focusing on convexity and compromise stability. However, their model for NTU-bankruptcy games does not straightforwardly generalize the original TU-bankruptcy games in the sense that NTU-bankruptcy games corresponding to NTU-bankruptcy problems induced by TU-bankruptcy problems are different from the NTU-games induced by TU-bankruptcy games.

This chapter, based on Dietzenbacher (2018), introduces a slightly modified version of the model of Orshan, Valenciano, and Zarzuelo (2003) for NTU-bankruptcy games which both generalizes the model for TU-bankruptcy games and stays in line with the idea of O'Neill (1982). Focusing on the structure of the core, we analyze NTU-bankruptcy games along the lines of Curiel, Maschler, and Tijs (1987). In particular, generalizing the core cover and the reasonable set, we define compromise stability and reasonable stability for NTU-games. Contrary to TU-games, the generalized reasonable set does not necessarily contain the core of an NTU-game and consequently reasonable stability does not imply compromise stability. Interestingly, the core of an NTU-bankruptcy game still coincides with the core cover and the reasonable set, which leads to a compact expression for the core of bankruptcy games.

Besides, we show that a bankruptcy rule is game theoretic if and only if it satisfies truncation invariance. This means that there exists a solution for NTU-games which coincides on the class of bankruptcy games with a certain bankruptcy rule if and only if this bankruptcy rule satisfies truncation invariance.

This chapter is organized in the following way. In Section 6.2, we generalize some notions for transferable utility games to nontransferable utility games. Section 6.3 discusses the modeling of NTU-bankruptcy games. In Section 6.4, we study the core and some properties of a modified model for NTU-bankruptcy games.

6.2 Compromise stability and reasonable stability

Let $V \in \mathrm{NTU}^N$ be a monotonic nontransferable utility game. Similar to Otten, Borm, Peleg, and Tijs (1998), we define $M^\sigma(V) \in \mathbb{R}^N_+$ corresponding to $\sigma \in \Pi(N)$ by

$$M^\sigma_{\sigma(k)}(V) = \max\left\{x \in \mathbb{R}_+ \;\middle|\; (M^\sigma_{\sigma(1)}(V), \ldots, M^\sigma_{\sigma(k-1)}(V), x) \in V(\{\sigma(1), \ldots, \sigma(k)\})\right\}$$

for all $k \in \{1, \ldots, |N|\}$. Note that the conditions on V imply that this maximum exists. As in the context of TU-games, $M^\sigma_{\sigma(k)}(V)$ can be interpreted as the maximal payoff of player $\sigma(k) \in N$ in a certain order $\sigma \in \Pi(N)$ when joining the predecessors, which have already been allocated their contributions. For any game $V \in \mathrm{NTU}^N$ for which $V(S) = \{x \in \mathbb{R}^S_+ \mid \sum_{i\in S} x_i \leq v(S)\}$ for all $S \in 2^N \setminus \{\emptyset\}$, induced by a nonnegative monotonic game $v \in \mathrm{TU}^N$, $M^\sigma(V) = M^\sigma(v)$ for all $\sigma \in \Pi(N)$.

Inspired by Borm, Keiding, McLean, Oortwijn, and Tijs (1992), we define $K(V) \in \mathbb{R}^N_+$ by

$$K_i(V) = \max\left\{x_i \;\middle|\; x \in V(N), x_{N\setminus\{i\}} \in \mathrm{SUC}(V(N \setminus \{i\}))\right\}$$

for all $i \in N$, and $k(V) \in \mathbb{R}^N_+$ by

$$k_i(V) = \max_{S\in 2^N : i\in S} \sup\left\{x \in \mathbb{R}_+ \;\middle|\; (x, K_{S\setminus\{i\}}(V)) \in V(S)\right\}$$

for all $i \in N$. Note that the conditions on V imply that these maxima exist. As in the context of TU-games, $K_i(V)$ can be interpreted as the maximal payoff of player $i \in N$ within an allocation of $V(N)$ which is stable against a coalitional deviation of the other players together. Moreover, $k_i(V)$ can be interpreted as the maximal payoff of player $i \in N$ which can be obtained within some coalition $S \in 2^N$ for which $i \in S$ when each other member $j \in S \setminus \{i\}$ is allocated $K_j(V)$. For any game $V \in \mathrm{NTU}^N$ for which $V(S) = \{x \in \mathbb{R}^S_+ \mid \sum_{i\in S} x_i \leq v(S)\}$ for all $S \in 2^N \setminus \{\emptyset\}$, induced by a nonnegative monotonic game $v \in \mathrm{TU}^N$, $K(V) = K(v)$ and $k(V) = k(v)$.

Using these notions, we can generalize the core cover and the reasonable set to the class of nontransferable utility games. Let $V \in \text{NTU}^N$ be a monotonic nontransferable utility game. The *core cover* is defined by

$$\mathcal{CC}(V) = \{x \in \text{SP}(V(N)) \mid k(V) \leq x \leq K(V)\},$$

and the *reasonable set* is defined by

$$\mathcal{R}(V) = \left\{ x \in \text{SP}(V(N)) \;\middle|\; \forall_{i \in N} : \min_{\sigma \in \Pi(N)} M_i^\sigma(V) \leq x_i \leq \max_{\sigma \in \Pi(N)} M_i^\sigma(V) \right\}.$$

Lemma 6.2.1
Let $V \in \text{NTU}^N$ be monotonic. Then $\mathcal{C}^{\mathcal{S}}(V) \subseteq \mathcal{CC}(V)$.

Proof. Let $x \in \mathcal{C}^{\mathcal{S}}(V)$. Then

$$\begin{aligned} x_i &\leq \max\{y_i \mid y \in \mathcal{C}^{\mathcal{S}}(V)\} \\ &= \max\{y_i \mid y \in V(N), \forall_{S \in 2^N \setminus \{\emptyset\}} : y_S \in \text{SUC}(V(S))\} \\ &\leq \max\{y_i \mid y \in V(N), y_{N \setminus \{i\}} \in \text{SUC}(V(N \setminus \{i\}))\} \\ &= K_i(V) \end{aligned}$$

for all $i \in N$. Suppose that there exists an $i \in N$ such that $x_i < k_i(V)$. Let $S \in 2^N$ be such that $i \in S$ and $(k_i(V), K_{S \setminus \{i\}}(V)) \in V(S)$. Then $x_S \leq (k_i(V), K_{S \setminus \{i\}}(V))$ and $x_S \neq (k_i(V), K_{S \setminus \{i\}}(V))$. This means that $x_S \notin \text{SUC}(V(S))$, which contradicts that $x \in \mathcal{C}^{\mathcal{S}}(V)$. Hence, $k(V) \leq x \leq K(V)$ and $x \in \mathcal{CC}(V)$. □

Lemma 6.2.1 shows that the core cover indeed contains the strong core. NTU-games for which the core cover coincides with the nonempty strong core are called compromise stable.

Definition (Compromise Stability)
A monotonic game $V \in \text{NTU}^N$ is *compromise stable* if $\mathcal{C}^{\mathcal{S}}(V) \neq \emptyset$ and $\mathcal{C}^{\mathcal{S}}(V) = \mathcal{CC}(V)$.

Contrary to TU-games, the following example shows that the reasonable set does not necessarily contain the strong core of an NTU-game.

Example 6.1
Let $N = \{1, 2, 3\}$ and consider the monotonic game $V \in \text{NTU}^N$ given by

$$V(S) = \begin{cases} \{x \in \mathbb{R}_+^S \mid x_1^2 + x_2^2 \leq (9 - x_3)^2, x_3 \leq 9\} & \text{if } S = N; \\ \{x \in \mathbb{R}_+^S \mid x_1 + x_2 \leq 4\} & \text{if } S = \{1, 2\}; \\ \{0_S\} & \text{otherwise} \end{cases}$$

for all $S \in 2^N \setminus \{\emptyset\}$.

The vectors $M^\sigma(V) \in \mathbb{R}^N_+$ corresponding to all $\sigma \in \Pi(N)$ are presented in the following table.

σ	$M^\sigma_1(V)$	$M^\sigma_2(V)$	$M^\sigma_3(V)$
$(1,2,3)$	0	4	5
$(1,3,2)$	0	9	0
$(2,1,3)$	4	0	5
$(2,3,1)$	9	0	0
$(3,1,2)$	0	9	0
$(3,2,1)$	9	0	0

This means that the reasonable set is given by

$$\mathcal{R}(V) = \{x \in \mathrm{SP}(V(N)) \mid 0 \leq x_1 \leq 9, 0 \leq x_2 \leq 9, 0 \leq x_3 \leq 5\}\,.$$

One can verify that $(2, 2, 9 - 2\sqrt{2}) \in \mathcal{C}^{\mathcal{S}}(V) \setminus \mathcal{R}(V)$. Hence, $\mathcal{C}^{\mathcal{S}}(V) \not\subseteq \mathcal{R}(V)$. △

The minimal and maximal contributions can still be considered as reasonable bounds for payoff allocations. NTU-games for which the strong core coincides with the reasonable set are called reasonable stable.

Definition (Reasonable Stability)
A monotonic game $V \in \mathrm{NTU}^N$ is *reasonable stable* if $\mathcal{C}^{\mathcal{S}}(V) = \mathcal{R}(V)$.

Note that reasonable stability is stronger than *marginal convexity* (cf. Hendrickx, Borm, and Timmer (2002)), which requires that $M^\sigma(V) \in \mathcal{C}^{\mathcal{S}}(V)$ for all $\sigma \in \Pi(N)$.

6.3 Modeling bankruptcy games

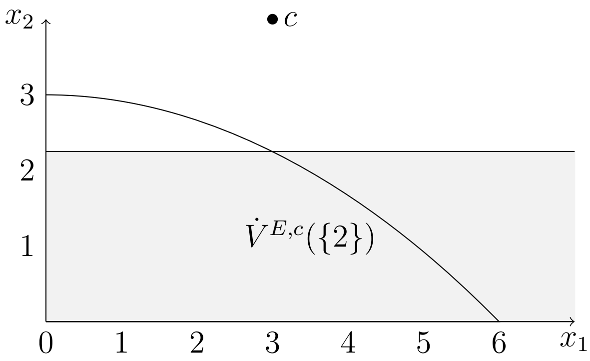
This section discusses the modeling of bankruptcy games with nontransferable utility. Since NTU-bankruptcy problems generalize TU-bankruptcy problems, and NTU-games generalize TU-games, an appropriate model for NTU-bankruptcy games would generalize TU-bankruptcy games.

One model for NTU-bankruptcy games was introduced by Orshan, Valenciano, and Zarzuelo (2003). Their bankruptcy game with nontransferable utility $\dot{V}^{E,c}$ corresponding to the bankruptcy problem $(E, c) \in \mathrm{BR}^N$ boils down to

$$\dot{V}^{E,c}(S) = \mathrm{comp}\left(\left\{x \in \mathbb{R}^N_+ \;\middle|\; (x_S, c_{N \setminus S}) \in E \text{ or } x_S = 0_S\right\}\right)$$

for all $S \in 2^N \setminus \{\emptyset\}$.

Example 6.2
Let $N = \{1,2\}$ and consider the bankruptcy problem $(E,c) \in \mathrm{BR}^N$ given by $E = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 12x_2 \leq 36\}$ and $c = (3,4)$. Then $\dot{V}^{E,c}(\{1\}) = \{x \in \mathbb{R}^N_+ \mid x_1 = 0\}$, $\dot{V}^{E,c}(\{2\}) = \{x \in \mathbb{R}^N_+ \mid x_2 \leq 2\frac{1}{4}\}$, and $\dot{V}^{E,c}(N) = E$. This is illustrated as follows.



△

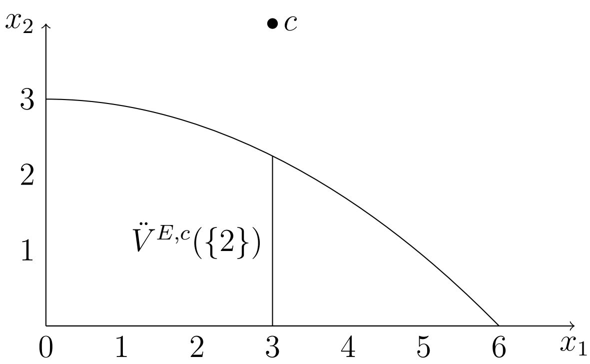
Estévez-Fernández, Borm, and Fiestras-Janeiro (2014) pointed out that coalitions can attain payoff allocations outside the estate in this game, as in Example 6.2, which contradicts the original idea of O'Neill (1982). They redefined NTU-bankruptcy games to stay in line with this original idea. Their bankruptcy game with nontransferable utility $\ddot{V}^{E,c}$ corresponding to the bankruptcy problem $(E,c) \in \mathrm{BR}^N$ boils down to $\ddot{V}^{E,c}(N) = E$ and

$$\ddot{V}^{E,c}(S) = \begin{cases} \mathrm{comp}_S(\{x \in \mathrm{SP}(E) \mid x_S \leq c_S, x_{N\setminus S} = c_{N\setminus S}\}) & \text{if } (0_S, c_{N\setminus S}) \in E; \\ \mathrm{comp}_S(\{x \in \mathrm{SP}(E) \mid x_S = 0_S, x_{N\setminus S} \leq c_{N\setminus S}\}) & \text{if } (0_S, c_{N\setminus S}) \notin E \end{cases}$$

for all $S \in 2^N \setminus \{\emptyset, N\}$, where

$$\mathrm{comp}_S(E) = \left\{x \in \mathbb{R}^N_+ \,\middle|\, \exists_{y\in E} : y_S \geq x_S, y_{N\setminus S} = x_{N\setminus S}\right\}.$$

Example 6.3
Let $N = \{1,2\}$ and consider the bankruptcy problem $(E,c) \in \mathrm{BR}^N$ given by $E = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 12x_2 \leq 36\}$ and $c = (3,4)$ as in Example 6.2. Then $\ddot{V}^{E,c}(\{1\}) = \{x \in \mathbb{R}^N_+ \mid x_1 = 0, x_2 = 3\}$, $\ddot{V}^{E,c}(\{2\}) = \{x \in \mathbb{R}^N_+ \mid x_1 = 3, x_2 \leq 2\frac{1}{4}\}$, and $\ddot{V}^{E,c}(N) = E$. This is illustrated as follows.



△

However, the following example shows that their model for NTU-bankruptcy games does not straightforwardly generalize the original TU-bankruptcy games in the sense that NTU-bankruptcy games corresponding to NTU-bankruptcy problems induced by TU-bankruptcy problems are different from the NTU-games induced by TU-bankruptcy games.

Example 6.4
Let $N = \{1, 2, 3\}$ and consider the bankruptcy problem $(e, c) \in \text{TUBR}^N$ given by $e = 4$ and $c = (1, 2, 3)$. The corresponding bankruptcy game $v^{e,c} \in \text{TU}^N$ is presented in the following table.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v^{e,c}(S)$	0	0	1	1	2	3	4

This induces the game $V \in \text{NTU}^N$ in which $V(\{2, 3\}) = \{x \in \mathbb{R}^{\{2,3\}}_+ \mid x_2 + x_3 \leq 3\}$. However, $\ddot{V}^{E,c}(\{2, 3\}) = \{x \in \mathbb{R}^N_+ \mid x_1 = 1, x_2 \leq 2, x_2 + x_3 \leq 3\}$ in the bankruptcy game corresponding to the induced bankruptcy problem $(E, c) \in \text{BR}^N$ in which $E = \{x \in \mathbb{R}^N_+ \mid x_1 + x_2 + x_3 \leq 4\}$. △

Next, we introduce a slightly modified version of the model of Orshan, Valenciano, and Zarzuelo (2003) for NTU-bankruptcy games, which generalizes TU-bankruptcy games and simultaneously stays in line with the original idea of O'Neill (1982).

Definition (Bankruptcy Game with Nontransferable Utility)
The *bankruptcy game with nontransferable utility* $V^{E,c} \in \text{NTU}^N$ corresponding to the bankruptcy problem $(E, c) \in \text{BR}^N$ is given by

$$V^{E,c}(S) = \begin{cases} \{x \in \mathbb{R}^S_+ \mid (x, c_{N\setminus S}) \in E\} & \text{if } (0_S, c_{N\setminus S}) \in E; \\ \{0_S\} & \text{if } (0_S, c_{N\setminus S}) \notin E \end{cases}$$

for all $S \in 2^N \setminus \{\emptyset\}$.

Note that $V^{E,c}$ is monotonic, $V^{E,c}(N) = E$, and $V^{E,c}(\{i\}) = [0, m_i(E, c)]$ for all $i \in N$. Moreover, $V^{E,c}(S)$ is nonleveled for all $S \in 2^N \setminus \{\emptyset\}$, which means that $\mathcal{C}^{\mathcal{S}}(V^{E,c}) = \mathcal{C}^{\mathcal{W}}(V^{E,c})$.

Example 6.5
Let $N = \{1, 2\}$ and consider the bankruptcy problem $(E, c) \in \text{BR}^N$ given by $E = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 12x_2 \leq 36\}$ and $c = (3, 4)$ as in Example 6.2 and Example 6.3. Then $V^{E,c}(\{1\}) = \{0\}$, $V^{E,c}(\{2\}) = [0, 2\frac{1}{4}]$, and $V^{E,c}(N) = E$. △

An interesting feature of this model is that any subgame of this new NTU-bankruptcy game is a bankruptcy game too, as is the case for TU-bankruptcy games.

Proposition 6.3.1
All subgames of bankruptcy games are bankruptcy games.

Proof. Let $(E, c) \in \mathrm{BR}^N$ and let $S \in 2^N \setminus \{\emptyset\}$. Then $V^{E,c}(S)$ is nonempty, closed, bounded, comprehensive, and nonleveled, and $c_S \in \mathrm{WUC}(V^{E,c}(S))$. This means that $(V^{E,c}(S), c_S) \in \mathrm{BR}^S$. We have

$$\begin{aligned}
V^{V^{E,c}(S),c_S}(R) &= \begin{cases} \{x \in \mathbb{R}^R_+ \mid (x, c_{S\setminus R}) \in V^{E,c}(S)\} & \text{if } (0_R, c_{S\setminus R}) \in V^{E,c}(S); \\ \{0_R\} & \text{if } (0_R, c_{S\setminus R}) \notin V^{E,c}(S) \end{cases} \\
&= \begin{cases} \{x \in \mathbb{R}^R_+ \mid (x, c_{S\setminus R}, c_{N\setminus S}) \in E\} & \text{if } (0_R, c_{S\setminus R}, c_{N\setminus S}) \in E; \\ \{0_R\} & \text{if } (0_R, c_{S\setminus R}, c_{N\setminus S}) \notin E \end{cases} \\
&= \begin{cases} \{x \in \mathbb{R}^R_+ \mid (x, c_{N\setminus R}) \in E\} & \text{if } (0_R, c_{N\setminus R}) \in E; \\ \{0_R\} & \text{if } (0_R, c_{N\setminus R}) \notin E \end{cases} \\
&= V^{E,c}(R) \\
&= V^{E,c}_S(R)
\end{aligned}$$

for all $R \in 2^S \setminus \{\emptyset\}$. Hence, $V^{E,c}_S \in \mathrm{NTU}^S$ is a bankruptcy game. □

6.4 Core structures

This section studies the relationship between the core, the core cover, and the reasonable set of bankruptcy games. A useful observation for this analysis is that NTU-bankruptcy games are invariant under claim truncation, as is the case for TU-bankruptcy games.

Lemma 6.4.1
Let $(E, c) \in \mathrm{BR}^N$. Then $V^{E,c} = V^{E,\hat{c}^E}$.

Proof. Let $S \in 2^N \setminus \{\emptyset\}$. If $\hat{c}^E_{N\setminus S} = c_{N\setminus S}$, then $V^{E,c}(S) = V^{E,\hat{c}^E}(S)$. Suppose that $\hat{c}^E_{N\setminus S} \neq c_{N\setminus S}$. Then there exists an $i \in N \setminus S$ for which $\hat{c}^E_i = u^E_i < c_i$. This means that $(0_S, c_{N\setminus S}) \notin E$, so $V^{E,c}(S) = \{0_S\}$. Since E is nonleveled, $V^{E,\hat{c}^E}(S) = \{x \in \mathbb{R}^S_+ \mid (x, \hat{c}^E_{N\setminus S}) \in E\} = \{0_S\}$ if $(0_S, \hat{c}^E_{N\setminus S}) \in E$. Hence, $V^{E,c}(S) = V^{E,\hat{c}^E}(S)$. □

The vector of truncated claims and the vector of minimal rights determine the upper and lower bound for the core cover of bankruptcy games, respectively.

Lemma 6.4.2
Let $(E, c) \in \mathrm{BR}^N$. *Then*

(i) $K(V^{E,c}) = \hat{c}^E$;

(ii) $k(V^{E,c}) = m(E, c)$.

Proof. (i) By Lemma 6.4.1, $V^{E,c} = V^{E,\hat{c}^E}$, so $K(V^{E,c}) = K(V^{E,\hat{c}^E})$. Let $i \in N$. Then $(\hat{c}_i^E, x) \in E$ for all $x \in V^{E,\hat{c}^E}(N \setminus \{i\})$, so $(\hat{c}_i^E, x) \in V^{E,\hat{c}^E}(N)$ for all $x \in \mathrm{SP}(V^{E,\hat{c}^E}(N \setminus \{i\}))$. This means that $K_i(V^{E,\hat{c}^E}) \geq \hat{c}_i^E$. Suppose that $K_i(V^{E,\hat{c}^E}) > \hat{c}_i^E$. Let $x \in \mathrm{SUC}(V^{E,\hat{c}^E}(N \setminus \{i\}))$ be such that $(K_i(V^{E,\hat{c}^E}), x) \in V^{E,\hat{c}^E}(N)$. Since $V^{E,\hat{c}^E}(N)$ is comprehensive, $(\hat{c}_i^E, x) \in V^{E,\hat{c}^E}(N) \setminus \mathrm{SP}(V^{E,\hat{c}^E}(N))$. This means that $x \in V^{E,\hat{c}^E}(N \setminus \{i\})$, so $x \in \mathrm{SP}(V^{E,\hat{c}^E}(N \setminus \{i\}))$. Moreover, since $V^{E,\hat{c}^E}(N)$ is nonleveled, this implies that $(\hat{c}_i^E, x) \notin \mathrm{WP}(V^{E,\hat{c}^E}(N))$. Then there exists a $y \in V^{E,\hat{c}^E}(N)$ such that $y > (\hat{c}_i^E, x)$. Since $V^{E,\hat{c}^E}(N)$ is comprehensive, $(\hat{c}_i^E, y_{N\setminus\{i\}}) \in V^{E,\hat{c}^E}(N)$. This means that $y_{N\setminus\{i\}} \in V^{E,\hat{c}^E}(N \setminus \{i\})$, which contradicts that $x \in \mathrm{SP}(V^{E,\hat{c}^E}(N \setminus \{i\}))$. Hence, $K_i(V^{E,c}) = K_i(V^{E,\hat{c}^E}) = \hat{c}_i^E$.

(ii) Let $i \in N$. Then

$$k_i(V^{E,c}) \geq \sup\left\{x \in \mathbb{R}_+ \mid x \in V^{E,c}(\{i\})\right\} = \max\left\{x \in V^{E,c}(\{i\})\right\} = m_i(E, c).$$

Suppose that $k_i(V^{E,c}) > m_i(E, c)$. Let $S \in 2^N$ be such that $i \in S$ and

$$(k_i(V^{E,c}), K_{S\setminus\{i\}}(V^{E,c})) \in V^{E,c}(S).$$

Then $(k_i(V^{E,\hat{c}^E}), \hat{c}^E_{S\setminus\{i\}}) \in V^{E,\hat{c}^E}(S)$ by Lemma 6.4.1 and Lemma 6.4.2(i). This means that $(k_i(V^{E,\hat{c}^E}), \hat{c}^E_{S\setminus\{i\}}, \hat{c}^E_{N\setminus S}) \in E$, which implies that $k_i(V^{E,\hat{c}^E}) \in V^{E,\hat{c}^E}(\{i\})$. This contradicts that $k_i(V^{E,\hat{c}^E}) > m_i(E, c)$. Hence, $k_i(V^{E,c}) = k_i(V^{E,\hat{c}^E}) = m_i(E, c)$. □

By Lemma 6.2.1 and Example 6.1, the core cover is not necessarily contained in the reasonable set of an NTU-game. Surprisingly, for the reasonable set of an NTU-bankruptcy game we find the same upper bound and lower bound as for the core cover, which means that the core cover and the reasonable set of an NTU-bankruptcy game coincide.

Lemma 6.4.3
Let $(E, c) \in \mathrm{BR}^N$ *and let* $i \in N$. *Then*

(i) $\max_{\sigma \in \Pi(N)} M_i^\sigma(V^{E,c}) = \hat{c}_i^E$;

(ii) $\min_{\sigma \in \Pi(N)} M_i^\sigma(V^{E,c}) = m_i(E, c)$.

Proof. (i) By Lemma 6.4.1, $V^{E,c} = V^{E,\hat{c}^E}$, so

$$\max_{\sigma\in\Pi(N)} M_i^{\sigma}(V^{E,c}) = \max_{\sigma\in\Pi(N)} M_i^{\sigma}(V^{E,\hat{c}^E}).$$

Let $\hat{\sigma} \in \Pi(N)$ be such that $\hat{\sigma}(|N|) = i$. Then $(x, \hat{c}_i^E) \in E$ for all $x \in V^{E,\hat{c}^E}(N \setminus \{i\})$, so

$$\left(M^{\hat{\sigma}}_{\hat{\sigma}(1)}(V^{E,\hat{c}^E}), \ldots, M^{\hat{\sigma}}_{\hat{\sigma}(|N|-1)}(V^{E,\hat{c}^E}), \hat{c}_i^E\right) \in V^{E,\hat{c}^E}(N).$$

This means that $\max_{\sigma\in\Pi(N)} M_i^{\sigma}(V^{E,\hat{c}^E}) \geq \hat{c}_i^E$. Suppose that $\max_{\sigma\in\Pi(N)} M_i^{\sigma}(V^{E,\hat{c}^E}) > \hat{c}_i^E$. Let $\hat{\sigma} \in \Pi(N)$ be such that $M_i^{\hat{\sigma}}(V^{E,\hat{c}^E}) = \max_{\sigma\in\Pi(N)} M_i^{\sigma}(V^{E,\hat{c}^E})$ and let $k \in \{2, \ldots, |N|\}$ be such that $\hat{\sigma}(k) = i$. Then

$$\left(M^{\hat{\sigma}}_{\hat{\sigma}(1)}(V^{E,\hat{c}^E}), \ldots, M^{\hat{\sigma}}_{\hat{\sigma}(k)}(V^{E,\hat{c}^E})\right) \in V^{E,\hat{c}^E}(\{\hat{\sigma}(1), \ldots, \hat{\sigma}(k)\}).$$

This means that

$$\left(M^{\hat{\sigma}}_{\hat{\sigma}(1)}(V^{E,\hat{c}^E}), \ldots, M^{\hat{\sigma}}_{\hat{\sigma}(k)}(V^{E,\hat{c}^E}), \hat{c}^E_{\hat{\sigma}(k+1)}, \ldots, \hat{c}^E_{\hat{\sigma}(|N|)}\right) \in E.$$

Since E is comprehensive,

$$\left(M^{\hat{\sigma}}_{\hat{\sigma}(1)}(V^{E,\hat{c}^E}), \ldots, M^{\hat{\sigma}}_{\hat{\sigma}(k-1)}(V^{E,\hat{c}^E}), \hat{c}^E_{\hat{\sigma}(k)}, \ldots, \hat{c}^E_{\hat{\sigma}(|N|)}\right) \in E \setminus \mathrm{SP}(E).$$

Since E is nonleveled,

$$\left(M^{\hat{\sigma}}_{\hat{\sigma}(1)}(V^{E,\hat{c}^E}), \ldots, M^{\hat{\sigma}}_{\hat{\sigma}(k-1)}(V^{E,\hat{c}^E}), \hat{c}^E_{\hat{\sigma}(k)}, \ldots, \hat{c}^E_{\hat{\sigma}(|N|)}\right) \in E \setminus \mathrm{WP}(E).$$

This means that there exists a $y \in E$ such that

$$y > \left(M^{\hat{\sigma}}_{\hat{\sigma}(1)}(V^{E,\hat{c}^E}), \ldots, M^{\hat{\sigma}}_{\hat{\sigma}(k-1)}(V^{E,\hat{c}^E}), \hat{c}^E_{\hat{\sigma}(k)}, \ldots, \hat{c}^E_{\hat{\sigma}(|N|)}\right).$$

Since E is comprehensive,

$$\left(M^{\hat{\sigma}}_{\hat{\sigma}(1)}(V^{E,\hat{c}^E}), \ldots, M^{\hat{\sigma}}_{\hat{\sigma}(k-2)}(V^{E,\hat{c}^E}), y_{\hat{\sigma}(k-1)}, \hat{c}^E_{\hat{\sigma}(k)}, \ldots, \hat{c}^E_{\hat{\sigma}(|N|)}\right) \in E.$$

This means that

$$\left(M^{\hat{\sigma}}_{\hat{\sigma}(1)}(V^{E,\hat{c}^E}), \ldots, M^{\hat{\sigma}}_{\hat{\sigma}(k-2)}(V^{E,\hat{c}^E}), y_{\hat{\sigma}(k-1)}\right) \in V^{E,\hat{c}^E}(\{\hat{\sigma}(1), \ldots, \hat{\sigma}(k-1)\}),$$

which contradicts the definition of $M^{\hat{\sigma}}_{\hat{\sigma}(k-1)}(V^{E,\hat{c}^E})$. Hence, $\max_{\sigma\in\Pi(N)} M_i^{\sigma}(V^{E,c}) = \max_{\sigma\in\Pi(N)} M_i^{\sigma}(V^{E,\hat{c}^E}) = \hat{c}_i^E$.

(*ii*) Let $\hat{\sigma} \in \Pi(N)$ be such that $\hat{\sigma}(1) = i$. Then

$$M_i^{\hat{\sigma}}(V^{E,c}) = \max\left\{x \in \mathbb{R}_+ \mid x \in V^{E,c}(\{i\})\right\} = \max\left\{x \in V^{E,c}(\{i\})\right\} = m_i(E,c).$$

This means that $\min_{\sigma\in\Pi(N)} M_i^{\sigma}(V^{E,c}) \leq m_i(E,c)$. Suppose that

$$\min_{\sigma\in\Pi(N)} M_i^{\sigma}(V^{E,c}) < m_i(E,c).$$

By Lemma 6.4.1, $V^{E,c} = V^{E,\hat{c}^E}$, so $\min_{\sigma\in\Pi(N)} M_i^{\sigma}(V^{E,c}) = \min_{\sigma\in\Pi(N)} M_i^{\sigma}(V^{E,\hat{c}^E})$. Let $\hat{\sigma} \in \Pi(N)$ be such that $M_i^{\hat{\sigma}}(V^{E,\hat{c}^E}) = \min_{\sigma\in\Pi(N)} M_i^{\sigma}(V^{E,\hat{c}^E})$ and let $k \in \{2,\ldots,|N|\}$ be such that $\hat{\sigma}(k) = i$. Then

$$\left(M_{\hat{\sigma}(1)}^{\hat{\sigma}}(V^{E,\hat{c}^E}),\ldots,M_{\hat{\sigma}(k-1)}^{\hat{\sigma}}(V^{E,\hat{c}^E}),m_i(E,c)\right) \notin V^{E,\hat{c}^E}(\{\hat{\sigma}(1),\ldots,\hat{\sigma}(k)\}).$$

This means that

$$\left(M_{\hat{\sigma}(1)}^{\hat{\sigma}}(V^{E,\hat{c}^E}),\ldots,M_{\hat{\sigma}(k-1)}^{\hat{\sigma}}(V^{E,\hat{c}^E}),m_i(E,c),\hat{c}^E_{\hat{\sigma}(k+1)},\ldots,\hat{c}^E_{\hat{\sigma}(|N|)}\right) \notin E.$$

Since E is comprehensive, $(m_i(E,c),\hat{c}^E_{N\setminus\{i\}}) \notin E$ by Lemma 6.4.3(*i*). This contradicts that $m_i(E,c) \in V^{E,\hat{c}^E}(\{i\})$. Hence, $\min_{\sigma\in\Pi(N)} M_i^{\sigma}(V^{E,c}) = \min_{\sigma\in\Pi(N)} M_i^{\sigma}(V^{E,\hat{c}^E}) = m_i(E,c)$. □

By Lemma 6.2.1, Lemma 6.4.2, and Lemma 6.4.3, the core of a bankruptcy game is contained in the core cover and in the reasonable set. Next, we generalize the result for TU-bankruptcy games which states that the core of a bankruptcy game coincides with the core cover and the reasonable set.

Theorem 6.4.4
Bankruptcy games are compromise stable and reasonable stable.

Proof. Let $(E,c) \in \mathrm{BR}^N$. Then $\mathcal{CC}(V^{E,c}) = \mathcal{R}(V^{E,c})$ by Lemma 6.4.2 and Lemma 6.4.3, so it suffices to show that $\mathcal{CC}(V^{E,c}) \neq \emptyset$ and $\mathcal{C}^{\mathcal{S}}(V^{E,c}) = \mathcal{CC}(V^{E,c})$. By Lemma 6.4.2, $\mathcal{CC}(V^{E,c}) = \{x \in \mathrm{SP}(E) \mid m(E,c) \leq x \leq \hat{c}^E\}$. Since $m(E,c) \in E$, $\hat{c}^E \in \mathrm{WUC}(E)$, $m(E,c) \leq \hat{c}^E$, and E is nonleveled, there exists an $x \in \mathrm{SP}(E)$ for which $m(E,c) \leq x \leq \hat{c}^E$. This means that $\mathcal{CC}(V^{E,c}) \neq \emptyset$.

Let $x \in \mathcal{CC}(V^{E,c})$. Then $x \leq \hat{c}^E \leq c$. Suppose that $x \notin \mathcal{C}^{\mathcal{S}}(V^{E,c})$. Then there exists an $S \in 2^N \setminus \{\emptyset\}$ such that $x_S \notin \mathrm{SUC}(V^{E,c}(S))$. This means that there exists a $y \in V^{E,c}(S)$ for which $y \geq x_S$ and $y \neq x_S$. Then $(y,c_{N\setminus S}) \in E$. Since $x \leq (y,c_{N\setminus S})$ and $x \neq (y,c_{N\setminus S})$, this means that $x \notin \mathrm{SP}(E)$. This contradicts that $x \in \mathcal{CC}(V^{E,c})$, so $x \in \mathcal{C}^{\mathcal{S}}(V^{E,c})$ and $\mathcal{CC}(V^{E,c}) \subseteq \mathcal{C}^{\mathcal{S}}(V^{E,c})$. By Lemma 6.2.1, $\mathcal{C}^{\mathcal{S}}(V^{E,c}) \subseteq \mathcal{CC}(V^{E,c})$. Hence, $\mathcal{C}^{\mathcal{S}}(V^{E,c}) = \mathcal{CC}(V^{E,c})$. □

Using Lemma 6.4.2, Lemma 6.4.3, Theorem 6.4.4, and using that E is nonleveled and $x \leq \hat{c}^E$ implies that $x \geq m(E,c)$ for all $(E,c) \in \mathrm{BR}^N$ and any $x \in \mathrm{SP}(E)$, we derive a compact expression for the core of a bankruptcy game.

Corollary 6.4.5
Let $(E,c) \in \mathrm{BR}^N$. *Then* $\mathcal{C}^{\mathcal{S}}(V^{E,c}) = \{x \in \mathrm{WP}(E) \mid x \leq c\}$.

In other words, bankruptcy rules assign to bankruptcy problems a core element of the corresponding bankruptcy game. This means that a solution for NTU-games corresponds on the class of bankruptcy games to some bankruptcy rule if and only if it assigns to any bankruptcy game a core element. The other way around, the question arises which bankruptcy rules correspond to a solution for NTU-games on the class of bankruptcy games. These bankruptcy rules are called game theoretic.

Definition (Game Theoretic Bankruptcy Rule)
A bankruptcy rule $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ is *game theoretic* if there exists a solution $F : \mathrm{NTU}^N \to \mathbb{R}^N_+$ such that $f(E,c) = F(V^{E,c})$ for all $(E,c) \in \mathrm{BR}^N$.

A necessary and sufficient condition for an NTU-bankruptcy rule to be game theoretic is to satisfy truncation invariance, as is the case for TU-bankruptcy rules (cf. Curiel, Maschler, and Tijs (1987)).

Theorem 6.4.6
A bankruptcy rule is game theoretic if and only if it satisfies truncation invariance.

Proof. Let $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ be a game theoretic bankruptcy rule. Let $(E,c) \in \mathrm{BR}^N$. Then $V^{E,c} = V^{E,\hat{c}^E}$ by Lemma 6.4.1 and

$$f(E,c) = F(V^{E,c}) = F(V^{E,\hat{c}^E}) = f(E,\hat{c}^E)$$

for some solution $F : \mathrm{NTU}^N \to \mathbb{R}^N_+$. Hence, f satisfies truncation invariance.

Let $f : \mathrm{BR}^N \to \mathbb{R}^N_+$ be a bankruptcy rule satisfying truncation invariance. Let $F : \mathrm{NTU}^N \to \mathbb{R}^N_+$ be a solution such that $F(V^{E,c}) = f(V^{E,c}(N), K(V^{E,c}))$ for any bankruptcy game $V^{E,c} \in \mathrm{NTU}^N$. Let $(E,c) \in \mathrm{BR}^N$. Then $K(V^{E,c}) = \hat{c}^E$ by Lemma 6.4.2 and

$$f(E,c) = f(E,\hat{c}^E) = f(V^{E,c}(N), K(V^{E,c})) = F(V^{E,c}).$$

Hence, f is game theoretic. □

The constrained relative equal awards rule, the truncated proportional rule, and all adjusted bankruptcy rules are examples of game theoretic bankruptcy rules. Future research could study the solutions for nontransferable utility games to which they correspond in order to further extend the relation between NTU-bankruptcy problems and NTU-games.

Part II

Cooperative Games

7 Egalitarianism in Transferable Utility Games

7.1 Introduction

Egalitarianism is a paradigm of economic thought that favors the idea of equality. Economic equality, or equity, refers to the concept of fairness in economics and underlies many theories of distributive justice. In the seminal work of Rawls (1971), equality plays a central role in two fundamental principles of justice. This has inspired scientists within several areas, e.g. social philosophy and welfare economics. Young (1995) provides a rich survey on equity concepts in both theoretical and practical contexts. This chapter, based on Dietzenbacher, Borm, and Hendrickx (2017c), focusses on the role of egalitarianism in distributive justice applied to coalitional arrangements which affect the distribution of joint revenues among cooperating participants.

Dutta and Ray (1989) introduced a concept of egalitarianism under participation constraints for transferable utility games. A transferable utility game describes an allocation problem for a set of cooperating players in which the economic possibilities of all subcoalitions are taken into account. The egalitarian solution of Dutta and Ray (1989) applies a Lorenz criterion to select a payoff allocation. Their most important result states that their egalitarian solution selects at most one feasible allocation, despite the partial ordering generated by the Lorenz criterion. However, existence of their egalitarian solution is only shown to be guaranteed for the special class of convex games.

The egalitarian solution of Dutta and Ray (1989) is well-studied on the class of convex games. Dutta and Ray (1989) showed that the egalitarian solution of a convex game cannot be blocked by any subcoalition, i.e. it is an element of the core. Dutta (1990) axiomatically characterized this solution on the class of convex games using consistency properties for reduced games of Davis and Maschler (1965) and Hart and Mas-Colell (1989). Other characterizations of this egalitarian solution on the class of convex games are provided by Klijn, Slikker, Tijs, and Zarzuelo (2000) and Arin, Kuipers, and Vermeulen (2003).

Another line of research studies egalitarian concepts similar to the egalitarian solution of Dutta and Ray (1989) for a larger class of transferable utility games. Branzei, Dimitrov, and Tijs (2006) extended the corresponding computational algorithm for convex games to superadditive games by introducing the equal split-off set. Arin and Iñarra (2001) applied an egalitarian criterion to the core of balanced games by introducing the egalitarian core which satisfies the consistency property for reduced games of Davis and Maschler (1965). Both the equal split-off set and the egalitarian core coincide with the egalitarian solution of Dutta and Ray (1989) on the class of convex games. The most important shortcoming of these notions is that they generally lack the fundamental uniqueness property. To our knowledge, no appropriate egalitarian, single-valued solution concept has been defined in the literature which coincides with the egalitarian solution of Dutta and Ray (1989) on the class of convex games and exists for any transferable utility game.

In this chapter, we introduce the *procedural egalitarian solution* as an egalitarian solution concept for which existence and uniqueness is guaranteed for any transferable utility game. Moreover, it coincides with the egalitarian solution of Dutta and Ray (1989) on the class of convex games. The procedural egalitarian solution follows from an egalitarian procedure which is inspired by ideas underlying the average rules for cooperative TU-games of Sugumaran, Thangaraj, and Ravindran (2013). In a model where utility is transferable, our interpretation of egalitarianism boils down to equal division. However, in a coalitional game, simple equal division of the worth of the grand coalition is not satisfactory. The egalitarian procedure models a natural way of negotiating by members of coalitions about an egalitarian distribution of their worth, taking into account their coalitional egalitarian externalities. This egalitarian procedure converges to a steady state in which each player has acquired a claim attainable in one or more egalitarian admissible coalitions. Using the constrained equal awards rule, the procedural egalitarian solution allocates the worth of the grand coalition in an egalitarian way among the players, taking into account their claims. In this way, the procedural egalitarian solution can be considered as a trade-off between egalitarianism and coalitional rationality.

Selten (1972) showed that egalitarian allocations successfully explain outcomes of experimental cooperative games. Experimental evidence clearly suggests that equity considerations have a strong influence on observed payoff divisions. Coalition members look for easily accessible cues like equitable shares in order to form aspiration levels for their payoffs (cf. Selten (1987)). The egalitarian procedure seamlessly connects this phenomenon with transferable utility games.

This chapter is organized in the following way. Section 7.2 formally introduces the egalitarian procedure and studies its underlying structure. In Section 7.3, we introduce the procedural egalitarian solution, derive some of its properties, and show that it coincides with the egalitarian solution of Dutta and Ray (1989) on the class of convex games.

7.2 The egalitarian procedure

In this section we introduce the egalitarian procedure for transferable utility games. This iterative procedure models negotiations between members of coalitions about the allocation of their worth, taking into account their coalitional egalitarian externalities. We formally define the egalitarian procedure after an illustrative example.

Example 7.1
Let $N = \{1, 2, 3\}$ and consider the game $v \in \text{TU}^N$ for which the worth of each coalition and the egalitarian distribution in all iterations of the egalitarian procedure are presented in the following table.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	5	0	0	8	0	0	9
$\chi^{v,1}(S)$	$(\underline{5},\cdot,\cdot)$	$(\cdot,0,\cdot)$	$(\cdot,\cdot,0)$	$(4,4,\cdot)$	$(0,\cdot,0)$	$(\cdot,0,0)$	$(3,3,3)$
$\chi^{v,2}(S)$	$(\mathbf{5},\cdot,\cdot)$	$(\cdot,0,\cdot)$	$(\cdot,\cdot,0)$	$(\mathbf{5},\underline{3},\cdot)$	$(\mathbf{5},\cdot,\text{-}5)$	$(\cdot,0,0)$	$(\mathbf{5},2,2)$
$\chi^{v,3}(S)$	$(\mathbf{5},\cdot,\cdot)$	$(\cdot,\mathbf{3},\cdot)$	$(\cdot,\cdot,0)$	$(\mathbf{5},\mathbf{3},\cdot)$	$(\mathbf{5},\cdot,\text{-}5)$	$(\cdot,\mathbf{3},\text{-}3)$	$(\mathbf{5},\mathbf{3},\underline{1})$
$\chi^{v,k}(S)$ $(k \geq 4)$	$(\mathbf{5},\cdot,\cdot)$	$(\cdot,\mathbf{3},\cdot)$	$(\cdot,\cdot,\mathbf{1})$	$(\mathbf{5},\mathbf{3},\cdot)$	$(\mathbf{5},\cdot,\mathbf{1})$	$(\cdot,\mathbf{3},\mathbf{1})$	$(\mathbf{5},\mathbf{3},\mathbf{1})$

A natural way to start negotiating about the allocation of the worth of a coalition among its members is to divide it equally, i.e. in the first iteration, the egalitarian distribution $\chi^{v,1}$ allocates in any coalition $S \in 2^N \setminus \{\emptyset\}$ the average worth $\frac{v(S)}{|S|}$ to each member $i \in S$. Players can only claim their highest allocated payoff if no other member of the corresponding coalition is allocated a higher payoff in any other coalition. All such players constitute the set of egalitarian claimants $P^{v,1}$ with corresponding claims $\gamma^{v,1}$, and the coalitions in which they obtained their claims are contained in the collection of egalitarian admissible coalitions $\mathcal{A}^{v,1}$.

The highest payoff allocated by $\chi^{v,1}$ to player 1 is 5 in coalition $\{1\}$, which player 1 can claim since this coalition contains no other members. The highest payoff allocated to player 2 is 4 in coalition $\{1,2\}$, which player 2 cannot claim since player 1 is allocated a higher payoff in another coalition. The highest payoff allocated to player 3 is 3 in coalition $\{1,2,3\}$, which player 3 cannot claim since players 1 and 2 are allocated a higher payoff in other coalitions. This means that the set of 1-egalitarian claimants is given by $P^{v,1} = \{1\}$, the corresponding vector of 1-egalitarian claims is given by $\gamma^{v,1} = (5, \cdot, \cdot)$, and the collection of 1-egalitarian admissible coalitions is given by $\mathcal{A}^{v,1} = \{\{1\}\}$.

In the next iteration, the claimants claim their egalitarian claim in any coalition of which they are member and $\chi^{v,2}$ divides the remaining worth equally among the other members. The claimants in $P^{v,2}$ and their corresponding claims $\gamma^{v,2}$ are constructed similarly to the first iteration, and $\mathcal{A}^{v,2}$ contains the coalitions in which all members can obtain their claims. In this way, the players continue negotiating in further iterations. Note that, once a player has acquired an egalitarian claim, it remains fixed in all further iterations.

In particular, the highest payoff allocated by $\chi^{v,2}$ to player 2 is 3 in coalition $\{1,2\}$, which player 2 can claim since no other member is allocated a higher payoff in any other coalition. The highest payoff allocated to player 3 is 2 in coalition $\{1,2,3\}$, which player 3 cannot claim since player 2 is allocated a higher payoff in another coalition. This means that $P^{v,2} = \{1,2\}$, $\gamma^{v,2} = (5,3,\cdot)$, and $\mathcal{A}^{v,2} = \{\{1\},\{1,2\}\}$. In the third iteration, the highest payoff allocated by $\chi^{v,3}$ to player 3 is 1 in coalition $\{1,2,3\}$, which player 3 can claim. This means that $P^{v,3} = \{1,2,3\}$, $\gamma^{v,3} = (5,3,1)$, and $\mathcal{A}^{v,3} = \{\{1\},\{1,2\},\{1,2,3\}\}$. In all further iterations, all players are allocated their claims in all coalitions of which they are member and the collection of egalitarian admissible coalitions remains unchanged. △

Definition (Egalitarian Procedure)
Let $v \in \mathrm{TU}^N$ be a transferable utility game. The set of 0-*egalitarian claimants* is given by $P^{v,0} = \emptyset$. Let $k \in \mathbb{N}$. The k-*egalitarian distribution* $\chi^{v,k}$ assigns to each $S \in 2^N \setminus \{\emptyset\}$ the payoff allocation $\chi^{v,k}(S) \in \mathbb{R}^S$ given by

$$\chi_i^{v,k}(S) = \begin{cases} \gamma_i^{v,k-1} & \text{for all } i \in S \cap P^{v,k-1}; \\ \frac{v(S) - \sum_{j \in S \cap P^{v,k-1}} \gamma_j^{v,k-1}}{|S \setminus P^{v,k-1}|} & \text{for all } i \in S \setminus P^{v,k-1}. \end{cases}$$

The collection of k-*egalitarian admissible coalitions* is given by $\mathcal{A}^{v,k} = \{S \in 2^N \setminus \{\emptyset\} \mid \sum_{i \in S} \chi_i^{v,k}(S) = v(S), \forall_{i \in S} \forall_{T \in 2^N : i \in T} : \chi_i^{v,k}(T) \leq \chi_i^{v,k}(S)\}$. The set of k-*egalitarian claimants* $P^{v,k} \in 2^N \setminus \{\emptyset\}$ is given by $P^{v,k} = \bigcup_{S \in \mathcal{A}^{v,k}} S$. The vector of k-*egalitarian claims* $\gamma^{v,k} \in \mathbb{R}^{P^{v,k}}$ is given by $\gamma_i^{v,k} = \chi_i^{v,k}(S)$ for all $i \in P^{v,k}$, where $S \in \mathcal{A}^{v,k}$ and $i \in S$.

The payoff $\chi_i^{v,k}(S)$ allocated to a player $i \in S \setminus P^{v,k-1}$ is the *average remaining worth* of $S \in 2^N \setminus \{\emptyset\}$. In the first step, the worth of each coalition is equally distributed among its members. A coalition is called admissible if all its members are allocated their highest payoff. The players which are member of these egalitarian admissible coalitions are called claimants and their allocated payoffs form their claims. Although egalitarian admissible coalitions need not be unique, the egalitarian claims of their members are uniquely defined. In the next iteration, the claimants are allocated their claims and the other players are allocated the average remaining worth in all coalitions. A typical observation is that the egalitarian distribution is in general overefficient, i.e. it allocates at least the worth of a coalition.

Lemma 7.2.1
Let $v \in \mathrm{TU}^N$ and let $S \in 2^N \setminus \{\emptyset\}$. Then $\sum_{i\in S} \chi_i^{v,k}(S) \geq v(S)$ for all $k \in \mathbb{N}$. Moreover, if $S \not\subseteq P^{v,k-1}$ for some $k \in \mathbb{N}$, then $\sum_{i\in S} \chi_i^{v,k}(S) = v(S)$.

Proof. We show the statement by induction. Since $P^{v,0} = \emptyset$,

$$\sum_{i\in S} \chi_i^{v,1}(S) = \sum_{i\in S\setminus P^{v,0}} \left(\frac{v(S) - \sum_{j\in S\cap P^{v,0}} \gamma_j^{v,0}}{|S \setminus P^{v,0}|}\right) = \sum_{i\in S} \left(\frac{v(S)}{|S|}\right) = |S| \left(\frac{v(S)}{|S|}\right) = v(S).$$

Let $k \in \mathbb{N}$ and assume that $\sum_{i\in S} \chi_i^{v,k}(S) \geq v(S)$. If $S \subseteq P^{v,k}$, then

$$\sum_{i\in S} \chi_i^{v,k+1}(S) = \sum_{i\in S} \gamma_i^{v,k} \geq \sum_{i\in S} \chi_i^{v,k}(S) \geq v(S).$$

If $S \not\subseteq P^{v,k}$, then

$$\begin{aligned}
\sum_{i\in S} \chi_i^{v,k+1}(S) &= \sum_{i\in S\cap P^{v,k}} \chi_i^{v,k+1}(S) + \sum_{i\in S\setminus P^{v,k}} \chi_i^{v,k+1}(S) \\
&= \sum_{i\in S\cap P^{v,k}} \gamma_i^{v,k} + \sum_{i\in S\setminus P^{v,k}} \left(\frac{v(S) - \sum_{j\in S\cap P^{v,k}} \gamma_j^{v,k}}{|S \setminus P^{v,k}|}\right) \\
&= \sum_{i\in S\cap P^{v,k}} \gamma_i^{v,k} + |S \setminus P^{v,k}| \left(\frac{v(S) - \sum_{j\in S\cap P^{v,k}} \gamma_j^{v,k}}{|S \setminus P^{v,k}|}\right) \\
&= \sum_{i\in S\cap P^{v,k}} \gamma_i^{v,k} + v(S) - \sum_{j\in S\cap P^{v,k}} \gamma_j^{v,k} \\
&= v(S).
\end{aligned}$$

□

Only coalitions for which the egalitarian distribution allocates exactly the worth among its members can be egalitarian admissible. The question arises whether egalitarian admissible coalitions exist in all iterations for any transferable utility game. We show that in all iterations of the egalitarian procedure at least one additional player becomes an egalitarian claimant as long as the collection of egalitarian admissible coalitions is not a cover, which implies that egalitarian admissible coalitions indeed always exist.

Lemma 7.2.2
Let $v \in \mathrm{TU}^N$ and let $k \in \mathbb{N}$. Then $\mathcal{A}^{v,k} \subseteq \mathcal{A}^{v,k+1}$. Moreover, if $P^{v,k-1} \neq N$, then $P^{v,k-1} \subset P^{v,k}$.

Proof. Let $S \in \mathcal{A}^{v,k}$. Then $S \subseteq P^{v,k}$ and

$$\sum_{i \in S} \chi_i^{v,k+1}(S) = \sum_{i \in S} \gamma_i^{v,k} = \sum_{i \in S} \chi_i^{v,k}(S) = v(S).$$

Moreover, $\chi_i^{v,k+1}(T) \leq \chi_i^{v,k+1}(S)$ for all $i \in S$ and any $T \in 2^N$ for which $i \in T$. This means that $S \in \mathcal{A}^{v,k+1}$. Hence, $\mathcal{A}^{v,k} \subseteq \mathcal{A}^{v,k+1}$.

Assume that $P^{v,k-1} \neq N$. Let $S \in 2^N$ with $S \not\subseteq P^{v,k-1}$ be a coalition with highest average remaining worth. Then $\sum_{i \in S} \chi_i^{v,k}(S) = v(S)$ by Lemma 7.2.1. Moreover, $\chi_i^{v,k}(T) \leq \chi_i^{v,k}(S)$ for all $i \in S$ and any $T \in 2^N$ for which $i \in T$. This means that $S \in \mathcal{A}^{v,k}$ and $S \subseteq P^{v,k}$. Hence, $P^{v,k-1} \subset P^{v,k}$. □

Lemma 7.2.2 not only tells us that egalitarian admissible coalitions always exist, but also that the collection of egalitarian admissible coalitions weakly expands in all iterations. The structure of this collection is determined by the structure of the underlying transferable utility game. Well-known properties for TU-games have interesting implications for the structure of the collection of egalitarian admissible coalitions in all iterations. We derive those implications for superadditive, convex, and balanced transferable utility games.

Proposition 7.2.3
Let $v \in \mathrm{TU}^N$ and let $k \in \mathbb{N}$.

(i) If v is superadditive, then $S \cup T \in \mathcal{A}^{v,k}$ for all $S, T \in \mathcal{A}^{v,k}$ for which $S \cap T = \emptyset$.

(ii) If v is convex, then $S \cup T \in \mathcal{A}^{v,k}$ and $S \cap T \in \mathcal{A}^{v,k}$ for all $S, T \in \mathcal{A}^{v,k}$ for which $S \cap T \neq \emptyset$.

(iii) If v is balanced, then $N \in \mathcal{A}^{v,k}$ if there exists a balanced collection $\mathcal{B} \subseteq \mathcal{A}^{v,k}$.

Proof. (*i*) Assume that v is superadditive. Let $S, T \in \mathcal{A}^{v,k}$ be such that $S \cap T = \emptyset$. Then $\sum_{i \in S \cup T} \chi_i^{v,k}(S \cup T) \geq v(S \cup T)$ by Lemma 7.2.1, and

$$\begin{aligned} v(S \cup T) &\geq v(S) + v(T) \\ &= \sum_{i \in S} \chi_i^{v,k}(S) + \sum_{i \in T} \chi_i^{v,k}(T) \\ &\geq \sum_{i \in S} \chi_i^{v,k}(S \cup T) + \sum_{i \in T} \chi_i^{v,k}(S \cup T) \\ &= \sum_{i \in S \cup T} \chi_i^{v,k}(S \cup T) \\ &\geq v(S \cup T). \end{aligned}$$

This means that $\sum_{i \in S \cup T} \chi_i^{v,k}(S \cup T) = v(S \cup T)$, $\chi_S^{v,k}(S \cup T) = \chi_S^{v,k}(S)$, and $\chi_T^{v,k}(S \cup T) = \chi_T^{v,k}(T)$. Hence, $S \cup T \in \mathcal{A}^{v,k}$.

(*ii*) Assume that v is convex. Let $S, T \in \mathcal{A}^{v,k}$ be such that $S \cap T \neq \emptyset$. Then $\sum_{i \in S \cup T} \chi_i^{v,k}(S \cup T) \geq v(S \cup T)$ and $\sum_{i \in S \cap T} \chi_i^{v,k}(S \cap T) \geq v(S \cap T)$ by Lemma 7.2.1, and

$$\begin{aligned} v(S \cup T) + v(S \cap T) &\geq v(S) + v(T) \\ &= \sum_{i \in S} \chi_i^{v,k}(S) + \sum_{i \in T} \chi_i^{v,k}(T) \\ &= \sum_{i \in S} \gamma_i^{v,k} + \sum_{i \in T} \gamma_i^{v,k} \\ &= \sum_{i \in S \cup T} \gamma_i^{v,k} + \sum_{i \in S \cap T} \gamma_i^{v,k} \\ &\geq \sum_{i \in S \cup T} \chi_i^{v,k}(S \cup T) + \sum_{i \in S \cap T} \chi_i^{v,k}(S \cap T) \\ &\geq v(S \cup T) + v(S \cap T). \end{aligned}$$

This means that $\sum_{i \in S \cup T} \chi_i^{v,k}(S \cup T) = v(S \cup T)$ and $\sum_{i \in S \cap T} \chi_i^{v,k}(S \cap T) = v(S \cap T)$. Moreover, $\chi_{S \cup T}^{v,k}(S \cup T) = \gamma_{S \cup T}^{v,k}$ and $\chi_{S \cap T}^{v,k}(S \cap T) = \gamma_{S \cap T}^{v,k}$. Hence, $S \cup T \in \mathcal{A}^{v,k}$ and $S \cap T \in \mathcal{A}^{v,k}$.

(*iii*) Assume that v is balanced. Let $\mathcal{B} \subseteq \mathcal{A}^{v,k}$ be a balanced collection and let $\delta : \mathcal{B} \to \mathbb{R}_{++}$ be such that $\sum_{S \in \mathcal{B}: i \in S} \delta(S) = 1$ for all $i \in N$. Then $\sum_{i \in N} \chi_i^{v,k}(N) \geq v(N)$ by Lemma 7.2.1, and

$$\begin{aligned} v(N) &\geq \sum_{S \in \mathcal{B}} \delta(S) v(S) = \sum_{S \in \mathcal{B}} \delta(S) \sum_{i \in S} \chi_i^{v,k}(S) = \sum_{i \in N} \sum_{S \in \mathcal{B}: i \in S} \delta(S) \chi_i^{v,k}(S) \\ &\geq \sum_{i \in N} \sum_{S \in \mathcal{B}: i \in S} \delta(S) \chi_i^{v,k}(N) = \sum_{i \in N} \chi_i^{v,k}(N) \sum_{S \in \mathcal{B}: i \in S} \delta(S) = \sum_{i \in N} \chi_i^{v,k}(N) \geq v(N). \end{aligned}$$

This means that $\sum_{i \in N} \chi_i^{v,k}(N) = v(N)$ and $\chi_i^{v,k}(N) = \chi_i^{v,k}(S)$ for all $i \in N$ and any $S \in \mathcal{B}$ for which $i \in S$. Hence, $N \in \mathcal{A}^{v,k}$. □

The egalitarian procedure is an egalitarian bargaining model that takes participation constraints explicitly into account. The egalitarian admissible coalitions can be considered as the coalitions in which members prefer to participate, concerning the corresponding allocation prescribed by the egalitarian distribution. This consideration suggests that the assigned allocation, consisting of the egalitarian claims for all members, is stable against subcoalitional deviations. Indeed, the vector of egalitarian claims corresponding to the members of an egalitarian admissible coalition is an element of the core of the induced subgame.

Proposition 7.2.4
Let $v \in \mathrm{TU}^N$ *and let* $k \in \mathbb{N}$. *Then* $\gamma_S^{v,k} \in \mathcal{C}(v_S)$ *for all* $S \in \mathcal{A}^{v,k}$.

Proof. Let $S \in \mathcal{A}^{v,k}$. Then

$$\sum_{i \in S} \gamma_i^{v,k} = \sum_{i \in S} \chi_i^{v,k}(S) = v(S) = v_S(S).$$

Moreover, by Lemma 7.2.1,

$$\sum_{i \in R} \gamma_i^{v,k} = \sum_{i \in R} \chi_i^{v,k}(S) \geq \sum_{i \in R} \chi_i^{v,k}(R) \geq v(R) = v_S(R)$$

for all $R \in 2^S$. Hence, $\gamma_S^{v,k} \in \mathcal{C}(v_S)$. □

The egalitarian procedure reaches a steady state when the collection of egalitarian admissible coalitions is a cover, i.e. all players have become egalitarian claimants. By Lemma 7.2.2, the egalitarian procedure converges to this steady state within a number of iterations which is bounded by the number of players in the underlying transferable utility game.

The players stop negotiating when they all have acquired an egalitarian claim. Although this egalitarian claim is bounded from below by the individual worth of the player, it is possibly negative. In any case, the egalitarian claims can be obtained in one or more egalitarian admissible coalitions. They form aspiration levels for the allocation of the worth of the grand coalition. A special situation arises when the grand coalition is egalitarian admissible. In the next section, we further describe the egalitarian steady state and define the procedural egalitarian solution which allocates the worth of the grand coalition in an egalitarian way among the players, taking into account their (generally overefficient) egalitarian claims.

7.3 The procedural egalitarian solution

In this section, we introduce the procedural egalitarian solution for transferable utility games. This solution is based on the egalitarian steady state to which the egalitarian procedure converges.

Definition
Let $v \in \mathrm{TU}^N$ be a transferable utility game. The iteration $n^v \in \{1, \ldots, |N|\}$ is given by $n^v = \min\{k \in \mathbb{N} \mid P^{v,k} = N\}$. The vector of *egalitarian claims* $\widehat{\gamma}^v \in \mathbb{R}^N$ is given by $\widehat{\gamma}^v = \gamma^{v,n^v}$. The collection $\widehat{\mathcal{A}}^v \subseteq 2^N \setminus \{\emptyset\}$ is given by $\widehat{\mathcal{A}}^v = \{S \in \mathcal{A}^{v,n^v} \mid \forall_{T \in \mathcal{A}^{v,n^v}} : S \not\subset T\}$. The set of *strong egalitarian claimants* $D^v \in 2^N$ is given by $D^v = \bigcap_{S \in \widehat{\mathcal{A}}^v} S$.

Note that $\sum_{i \in S} \widehat{\gamma}_i^v \geq v(S)$ for all $S \in 2^N$ and $\mathcal{A}^{v,n^v} = \{S \in 2^N \setminus \{\emptyset\} \mid \sum_{i \in S} \widehat{\gamma}_i^v = v(S)\}$. Players can obtain their egalitarian claim in the egalitarian admissible coalitions of which they are member. We only consider the inclusion-wise maximal egalitarian admissible coalitions. Players which are member of all maximal egalitarian admissible coalitions are called *strong* egalitarian claimants. The procedural egalitarian solution assigns to the strong egalitarian claimants their claims and divides the remaining worth of the grand coalition among the other players according to the constrained equal awards rule, the standard concept of egalitarianism in the context of bankruptcy problems with transferable utility.

Definition (Procedural Egalitarian Solution)
The *procedural egalitarian solution* $\Gamma : \mathrm{TU}^N \to \mathbb{R}^N$ is the solution which assigns to any $v \in \mathrm{TU}^N$ the payoff allocation

$$\Gamma(v) = \left(\widehat{\gamma}^v_{D^v}, \mathrm{CEA}_{N \setminus D^v}\left(v(N) - \sum_{i \in D^v} \widehat{\gamma}_i^v, \widehat{\gamma}^v_{N \setminus D^v}\right)\right).$$

Note that the procedural egalitarian solution is well-defined by extending the domain of CEA to bankruptcy problems with negative estate or claims and allowing $a^{v(N) - \sum_{i \in D^v} \widehat{\gamma}_i^v, \widehat{\gamma}^v_{N \setminus D^v}}$ to take a negative value for any $v \in \mathrm{TU}^N$.

Example 7.2
Let $N = \{1, 2, 3\}$ and consider the game $v \in \mathrm{TU}^N$ from Example 7.1. Then $n^v = 3$, $\widehat{\gamma}^v = (5, 3, 1)$, $\widehat{\mathcal{A}}^v = \{N\}$, and $D^v = N$. Hence, $\Gamma(v) = (5, 3, 1)$. △

As in Example 7.1 and Example 7.2, an interesting situation arises when the grand coalition is egalitarian admissible and all players are strong egalitarian claimants. We introduce the notion of egalitarian stability to describe these games and we show that egalitarian stability characterizes the class of games for which the procedural egalitarian solution is a core element.

Definition (Egalitarian Stability)
A transferable utility game $v \in \mathrm{TU}^N$ is *egalitarian stable* if $\widehat{\mathcal{A}}^v = \{N\}$.

Theorem 7.3.1
A transferable utility game $v \in \mathrm{TU}^N$ is egalitarian stable if and only if $\Gamma(v) \in \mathcal{C}(v)$.

Proof. Assume that $v \in \mathrm{TU}^N$ is egalitarian stable. Then $N \in \mathcal{A}^{v,n^v}$ and $D^v = N$. This means that $\Gamma(v) = \widehat{\gamma}^v = \gamma^{v,n^v}$. Moreover, Proposition 7.2.4 implies that $\gamma^{v,n^v} \in \mathcal{C}(v)$. Hence, $\Gamma(v) \in \mathcal{C}(v)$.

Assume that $\Gamma(v) \in \mathcal{C}(v)$. Suppose that v is not egalitarian stable. Then $\sum_{i \in N} \widehat{\gamma}_i^v > v(N)$ and there exists an $i \in N \setminus D^v$ such that $\Gamma_i(v) < \widehat{\gamma}_i^v$. This means that

$$\sum_{i \in S} \Gamma_i(v) < \sum_{i \in S} \widehat{\gamma}_i^v = v(S)$$

for all $S \in \mathcal{A}^{v,n^v}$ for which $i \in S$. This contradicts that $\Gamma(v) \in \mathcal{C}(v)$. Hence, v is egalitarian stable. □

Since the collection of egalitarian admissible coalitions is a cover, convexity is a sufficient condition for egalitarian stability by Proposition 7.2.3. Examples of convex games are bankruptcy games. Interestingly, we show that the procedural egalitarian solution of a bankruptcy game coincides with the constrained equal awards rule of the underlying bankruptcy problem. This illustrates the strong connection between the procedural egalitarian solution and the constrained equal awards rule. Besides, it justifies the use of the latter in the definition of the procedural egalitarian solution for transferable utility games which are not egalitarian stable.

Theorem 7.3.2
Let $(e, c) \in \mathrm{TUBR}^N$ be a bankruptcy problem. Then $\Gamma(v^{e,c}) = \mathrm{CEA}(e, c)$.

Proof. Since bankruptcy games are convex, bankruptcy games are egalitarian stable by Proposition 7.2.3. Then $\Gamma(v^{e,c}) \in \mathcal{C}(v^{e,c})$ by Theorem 7.3.1. First, we show that $\Gamma(v^{e,c}) \leq c$. Suppose that there exists an $i \in N$ such that $\Gamma_i(v^{e,c}) > c_i \geq \min\{c_i, e\}$. Then

$$\begin{aligned}
\sum_{j \in N \setminus \{i\}} \Gamma_j(v^{e,c}) = \sum_{j \in N} \Gamma_j(v^{e,c}) - \Gamma_i(v^{e,c}) &= v^{e,c}(N) - \Gamma_i(v^{e,c}) \\
< e - \min\{c_i, e\} \qquad &= v^{e,c}(N \setminus \{i\}).
\end{aligned}$$

This contradicts that $\Gamma(v^{e,c}) \in \mathcal{C}(v^{e,c})$. Hence, $\Gamma(v^{e,c}) \leq c$.

If $\sum_{j\in N} c_j = e$, then $\Gamma(v^{e,c}) = c = \text{CEA}(e,c)$. Suppose that $\sum_{j\in N} c_j > e$. We show that $\chi_i^{v^{e,c},1}(S) \leq a^{e,c}$ for all $S \in 2^N \setminus \{\emptyset\}$ and any $i \in S$. Let $S \in 2^N \setminus \{\emptyset\}$ and let $i \in S$. If $\sum_{j\in N\setminus S} c_j > e$, then $\chi_i^{v^{e,c},1}(S) = \frac{v^{e,c}(S)}{|S|} = 0 \leq a^{e,c}$. If $\sum_{j\in N\setminus S} c_j \leq e$, then

$$\begin{aligned}\chi_i^{v^{e,c},1}(S) = \frac{v^{e,c}(S)}{|S|} &= \frac{e - \sum_{j\in N\setminus S} c_j}{|S|} = \frac{\sum_{j\in N} \text{CEA}_j(e,c) - \sum_{j\in N\setminus S} c_j}{|S|} \\ \leq \frac{\sum_{j\in S} \text{CEA}_j(e,c)}{|S|} &\leq \frac{\sum_{j\in S} a^{e,c}}{|S|} = \frac{|S|a^{e,c}}{|S|} = a^{e,c}.\end{aligned}$$

Hence, $\chi_i^{v^{e,c},1}(S) \leq a^{e,c}$ for all $S \in 2^N \setminus \{\emptyset\}$ and any $i \in S$.

Now, let $H^{e,c} \in 2^N \setminus \{\emptyset\}$ be defined by

$$H^{e,c} = \{i \in N \mid \text{CEA}_i(e,c) = a^{e,c}\}.$$

Then

$$\begin{aligned}\chi_i^{v^{e,c},1}(H^{e,c}) = \frac{v^{e,c}(H^{e,c})}{|H^{e,c}|} &= \frac{e - \sum_{j\in N\setminus H^{e,c}} c_j}{|H^{e,c}|} = \frac{\sum_{j\in H^{e,c}} \text{CEA}_j(e,c)}{|H^{e,c}|} \\ = \frac{\sum_{j\in H^{e,c}} a^{e,c}}{|H^{e,c}|} &= \frac{|H^{e,c}|a^{e,c}}{|H^{e,c}|} = a^{e,c}\end{aligned}$$

for all $i \in H^{e,c}$. This means that $\Gamma_i(v^{e,c}) = \widehat{\gamma}_i^{v^{e,c}} = \gamma_i^{v^{e,c},1} = \chi_i^{v^{e,c},1}(H^{e,c}) = a^{e,c}$ for all $i \in H^{e,c}$. Since $\Gamma(v^{e,c}) \leq c$, this implies that $\Gamma_{N\setminus H^{e,c}}(v^{e,c}) = c_{N\setminus H^{e,c}}$. Hence, $\Gamma(v^{e,c}) = \text{CEA}(e,c)$. □

Example 7.3
Let $N = \{1,2,3\}$ and consider the bankruptcy problem $(e,c) \in \text{TUBR}^N$ given by $e = 12$ and $c = (2,6,8)$. Then $\text{CEA}(e,c) = (2,5,5)$. The worth of each coalition in the corresponding bankruptcy game $v^{e,c} \in \text{TU}^N$ and the egalitarian distribution in the first two iterations of the egalitarian procedure are presented in the following table.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v^{e,c}(S)$	0	2	4	4	6	10	12
$\chi^{v^{e,c},1}(S)$	$(0,\cdot,\cdot)$	$(\cdot,2,\cdot)$	$(\cdot,\cdot,4)$	$(2,2,\cdot)$	$(3,\cdot,3)$	$(\cdot,\underline{5},\underline{5})$	$(4,4,4)$
$\chi^{v^{e,c},2}(S)$	$(0,\cdot,\cdot)$	$(\cdot,\mathbf{5},\cdot)$	$(\cdot,\cdot,\mathbf{5})$	$(-1,\mathbf{5},\cdot)$	$(1,\cdot,\mathbf{5})$	$(\cdot,\mathbf{5},\mathbf{5})$	$(\underline{2},\mathbf{5},\mathbf{5})$

In the first iteration, $\mathcal{A}^{v^{e,c},1} = \{\{2,3\}\}$, $P^{v^{e,c},1} = \{2,3\}$, and $\gamma^{v^{e,c},1} = (\cdot,5,5)$. In the second iteration, $\mathcal{A}^{v^{e,c},2} = \{\{2,3\},\{1,2,3\}\}$, $P^{v^{e,c},2} = N$, and $\gamma^{v^{e,c},2} = (2,5,5)$. This means that $n^{v^{e,c}} = 2$, $\widehat{\gamma}^{v^{e,c}} = (2,5,5)$, $\widehat{\mathcal{A}}^{v^{e,c}} = \{N\}$, and $D^{v^{e,c}} = N$. Hence, $\Gamma(v^{e,c}) = (2,5,5)$. △

Although convexity is a sufficient condition for egalitarian stability, Example 7.1 shows that this condition is not necessary. Balancedness is a necessary condition for egalitarian stability and it is sufficient if there exists a balanced collection of egalitarian admissible coalitions by Proposition 7.2.3. The next example shows that balancedness is not a sufficient condition for egalitarian stability.

Example 7.4
In a glove game $v \in \mathrm{TU}^N$, there exist $L, R \in 2^N \setminus \{\emptyset\}$ such that $N = L \cup R$ and $L \cap R = \emptyset$. Players in L are each endowed with a left-hand glove and players in R are each endowed with a right-hand glove, but only pairs of one left-hand and one right-hand glove have value. The worth of a coalition $S \in 2^N$ can therefore be described by $v(S) = \min\{|L \cap S|, |R \cap S|\}$. In a glove game, the egalitarian steady state is reached in the first iteration, i.e. $n^v = 1$. Moreover, $\widehat{\mathcal{A}}^v = \{S \in 2^N \mid v(S) = v(N), |L \cap S| = |R \cap S|\}$ and $\widehat{\gamma}_i^v = \frac{1}{2}$ for all $i \in N$. This means that

$$D^v = \begin{cases} L & \text{if } |L| < |R|; \\ N & \text{if } |L| = |R|; \\ R & \text{if } |L| > |R|. \end{cases}$$

This means that a glove game is egalitarian stable if and only if $|L| = |R|$. The procedural egalitarian solution divides a half per pair of gloves equally among the left-hand glove players, and the other half per pair of gloves equally among the right-hand glove players.

Let $N = \{1, 2, 3\}$ and consider the glove game $v \in \mathrm{TU}^N$ in which $L = \{1\}$ and $R = \{2, 3\}$. The worth of each coalition and the egalitarian distribution in the first iteration of the egalitarian procedure are presented in the following table.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	1	1	0	1
$\chi^{v,1}(S)$	$(0,\cdot,\cdot)$	$(\cdot,0,\cdot)$	$(\cdot,\cdot,0)$	$(\underline{\frac{1}{2}},\underline{\frac{1}{2}},\cdot)$	$(\underline{\frac{1}{2}},\cdot,\underline{\frac{1}{2}})$	$(\cdot,0,0)$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$

Then $\mathcal{A}^{v,1} = \{\{1,2\},\{1,3\}\}$, $P^{v,1} = N$, and $\gamma^{v,1} = (\frac{1}{2},\frac{1}{2},\frac{1}{2})$. This means that $n^v = 1$, $\widehat{\gamma}^v = (\frac{1}{2},\frac{1}{2},\frac{1}{2})$, $\widehat{\mathcal{A}}^v = \{\{1,2\},\{1,3\}\}$, and $D^v = \{1\}$. Hence, $\Gamma(v) = (\frac{1}{2},\frac{1}{4},\frac{1}{4})$.

Besides, the Shapley value (cf. Shapley (1953)) is given by $(\frac{2}{3},\frac{1}{6},\frac{1}{6})$ and the nucleolus (cf. Schmeidler (1969)) is given by $(1,0,0)$. The egalitarian solution of Dutta and Ray (1989) does not exist, since the set $\{x \in \mathbb{R}^N \mid x_1 > \frac{1}{2}, x_1 + x_2 + x_3 = 1\}$ does not have a Lorenz maximal element. Contrary to these solution concepts, the procedural egalitarian solution treats not only the players within L or R symmetrically, but also L and R as groups symmetrically. △

Future research could look for a characterization of the class of egalitarian stable transferable utility games. This will contribute to a better understanding of situations in which egalitarianism and coalitional rationality do not conflict.

Next, we discuss some properties of the procedural egalitarian solution. We show that the procedural egalitarian solution satisfies the properties symmetry, dummy invariance, weak strategic covariance, and aggregate monotonicity. The proof is provided in the appendix.

Proposition 7.3.3

(i) The procedural egalitarian solution satisfies symmetry.

(ii) The procedural egalitarian solution satisfies dummy invariance.

(iii) The procedural egalitarian solution satisfies weak strategic covariance.

(iv) The procedural egalitarian solution satisfies aggregate monotonicity.

Contrary to the Shapley value (cf. Shapley (1953)) and the the nucleolus (cf. Schmeidler (1969)), the procedural egalitarian solution does not satisfy strong strategic covariance. We refer to Dutta and Ray (1989) for a discussion on why egalitarian solution concepts actually *should* fail to satisfy this strong property.

Megiddo (1974) showed that the nucleolus does not satisfy aggregate monotonicity. Young (1985) showed that the Shapley value is the unique solution satisfying symmetry and marginal monotonicity. This means that the procedural egalitarian solution does not satisfy marginal monotonicity. The following example shows that the procedural egalitarian does not satisfy coalitional monotonicity either.

Example 7.5
Let $N = \{1, 2, 3\}$ and consider the game $v' \in \mathrm{TU}^N$ for which the worth of each coalition and the egalitarian distribution in the first two iterations of the egalitarian procedure are presented in the following table.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v'(S)$	5	0	0	10	0	0	9
$\chi^{v',1}(S)$	$(\underline{5},\cdot,\cdot)$	$(\cdot,0,\cdot)$	$(\cdot,\cdot,0)$	$(\underline{5},\underline{5},\cdot)$	$(0,\cdot,0)$	$(\cdot,0,0)$	$(3,3,3)$
$\chi^{v',2}(S)$	$(\mathbf{5},\cdot,\cdot)$	$(\cdot,\mathbf{5},\cdot)$	$(\cdot,\cdot,\underline{0})$	$(\mathbf{5},\mathbf{5},\cdot)$	$(\mathbf{5},\cdot,\text{-}5)$	$(\cdot,\mathbf{5},\text{-}5)$	$(\mathbf{5},\mathbf{5},\text{-}1)$

In the first iteration, $\mathcal{A}^{v',1} = \{\{1\},\{1,2\}\}$, $P^{v',1} = \{1,2\}$, and $\gamma^{v',1} = (5,5,\cdot)$. In the second iteration, $\mathcal{A}^{v',2} = \{\{1\},\{3\},\{1,2\}\}$, $P^{v',2} = N$, and $\gamma^{v',2} = (5,5,0)$. This means that $n^{v'} = 2$, $\widehat{\gamma}^{v'} = (5,5,0)$, $\widehat{\mathcal{A}}^{v'} = \{\{3\},\{1,2\}\}$, and $D^{v'} = \emptyset$. Hence, $\Gamma(v') = (4\frac{1}{2}, 4\frac{1}{2}, 0)$.

Consider the game $v \in \mathrm{TU}^N$ from Example 7.1 and Example 7.2. We have $v(\{1,2\}) \leq v'(\{1,2\})$ and $v(S) = v'(S)$ for all $S \in 2^N$ for which $S \neq \{1,2\}$. However, $\Gamma_{\{1,2\}}(v) \nleq \Gamma_{\{1,2\}}(v')$. Hence, the procedural egalitarian solution does not satisfy coalitional monotonicity. △

Besides, contrary to both the Shapley value and the nucleolus, the procedural egalitarian solution prescribes equal division of the worth of the grand coalition when the grand coalition has highest average worth, i.e.

$$\left(\frac{v(N)}{|N|}\right)_{i\in N} \in \mathcal{C}(v) \text{ implies that } \Gamma(v) = \left(\frac{v(N)}{|N|}\right)_{i\in N}.$$

Finally, we show that the procedural egalitarian solution coincides with the egalitarian solution of Dutta and Ray (1989) on the class of convex games.

Definition (Egalitarian Solution of Dutta and Ray (1989))
Let $v \in \mathrm{TU}^N$ be a convex transferable utility game. Let $v_0 = v$ and $T_0^v = \emptyset$. For any $k \in \mathbb{N}$, let v_k assign to each $S \subseteq N \setminus (\bigcup_{s=0}^{k-1} T_s^v)$ the worth $v_k(S) = v_{k-1}(S \cup T_{k-1}^v) - v_{k-1}(T_{k-1}^v)$ and let $T_k^v \in 2^N \setminus \{\emptyset\}$ be the largest coalition having the highest average worth in v_k. Let $i \in N$ and let $k \in \mathbb{N}$ be such that $i \in T_k^v$. The *egalitarian solution* ES is given by $\mathrm{ES}_i(v) = \frac{v_k(T_k^v)}{|T_k^v|}$.

Theorem 7.3.4
The procedural egalitarian solution coincides with the egalitarian solution of Dutta and Ray (1989) on the class of convex games.

Proof. Let $v \in \mathrm{TU}^N$ be a convex transferable utility game. Since v is egalitarian stable, $\Gamma(v) = \widehat{\gamma}^v$. First, we show by induction that $v_k(S) = v(S \cup P^{v,k-1}) - \sum_{j\in P^{v,k-1}} \gamma_j^{v,k-1}$ for all $k \in \mathbb{N}$ and any $S \subseteq N \setminus (\bigcup_{s=0}^{k-1} T_s^v)$. We have

$$v_1(S) = v_0(S \cup T_0^v) - v_0(T_0^v) = v(S) - v(\emptyset) = v(S) = v(S \cup P^{v,0}) - \sum_{j\in P^{v,0}} \gamma_j^{v,0}$$

for all $S \subseteq N$.

Let $k \in \mathbb{N}$ and assume that $v_k(S) = v(S \cup P^{v,k-1}) - \sum_{j\in P^{v,k-1}} \gamma_j^{v,k-1}$ for all $S \subseteq N \setminus (\bigcup_{s=0}^{k-1} T_s^v)$. Then

$$\begin{aligned} v_{k+1}(S) &= v_k(S \cup T_k^v) - v_k(T_k^v) \\ &= v(S \cup T_k^v \cup P^{v,k-1}) - \sum_{j\in P^{v,k-1}} \gamma_j^{v,k-1} - v(T_k^v \cup P^{v,k-1}) + \sum_{j\in P^{v,k-1}} \gamma_j^{v,k-1} \\ &= v(S \cup P^{v,k}) - v(P^{v,k}) \\ &= v(S \cup P^{v,k}) - \sum_{j\in P^{v,k}} \gamma_j^{v,k} \end{aligned}$$

for all $S \subseteq N \setminus (\bigcup_{s=0}^{k} T_s^v)$, where the last equality follows from Proposition 7.2.3.

Hence, $v_k(S) = v(S \cup P^{v,k-1}) - \sum_{j \in P^{v,k-1}} \gamma_j^{v,k-1}$ for all $k \in \mathbb{N}$ and any $S \subseteq N \setminus (\bigcup_{s=0}^{k-1} T_s^v)$. This means that $T_k^v = P^{v,k} \setminus P^{v,k-1}$ for all $k \in N$ and $\text{ES}_i(v) = \widehat{\gamma}_i^v$ for all $i \in P^{v,k}$. Hence, $\text{ES}(v) = \Gamma(v)$. □

This means that the procedural egalitarian solution for convex games is axiomatically characterized by Dutta (1990), Klijn, Slikker, Tijs, and Zarzuelo (2000), and Arin, Kuipers, and Vermeulen (2003). Future research could look for properties which extend axiomatic characterizations of the procedural egalitarian solution for convex games to a more general class of games.

7.A Appendix

Proposition 7.3.3

Proof. (i) Let $v \in \text{TU}^N$ and let $i, j \in N$ be such that $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$. First, we show by induction that $\chi_i^{v,k}(S \cup \{i\}) = \chi_j^{v,k}(S \cup \{j\})$ and $\chi_h^{v,k}(S \cup \{i\}) = \chi_h^{v,k}(S \cup \{j\})$ for all $k \in \mathbb{N}$, any $S \in 2^N$ for which $i, j \in S$ or $i, j \notin S$, and any $h \in S$. Then

$$\begin{aligned}\chi_h^{v,1}(S \cup \{i\}) = \chi_i^{v,1}(S \cup \{i\}) &= \frac{v(S \cup \{i\})}{|S \cup \{i\}|} \\ &= \frac{v(S \cup \{j\})}{|S \cup \{j\}|} = \chi_j^{v,1}(S \cup \{j\}) = \chi_h^{v,1}(S \cup \{j\})\end{aligned}$$

for all $S \in 2^N$ for which $i, j \in S$ or $i, j \notin S$, and any $h \in S$.

Let $k \in \mathbb{N}$ and assume that $\chi_i^{v,k}(S \cup \{i\}) = \chi_j^{v,k}(S \cup \{j\})$ and $\chi_h^{v,k}(S \cup \{i\}) = \chi_h^{v,k}(S \cup \{j\})$ for all $S \in 2^N$ for which $i, j \in S$ or $i, j \notin S$, and any $h \in S$. Then $S \cup \{i\} \in \mathcal{A}^{v,k}$ if and only if $S \cup \{j\} \in \mathcal{A}^{v,k}$ for all $S \in 2^N$ for which $i, j \in S$ or $i, j \notin S$, which means that $i \in P^{v,k}$ if and only if $j \in P^{v,k}$. Moreover, $\gamma_i^{v,k} = \gamma_j^{v,k}$ if $i, j \in P^{v,k}$. We have

$$\begin{aligned}\chi_i^{v,k+1}(S \cup \{i\}) &= \begin{cases} \gamma_i^{v,k} & \text{if } i \in P^{v,k}; \\ \frac{v(S \cup \{i\}) - \sum_{p \in S \cap P^{v,k}} \gamma_p^{v,k}}{|(S \cup \{i\}) \setminus P^{v,k}|} & \text{if } i \notin P^{v,k} \end{cases} \\ &= \begin{cases} \gamma_j^{v,k} & \text{if } j \in P^{v,k}; \\ \frac{v(S \cup \{j\}) - \sum_{p \in S \cap P^{v,k}} \gamma_p^{v,k}}{|(S \cup \{j\}) \setminus P^{v,k}|} & \text{if } j \notin P^{v,k} \end{cases} \\ &= \chi_j^{v,k+1}(S \cup \{j\})\end{aligned}$$

and

$$\begin{aligned}\chi_h^{v,k+1}(S\cup\{i\}) &= \begin{cases}\gamma_h^{v,k} & \text{if } h\in P^{v,k};\\ \frac{v(S\cup\{i\})-\sum_{p\in(S\cup\{i\})\cap P^{v,k}}\gamma_p^{v,k}}{|(S\cup\{i\})\setminus P^{v,k}|} & \text{if } h\notin P^{v,k}\end{cases}\\ &= \begin{cases}\gamma_h^{v,k} & \text{if } h\in P^{v,k};\\ \frac{v(S\cup\{j\})-\sum_{p\in(S\cup\{j\})\cap P^{v,k}}\gamma_p^{v,k}}{|(S\cup\{j\})\setminus P^{v,k}|} & \text{if } h\notin P^{v,k}\end{cases}\\ &= \chi_h^{v,k+1}(S\cup\{j\})\end{aligned}$$

for all $S\in 2^N$ for which $i,j\in S$ or $i,j\notin S$, and any $h\in S$. Hence, $\chi_i^{v,k}(S\cup\{i\}) = \chi_j^{v,k}(S\cup\{j\})$ and $\chi_h^{v,k}(S\cup\{i\}) = \chi_h^{v,k}(S\cup\{j\})$ for all $k\in\mathbb{N}$, any $S\in 2^N$ for which $i,j\in S$ or $i,j\notin S$, and any $h\in S$.

Then $S\cup\{i\}\in\widehat{\mathcal{A}}^v$ if and only if $S\cup\{j\}\in\widehat{\mathcal{A}}^v$ for all $S\in 2^N$ for which $i,j\in S$ or $i,j\notin S$, which means that $i\in D^v$ if and only if $j\in D^v$. Moreover, $\widehat{\gamma}_i^v = \widehat{\gamma}_j^v$. This implies that

$$\begin{aligned}\Gamma_i(v) &= \begin{cases}\widehat{\gamma}_i^v & \text{if } i\in D^v;\\ \text{CEA}_i\left(v(N)-\sum_{k\in D^v}\widehat{\gamma}_k^v, \widehat{\gamma}_{N\setminus D^v}^v\right) & \text{if } i\notin D^v\end{cases}\\ &= \begin{cases}\widehat{\gamma}_j^v & \text{if } j\in D^v;\\ \text{CEA}_j\left(v(N)-\sum_{k\in D^v}\widehat{\gamma}_k^v, \widehat{\gamma}_{N\setminus D^v}^v\right) & \text{if } j\notin D^v\end{cases}\\ &= \Gamma_j(v).\end{aligned}$$

Hence, the procedural egalitarian solution satisfies symmetry.

(*ii*) Let $v\in\text{TU}^N$ and let $i\in N$ be such that $v(S\cup\{i\}) = v(S)+v(\{i\})$ for all $S\subseteq N\setminus\{i\}$. Then

$$v(S\setminus\{i\}) = v(S)-v(\{i\}) = \sum_{j\in S}\widehat{\gamma}_j^v - v(\{i\}) \geq \sum_{j\in S}\widehat{\gamma}_j^v - \widehat{\gamma}_i^v = \sum_{j\in S\setminus\{i\}}\widehat{\gamma}_j^v \geq v(S\setminus\{i\})$$

for all $S\in\mathcal{A}^{v,n^v}$ for which $i\in S$. This means that $\widehat{\gamma}_i^v = v(\{i\})$. Then

$$v(S\cup\{i\}) = v(S)+v(\{i\}) = \sum_{j\in S}\widehat{\gamma}_j^v + \widehat{\gamma}_i^v = \sum_{j\in S\cup\{i\}}\widehat{\gamma}_j^v$$

for all $S\in\mathcal{A}^{v,n^v}$ for which $i\notin S$. This means that $S\cup\{i\}\in\mathcal{A}^{v,n^v}$ for all $S\in\mathcal{A}^{v,n^v}$. This implies that $i\in D^v$, so $\Gamma_i(v) = \widehat{\gamma}_i^v = v(\{i\})$. Hence, the procedural egalitarian solution satisfies dummy invariance.

(iii) Let $v, v' \in \mathrm{TU}^N$, let $\alpha \in \mathbb{R}_{++}$, and let $\beta \in \mathbb{R}$ be such that $v(S) = \alpha v'(S) + \beta|S|$ for all $S \in 2^N$. First, we show by induction that $\chi_i^{v,k}(S) = \alpha\chi_i^{v',k}(S) + \beta$ for all $k \in \mathbb{N}$, any $S \in 2^N \setminus \{\emptyset\}$, and any $i \in S$. We have

$$\chi_i^{v,1}(S) = \frac{v(S)}{|S|} = \frac{\alpha v'(S) + \beta|S|}{|S|} = \alpha\frac{v'(S)}{|S|} + \beta = \alpha\chi_i^{v',1}(S) + \beta$$

for all $S \in 2^N \setminus \{\emptyset\}$ and any $i \in S$.

Let $k \in \mathbb{N}$ and assume that $\chi_i^{v,k}(S) = \alpha\chi_i^{v',k}(S) + \beta$ for all $S \in 2^N \setminus \{\emptyset\}$ and any $i \in S$. Then $\mathcal{A}^{v,k} = \mathcal{A}^{v',k}$, which means that $P^{v,k} = P^{v',k}$. Moreover, $\gamma^{v,k} = (\alpha\gamma_j^{v',k} + \beta)_{j \in P^{v',k}}$. We have

$$\begin{aligned}
\chi_i^{v,k+1}(S) &= \begin{cases} \gamma_i^{v,k} & \text{if } i \in P^{v,k}; \\ \frac{v(S) - \sum_{j \in S \cap P^{v,k}} \gamma_j^{v,k}}{|S \setminus P^{v,k}|} & \text{if } i \notin P^{v,k} \end{cases} \\
&= \begin{cases} \alpha\gamma_i^{v',k} + \beta & \text{if } i \in P^{v',k}; \\ \frac{\alpha v'(S) + \beta|S| - \sum_{j \in S \cap P^{v',k}} (\alpha\gamma_j^{v',k} + \beta)}{|S \setminus P^{v',k}|} & \text{if } i \notin P^{v',k} \end{cases} \\
&= \begin{cases} \alpha\gamma_i^{v',k} + \beta & \text{if } i \in P^{v',k}; \\ \alpha\frac{v'(S) - \sum_{j \in S \cap P^{v',k}} \gamma_j^{v',k}}{|S \setminus P^{v',k}|} + \beta & \text{if } i \notin P^{v',k} \end{cases} \\
&= \alpha\chi_i^{v',k+1}(S) + \beta
\end{aligned}$$

for all $S \in 2^N \setminus \{\emptyset\}$ and any $i \in S$. Hence, $\chi_i^{v,k}(S) = \alpha\chi_i^{v',k}(S) + \beta$ for all $k \in \mathbb{N}$, any $S \in 2^N \setminus \{\emptyset\}$, and any $i \in S$.

Then $n^v = n^{v'}$ and $\widehat{\mathcal{A}}^v = \widehat{\mathcal{A}}^{v'}$, which means that $D^v = D^{v'}$. Moreover, $\widehat{\gamma}^v = (\alpha\widehat{\gamma}_j^{v'} + \beta)_{j \in N}$. This implies that

$$\begin{aligned}
\Gamma_i(v) &= \begin{cases} \widehat{\gamma}_i^v & \text{if } i \in D^v; \\ \mathrm{CEA}_i\left(v(N) - \sum_{j \in D^v} \widehat{\gamma}_j^v, \widehat{\gamma}_{N \setminus D^v}^v\right) & \text{if } i \notin D^v \end{cases} \\
&= \begin{cases} \alpha\widehat{\gamma}_i^{v'} + \beta & \text{if } i \in D^{v'}; \\ \mathrm{CEA}_i\left(\alpha v'(N) + \beta|N| - \sum_{j \in D^{v'}} (\alpha\widehat{\gamma}_j^{v'} + \beta), (\alpha\widehat{\gamma}_j^{v'} + \beta)_{j \in N \setminus D^{v'}}\right) & \text{if } i \notin D^{v'} \end{cases} \\
&= \begin{cases} \alpha\widehat{\gamma}_i^{v'} + \beta & \text{if } i \in D^{v'}; \\ \mathrm{CEA}_i\left(\alpha(v'(N) - \sum_{j \in D^{v'}} \widehat{\gamma}_j^{v'}) + \beta|N \setminus D^{v'}|, (\alpha\widehat{\gamma}_j^{v'} + \beta)_{j \in N \setminus D^{v'}}\right) & \text{if } i \notin D^{v'} \end{cases} \\
&= \begin{cases} \alpha\widehat{\gamma}_i^{v'} + \beta & \text{if } i \in D^{v'}; \\ \alpha\mathrm{CEA}_i\left(v'(N) - \sum_{j \in D^{v'}} \widehat{\gamma}_j^{v'}, \widehat{\gamma}_{N \setminus D^{v'}}^{v'}\right) + \beta & \text{if } i \notin D^{v'} \end{cases} \\
&= \alpha\Gamma_i(v') + \beta
\end{aligned}$$

for all $i \in N$. Hence, the procedural egalitarian solution satisfies weak covariance.

(iv) First, we show that $\chi_i^{v,k+1}(S) \leq \chi_i^{v,k}(S)$ for all $v \in \mathrm{TU}^N$, any $k \in \mathbb{N}$, any $i \in N \backslash P^{v,k}$, and any $S \in 2^N$ for which $i \in S$. Let $v \in \mathrm{TU}^N$, let $k \in \mathbb{N}$, let $i \in N \backslash P^{v,k}$, and let $S \in 2^N$ be such that $i \in S$. Then $i \notin P^{v,k-1}$ by Lemma 7.2.2 and

$$\begin{aligned}
\chi_i^{v,k+1}(S) &= \frac{v(S) - \sum_{j \in S \cap P^{v,k}} \gamma_j^{v,k}}{|S \setminus P^{v,k}|} \\
&= \frac{v(S) - \sum_{j \in S \cap P^{v,k-1}} \gamma_j^{v,k} - \sum_{j \in S \cap (P^{v,k} \setminus P^{v,k-1})} \gamma_j^{v,k}}{|S \setminus P^{v,k}|} \\
&\leq \frac{v(S) - \sum_{j \in S \cap P^{v,k-1}} \gamma_j^{v,k-1} - \sum_{j \in S \cap (P^{v,k} \setminus P^{v,k-1})} \chi_j^{v,k}(S)}{|S \setminus P^{v,k}|} \\
&= \frac{|S \setminus P^{v,k-1}|\chi_i^{v,k}(S) - |S \cap (P^{v,k} \setminus P^{v,k-1})|\chi_i^{v,k}(S)}{|S \setminus P^{v,k}|} \\
&= \frac{|S \setminus P^{v,k}|\chi_i^{v,k}(S)}{|S \setminus P^{v,k}|} \\
&= \chi_i^{v,k}(S).
\end{aligned}$$

Hence, $\chi_i^{v,k+1}(S) \leq \chi_i^{v,k}(S)$ for all $v \in \mathrm{TU}^N$, any $k \in \mathbb{N}$, any $i \in N \setminus P^{v,k}$, and any $S \in 2^N$ for which $i \in S$. This also means that, for all $v \in \mathrm{TU}^N$ and any $k \in \mathbb{N}$, $\gamma_i^{v,k} \leq \gamma_j^{v,k}$ for all $i \in P^{v,k} \setminus P^{v,k-1}$ and any $j \in P^{v,k}$ for which $i, j \in S$ for some $S \in \mathcal{A}^{v,k}$. In other words, for all $v \in \mathrm{TU}^N$ and any $i \in N$, there is a coalition $S \in \mathcal{A}^{v,n^v}$ for which $\widehat{\gamma}_i^v \leq \widehat{\gamma}_j^v$ for all $j \in S$.

Now, let $v, v' \in \mathrm{TU}^N$ be such that $v(N) \leq v'(N)$ and $v(S) = v'(S)$ for all $S \subset N$. We show by induction that, for all $k \in \mathbb{N}$, $\gamma^{v',k} \geq \widehat{\gamma}^v$ if $N \in \mathcal{A}^{v',k}$, and $\gamma_i^{v',k} = \widehat{\gamma}_i^v$ for all $i \in P^{v',k}$ if $N \notin \mathcal{A}^{v',k}$. If $N \in \mathcal{A}^{v',1}$, then

$$\gamma_i^{v',1} = \chi_i^{v',1}(N) \geq \chi_i^{v',1}(S) = \frac{v'(S)}{|S|} \geq \frac{v(S)}{|S|} = \chi_i^{v,1}(S) \geq \chi_i^{v,k}(S) = \gamma_i^{v,k} = \widehat{\gamma}_i^v$$

for all $i \in N$, where $S \in 2^N$ and $k \in \mathbb{N}$ are such that $i \in S \cap (P^{v,k} \setminus P^{v,k-1})$ and $S \in \mathcal{A}^{v,k}$. If $N \notin \mathcal{A}^{v',1}$, then

$$\chi_i^{v,1}(S) = \frac{v(S)}{|S|} = \frac{v'(S)}{|S|} = \chi_i^{v',1}(S) \geq \chi_i^{v',1}(T) = \frac{v'(T)}{|T|} \geq \frac{v(T)}{|T|} = \chi_i^{v,1}(T)$$

for all $S \in \mathcal{A}^{v',1}$, any $i \in S$, and any $T \in 2^N$ for which $i \in T$, which means that $\mathcal{A}^{v',1} \subseteq \mathcal{A}^{v,1}$ and $\gamma_i^{v',1} = \gamma_i^{v,1} = \widehat{\gamma}_i^v$ for all $i \in P^{v',1}$.

Let $k \in \mathbb{N}$ and assume that $\gamma^{v',k} \geq \widehat{\gamma}^v$ if $N \in \mathcal{A}^{v',k}$, and $\gamma_i^{v',k} = \widehat{\gamma}_i^v$ for all $i \in P^{v',k}$ if $N \notin \mathcal{A}^{v',k}$. If $N \in \mathcal{A}^{v',k}$, then $N \in \mathcal{A}^{v',k+1}$ by Lemma 7.2.2 and $\gamma^{v',k+1} = \gamma^{v',k} \geq \widehat{\gamma}^v$. Suppose that $N \notin \mathcal{A}^{v',k}$. Then $\gamma_i^{v',k+1} = \gamma_i^{v',k} = \widehat{\gamma}_i^v$ for all $i \in P^{v',k}$ and

$$\begin{aligned}\gamma_i^{v',k+1} \geq \chi_i^{v',k+1}(S) = \frac{v'(S) - \sum_{j \in S \cap P^{v',k}} \gamma_j^{v',k}}{|S \setminus P^{v',k}|} &\geq \frac{v(S) - \sum_{j \in S \cap P^{v',k}} \widehat{\gamma}_j^v}{|S \setminus P^{v',k}|} \\ \geq v(S) - \sum_{j \in S \setminus \{i\}} \widehat{\gamma}_j^v \qquad &= \widehat{\gamma}_i^v\end{aligned}$$

for all $i \in P^{v',k+1} \setminus P^{v',k}$, where $S \in \mathcal{A}^{v,n^v}$ is such that $\widehat{\gamma}_i^v \leq \widehat{\gamma}_j^v$ for all $j \in S$. If $N \in \mathcal{A}^{v',k+1}$, this means that $\gamma^{v',k+1} \geq \widehat{\gamma}^v$. If $N \notin \mathcal{A}^{v',k+1}$, then

$$v'(S) = \sum_{i \in S} \gamma_i^{v',k+1} \geq \sum_{i \in S} \widehat{\gamma}_i^v \geq v(S) = v'(S)$$

for all $S \in \mathcal{A}^{v',k+1}$, which means that $\gamma_i^{v',k+1} = \widehat{\gamma}_i^v$ for all $i \in P^{v',k+1}$. Hence, for all $k \in \mathbb{N}$, $\gamma^{v',k} \geq \widehat{\gamma}^v$ if $N \in \mathcal{A}^{v',k}$, and $\gamma_i^{v',k} = \widehat{\gamma}_i^v$ for all $i \in P^{v',k}$ if $N \notin \mathcal{A}^{v',k}$.

In particular, $\widehat{\gamma}^{v'} = \gamma^{v',n^{v'}} \geq \widehat{\gamma}^v$ if $N \in \mathcal{A}^{v',n^{v'}}$, and $\widehat{\gamma}^{v'} = \gamma^{v',n^{v'}} = \widehat{\gamma}^v$ if $N \notin \mathcal{A}^{v',n^{v'}}$. This means that $\Gamma(v') = \widehat{\gamma}^{v'} \geq \widehat{\gamma}^v \geq \Gamma(v)$ if v' is egalitarian stable, and

$$\begin{aligned}\Gamma(v') &= \left(\widehat{\gamma}^{v'}_{D^{v'}}, \mathrm{CEA}_{N \setminus D^{v'}}\left(v'(N) - \sum_{i \in D^{v'}} \widehat{\gamma}_i^{v'}, \widehat{\gamma}^{v'}_{N \setminus D^{v'}}\right)\right) \\ &= \left(\widehat{\gamma}^{v}_{D^{v}}, \mathrm{CEA}_{N \setminus D^{v}}\left(v'(N) - \sum_{i \in D^{v}} \widehat{\gamma}_i^{v}, \widehat{\gamma}^{v}_{N \setminus D^{v}}\right)\right) \\ &\geq \left(\widehat{\gamma}^{v}_{D^{v}}, \mathrm{CEA}_{N \setminus D^{v}}\left(v(N) - \sum_{i \in D^{v}} \widehat{\gamma}_i^{v}, \widehat{\gamma}^{v}_{N \setminus D^{v}}\right)\right) \\ &= \Gamma(v)\end{aligned}$$

if v' is not egalitarian stable. Hence, the procedural egalitarian solution satisfies aggregate monotonicity. □

8 Egalitarianism in Nontransferable Utility Games

8.1 Introduction

This chapter, based on Dietzenbacher, Borm, and Hendrickx (2017b), focusses on egalitarianism in the context of nontransferable utility games. In a general payoff space where utility levels are individually measured, egalitarianism cannot be applied straightforwardly. To do so, it is necessary to impose assumptions which allow to compare utility not only intrapersonally, but to some extent also interpersonally. In a general allocation problem, a natural and helpful operation is normalization. In particular, zero-normalization requires transforming individual utility in such a way that allocating nothing corresponds to a utility level of zero. In other words, a payoff of zero utility generates the same well-being for an involved agent as the event in which the allocation problem is not solved. This implies that, in case of allocating revenues, it is convenient to restrict to feasible payoff allocations that are nonnegative.

After zero-normalization, the zero vector plays a fundamental role. There, agents are comparable in terms of well-being and the allocation is in that sense egalitarian. The zero vector actually serves as a benchmark for egalitarian allocations. However, in order to study efficient egalitarianism, this single point is not sufficient. For this, at least a second reference point is necessary. The maximal individual payoffs within the feasible allocations, or utopia values, constitute a natural candidate. There, agents are comparable in terms of maximal satisfaction on the basis of feasible allocations and the corresponding vector of utopia values is in that sense egalitarian. The utopia vector relative to the zero vector can be interpreted as an egalitarian direction. It is important to note that this direction and the subsequent results are covariant under individual rescaling of utility.

In a cooperative game, solutions focus on allocations for the grand coalition while taking the opportunities of subcoalitions into account. To allow for an appropriate egalitarian comparison of subcoalitions, it is required to consistently apply a fixed interpretation of egalitarianism. Therefore, the utopia values of the grand coalition are used as a common benchmark within *any* subcoalition. Applying this approach to nontransferable utility games, we define an egalitarian procedure in which players iteratively consider their egalitarian opportunities within subcoalitions in a similar way as in the previous chapter. We introduce the constrained egalitarian solution for nontransferable utility games which takes the result of this egalitarian procedure into account to prescribe a unique egalitarian allocation for the grand coalition. The constrained egalitarian solution generalizes the nonnegative procedural egalitarian solution for transferable utility games.

We compare the constrained egalitarian solution with other well-known solution concepts for nontransferable utility games like the Shapley value (cf. Shapley (1969)), the Harsanyi value (cf. Harsanyi (1963)), and the monotonic solution of Kalai and Samet (1985) using the famous examples introduced by Roth (1980) and Shafer (1980). Contrary to the other solution concepts, the constrained egalitarian solution exactly prescribes the allocation which was proposed by Roth (1980). Moreover, it neatly follows the line of reasoning stated by Shafer (1980).

Interestingly, we show that the constrained egalitarian solution of a bankruptcy game corresponds to the constrained relative equal awards rule of the underlying bankruptcy problem. This illustrates the strong connection between the constrained egalitarian solution and the constrained relative equal awards rule. On the class of bargaining games (cf. Nash (1950)) corresponding to bargaining problems with the zero vector as disagreement point, the constrained egalitarian solution corresponds to the solutions introduced by Kalai and Smorodinsky (1975) and Kalai and Rosenthal (1978). For bargaining games with nonzero disagreement point, it is illustrated that the constrained egalitarian solution offers a new, interesting way to solve bargaining problems.

This chapter is organized in the following way. Section 8.2 formally introduces the constrained egalitarian solution and the underlying egalitarian procedure. Section 8.3 studies the new solution for the Roth-Shafer examples. In Section 8.4 and Section 8.5, the constrained egalitarian solution is analyzed on the class of bankruptcy games and the class of bargaining games, respectively.

8.2 The constrained egalitarian solution

In this section, we introduce the constrained egalitarian solution as an egalitarian solution concept for nontransferable utility games. The constrained egalitarian solution is based on an egalitarian procedure in which coalitional opportunities are explicitly taken into account. By applying the utopia values of the grand coalition as an egalitarian direction in any subcoalition, the procedure starts assigning to any coalition the maximally feasible egalitarian allocation. Players can fix their allocated payoff in a coalition if no member is allocated a higher payoff in any other coalition. These players would still be willing to cooperate within other coalitions provided that they are compensated. Therefore, they claim their fixed payoff in any coalition and the other members are assigned the maximally feasible egalitarian allocation. This iterative procedure continues and eventually all players acquire a claim which is attainable in at least one coalition.

Definition (Egalitarian Procedure)
Let $V \in \mathrm{NTU}^N$ be a nontransferable utility game such that $V(N)$ is nontrivial. The set of 0-*egalitarian claimants* is given by $P^{V,0} = \emptyset$. Let $k \in \mathbb{N}$. The k-*egalitarian distribution* $\chi^{V,k}$ assigns to each $S \in 2^N \setminus \{\emptyset\}$ the payoff allocation $\chi^{V,k}(S) \in \mathbb{R}^S_+$ given by

$$\chi^{V,k}(S) = \left(\gamma^{V,k-1}_{S\cap P^{V,k-1}}, \lambda^{V,k}(S) u^{V(N)}_{S\setminus P^{V,k-1}}\right),$$

where $\lambda^{V,k}$ assigns to each $S \in 2^N \setminus \{\emptyset\}$ for which $S \not\subseteq P^{V,k-1}$ the scalar

$$\lambda^{V,k}(S) = \begin{cases} \max\{t \in \mathbb{R}_+ \mid (\gamma^{V,k-1}_{S\cap P^{V,k-1}}, t u^{V(N)}_{S\setminus P^{V,k-1}}) \in \mathrm{WP}(V(S))\} \\ \qquad\qquad\qquad\qquad \text{if } (\gamma^{V,k-1}_{S\cap P^{V,k-1}}, 0_{S\setminus P^{V,k-1}}) \in V(S); \\ 0 \qquad\qquad\qquad\qquad \text{if } (\gamma^{V,k-1}_{S\cap P^{V,k-1}}, 0_{S\setminus P^{V,k-1}}) \notin V(S). \end{cases}$$

The collection of k-*egalitarian admissible coalitions* is given by

$$\mathcal{A}^{V,k} = \left\{S \in 2^N \setminus \{\emptyset\} \;\middle|\; \chi^{V,k}(S) \in \mathrm{WP}(V(S)), \forall_{i\in S}\forall_{T\in 2^N: i\in T} : \chi^{V,k}_i(T) \leq \chi^{V,k}_i(S)\right\}.$$

The set of k-*egalitarian claimants* $P^{V,k} \in 2^N \setminus \{\emptyset\}$ is given by $P^{V,k} = \bigcup_{S\in\mathcal{A}^{V,k}} S$. The vector of k-*egalitarian claims* $\gamma^{V,k} \in \mathbb{R}^{P^{V,k}}_+$ is given by $\gamma^{V,k}_i = \chi^{V,k}_i(S)$ for all $i \in P^{V,k}$, where $S \in \mathcal{A}^{V,k}$ and $i \in S$.

Later, we show that this procedure is well-defined. First, we provide an illustrative example.

Example 8.1

Let $N = \{1,2,3\}$ and consider the game $V \in \mathrm{NTU}^N$ given by

$$\begin{aligned}
V(\{1\}) &= \left\{x \in \mathbb{R}_+^{\{1\}} \;\middle|\; x \leq 4\right\};\\
V(\{2\}) &= \left\{x \in \mathbb{R}_+^{\{2\}} \;\middle|\; x \leq 1\right\};\\
V(\{3\}) &= \left\{x \in \mathbb{R}_+^{\{3\}} \;\middle|\; x \leq 0\right\};\\
V(\{1,2\}) &= \left\{x \in \mathbb{R}_+^{\{1,2\}} \;\middle|\; x_1 \leq 4, x_2 \leq 2\right\};\\
V(\{1,3\}) &= \left\{x \in \mathbb{R}_+^{\{1,3\}} \;\middle|\; x_1 \leq 2, x_3 \leq 2\right\};\\
V(\{2,3\}) &= \left\{x \in \mathbb{R}_+^{\{2,3\}} \;\middle|\; 2x_2 + x_3 \leq 4\right\};\\
V(\{1,2,3\}) &= \left\{x \in \mathbb{R}_+^{\{1,2,3\}} \;\middle|\; 2x_1 + 2x_2 + x_3 \leq 12\right\}.
\end{aligned}$$

Then $u^{V(N)} = (6,6,12)$. The egalitarian distribution in all iterations of the egalitarian procedure is presented in the following table.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\chi^{V,1}(S)$	$(\underline{4},\cdot,\cdot)$	$(\cdot,1,\cdot)$	$(\cdot,\cdot,0)$	$(2,2,\cdot)$	$(1,\cdot,2)$	$(\cdot,1,2)$	$(2,2,4)$
$\chi^{V,2}(S)$	$(\mathbf{4},\cdot,\cdot)$	$(\cdot,1,\cdot)$	$(\cdot,\cdot,0)$	$(\mathbf{4},\underline{2},\cdot)$	$(\mathbf{4},\cdot,0)$	$(\cdot,1,2)$	$(\mathbf{4},1,2)$
$\chi^{V,3}(S)$	$(\mathbf{4},\cdot,\cdot)$	$(\cdot,\mathbf{2},\cdot)$	$(\cdot,\cdot,\underline{0})$	$(\mathbf{4},\mathbf{2},\cdot)$	$(\mathbf{4},\cdot,0)$	$(\cdot,\mathbf{2},\underline{0})$	$(\mathbf{4},\mathbf{2},\underline{0})$
$\chi^{V,k}(S)$ $(k \geq 4)$	$(\mathbf{4},\cdot,\cdot)$	$(\cdot,\mathbf{2},\cdot)$	$(\cdot,\cdot,\mathbf{0})$	$(\mathbf{4},\mathbf{2},\cdot)$	$(\mathbf{4},\cdot,\mathbf{0})$	$(\cdot,\mathbf{2},\mathbf{0})$	$(\mathbf{4},\mathbf{2},\mathbf{0})$

In the first iteration, $\mathcal{A}^{V,1} = \{\{1\}\}$, $P^{V,1} = \{1\}$, and $\gamma^{V,1} = (4,\cdot,\cdot)$. In the second iteration, $\mathcal{A}^{V,2} = \{\{1\},\{1,2\}\}$, $P^{V,2} = \{1,2\}$, and $\gamma^{V,2} = (4,2,\cdot)$. In all subsequent iterations $k \geq 3$, $\mathcal{A}^{V,k} = \{\{1\},\{3\},\{1,2\},\{2,3\},\{1,2,3\}\}$, $P^{V,k} = N$, and $\gamma^{V,k} = (4,2,0)$. △

Lemma 8.2.1

Let $V \in \mathrm{NTU}^N$ be such that $V(N)$ is nontrivial and let $S \in 2^N \setminus \{\emptyset\}$. Then $\chi^{V,k}(S) \in \mathrm{WUC}(V(S))$ for all $k \in \mathbb{N}$.

Proof. We show the statement by induction. Suppose that $\chi^{V,1}(S) \notin \mathrm{WUC}(V(S))$. Then there exists an $x \in V(S)$ for which $x > \chi^{V,1}(S)$. Since $V(S)$ is comprehensive, this means that there exists a $y \in V(S)$ with $y > \chi^{V,1}(S)$ for which $y = tu_S^{V(N)}$ for some $t \in \mathbb{R}_+$. Since $P^{V,0} = \emptyset$, this means that $t > \lambda^{V,1}(S)$, which contradicts the definition of $\lambda^{V,1}(S)$. Hence, $\chi^{V,1}(S) \in \mathrm{WUC}(V(S))$.

Let $k \in \mathbb{N}$ and assume that $\chi^{V,k}(S) \in \mathrm{WUC}(V(S))$. If $S \subseteq P^{V,k}$, then $\chi^{V,k+1}(S) = \gamma_S^{V,k} \geq \chi^{V,k}(S)$, so $\chi^{V,k+1}(S) \in \mathrm{WUC}(V(S))$. Assume that $S \nsubseteq P^{V,k}$ and suppose that $\chi^{V,k+1}(S) \notin \mathrm{WUC}(V(S))$. Then there exists an $x \in V(S)$ for which $x > \chi^{V,k+1}(S)$. Since $V(S)$ is comprehensive, this means that there exists a $y \in V(S)$ with $y \geq \chi^{V,k+1}(S)$ and $y \neq \chi^{V,k+1}(S)$ for which $y = (\gamma^{V,k}_{S \cap P^{V,k}}, tu^{V(N)}_{S \setminus P^{V,k}})$ for some $t \in \mathbb{R}_+$. This means that $t > \lambda^{V,k+1}(S)$, which contradicts the definition of $\lambda^{V,k+1}(S)$. Hence, $\chi^{V,k+1}(S) \in \mathrm{WUC}(V(S))$. □

Lemma 8.2.1 shows that the egalitarian distribution generally assigns an overefficient allocation to each coalition. Only coalitions which are assigned an efficient allocation can be egalitarian admissible. There, members fix their allocated payoff and claim it in all further iterations. Efficiency can only be achieved when it is possible to allocate to the egalitarian claimants which are member of the coalition their corresponding egalitarian claims. Formally, for all $S \in 2^N \setminus \{\emptyset\}$ and any $k \in \mathbb{N}$, $\chi^{V,k}(S) \in \mathrm{WP}(V(S))$ if and only if $(\gamma^{V,k-1}_{S \cap P^{V,k-1}}, 0_{S \setminus P^{V,k-1}}) \in V(S)$. In particular, this means that the egalitarian distribution assigns in the first iteration an efficient allocation to each coalition.

To an egalitarian admissible coalition, the egalitarian distribution assigns an efficient allocation for which no member is allocated a higher payoff in any other coalition. This suggests that the payoff allocation is an element of the core. Indeed, for each egalitarian admissible coalition, the corresponding vector of egalitarian claims is a core element of the induced subgame.

Proposition 8.2.2
Let $V \in \mathrm{NTU}^N$ be such that $V(N)$ is nontrivial and let $k \in \mathbb{N}$. Then $\gamma^{V,k}_S \in \mathcal{C}^{\mathcal{W}}(V_S)$ for all $S \in \mathcal{A}^{V,k}$.

Proof. Let $S \in \mathcal{A}^{V,k}$. Then $\gamma^{V,k}_S = \chi^{V,k}(S)$ and $\chi^{V,k}(S) \in V_S(S)$. Suppose that $\gamma^{V,k}_S \notin \mathcal{C}^{\mathcal{W}}(V_S)$. Then there exists an $R \in 2^S \backslash \{\emptyset\}$ for which $\gamma^{V,k}_R \in V_S(R) \backslash \mathrm{WP}(V_S(R))$. We have $\gamma^{V,k}_R = \chi^{V,k}_R(S) \geq \chi^{V,k}(R)$. Since $V_S(R)$ is comprehensive, this means that $\chi^{V,k}(R) \in V_S(R) \backslash \mathrm{WP}(V_S(R))$. This contradicts Lemma 8.2.1. Hence, $\gamma^{V,k}_S \in \mathcal{C}^{\mathcal{W}}(V_S)$. $\square$

The question arises whether egalitarian admissible coalitions and egalitarian claimants exist in every nontransferable utility game. Are players always able to acquire an egalitarian claim? The answer is affirmative.

Lemma 8.2.3
Let $V \in \mathrm{NTU}^N$ be such that $V(N)$ is nontrivial and let $k \in \mathbb{N}$. Then $\mathcal{A}^{V,k} \subseteq \mathcal{A}^{V,k+1}$. Moreover, if $P^{V,k-1} \neq N$, then $P^{V,k-1} \subset P^{V,k}$.

Proof. Let $S \in \mathcal{A}^{V,k}$. Then $\chi^{V,k}(S) \in \mathrm{WP}(V(S))$, $S \subseteq P^{V,k}$, and $\chi^{V,k+1}(S) = \gamma^{V,k}_S = \chi^{V,k}(S)$. This means that $\chi^{V,k+1}(S) \in \mathrm{WP}(V(S))$ and $\chi^{V,k+1}_i(T) = \gamma^{V,k}_i \leq \chi^{V,k+1}_i(S)$ for all $i \in S$ and any $T \in 2^N$ for which $i \in T$, so $S \in \mathcal{A}^{V,k+1}$. Hence, $\mathcal{A}^{V,k} \subseteq \mathcal{A}^{V,k+1}$.

Assume that $P^{V,k-1} \neq N$. Let $S \in 2^N$ with $S \not\subseteq P^{V,k-1}$ and $(\gamma^{V,k-1}_{S \cap P^{V,k-1}}, 0_{S \setminus P^{V,k-1}}) \in V(S)$ be a coalition such that $\lambda^{V,k}(S)$ equals the highest $\lambda^{V,k}(R)$ over all coalitions $R \in 2^N$ with $R \not\subseteq P^{V,k-1}$. Then $\chi^{V,k}(S) \in \mathrm{WP}(V(S))$ and $\chi^{V,k}_i(T) \leq \chi^{V,k}_i(S)$ for all $i \in S$ and any $T \in 2^N$ for which $i \in T$. This means that $S \in \mathcal{A}^{V,k}$ and $S \subseteq P^{V,k}$. Hence, $P^{V,k-1} \subset P^{V,k}$. $\square$

Lemma 8.2.3 shows that the collection of egalitarian admissible coalitions weakly expands in each iteration and eventually covers all players. Interestingly, some well-known properties for nontransferable utility games have implications for the relation of the collections of egalitarian admissible coalitions in two subsequent iterations.

Proposition 8.2.4
Let $V \in \text{NTU}^N$ be such that $V(N)$ is nontrivial and let $k \in \mathbb{N}$.

(i) If V is superadditive, then $S \cup T \in \mathcal{A}^{V,k+1}$ for all $S, T \in \mathcal{A}^{V,k}$ for which $S \cap T = \emptyset$.

(ii) If V is ordinal convex, then $S \cup T \in \mathcal{A}^{V,k+1}$ or $S \cap T \in \mathcal{A}^{V,k+1}$ for all $S, T \in \mathcal{A}^{V,k}$ for which $S \cap T \neq \emptyset$.

(iii) If V is coalitional merge convex, then $S \cup T \in \mathcal{A}^{V,k+1}$ for all $S, T \in \mathcal{A}^{V,k}$.

(iv) If V is balanced, then $N \in \mathcal{A}^{V,k+1}$ if there exists a balanced collection $\mathcal{B} \subseteq \mathcal{A}^{V,k}$.

Proof. (i) Assume that V is superadditive. Let $S, T \in \mathcal{A}^{V,k}$ be such that $S \cap T = \emptyset$. Then $\gamma_S^{V,k} \in V(S)$ and $\gamma_T^{V,k} \in V(T)$. Since V is superadditive, this means that $\gamma_{S \cup T}^{V,k} \in V(S \cup T)$. By Lemma 8.2.1, $\chi^{V,k+1}(S \cup T) \in \text{WUC}(V(S \cup T))$. Since $\chi^{V,k+1}(S \cup T) = \gamma_{S \cup T}^{V,k}$, this implies that $\chi^{V,k+1}(S \cup T) \in \text{WP}(V(S \cup T))$. Hence, $S \cup T \in \mathcal{A}^{V,k+1}$.

(ii) Assume that V is ordinal convex. Let $S, T \in \mathcal{A}^{V,k}$ be such that $S \cap T \neq \emptyset$. Then $\gamma_S^{V,k} \in V(S)$ and $\gamma_T^{V,k} \in V(T)$. Since V is ordinal convex, this means that $\gamma_{S \cup T}^{V,k} \in V(S \cup T)$ or $\gamma_{S \cap T}^{V,k} \in V(S \cap T)$. By Lemma 8.2.1, $\chi^{V,k+1}(S \cup T) \in \text{WUC}(V(S \cup T))$ and $\chi^{V,k+1}(S \cap T) \in \text{WUC}(V(S \cap T))$. Since $\chi^{V,k+1}(S \cup T) = \gamma_{S \cup T}^{V,k}$ and $\chi^{V,k+1}(S \cap T) = \gamma_{S \cap T}^{V,k}$, this implies that $\chi^{V,k+1}(S \cup T) \in \text{WP}(V(S \cup T))$ or $\chi^{V,k+1}(S \cap T) \in \text{WP}(V(S \cap T))$. Hence, $S \cup T \in \mathcal{A}^{V,k+1}$ or $S \cap T \in \mathcal{A}^{V,k+1}$.

(iii) Assume that V is coalitional merge convex. Let $S, T \in \mathcal{A}^{V,k}$. If $S \cap T = \emptyset$, $S \subseteq T$, or $T \subseteq S$, then $S \cup T \in \mathcal{A}^{V,k+1}$ by (i) and Lemma 8.2.3. Suppose that $S \cap T \neq \emptyset$, $S \not\subseteq T$, and $T \not\subseteq S$. Then $\gamma_S^{V,k} \in V(S)$ and $\gamma_T^{V,k} \in V(T)$. Since V is coalitional merge convex, there exists a $y \in V(S \cup T)$ for which $y_S \geq \gamma_S^{V,k}$ and $y_{T \setminus S} \geq \gamma_{T \setminus S}^{V,k}$, i.e. $y \geq \gamma_{S \cup T}^{V,k}$. Since $V(S \cup T)$ is comprehensive, this means that $\gamma_{S \cup T}^{V,k} \in V(S \cup T)$. By Lemma 8.2.1, $\chi^{V,k+1}(S \cup T) \in \text{WUC}(V(S \cup T))$. Since $\chi^{V,k+1}(S \cup T) = \gamma_{S \cup T}^{V,k}$, this implies that $\chi^{V,k+1}(S \cup T) \in \text{WP}(V(S \cup T))$. Hence, $S \cup T \in \mathcal{A}^{V,k+1}$.

(iv) Assume that V is balanced. Let $\mathcal{B} \subseteq \mathcal{A}^{V,k}$ be a balanced collection. Then $\gamma_S^{V,k} \in V(S)$ for all $S \in \mathcal{B}$. Since V is balanced, this means that $\gamma^{V,k} \in V(N)$. By Lemma 8.2.1, $\chi^{V,k+1}(N) \in \text{WUC}(V(N))$. Since $\chi^{V,k+1}(N) = \gamma^{V,k}$, this implies that $\chi^{V,k+1}(N) \in \text{WP}(V(N))$. Hence, $N \in \mathcal{A}^{V,k+1}$. □

Furthermore, Lemma 8.2.3 also shows that in each iteration of the egalitarian procedure, at least one additional player acquires an egalitarian claim as long as the collection of egalitarian admissible coalitions does not cover all players. The egalitarian procedure reaches a steady state when all players are egalitarian claimants. This means that the number of iterations needed to converge to a steady state is bounded by the number of players. Example 8.1 shows that this bound is tight.

Definition
Let $V \in \mathrm{NTU}^N$ be a nontransferable utility game such that $V(N)$ is nontrivial. The iteration $n^V \in \{1, \dots, |N|\}$ is given by $n^V = \min\{k \in \mathbb{N} \mid P^{V,k} = N\}$. The vector of *egalitarian claims* $\widehat{\gamma}^V \in \mathbb{R}^N_+$ is given by $\widehat{\gamma}^V = \gamma^{V,n^V}$. The collection $\widehat{\mathcal{A}}^V \subseteq 2^N \setminus \{\emptyset\}$ is given by $\widehat{\mathcal{A}}^V = \{S \in 2^N \setminus \{\emptyset\} \mid \widehat{\gamma}^V_S \in V(S), \forall_{T \in 2^N \setminus \{\emptyset\}: \widehat{\gamma}^V_T \in V(T)} : S \not\subset T\}$. The set of *strong egalitarian claimants* $D^V \in 2^N$ is given by $D^V = \bigcap_{S \in \widehat{\mathcal{A}}^V} S$.

The constrained egalitarian solution is a solution concept which takes both the set of strong egalitarian claimants and the vector of egalitarian claims into account to prescribe a payoff allocation for the grand coalition. The egalitarian claims can be interpreted as aspiration levels for such an allocation, which are based on egalitarian opportunities within subcoalitions. The constrained egalitarian solution first allocates to all strong egalitarian claimants their claims and subsequently allocates to all other players their claims. The possibly resulting infeasibility is modeled as a bankruptcy problem in which the egalitarian claims are adopted.

Taking the egalitarian claims and the set of strong egalitarian claimants into account, the constrained egalitarian solution for nontransferable utility games uses the constrained relative equal awards rule to prescribe a payoff allocation for the grand coalition. In Section 8.4, we further elaborate on the choice of this specific bankruptcy rule.

Definition (Constrained Egalitarian Solution)
The *constrained egalitarian solution* $\Gamma : \mathrm{NTU}^N \to \mathbb{R}^N_+$ is the solution which assigns to any $V \in \mathrm{NTU}^N$ for which $V(N)$ is nontrivial the payoff allocation

$$\Gamma(V) = \begin{cases} \left(\widehat{\gamma}^V_{D^V}, \mathrm{CREA}(\{x \in \mathbb{R}^{N \setminus D^V}_+ \mid (\widehat{\gamma}^V_{D^V}, x) \in V(N)\}, \widehat{\gamma}^V_{N \setminus D^V})\right) \\ \qquad\qquad\qquad\qquad \text{if } (\widehat{\gamma}^V_{D^V}, 0_{N \setminus D^V}) \in V(N); \\ \left(\mathrm{CREA}(\{x \in \mathbb{R}^{D^V}_+ \mid (x, 0_{N \setminus D^V}) \in V(N)\}, \widehat{\gamma}^V_{D^V}), 0_{N \setminus D^V}\right) \\ \qquad\qquad\qquad\qquad \text{if } (\widehat{\gamma}^V_{D^V}, 0_{N \setminus D^V}) \notin V(N). \end{cases}$$

Note that the constrained egalitarian solution is well-defined by extending the domain of CREA to bankruptcy problems for which the estate is not necessarily nonleveled.

Moreover, for any NTU-game $V \in \text{NTU}^N$ for which $V(S) = \{x \in \mathbb{R}^S_+ \mid \sum_{i \in S} x_i \leq v(S)\}$ for all $S \in 2^N \setminus \{\emptyset\}$, induced by a nonnegative TU-game $v \in \text{TU}^N$ for which $v(N) > 0$, $\Gamma(V) = \Gamma(v)$ if $\Gamma(v) \in \mathbb{R}^N_+$, i.e. the constrained egalitarian solution coincides with the procedural egalitarian solution if the latter is nonnegative.

Example 8.2
Let $N = \{1, 2, 3\}$ and consider the game $V \in \text{NTU}^N$ from Example 8.1. Then $n^V = 3$, $\widehat{\gamma}^V = (4, 2, 0)$, $\widehat{\mathcal{A}}^V = \{N\}$, and $D^V = N$. Hence, $\Gamma(V) = (4, 2, 0)$. △

As in Example 8.1 and Example 8.2, an interesting situation arises when the grand coalition is egalitarian admissible. Then all players are strong egalitarian claimants, there is no infeasibility, and the constrained egalitarian solution assigns to all players their egalitarian claims. Moreover, by Proposition 8.2.2, the constrained egalitarian solution constitutes a core element. Therefore, such nontransferable utility games are called egalitarian stable.

Definition (Egalitarian Stability)
A nontransferable utility game $V \in \text{NTU}^N$ for which $V(N)$ is nontrivial is *egalitarian stable* if $\widehat{\mathcal{A}}^V = \{N\}$.

Egalitarian stability is a sufficient condition for nontransferable utility games to contain the constrained egalitarian solution in the core. The following example shows that this condition is not necessary.

Example 8.3
Let $N = \{1, 2, 3\}$ and consider the game $V \in \text{NTU}^N$ given by

$$\begin{aligned} V(\{i\}) &= \left\{x \in \mathbb{R}^{\{i\}}_+ \;\middle|\; x \leq 0\right\} \text{ for } i \in N; \\ V(\{1,i\}) &= \left\{x \in \mathbb{R}^{\{1,i\}}_+ \;\middle|\; x_1 \leq 4, x_i \leq 4\right\} \text{ for } i \in \{2,3\}; \\ V(\{2,3\}) &= \left\{x \in \mathbb{R}^{\{2,3\}}_+ \;\middle|\; x_2 \leq 0, x_3 \leq 0\right\}; \\ V(\{1,2,3\}) &= \left\{x \in \mathbb{R}^{\{1,2,3\}}_+ \;\middle|\; x_1 + x_2 + x_3 \leq 6\right\}. \end{aligned}$$

Then $u^{V(N)} = (6, 6, 6)$. The egalitarian distribution in the first iteration of the egalitarian procedure is presented in the following table.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\chi^{V,1}(S)$	$(0,\cdot,\cdot)$	$(\cdot,0,\cdot)$	$(\cdot,\cdot,0)$	$(\underline{4},\underline{4},\cdot)$	$(\underline{4},\cdot,\underline{4})$	$(\cdot,0,0)$	$(2,2,2)$

In the first iteration, $\mathcal{A}^{V,1} = \{\{1,2\},\{1,3\}\}$, $P^{V,1} = N$, and $\gamma^{V,1} = (4, 4, 4)$. This means that $n^V = 1$, $\widehat{\gamma}^V = (4, 4, 4)$, $\widehat{\mathcal{A}}^V = \{\{1,2\},\{1,3\}\}$, and $D^V = \{1\}$. Hence,

$$\Gamma(V) = \Big(4, \text{CREA}(\{x \in \mathbb{R}^{\{2,3\}}_+ \mid x_2 + x_3 \leq 2\}, (\cdot, 4, 4))\Big) = (4, 1, 1).$$

Note that $\Gamma(V) \in \mathcal{C}^{\mathcal{W}}(V)$. △

In Example 8.3, the sets of payoff allocations $V(\{1,2\})$ and $V(\{1,3\})$ are not nonleveled. For nontransferable utility games $V \in \mathrm{NTU}^N$ for which $V(N)$ is nontrivial and $V(S)$ is nonleveled for all $S \in 2^N \setminus \{\emptyset\}$, egalitarian stability is a necessary and sufficient condition to contain the constrained egalitarian solution in the core. The question arises which nontransferable utility games are egalitarian stable. By Proposition 8.2.4, coalitional merge convex games are egalitarian stable. In the next sections we show that the Roth-Shafer examples, bankruptcy games, and bargaining games are all egalitarian stable as well.

8.3 Roth-Shafer examples

In this section, we study the constrained egalitarian solution for the examples introduced by Roth (1980) and Shafer (1980). These examples initiated an interesting and extensive discussion on the interpretation of solutions for nontransferable utility games. Along the lines of this discussion, we compare the constrained egalitarian solution with the Shapley value (cf. Shapley (1969)), the Harsanyi value (cf. Harsanyi (1963)), and the monotonic solution of Kalai and Samet (1985). For more details, we refer to Harsanyi (1980), Aumann (1985), Hart (1985), Roth (1986), and Aumann (1986).

Example 8.4 (cf. Roth (1980))
Let $N = \{1,2,3\}$ and consider the game $V_p \in \mathrm{NTU}^N$ which is for all $p \in (0, \frac{1}{2})$ given by

$$\begin{aligned}
V_p(\{i\}) &= \left\{x \in \mathbb{R}_+^{\{i\}} \;\middle|\; x \leq 0\right\} \text{ for } i \in N;\\
V_p(\{1,2\}) &= \left\{x \in \mathbb{R}_+^{\{1,2\}} \;\middle|\; x_1 \leq \tfrac{1}{2}, x_2 \leq \tfrac{1}{2}\right\};\\
V_p(\{i,3\}) &= \left\{x \in \mathbb{R}_+^{\{i,3\}} \;\middle|\; x_i \leq p, x_3 \leq 1-p\right\} \text{ for } i \in \{1,2\};\\
V_p(\{1,2,3\}) &= \left\{x \in \mathbb{R}_+^{\{1,2,3\}} \;\middle|\; x \in \mathrm{comp}(\mathrm{conv}(\{(\tfrac{1}{2},\tfrac{1}{2},0),(p,0,1-p),(0,p,1-p)\}))\right\}.
\end{aligned}$$

Then $u^{V_p(N)} = (\frac{1}{2}, \frac{1}{2}, 1-p)$. A part of the egalitarian distribution in the first two iterations of the egalitarian procedure is presented in the following table.

S	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\chi^{V_p,1}(S)$	$(\cdot,\cdot,0)$	$(\frac{1}{2},\frac{1}{2},\cdot)$	$(p,\cdot,2p(1-p))$	$(\cdot,p,2p(1-p))$	$\lambda^{V_p,1}(N)u^{V_p(N)}$
$\chi^{V_p,2}(S)$	$(\cdot,\cdot,\underline{0})$	$(\boldsymbol{\frac{1}{2}},\boldsymbol{\frac{1}{2}},\cdot)$	$(\boldsymbol{\frac{1}{2}},\cdot,0)$	$(\cdot,\boldsymbol{\frac{1}{2}},0)$	$(\boldsymbol{\frac{1}{2}},\boldsymbol{\frac{1}{2}},\underline{0})$

In the first iteration, $\mathcal{A}^{V_p,1} = \{\{1,2\}\}$, $P^{V_p,1} = \{1,2\}$, and $\gamma^{V_p,1} = (\frac{1}{2},\frac{1}{2},\cdot)$. In the second iteration, $\mathcal{A}^{V_p,2} = \{\{3\},\{1,2\},\{1,2,3\}\}$, $P^{V_p,2} = N$, and $\gamma^{V_p,2} = (\frac{1}{2},\frac{1}{2},0)$. This means that $n^{V_p} = 2$, $\widehat{\gamma}^{V_p} = (\frac{1}{2},\frac{1}{2},0)$, $\widehat{\mathcal{A}}^{V_p} = \{N\}$, and $D^{V_p} = N$. Hence, $\Gamma(V_p) = (\frac{1}{2},\frac{1}{2},0)$.

Besides, the Shapley value equals $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and the Harsanyi value and the monotonic solution both equal $(\frac{1}{2} - \frac{1}{3}p, \frac{1}{2} - \frac{1}{3}p, \frac{2}{3}p)$. Note that, contrary to the constrained egalitarian solution, these solutions do not belong to the core. Roth argues that the payoff allocation $(\frac{1}{2}, \frac{1}{2}, 0)$ is the unique outcome of this game which is consistent with the hypothesis that the players are rational utility maximizers, since this payoff allocation is strictly preferred by both players 1 and 2, and it can be achieved without player 3. The constrained egalitarian solution perfectly matches this idea. △

Example 8.5 (cf. Shafer (1980) and Hart and Kurz (1983))
Let $N = \{1, 2, 3\}$ and consider the game $V_\varepsilon \in \mathrm{NTU}^N$ which is for all $\varepsilon \in [0, \frac{1}{6})$ given by

$$\begin{aligned} V_\varepsilon(\{i\}) &= \left\{x \in \mathbb{R}_+^{\{i\}} \;\middle|\; x \leq 0\right\} \text{ for } i \in \{1,2\}; \\ V_\varepsilon(\{3\}) &= \left\{x \in \mathbb{R}_+^{\{3\}} \;\middle|\; x \leq \varepsilon\right\}; \\ V_\varepsilon(\{1,2\}) &= \left\{x \in \mathbb{R}_+^{\{1,2\}} \;\middle|\; x_1 + x_2 \leq 1 - \varepsilon\right\}; \\ V_\varepsilon(\{i,3\}) &= \left\{x \in \mathbb{R}_+^{\{i,3\}} \;\middle|\; x_i \leq \varepsilon, x_i + x_3 \leq \tfrac{1}{2} + \tfrac{1}{2}\varepsilon\right\} \text{ for } i \in \{1,2\}; \\ V_\varepsilon(\{1,2,3\}) &= \left\{x \in \mathbb{R}_+^{\{1,2,3\}} \;\middle|\; x_1 + x_2 + x_3 \leq 1\right\}. \end{aligned}$$

Then $u^{V_\varepsilon(N)} = (1, 1, 1)$. A part of the egalitarian distribution in the first two iterations of the egalitarian procedure is presented in the following table.

S	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\chi^{V_\varepsilon,1}(S)$	$(\cdot,\cdot,\varepsilon)$	$(\underline{\frac{1-\varepsilon}{2}}, \underline{\frac{1-\varepsilon}{2}}, \cdot)$	$(\varepsilon,\cdot,\varepsilon)$	$(\cdot,\varepsilon,\varepsilon)$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$
$\chi^{V_\varepsilon,2}(S)$	$(\cdot,\cdot,\underline{\varepsilon})$	$(\boldsymbol{\frac{1-\varepsilon}{2}}, \boldsymbol{\frac{1-\varepsilon}{2}}, \cdot)$	$(\boldsymbol{\frac{1-\varepsilon}{2}},\cdot,0)$	$(\cdot,\boldsymbol{\frac{1-\varepsilon}{2}},0)$	$(\boldsymbol{\frac{1-\varepsilon}{2}},\boldsymbol{\frac{1-\varepsilon}{2}},\underline{\varepsilon})$

In the first iteration, $\mathcal{A}^{V_\varepsilon,1} = \{\{1,2\}\}$, $P^{V_\varepsilon,1} = \{1,2\}$, and $\gamma^{V_\varepsilon,1} = (\frac{1-\varepsilon}{2}, \frac{1-\varepsilon}{2}, \cdot)$. In the second iteration, $\mathcal{A}^{V_\varepsilon,2} = \{\{3\},\{1,2\},\{1,2,3\}\}$, $P^{V_\varepsilon,2} = N$, and $\gamma^{V_\varepsilon,2} = (\frac{1-\varepsilon}{2}, \frac{1-\varepsilon}{2}, \varepsilon)$. This means that $n^{V_\varepsilon} = 2$, $\widehat{\gamma}^{V_\varepsilon} = (\frac{1-\varepsilon}{2}, \frac{1-\varepsilon}{2}, \varepsilon)$, $\widehat{\mathcal{A}}^{V_\varepsilon} = \{N\}$, and $D^{V_\varepsilon} = N$. Hence, $\Gamma(V_\varepsilon) = (\frac{1}{2} - \frac{1}{2}\varepsilon, \frac{1}{2} - \frac{1}{2}\varepsilon, \varepsilon)$.

Besides, the Shapley value equals $(\frac{5}{12} - \frac{5}{12}\varepsilon, \frac{5}{12} - \frac{5}{12}\varepsilon, \frac{2}{12} + \frac{10}{12}\varepsilon)$, and the Harsanyi value and the monotonic solution both equal $(\frac{1}{2} - \frac{5}{6}\varepsilon, \frac{1}{2} - \frac{5}{6}\varepsilon, \frac{10}{6}\varepsilon)$. Note that, contrary to the constrained egalitarian solution, these solutions do not belong to the core. Shafer states that it is unreasonable to allocate at least $\frac{1}{6}$ to player 3, independent of ε and especially in the case $\varepsilon = 0$. The constrained egalitarian solution seamlessly connects with this idea. △

8.4 Bankruptcy games

In this section, we analyze the constrained egalitarian solution on the class of bankruptcy games with nontransferable utility. We show that bankruptcy games are egalitarian stable, which means that the constrained egalitarian solution assigns to all players their egalitarian claims without having to rely on the constrained relative equal awards rule in its definition. Interestingly, we show that the constrained egalitarian solution of a bankruptcy game corresponds to the constrained relative equal awards rule of the underlying bankruptcy problem. This illustrates the strong connection between the constrained egalitarian solution and the constrained relative equal awards rule. Besides, it justifies the use of the latter in the definition of the constrained egalitarian solution for nontransferable utility games which are not egalitarian stable.

Theorem 8.4.1
Let $(E, c) \in \mathrm{BR}^N$ *be a bankruptcy problem such that* $E \neq \{0_N\}$*. Then* $\Gamma(V^{E,c}) = \mathrm{CREA}(E, c)$*.*

Proof. First, we show that $\widehat{\gamma}^{V^{E,c}} \leq c$. Suppose that there exists an $i \in N$ for which $\widehat{\gamma}_i^{V^{E,c}} > c_i$. Let $k \in \mathbb{N}$ be such that $i \in P^{V^{E,c},k} \setminus P^{V^{E,c},k-1}$ and let $S \in \mathcal{A}^{V^{E,c},k}$ be such that $i \in S$. Then $\widehat{\gamma}_i^{V^{E,c}} \notin V^{E,c}(\{i\})$ and $\chi^{V^{E,c},k}(S) \in \mathrm{WP}(V^{E,c}(S))$, so $S \neq \{i\}$ and

$$\left(\gamma^{V^{E,c},k-1}_{S\cap P^{V^{E,c},k-1}}, \lambda^{V^{E,c},k}(S) u^{V^{E,c}(N)}_{S\setminus P^{V^{E,c},k-1}}\right) \in \mathrm{WP}(V^{E,c}(S)).$$

Since E is comprehensive and nonleveled, this means that

$$\left(\gamma^{V^{E,c},k-1}_{S\cap P^{V^{E,c},k-1}}, \lambda^{V^{E,c},k}(S) u^{V^{E,c}(N)}_{S\setminus P^{V^{E,c},k-1}}, c_{N\setminus S}\right) \in \mathrm{SP}(E).$$

Since E is comprehensive,

$$\left(\gamma^{V^{E,c},k-1}_{S\cap P^{V^{E,c},k-1}}, \lambda^{V^{E,c},k}(S) u^{V^{E,c}(N)}_{S\setminus (P^{V^{E,c},k-1}\cup\{i\})}, c_i, c_{N\setminus S}\right) \in E \setminus \mathrm{SP}(E).$$

Since E is nonleveled,

$$\left(\gamma^{V^{E,c},k-1}_{S\cap P^{V^{E,c},k-1}}, \lambda^{V^{E,c},k}(S) u^{V^{E,c}(N)}_{S\setminus (P^{V^{E,c},k-1}\cup\{i\})}\right) \in V^{E,c}(S\setminus\{i\}) \setminus \mathrm{WP}(V^{E,c}(S\setminus\{i\})).$$

By Lemma 8.2.1, $\chi^{V^{E,c},k}(S\setminus\{i\}) \in \mathrm{WUC}(V^{E,c}(S\setminus\{i\}))$. Then

$$\begin{aligned}\chi^{V^{E,c},k}_{S\setminus (P^{V^{E,c},k-1}\cup\{i\})}(S\setminus\{i\}) &= \lambda^{V^{E,c},k}(S\setminus\{i\}) u^{V^{E,c}(N)}_{S\setminus (P^{V^{E,c},k-1}\cup\{i\})} \\ &> \lambda^{V^{E,c},k}(S) u^{V^{E,c}(N)}_{S\setminus (P^{V^{E,c},k-1}\cup\{i\})} = \chi^{V^{E,c},k}_{S\setminus (P^{V^{E,c},k-1}\cup\{i\})}(S).\end{aligned}$$

This contradicts that $S \in \mathcal{A}^{V^{E,c},k}$. Hence, $\widehat{\gamma}^{V^{E,c}} \leq c$.

Suppose that $c \in E$. Then $\chi^{V^{E,c}, n^{V^{E,c}}}(N) \leq \gamma^{V^{E,c}, n^{V^{E,c}}} = \widehat{\gamma}^{V^{E,c}} \leq c$. By Lemma 8.2.1, $\chi^{V^{E,c}, n^{V^{E,c}}}(N) \in \mathrm{WUC}(E)$. Since E is nonleveled, this means that $\widehat{\gamma}^{V^{E,c}} = c$, $\widehat{\mathcal{A}}^{V^{E,c}} = \{N\}$, and $D^{V^{E,c}} = N$. Hence, $\Gamma(V^{E,c}) = c = \mathrm{CREA}(E, c)$.

Now suppose that $c \notin E$. First, we show that $\chi^{V^{E,c},1}(S) \leq \alpha^{E,c} u^E_S$ for all $S \in 2^N \setminus \{\emptyset\}$. Suppose that there exists an $S \in 2^N \setminus \{\emptyset\}$ for which $\chi_i^{V^{E,c},1}(S) > \alpha^{E,c} u^E_i$ for some $i \in S$. Then $\chi^{V^{E,c},1}(S) \in \mathrm{WP}(V^{E,c}(S))$ and

$$\chi^{V^{E,c},1}(S) = \lambda^{V^{E,c},1}(S) u^E_S > \alpha^{E,c} u^E_S \geq \mathrm{CREA}_S(E, c).$$

Since E is comprehensive and nonleveled, we have $(\chi^{V^{E,c},1}(S), c_{N \setminus S}) \in \mathrm{WP}(E)$, $(\chi^{V^{E,c},1}(S), c_{N \setminus S}) \geq \mathrm{CREA}(E, c)$, and $(\chi^{V^{E,c},1}(S), c_{N \setminus S}) \neq \mathrm{CREA}(E, c)$. Since E is nonleveled, this contradicts that $\mathrm{CREA}(E, c) \in \mathrm{WP}(E)$. Hence, $\chi^{V^{E,c},1}(S) \leq \alpha^{E,c} u^E_S$ for all $S \in 2^N \setminus \{\emptyset\}$. Next, let $H^{E,c} \in 2^N \setminus \{\emptyset\}$ be defined by

$$H^{E,c} = \left\{ i \in N \;\middle|\; \mathrm{CREA}_i(E, c) = \alpha^{E,c} u^E_i \right\}.$$

Then $\chi^{V^{E,c},1}(H^{E,c}) \in \mathrm{WP}(V^{E,c}(H^{E,c}))$ and

$$\chi^{V^{E,c},1}(H^{E,c}) = \lambda^{V^{E,c},1}(H^{E,c}) u^E_{H^{E,c}} = \alpha^{E,c} u^E_{H^{E,c}} = \mathrm{CREA}_{H^{E,c}}(E, c).$$

This means that $H^{E,c} \in \mathcal{A}^{V^{E,c},1}$, $H^{E,c} \subseteq P^{V^{E,c},1}$ and $\gamma_{H^{E,c}}^{V^{E,c},1} = \mathrm{CREA}_{H^{E,c}}(E, c)$. Now,

$$\chi^{V^{E,c}, n^{V^{E,c}}}(N) \leq \gamma^{V^{E,c}, n^{V^{E,c}}} = \widehat{\gamma}^{V^{E,c}} \leq \left(\mathrm{CREA}_{H^{E,c}}(E, c), c_{N \setminus H^{E,c}} \right) = \mathrm{CREA}(E, c).$$

By Lemma 8.2.1, $\chi^{V^{E,c}, n^{V^{E,c}}}(N) \in \mathrm{WUC}(E)$. Since $\mathrm{CREA}(E, c) \in \mathrm{WP}(E)$ and E is nonleveled, this means that $\widehat{\gamma}^{V^{E,c}} = \mathrm{CREA}(E, c)$, $\widehat{\mathcal{A}}^{V^{E,c}} = \{N\}$, and $D^{V^{E,c}} = N$. Hence, $\Gamma(V^{E,c}) = \mathrm{CREA}(E, c)$. □

8.5 Bargaining games

In this section, we analyze the constrained egalitarian solution on the class of bargaining games. A *bargaining problem* (cf. Nash (1950)) is a triple (N, F, d) in which N is a nonempty and finite set of *bargainers*, $F \subseteq \mathbb{R}^N_+$ is a nonempty, closed, bounded, nontrivial and comprehensive *feasible set*, and $d \in F$ is a *disagreement point*. Let BG^N denote the class of all bargaining problems with bargainer set N. For convenience, a bargaining problem on N is denoted by $(F, d) \in \mathrm{BG}^N$.

Kalai and Smorodinsky (1975) introduced the solution $\mathrm{KS} : \mathrm{BG}^N \to \mathbb{R}^N_+$ assigning to any bargaining problem $(F, d) \in \mathrm{BG}^N$ the payoff allocation

$$\mathrm{KS}(F, d) = d + \kappa_1^{F,d} \left(u^{F_d} - d \right),$$

where $F_d = \{x \in F \mid x \geq d\}$ and $\kappa_1^{F,d} = \max\{t \in [0, 1] \mid d + t(u^{F_d} - d) \in \mathrm{WP}(F)\}$.

Kalai and Rosenthal (1978) introduced the solution $\mathrm{KR} : \mathrm{BG}^N \to \mathbb{R}^N_+$ assigning to any bargaining problem $(F, d) \in \mathrm{BG}^N$ the payoff allocation

$$\mathrm{KR}(F, d) = d + \kappa_2^{F,d} \left(u^F - d\right),$$

where $\kappa_2^{F,d} = \max\{t \in [0, 1] \mid d + t(u^F - d) \in \mathrm{WP}(F)\}$.

The *bargaining game* $V^{F,d} \in \mathrm{NTU}^N$ corresponding to the bargaining problem $(F, d) \in \mathrm{BG}^N$ is given by

$$V^{F,d}(S) = \begin{cases} F & \text{for } S = N; \\ \{x \in \mathbb{R}^S_+ \mid x \leq d_S\} & \text{for all } S \in 2^N \setminus \{\emptyset, N\}. \end{cases}$$

The core of a bargaining game is given by $\mathcal{C}^{\mathcal{W}}(V^{F,d}) = \{x \in \mathrm{WP}(F) \mid x \geq d\}$. Note that bargaining games are coalitional merge convex, which implies that bargaining games are egalitarian stable and that the constrained egalitarian solution constitutes a core element.

Theorem 8.5.1
Let $(F, d) \in \mathrm{BG}^N$ be a bargaining problem such that $d = 0_N$. Then $\Gamma(V^{F,d}) = \mathrm{KS}(F, d) = \mathrm{KR}(F, d)$.

Proof. Since $d = 0_N$, $F_d = F$ and $\mathrm{KS}(F, d) = \kappa_1^{F,d} u^{F_d} = \kappa_2^{F,d} u^F = \mathrm{KR}(F, d)$. In the first iteration of the egalitarian procedure,

$$\chi^{V^{F,d},1}(S) = \begin{cases} \lambda^{V^{F,d},1}(N) u^F & \text{for } S = N; \\ 0_S & \text{for all } S \in 2^N \setminus \{\emptyset, N\}, \end{cases}$$

where $\lambda^{V^{F,d},1}(N) \in [0, 1]$ is such that $\lambda^{V^{F,d},1}(N) u^F \in \mathrm{WP}(F)$. This means that $N \in \mathcal{A}^{V^{F,d},1}$, $P^{V^{F,d},1} = N$, and $\gamma^{V^{F,d},1} = \lambda^{V^{F,d},1}(N) u^F$, which implies that $n^{V^{F,d}} = 1$, $\widehat{\gamma}^{V^{F,d}} = \lambda^{V^{F,d},1}(N) u^F$, $\widehat{\mathcal{A}}^{V^{F,d}} = \{N\}$, and $D^{V^{F,d}} = N$. Hence, $\Gamma(V^{F,d}) = \lambda^{V^{F,d},1}(N) u^F$. Moreover, the assumptions on F imply that $\lambda^{V^{F,d},1}(N) = \kappa_1^{F,d} = \kappa_2^{F,d}$. Hence, $\Gamma(V^{F,d}) = \mathrm{KS}(F, d) = \mathrm{KR}(F, d)$. □

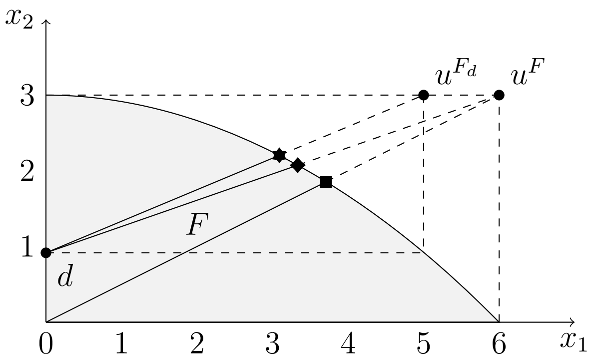
Theorem 8.5.1 shows that the constrained egalitarian solution of a bargaining game corresponds to the solutions of Kalai and Smorodinsky (1975) and Kalai and Rosenthal (1978) of the underlying bargaining problem if the disagreement point equals the zero vector. In general, the constrained egalitarian solution assigns to any bargaining problem $(F, d) \in \mathrm{BG}^N$ the payoff allocation

$$\Gamma(V^{F,d}) = \left(\max\{d_i, \alpha^{F,d} u_i^F\}\right)_{i \in N},$$

where $\alpha^{F,d} = \max\{t \in [0, 1] \mid (\max\{d_i, t u_i^F\})_{i \in N} \in \mathrm{WP}(F)\}$.

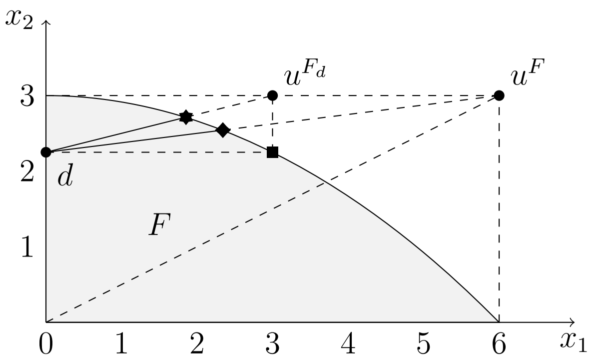
The following two examples illustrate the nature of the constrained egalitarian solution for bargaining problems with nonzero disagreement point.

Example 8.6
Let $N = \{1,2\}$ and consider the bargaining problem $(F,d) \in \mathrm{BG}^N$ given by $F = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 12x_2 \leq 36\}$ and $d = (0, \frac{11}{12})$. Then $u^F = (6,3)$ and $u^{F_d} = (5,3)$. This means that $\mathrm{KS}(F,d) = (\frac{5}{2}\sqrt{5} - \frac{5}{2}, \frac{25}{24}\sqrt{5} - \frac{1}{8})$ (✸), $\mathrm{KR}(F,d) = (3\frac{1}{3}, 2\frac{2}{27})$ (◆), and $\Gamma(V^{F,d}) = (3\sqrt{5} - 3, \frac{3}{2}\sqrt{5} - \frac{3}{2})$ (■). This is illustrated as follows.



△

Example 8.7
Let $N = \{1,2\}$ and consider the bargaining problem $(F,d) \in \mathrm{BG}^N$ given by $F = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 12x_2 \leq 36\}$ and $d = (0, 2\frac{1}{4})$. Then $u^F = (6,3)$ and $u^{F_d} = (3,3)$. This means that $\mathrm{KS}(F,d) = (\frac{3}{2}\sqrt{5} - \frac{3}{2}, \frac{3}{8}\sqrt{5} + \frac{15}{8})$ (✸), $\mathrm{KR}(F,d) = (\frac{3}{4}\sqrt{17} - \frac{3}{4}, \frac{3}{32}\sqrt{17} + \frac{69}{32})$ (◆), and $\Gamma(V^{F,d}) = (3, 2\frac{1}{4})$ (■). This is illustrated as follows.



△

Future research could further study the interpretation and axiomatic significance of this new egalitarian solution concept for bargaining problems.

Part III

Communication Situations

9 Decomposition of Network Communication Games

9.1 Introduction

In a cooperative game with communication structure, the players of a transferable utility game are subject to cooperation restrictions. Myerson (1977) introduced communication situations in which these cooperation restrictions are modeled by an undirected graph. Nodes of the undirected graph represent the players of the game and there is a link between two nodes if and only if the corresponding players are able to communicate directly. A coalition can attain its worth if its members are able to communicate, i.e. if their corresponding nodes induce a connected subgraph.

Myerson (1977) introduced the graph-restricted game corresponding to a communication situation in which each coalition of nodes is assigned the sum of the worths of the components in its induced subgraph. We refer to this game as the corresponding node game. Owen (1986) studied the decomposition into unanimity games of these node games for the special case that the communication network is cycle-free. The Myerson value of a communication situation is defined as the Shapley value of the corresponding node game.

Borm, Owen, and Tijs (1992) introduced a game on the links corresponding to a communication situation in which each coalition of links is assigned the sum of the worths of the components in its induced subgraph. We refer to this game as the corresponding link game. Borm, Owen, and Tijs (1992) also studied the decomposition into unanimity games of these link games for the special case that the communication network is cycle-free. The position value of a communication situation assigns to each player half of the payoffs allocated to the incident links by the Shapley value of the corresponding link game.

This chapter, which is based on Dietzenbacher, Borm, and Hendrickx (2017a), introduces a general class of network communication games and a corresponding class of network control values for communication situations. A network communication game is a transferable utility game integrating the features of a communication situation and a network control structure on a communication network. Here, a network control structure models the way in which the nodes and links of the graph control the communication network. Where Myerson (1977) considered the nodes and Borm, Owen, and Tijs (1992) considered the links as controllers of the network, a network control structure allows both the nodes and links to control the network in any way. In the corresponding network communication game, each coalition of nodes and links is assigned the sum of the worths of the components in the subgraph which the members control together.

Focusing on the decomposition into unanimity games of network communication games, communication situations with an underlying unanimity game induce simple network communication games for any network control structure. The minimal winning coalitions in this game play a central role in its decomposition. We obtain a relation between the dividends in the network communication game and in the underlying transferable utility game, which depends on the structure of the communication network. This relation extends the results of Owen (1986) and Borm, Owen, and Tijs (1992) for cycle-free networks to all undirected graphs.

For any network control structure, the corresponding network control value of a communication situation assigns to each player the payoff allocated by the Shapley value of the corresponding network communication game to its corresponding node and half of the payoff allocated to the incident links. The Myerson value and the position value are network control values which correspond to specific network control structures. We derive an explicit expression for any network control value in terms of the dividends in the underlying transferable utility game.

The main aim of this chapter is to develop the decomposition theory for network communication games as a mathematical tool which can be used to derive any network control value for communication situations in a structured way. Future research could study further interpretations and applications of this new framework.

This chapter is organized in the following way. Section 9.2 provides an overview of basic game theoretic and graph theoretic notions and notations. Section 9.3 formally introduces network control structures, network communication games, and network control values, and studies the decomposition into unanimity games. Section 9.4 discusses the Myerson value and the position value, and the decomposition of their corresponding node games and link games. Section 9.5 illustrates how the decomposition theory can be extended to more general communication structures.

9.2 Preliminaries

A transferable utility game $v \in \mathrm{TU}^N$ is *simple* if v is monotonic, $v(S) \in \{0,1\}$ for all $S \in 2^N$, and $v(N) = 1$. Let SI^N denote the class of simple games with player set N. A coalition $S \in 2^N$ is *winning* in $v \in \mathrm{SI}^N$ if $v(S) = 1$ and *losing* in $v \in \mathrm{SI}^N$ if $v(S) = 0$. The collection of *minimal winning* coalitions in $v \in \mathrm{SI}^N$ is given by

$$\mathcal{M}(v) = \left\{ S \in 2^N \;\middle|\; v(S) = 1, \forall_{R \subset S} : v(R) = 0 \right\}. \tag{9.1}$$

The *maximum game* $\max\{v \mid v \in \mathcal{V}\} \in \mathrm{TU}^N$ of a nonempty and finite set of transferable utility games $\mathcal{V} \subseteq \mathrm{TU}^N$ is given by $\max\{v \mid v \in \mathcal{V}\}(S) = \max\{v(S) \mid v \in \mathcal{V}\}$ for all $S \in 2^N$. The *minimum game* is defined similarly. Note that both the maximum game and the minimum game of a nonempty set of simple games are simple.

The *unanimity game* $u_R \in \mathrm{SI}^N$ on $R \in 2^N \setminus \{\emptyset\}$ is given by

$$u_R(S) = \begin{cases} 1 & \text{if } R \subseteq S; \\ 0 & \text{if } R \not\subseteq S \end{cases}$$

for all $S \in 2^N$. We have $v \in \mathrm{SI}^N$ and $\mathcal{M}(v) = \mathcal{B}$ if and only if $\mathcal{B} \subseteq 2^N \setminus \{\emptyset\}$ is independent and $v = \max\{u_R \mid R \in \mathcal{B}\}$.

A TU-game $v \in \mathrm{TU}^N$ can be uniquely decomposed into unanimity games,

$$v = \sum_{S \in 2^N \setminus \{\emptyset\}} \Delta^v(S) u_S, \tag{9.2}$$

where $\Delta^v : 2^N \setminus \{\emptyset\} \to \mathbb{R}$ assigns to each nonempty coalition $S \in 2^N \setminus \{\emptyset\}$ its *dividend* (cf. Harsanyi (1959))

$$\Delta^v(S) = \sum_{R \subseteq S} (-1)^{|S|-|R|} v(R). \tag{9.3}$$

The *Shapley value* (cf. Shapley (1953)) $\Phi : \mathrm{TU}^N \to \mathbb{R}^N$ assigns to any $v \in \mathrm{TU}^N$ the payoff allocation given by

$$\Phi_i(v) = \sum_{S \in 2^N : i \in S} \frac{1}{|S|} \Delta^v(S) \tag{9.4}$$

for all $i \in N$.

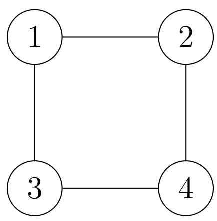
Let $L \subseteq \{S \in 2^N \mid |S| = 2\}$ be a set of unordered pairs of players. The pair (N, L) represents an *undirected graph* in which N is the set of *nodes* and L is the set of *links*. We denote $L_i = \{l \in L \mid i \in l\}$ for all $i \in N$, $L[S] = \{l \in L \mid l \subseteq S\}$ for all $S \in 2^N$, and $N[T] = \bigcup_{l \in T} l$ for all $T \in 2^L$.

A pair (S,T) is a *subgraph* of (N,L) if $S \in 2^N$, $T \in 2^L$, and $N[T] \subseteq S$. The collection of all subgraphs of (N,L) is denoted by $\mathcal{G}^{N,L}$. Let $N[H]$ denote the set of nodes and let $L[H]$ denote the set of links of a subgraph $H \in \mathcal{G}^{N,L}$, respectively. The subgraph of (N,L) *induced by* $S \in 2^N$ is $(S, L[S])$. The subgraph of (N,L) *induced by* $T \in 2^L$ is $(N[T], T)$.

A *path* in $(S,T) \in \mathcal{G}^{N,L}$ from $i_1 \in S$ to $i_n \in S$ is a sequence $(i_k)_{k=1}^n$ of $n \geq 2$ distinct nodes in S such that $\{i_k, i_{k+1}\} \in T$ for all $k \in \{1, \dots, n-1\}$. A subgraph $H \in \mathcal{G}^{N,L}$ *connects* $R \in 2^N \setminus \{\emptyset\}$ if for any $i, j \in R$, $i \neq j$, there exists a path in H from i to j. A coalition $C \in 2^N \setminus \{\emptyset\}$ is a *component* in $H \in \mathcal{G}^{N,L}$ if H connects C and H does not connect any $R \in 2^N \setminus \{\emptyset\}$ for which $C \subset R$. The collection of all components in $H \in \mathcal{G}^{N,L}$ is denoted by $\mathcal{K}(H)$. A subgraph $(S,T) \in \mathcal{G}^{N,L}$ is *connected* if it connects S. A connected subgraph $(S,T) \in \mathcal{G}^{N,L}$ is *cycle-free* if for any $i, j \in S$, $i \neq j$, there exists a unique path in (S,T) from i to j.

A subgraph $(S, L[S]) \in \mathcal{G}^{N,L}$ is a *minimal R-connecting node-induced subgraph* if it connects $R \in 2^N \setminus \{\emptyset\}$ and any $(S', L[S'])$ for which $S' \subset S$ does not connect R. The collection of coalitions of nodes that induce a minimal R-connecting node-induced subgraph of (N,L) is denoted by $\mathcal{N}_L^R \subseteq 2^N \setminus \{\emptyset\}$. A subgraph $(N[T], T) \in \mathcal{G}^{N,L}$ is a *minimal R-connecting link-induced subgraph* if it connects $R \in 2^N \setminus \{\emptyset\}$ and any $(N[T'], T')$ for which $T' \subset T$ does not connect R. The collection of coalitions of links that induce a minimal R-connecting link-induced subgraph of (N,L) is denoted by $\mathcal{L}_N^R \subseteq 2^L \setminus \{\emptyset\}$.

Example 9.1
Let $N = \{1,2,3,4\}$, let $L = \Big\{\{1,2\},\{1,3\},\{2,4\},\{3,4\}\Big\}$, and consider the graph (N,L). This is illustrated as follows.



Then

$$\mathcal{N}_L^{\{1,2,3\}} = \Big\{\{1,2,3\}\Big\}$$
$$\text{and} \quad \mathcal{L}_N^{\{1,2,3\}} = \Big\{\Big\{\{1,2\},\{1,3\}\Big\}, \Big\{\{1,2\},\{2,4\},\{3,4\}\Big\}, \Big\{\{1,3\},\{2,4\},\{3,4\}\Big\}\Big\}.$$

△

A *communication situation* (cf. Myerson (1977)) is a triple (N, v, L) in which $v \in \mathrm{TU}^N$ is a transferable utility game such that $v(\{i\}) = 0$ for all $i \in N$, and (N, L) is a connected undirected graph representing the communication possibilities between the players. Let $\mathrm{CS}^{N,L}$ denote the class of communication situations with communication network (N, L). For convenience, a communication situation on (N, L) is denoted by $v \in \mathrm{CS}^{N,L}$. A *solution* for communication situations $f : \mathrm{CS}^{N,L} \to \mathbb{R}^N$ assigns to any communication situation $v \in \mathrm{CS}^{N,L}$ a payoff allocation $f(v) \in \mathbb{R}^N$ for which $\sum_{i \in N} f_i(v) = v(N)$.

The *node game* $w_L^v \in \mathrm{TU}^N$ corresponding to $v \in \mathrm{CS}^{N,L}$ (cf. Myerson (1977)) is given by

$$w_L^v(S) = \sum_{C \in \mathcal{K}(S, L[S])} v(C)$$

for all $S \in 2^N$. The *Myerson value* $\mu : \mathrm{CS}^{N,L} \to \mathbb{R}^N$ assigns to any $v \in \mathrm{CS}^{N,L}$ the payoff allocation given by

$$\mu(v) = \Phi(w_L^v).$$

The *link game* $w_N^v \in \mathrm{TU}^L$ corresponding to $v \in \mathrm{CS}^{N,L}$ (cf. Borm, Owen, and Tijs (1992)) is given by

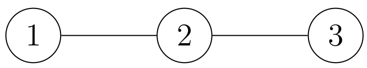
$$w_N^v(T) = \sum_{C \in \mathcal{K}(N[T], T)} v(C)$$

for all $T \in 2^L$. The *position value* $\pi : \mathrm{CS}^{N,L} \to \mathbb{R}^N$ assigns to any $v \in \mathrm{CS}^{N,L}$ the payoff allocation given by

$$\pi_i(v) = \frac{1}{2} \sum_{l \in L_i} \Phi_l(w_N^v)$$

for all $i \in N$.

Example 9.2
Let $N = \{1, 2, 3\}$, let $L = \Big\{\{1, 2\}, \{2, 3\}\Big\}$, and consider the graph (N, L). This is illustrated as follows.



Consider the communication situation $u_{\{1,3\}} \in \mathrm{CS}^{N,L}$. Then

$$w_L^{u_{\{1,3\}}} = u_N \quad \text{and} \quad w_N^{u_{\{1,3\}}} = u_L.$$

Hence,

$$\mu(u_{\{1,3\}}) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \quad \text{and} \quad \pi(u_{\{1,3\}}) = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right).$$

△

9.3 Network communication games

In this section, we introduce network communication games and study their decomposition into unanimity games. The corresponding network control structure explicitly models the control of the nodes and links in the underlying communication network.

Definition (Network Control Structure)
A *network control structure* is a triple (N, L, G) in which (N, L) is an undirected graph and $G : 2^{N\cup L} \to \mathcal{G}^{N,L}$ assigns to each coalition of nodes and links a subgraph of (N, L) such that $G(\emptyset) = (\emptyset, \emptyset)$, $G(N \cup L) = (N, L)$, and $N[G(Z)] \subseteq N[G(Z')]$ and $L[G(Z)] \subseteq L[G(Z')]$ for all $Z, Z' \in 2^{N\cup L}$ for which $Z \subseteq Z'$.

Let $\text{NCS}^{N,L}$ denote the class of network control structures on (N, L). For convenience, a network control structure on (N, L) is denoted by $G \in \text{NCS}^{N,L}$.

Example 9.3
Let $N = \{1, 2, 3\}$, let $L = \big\{\{1, 2\}, \{2, 3\}\big\}$, and consider the network control structure $G \in \text{NCS}^{N,L}$ given by $G(Z) = \big((Z\cap N)\cup N[Z\cap L], (Z\cap L)\cup L[Z\cap N]\big)$ for all $Z \in 2^{N\cup L}$. This means that each node is controlled by itself and its incident links. Moreover, each link is controlled by itself and its two endpoints together. This is presented in the following table.

Z	$G(Z)$
$\emptyset$	$(\emptyset, \emptyset)$
$\{1\}$	$(\{1\}, \emptyset)$
$\{2\}$	$(\{2\}, \emptyset)$
$\{3\}$	$(\{3\}, \emptyset)$
$\{1, 2\}, \big\{\{1, 2\}\big\}, \big\{1, \{1, 2\}\big\}, \big\{2, \{1, 2\}\big\}, \big\{1, 2, \{1, 2\}\big\}$	$(\{1, 2\}, \big\{\{1, 2\}\big\})$
$\{1, 3\}$	$(\{1, 3\}, \emptyset)$
$\{2, 3\}, \big\{\{2, 3\}\big\}, \big\{2, \{2, 3\}\big\}, \big\{3, \{2, 3\}\big\}, \big\{2, 3, \{2, 3\}\big\}$	$(\{2, 3\}, \big\{\{2, 3\}\big\})$
$\big\{1, \{2, 3\}\big\}, \big\{1, 3, \{2, 3\}\big\}$	$(\{1, 2, 3\}, \big\{\{2, 3\}\big\})$
$\big\{3, \{1, 2\}\big\}, \big\{1, 3, \{1, 2\}\big\}$	$(\{1, 2, 3\}, \big\{\{1, 2\}\big\})$
$\big\{\{1, 2\}, \{2, 3\}\big\}, \{1, 2, 3\}, \big\{1, 2, \{2, 3\}\big\}, \big\{2, 3, \{1, 2\}\big\}$	$(\{1, 2, 3\}, \big\{\{1, 2\}, \{2, 3\}\big\})$
$\big\{1, \{1, 2\}, \{2, 3\}\big\}, \big\{2, \{1, 2\}, \{2, 3\}\big\}, \big\{3, \{1, 2\}, \{2, 3\}\big\}$	$(\{1, 2, 3\}, \big\{\{1, 2\}, \{2, 3\}\big\})$
$\big\{1, 2, \{1, 2\}, \{2, 3\}\big\}, \big\{2, 3, \{1, 2\}, \{2, 3\}\big\}$	$(\{1, 2, 3\}, \big\{\{1, 2\}, \{2, 3\}\big\})$
$\big\{1, 3, \{1, 2\}, \{2, 3\}\big\}$	$(\{1, 2, 3\}, \big\{\{1, 2\}, \{2, 3\}\big\})$
$\big\{1, 2, 3, \{1, 2\}\big\}, \big\{1, 2, 3, \{2, 3\}\big\}, \big\{1, 2, 3, \{1, 2\}, \{2, 3\}\big\}$	$(\{1, 2, 3\}, \big\{\{1, 2\}, \{2, 3\}\big\})$

$\triangle$

A network communication game integrates the features of a network control structure $G \in \mathrm{NCS}^{N,L}$ and a communication situation $v \in \mathrm{CS}^{N,L}$ into a transferable utility game on $N \cup L$ in which the worth of a coalition of nodes and links equals the sum of the worths of the components in the subgraph which the members control together.

Definition (Network Communication Game)
Let $G \in \mathrm{NCS}^{N,L}$ be a network control structure and let $v \in \mathrm{CS}^{N,L}$ be a communication situation. In the corresponding *network communication game* $w_G^v \in \mathrm{TU}^{N \cup L}$, the worth of each coalition of nodes and links $Z \in 2^{N \cup L}$ is given by

$$w_G^v(Z) = \sum_{C \in \mathcal{K}(G(Z))} v(C). \tag{9.5}$$

For any network control structure, the network control value of a communication situation assigns to each player the payoff allocated by the Shapley value of the corresponding network communication game to the corresponding node and half of the payoff allocated to the incident links.

Definition (Network Control Value)
Let $G \in \mathrm{NCS}^{N,L}$ be a network control structure. The corresponding *network control value* $\phi^G : \mathrm{CS}^{N,L} \to \mathbb{R}^N$ is the solution which assigns to any communication situation $v \in \mathrm{CS}^{N,L}$ the payoff allocation given by

$$\phi_i^G(v) = \Phi_i(w_G^v) + \frac{1}{2} \sum_{l \in L_i} \Phi_l(w_G^v) \tag{9.6}$$

for all $i \in N$.

We focus on the decomposition of network communication games into unanimity games. For any network control structure, a communication situation with an underlying unanimity game induces a simple network communication game. The corresponding collection of minimal winning coalitions is given by $\mathcal{M}_G^R \subseteq 2^{N \cup L} \setminus \{\emptyset\}$, the collection of coalitions of nodes and links $Z \in 2^{N \cup L}$ for which $G(Z)$ connects $R \in 2^N \setminus \{\emptyset\}$ and any $G(Z')$ for which $Z' \subset Z$ does not connect R for any $G \in \mathrm{NCS}^{N,L}$.

Lemma 9.3.1
Let $G \in \mathrm{NCS}^{N,L}$ and let $R \in 2^N \setminus \{\emptyset\}$. Then $w_G^{u_R} \in \mathrm{SI}^{N \cup L}$ and $\mathcal{M}(w_G^{u_R}) = \mathcal{M}_G^R$.

Proof. Since for any coalition of nodes and links $Z \in 2^{N \cup L}$ there is at most one component $C \in \mathcal{K}(G(Z))$ for which $R \subseteq C$,

$$\begin{aligned} w_G^{u_R}(Z) &\overset{(9.5)}{=} \sum_{C \in \mathcal{K}(G(Z))} u_R(C) = |\{C \in \mathcal{K}(G(Z)) \mid R \subseteq C\}| \\ &= \begin{cases} 1 & \text{if } \exists_{C \in \mathcal{K}(G(Z))} : R \subseteq C; \\ 0 & \text{if } \forall_{C \in \mathcal{K}(G(Z))} : R \nsubseteq C \end{cases} = \begin{cases} 1 & \text{if } G(Z) \text{ connects } R; \\ 0 & \text{if } G(Z) \text{ does not connect } R \end{cases} \end{aligned}$$

for all $Z \in 2^{N \cup L}$.

Since (N, L) is connected, $G(N \cup L) = (N, L)$ connects R, so $w_G^{u_R}(N \cup L) = 1$. If $G(Z)$ connects R for some $Z \in 2^{N \cup L}$, then $G(Z')$ connects R for all $Z' \in 2^{N \cup L}$ for which $Z \subseteq Z'$, so $w_G^{u_R}(Z) \leq w_G^{u_R}(Z')$ for all $Z, Z' \in 2^{N \cup L}$ for which $Z \subseteq Z'$. This means that $w_G^{u_R}(Z) \in \{0, 1\}$ for all $Z \in 2^{N \cup L}$, $w_G^{u_R}(N \cup L) = 1$, and $w_G^{u_R}(Z) \leq w_G^{u_R}(Z')$ for all $Z, Z' \in 2^{N \cup L}$ for which $Z \subseteq Z'$. Hence, $w_G^{u_R} \in \mathrm{SI}^{N \cup L}$. Moreover, $\mathcal{M}(w_G^{u_R}) = \mathcal{M}_G^R$ is a direct consequence of (9.1). □

The following lemma presents the decomposition of simple games into unanimity games.

Lemma 9.3.2
Let $v \in \mathrm{SI}^N$. Then

$$v = \sum_{\mathcal{B} \subseteq \mathcal{M}(v) : \mathcal{B} \neq \emptyset} (-1)^{|\mathcal{B}|+1} u_{\bigcup_{R \in \mathcal{B}} R}. \tag{9.7}$$

Moreover,

$$\Delta^v(S) = \sum_{\mathcal{B} \subseteq \mathcal{M}(v) : \bigcup_{R \in \mathcal{B}} R = S} (-1)^{|\mathcal{B}|+1} \tag{9.8}$$

for all $S \in 2^N \setminus \{\emptyset\}$.

Proof. Since (9.8) is a direct consequence of (9.7), it suffices to show (9.7). We first show that

$$\min\{v', u_{R'}\} = \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^{v'}(R) u_{R \cup R'} \tag{9.9}$$

for all $v' \in \mathrm{SI}^N$ and for any $R' \in 2^N \setminus \{\emptyset\}$. Let $v' \in \mathrm{SI}^N$ and let $S \in 2^N$. Let $R' \in 2^N \setminus \{\emptyset\}$ and suppose that $R' \not\subseteq S$. Then $u_{R'}(S) = 0$ and $R \cup R' \not\subseteq S$ for any $R \in 2^N \setminus \{\emptyset\}$, which means that $u_{R \cup R'}(S) = 0$ for any $R \in 2^N \setminus \{\emptyset\}$. This implies that,

$$\begin{aligned} \min\{v', u_{R'}\}(S) &= \min\{v'(S), u_{R'}(S)\} = \min\{v'(S), 0\} \\ &= 0 = \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^{v'}(R) u_{R \cup R'}(S). \end{aligned}$$

Next, suppose that $R' \subseteq S$. Then $u_{R'}(S) = 1$, and $R \cup R' \subseteq S$ if and only if $R \subseteq S$ for any $R \in 2^N \setminus \{\emptyset\}$, which means that $u_{R \cup R'}(S) = u_R(S)$ for any $R \in 2^N \setminus \{\emptyset\}$. This implies that

$$\begin{aligned} \min\{v', u_{R'}\}(S) &= \min\{v'(S), u_{R'}(S)\} = \min\{v'(S), 1\} = v'(S) \\ &\overset{(9.2)}{=} \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^{v'}(R) u_R(S) = \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^{v'}(R) u_{R \cup R'}(S). \end{aligned}$$

Hence, (9.9) holds.

Next, we prove (9.7) by induction on $|\mathcal{M}(v)|$. Suppose that $|\mathcal{M}(v)| = 1$ and denote $\mathcal{M}(v) = \{R_1\}$. Then

$$v = \max\{u_R \mid R \in \mathcal{M}(v)\} = \max\{u_{R_1}\} = u_{R_1} = \sum_{\mathcal{B}\subseteq\mathcal{M}(v):\mathcal{B}\neq\emptyset} (-1)^{|\mathcal{B}|+1} u_{\bigcup_{R\in\mathcal{B}} R}.$$

Let $k \in \mathbb{N}$ and assume that $v' = \sum_{\mathcal{B}\subseteq\mathcal{M}(v'):\mathcal{B}\neq\emptyset}(-1)^{|\mathcal{B}|+1}u_{\bigcup_{R\in\mathcal{B}} R}$ for any simple game $v' \in \mathrm{SI}^N$ for which $|\mathcal{M}(v')| = k$. Suppose that $|\mathcal{M}(v)| = k+1$. Denote $\mathcal{M}(v) = \{R_1, \ldots, R_{k+1}\}$. Then

$$\begin{aligned}
v &= \max\{u_R \mid R \in \mathcal{M}(v)\}\\
&= \max\{u_{R_1}, \ldots, u_{R_{k+1}}\}\\
&= \max\{\max\{u_{R_1}, \ldots, u_{R_k}\}, u_{R_{k+1}}\}\\
&= \max\{u_{R_1}, \ldots, u_{R_k}\} + u_{R_{k+1}} - \min\{\max\{u_{R_1}, \ldots, u_{R_k}\}, u_{R_{k+1}}\}\\
&= \sum_{\mathcal{B}\subseteq\{R_1,\ldots,R_k\}:\mathcal{B}\neq\emptyset} (-1)^{|\mathcal{B}|+1}u_{\bigcup_{R\in\mathcal{B}} R} + u_{R_{k+1}} - \sum_{\mathcal{B}\subseteq\{R_1,\ldots,R_k\}:\mathcal{B}\neq\emptyset} (-1)^{|\mathcal{B}|+1}u_{\bigcup_{R\in\mathcal{B}} R\cup R_{k+1}}\\
&= \sum_{\mathcal{B}\subseteq\{R_1,\ldots,R_{k+1}\}:\mathcal{B}\neq\emptyset} (-1)^{|\mathcal{B}|+1}u_{\bigcup_{R\in\mathcal{B}} R}\\
&= \sum_{\mathcal{B}\subseteq\mathcal{M}(v):\mathcal{B}\neq\emptyset} (-1)^{|\mathcal{B}|+1}u_{\bigcup_{R\in\mathcal{B}} R},
\end{aligned}$$

where the fifth equality follows from (9.9). □

Example 9.4
Let $N = \{1, 2, 3\}$, let $L = \big\{\{1, 2\}, \{2, 3\}\big\}$, and consider the network control structure $G \in \mathrm{NCS}^{N,L}$ given by $G(Z) = \big((Z\cap N)\cup N[Z\cap L], (Z\cap L)\cup L[Z\cap N]\big)$ for all $Z \in 2^{N\cup L}$ as in Example 9.3. Then

$$\mathcal{M}_G^{\{1,3\}} = \Big\{\big\{1, 2, 3\big\}, \big\{1, 2, \{2, 3\}\big\}, \big\{2, 3, \{1, 2\}\big\}, \big\{\{1, 2\}, \{2, 3\}\big\}\Big\}.$$

Note that (N, L) is cycle-free and $\mathcal{M}_G^{\{1,3\}}$ contains multiple elements. Consider the communication situation $u_{\{1,3\}} \in \mathrm{CS}^{N,L}$ as in Example 9.2. By Lemma 9.3.1 and Lemma 9.3.2,

$$\begin{aligned}
w_G^{u_{\{1,3\}}} &= u_{\{1,2,3\}} + u_{\{1,2,\{2,3\}\}} + u_{\{2,3,\{1,2\}\}} + u_{\{\{1,2\},\{2,3\}\}}\\
&\quad - u_{\{1,2,3,\{2,3\}\}} - u_{\{1,2,3,\{1,2\}\}} - u_{\{1,2,3,\{1,2\},\{2,3\}\}}\\
&\quad - u_{\{1,2,3,\{1,2\},\{2,3\}\}} - u_{\{1,2,\{1,2\},\{2,3\}\}} - u_{\{2,3,\{1,2\},\{2,3\}\}}\\
&\quad + u_{\{1,2,3,\{1,2\},\{2,3\}\}} + u_{\{1,2,3,\{1,2\},\{2,3\}\}} + u_{\{1,2,3,\{1,2\},\{2,3\}\}} + u_{\{1,2,3,\{1,2\},\{2,3\}\}}\\
&\quad - u_{\{1,2,3,\{1,2\},\{2,3\}\}}\\
&= u_{\{1,2,3\}} + u_{\{1,2,\{2,3\}\}} + u_{\{2,3,\{1,2\}\}} + u_{\{\{1,2\},\{2,3\}\}} - u_{\{1,2,3,\{1,2\}\}} - u_{\{1,2,3,\{2,3\}\}}\\
&\quad - u_{\{1,2,\{1,2\},\{2,3\}\}} - u_{\{2,3,\{1,2\},\{2,3\}\}} + u_{\{1,2,3,\{1,2\},\{2,3\}\}}.
\end{aligned}$$

The corresponding network control value is given by

$$\phi^G(u_{\{1,3\}}) = \left(\frac{31}{120}, \frac{58}{120}, \frac{31}{120}\right).$$

△

For any network control structure, the dividends in a general network communication game can be derived from the dividends in the underlying transferable utility game and the dividends in network communication games with an underlying unanimity game.

Lemma 9.3.3
Let $G \in \mathrm{NCS}^{N,L}$, let $v \in \mathrm{CS}^{N,L}$, and let $Z \in 2^{N \cup L} \setminus \{\emptyset\}$. Then

$$\Delta^{w_G^v}(Z) = \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) \Delta^{w_G^{u_R}}(Z). \tag{9.10}$$

Proof. We have

$$\begin{aligned}
\Delta^{w_G^v}(Z) &\overset{(9.3)}{=} \sum_{Z' \subseteq Z} (-1)^{|Z|-|Z'|} w_G^v(Z') \\
&\overset{(9.5)}{=} \sum_{Z' \subseteq Z} (-1)^{|Z|-|Z'|} \sum_{C \in \mathcal{K}(G(Z'))} v(C) \\
&\overset{(9.2)}{=} \sum_{Z' \subseteq Z} (-1)^{|Z|-|Z'|} \sum_{C \in \mathcal{K}(G(Z'))} \left(\sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) u_R(C) \right) \\
&= \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) \sum_{Z' \subseteq Z} (-1)^{|Z|-|Z'|} \sum_{C \in \mathcal{K}(G(Z'))} u_R(C) \\
&\overset{(9.5)}{=} \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) \sum_{Z' \subseteq Z} (-1)^{|Z|-|Z'|} w_G^{u_R}(Z') \\
&\overset{(9.3)}{=} \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) \Delta^{w_G^{u_R}}(Z).
\end{aligned}$$

□

Using Lemma 9.3.3, we can extend the decomposition results for network communication games with an underlying unanimity game to general network communication games for any network control structure. Moreover, we derive an explicit expression for any network control value in terms of the dividends in the underlying transferable utility game.

Theorem 9.3.4
Let $G \in \mathrm{NCS}^{N,L}$ *be a network control structure and let* $v \in \mathrm{CS}^{N,L}$ *be a communication situation. Then*

$$w_G^v = \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) \sum_{\mathcal{B} \subseteq \mathcal{M}_G^R : \mathcal{B} \neq \emptyset} (-1)^{|\mathcal{B}|+1} u_{\bigcup_{Z \in \mathcal{B}} Z}.$$

Proof. By Lemma 9.3.1, Lemma 9.3.2, and Lemma 9.3.3,

$$\begin{aligned}
w_G^v &\overset{(9.2)}{=} \sum_{Z \in 2^{N \cup L} \setminus \{\emptyset\}} \Delta^{w_G^v}(Z) u_Z \\
&\overset{(9.10)}{=} \sum_{Z \in 2^{N \cup L} \setminus \{\emptyset\}} \left(\sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) \Delta^{w_G^{u_R}}(Z) u_Z \right) \\
&= \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) \sum_{Z \in 2^{N \cup L} \setminus \{\emptyset\}} \Delta^{w_G^{u_R}}(Z) u_Z \\
&\overset{(9.2)}{=} \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) w_G^{u_R} \\
&\overset{(9.7)}{=} \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) \sum_{\mathcal{B} \subseteq \mathcal{M}_G^R : \mathcal{B} \neq \emptyset} (-1)^{|\mathcal{B}|+1} u_{\bigcup_{Z \in \mathcal{B}} Z}.
\end{aligned}$$

□

Theorem 9.3.5
Let $G \in \mathrm{NCS}^{N,L}$ *be a network control structure, let* $v \in \mathrm{CS}^{N,L}$ *be a communication situation, and let* $i \in N$ *be a player. Then*

$$\phi_i^G(v) = \sum_{Z \in 2^{N \cup L}} \frac{|Z \cap \{i\}| + \frac{1}{2}|Z \cap L_i|}{|Z|} \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) \sum_{\mathcal{B} \subseteq \mathcal{M}_G^R : \bigcup_{Z' \in \mathcal{B}} Z' = Z} (-1)^{|\mathcal{B}|+1}.$$

Proof. By Lemma 9.3.1, Lemma 9.3.2, and Lemma 9.3.3,

$$\begin{aligned}
\phi_i^G(v) &\overset{(9.6)}{=} \Phi_i(w_G^v) + \frac{1}{2} \sum_{l \in L_i} \Phi_l(w_G^v) \\
&\overset{(9.4)}{=} \sum_{Z \in 2^{N \cup L} : i \in Z} \frac{1}{|Z|} \Delta^{w_G^v}(Z) + \frac{1}{2} \sum_{l \in L_i} \sum_{Z \in 2^{N \cup L} : l \in Z} \frac{1}{|Z|} \Delta^{w_G^v}(Z) \\
&= \sum_{Z \in 2^{N \cup L}} \frac{|Z \cap \{i\}| + \frac{1}{2}|Z \cap L_i|}{|Z|} \Delta^{w_G^v}(Z) \\
&\overset{(9.10)}{=} \sum_{Z \in 2^{N \cup L}} \frac{|Z \cap \{i\}| + \frac{1}{2}|Z \cap L_i|}{|Z|} \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) \Delta^{w_G^{u_R}}(Z) \\
&\overset{(9.8)}{=} \sum_{Z \in 2^{N \cup L}} \frac{|Z \cap \{i\}| + \frac{1}{2}|Z \cap L_i|}{|Z|} \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) \sum_{\mathcal{B} \subseteq \mathcal{M}_G^R : \bigcup_{Z' \in \mathcal{B}} Z' = Z} (-1)^{|\mathcal{B}|+1}.
\end{aligned}$$

□

9.4 Network control values

In this section, we discuss the Myerson value, the position value, and the decomposition of their corresponding node games and link games. Moreover, we focus on the special case that the underlying communication network is cycle-free.

From the viewpoint of Myerson (1977), the nodes of the graph control the network in such a way that each node controls itself and each link is controlled by its two endpoints together. In other words, each coalition of nodes controls its induced subgraph. This can be described by the network control structure $G \in \mathrm{NCS}^{N,L}$ given by $G(Z) = (Z \cap N, L[Z \cap N])$ for all $Z \in 2^{N \cup L}$. Then $\mathcal{M}_G^R = \mathcal{M}(w_G^{u_R}) = \mathcal{M}(w_L^{u_R}) = \mathcal{N}_L^R$ for any $R \in 2^N \setminus \{\emptyset\}$ and the corresponding network control value for communication situations coincides with the Myerson value.

From the viewpoint of Borm, Owen, and Tijs (1992), the links of the graph control the network in such a way that each link controls itself and both its endpoints. In other words, each coalition of links controls its induced subgraph. This can be described by the network control structure $G \in \mathrm{NCS}^{N,L}$ given by $G(Z) = (N[Z \cap L], Z \cap L)$ for all $Z \in 2^{N \cup L}$. Then $\mathcal{M}_G^R = \mathcal{M}(w_G^{u_R}) = \mathcal{M}(w_N^{u_R}) = \mathcal{L}_N^R$ for any $R \in 2^N \setminus \{\emptyset\}$ and the corresponding network control value for communication situations coincides with the position value.

Using Theorem 9.3.4, we find the decomposition into unanimity games of node games and link games in terms of the dividends in the transferable utility game underlying the corresponding communication situation.

Theorem 9.4.1
Let $v \in \mathrm{CS}^{N,L}$ be a communication situation. Then

$$\begin{aligned} w_L^v &= \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) \sum_{\mathcal{B} \subseteq \mathcal{N}_L^R : \mathcal{B} \neq \emptyset} (-1)^{|\mathcal{B}|+1} u_{\bigcup_{S \in \mathcal{B}} S} \\ \text{and} \quad w_N^v &= \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) \sum_{\mathcal{B} \subseteq \mathcal{L}_N^R : \mathcal{B} \neq \emptyset} (-1)^{|\mathcal{B}|+1} u_{\bigcup_{T \in \mathcal{B}} T}. \end{aligned}$$

Using Theorem 9.3.5, we obtain alternative expressions for the Myerson value and the position value in terms of the dividends in the transferable utility game underlying the corresponding communication situation.

Theorem 9.4.2
Let $v \in \mathrm{CS}^{N,L}$ be a communication situation and let $i \in N$ be a player. Then

$$\mu_i(v) = \sum_{S \in 2^N : i \in S} \frac{1}{|S|} \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) \sum_{\mathcal{B} \subseteq \mathcal{N}_L^R : \bigcup_{S' \in \mathcal{B}} S' = S} (-1)^{|\mathcal{B}|+1}$$

$$\textit{and} \quad \pi_i(v) = \sum_{T \in 2^L} \frac{|T \cap L_i|}{2|T|} \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) \sum_{\mathcal{B} \subseteq \mathcal{L}_N^R : \bigcup_{T' \in \mathcal{B}} T' = T} (-1)^{|\mathcal{B}|+1}.$$

If the underlying communication network is cycle-free, then it contains for any $R \in 2^N$ for which $|R| \geq 2$, a unique minimal R-connecting node-induced subgraph and a unique minimal R-connecting link-induced subgraph, which both coincide. This means that any node game or link game for which a unanimity game underlies the corresponding communication situation is a unanimity game as well.

If (N, L) is cycle-free, for any $R \in 2^N$ for which $|R| \geq 2$, let $S_L^R \in 2^N \setminus \{\emptyset\}$ denote the unique coalition of nodes for which $S_L^R \in \mathcal{N}_L^R$. Then $\mathcal{N}_L^R = \{S_L^R\}$ and $\mathcal{L}_N^R = \{L[S_L^R]\}$. Moreover, $w_L^{u_R} = u_{S_L^R}$ and $w_N^{u_R} = u_{L[S_L^R]}$. Combining these observations with Lemma 9.3.3, we obtain the following relations.

Corollary 9.4.3
Let $v \in \mathrm{CS}^{N,L}$ be a communication situation. If (N, L) is cycle-free, then

$$\Delta^{w_L^v}(S) \overset{1}{=} \sum_{R \in 2^N \setminus \{\emptyset\} : S_L^R = S} \Delta^v(R) \quad \textit{for all } S \in 2^N \setminus \{\emptyset\}$$

$$\textit{and} \quad \Delta^{w_N^v}(T) = \sum_{R \in 2^N \setminus \{\emptyset\} : L[S_L^R] = T} \Delta^v(R) \quad \textit{for all } T \in 2^L \setminus \{\emptyset\}.$$

Corollary 9.4.3 offers results which were also found by Owen (1986) and Borm, Owen, and Tijs (1992). The following results are derived from Theorem 9.4.1 and Theorem 9.4.2, respectively.

Corollary 9.4.4
Let $v \in \mathrm{CS}^{N,L}$ be a communication situation. If (N, L) is cycle-free, then

$$w_L^v \overset{1}{=} \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) u_{S_L^R}$$

$$\textit{and} \quad w_N^v = \sum_{R \in 2^N \setminus \{\emptyset\}} \Delta^v(R) u_{L[S_L^R]}.$$

[1] This relation even holds if (N, L) is cycle-complete, i.e. if each cycle in (N, L) induces a complete subgraph.

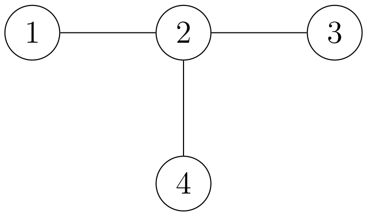
Corollary 9.4.5

Let $v \in \mathrm{CS}^{N,L}$ be a communication situation and let $i \in N$ be a player. If (N,L) is cycle-free, then

$$\begin{aligned} \mu_i(v) &= {}^1 \sum_{R \in 2^N \setminus \{\emptyset\} : i \in S_L^R} \frac{1}{|S_L^R|} \Delta^v(R) \\ \textit{and} \quad \pi_i(v) &= \sum_{R \in 2^N \setminus \{\emptyset\}} \frac{|L[S_L^R] \cap L_i|}{2|L[S_L^R]|} \Delta^v(R). \end{aligned} \tag{9.11}$$

Example 9.5

Let $N = \{1,2,3,4\}$, let $L = \big\{\{1,2\},\{2,3\},\{2,4\}\big\}$, and consider the cycle-free graph (N,L). This is illustrated as follows.



Consider the communication situation $v \in \mathrm{CS}^{N,L}$ given by

$$v = 2u_{\{1,2\}} + 3u_{\{1,3\}} - 3u_{\{1,2,3\}} + 5u_{\{1,3,4\}} + 7u_{\{1,2,3,4\}}.$$

By Corollary 9.4.4,

$$\begin{aligned} w_L^v &= 2u_{\{1,2\}} + 3u_{\{1,2,3\}} - 3u_{\{1,2,3\}} + 5u_{\{1,2,3,4\}} + 7u_{\{1,2,3,4\}} \\ &= 2u_{\{1,2\}} + 12u_{\{1,2,3,4\}} \end{aligned}$$

and

$$\begin{aligned} w_N^v &= 2u_{\{\{1,2\}\}} + 3u_{\{\{1,2\},\{2,3\}\}} - 3u_{\{\{1,2\},\{2,3\}\}} + 5u_{\{\{1,2\},\{2,3\},\{2,4\}\}} + 7u_{\{\{1,2\},\{2,3\},\{2,4\}\}} \\ &= 2u_{\{\{1,2\}\}} + 12u_{\{\{1,2\},\{2,3\},\{2,4\}\}}. \end{aligned}$$

By Corollary 9.4.5,

$$\begin{aligned} \mu(v) &= (1+3, 1+3, 3, 3) = (4,4,3,3) \\ \text{and} \quad \pi(v) &= (1+2, 1+6, 2, 2) = (3,7,2,2). \end{aligned}$$

△

The uniqueness relation in cycle-free communication networks not only holds for the Myerson value and the position value, but also for other network control values with a specific type of network control structure. In particular, for the network control structure $\widetilde{G} \in \mathrm{NCS}^{N,L}$ given by $\widetilde{G}(Z) = \big(Z \cap N[Z \cap L], Z \cap L[Z \cap N]\big)$ for all $Z \in 2^{N \cup L}$, in which each node and each link only controls itself. Then $\mathcal{M}_{\widetilde{G}}^R = \{N[T] \cup T \mid T \in \mathcal{L}_N^R\}$ for any $R \in 2^N$ for which $|R| \geq 2$. If (N,L) is cycle-free, $\mathcal{M}_{\widetilde{G}}^R = \{S_L^R \cup L[S_L^R]\}$ and $w_{\widetilde{G}}^{u_R} = u_{S_L^R \cup L[S_L^R]}$ for any $R \in 2^N$ for which $|R| \geq 2$.

Example 9.6
Let $N = \{1, 2, 3\}$, let $L = \Big\{\{1, 2\}, \{2, 3\}\Big\}$, and consider the cycle-free graph (N, L) as in Example 9.2. Then $S_L^{\{1,3\}} = N$,

$$w_{\widetilde{G}}^{u_{\{1,3\}}} = u_{N \cup L}, \quad \text{and} \quad \phi^{\widetilde{G}}(u_{\{1,3\}}) = \left(\frac{3}{10}, \frac{4}{10}, \frac{3}{10}\right).$$

Note that $\phi^{\widetilde{G}}(u_{\{1,3\}}) = \frac{3}{5}\mu(u_{\{1,3\}}) + \frac{2}{5}\pi(u_{\{1,3\}})$. △

The value $\phi^{\widetilde{G}}$ was introduced by Borm, Van den Nouweland, and Tijs (1994). In Example 9.6, we observe that the value $\phi^{\widetilde{G}}$ is a specific convex combination of the Myerson value μ and the position value π. This holds for any communication situation with an underlying unanimity game and a cycle-free communication network.

Theorem 9.4.6
Let $R \in 2^N$ be such that $|R| \geq 2$. If (N, L) is cycle-free, then

$$\phi^{\widetilde{G}}(u_R) = \frac{|S_L^R|}{2|S_L^R| - 1}\mu(u_R) + \frac{|S_L^R| - 1}{2|S_L^R| - 1}\pi(u_R).$$

Proof. Assume that (N, L) is cycle-free and let $i \in N$. If $i \notin S_L^R$, then $l \notin L[S_L^R]$ for all $l \in L_i$, so $\phi_i^{\widetilde{G}}(u_R) = \mu_i(u_R) = \pi_i(u_R) = 0$ and the statement follows. Suppose that $i \in S_L^R$. By Corollary 9.4.5 and $|L[S_L^R]| = |S_L^R| - 1$,

$$\begin{aligned}
\phi_i^{\widetilde{G}}(u_R) &= \Phi_i(w_{\widetilde{G}}^{u_R}) + \frac{1}{2}\sum_{l \in L_i} \Phi_e(w_{\widetilde{G}}^{u_R}) \\
&\overset{(9.4)}{=} \sum_{Z \in 2^{N \cup L}: i \in Z} \frac{1}{|Z|}\Delta^{w_{\widetilde{G}}^{u_R}}(Z) + \frac{1}{2}\sum_{l \in L_i} \sum_{Z \in 2^{N \cup L}: l \in Z} \frac{1}{|Z|}\Delta^{w_{\widetilde{G}}^{u_R}}(Z) \\
&= \frac{1}{|S_L^R \cup L[S_L^R]|} + \frac{|L[S_L^R] \cap L_i|}{2|S_L^R \cup L[S_L^R]|} \\
&= \frac{1}{2|S_L^R| - 1} + \frac{|L[S_L^R] \cap L_i|}{4|S_L^R| - 2} \\
&= \frac{|S_L^R|}{2|S_L^R| - 1}\left(\frac{1}{|S_L^R|}\right) + \frac{|S_L^R| - 1}{2|S_L^R| - 1}\left(\frac{|L[S_L^R] \cap L_i|}{2|L[S_L^R]|}\right) \\
&\overset{(9.11)}{=} \frac{|S_L^R|}{2|S_L^R| - 1}\mu_i(u_R) + \frac{|S_L^R| - 1}{2|S_L^R| - 1}\pi_i(u_R).
\end{aligned}$$

□

The value $\phi^{\widetilde{G}}$ is not necessarily a convex combination of the values μ and π in communication situations for which the underlying game is not a unanimity game or the underlying communication network is not cycle-free.

Example 9.7
Let $N = \{1,2,3,4\}$, let $L = \Big\{\{1,2\},\{1,3\},\{2,4\},\{3,4\}\Big\}$, and consider the graph (N,L) as in Example 9.1. Then

$$\begin{aligned} w_L^{u_{\{1,2,3\}}} &= u_{\{1,2,3\}}, \\ w_N^{u_{\{1,2,3\}}} &= u_{\{\{1,2\}\{1,3\}\}} + u_{\{\{1,2\},\{2,4\},\{3,4\}\}} + u_{\{\{1,3\},\{2,4\},\{3,4\}\}} - 2u_L, \\ \text{and} \quad w_{\widetilde{G}}^{u_{\{1,2,3\}}} &= u_{\{1,2,3,\{1,2\}\{1,3\}\}} + u_{N\cup\{\{1,2\},\{2,4\},\{3,4\}\}} + u_{N\cup\{\{1,3\},\{2,4\},\{3,4\}\}} - 2u_{N\cup L}. \end{aligned}$$

This means that

$$\begin{aligned} \mu(u_{\{1,2,3\}}) &= \left(\frac{1}{3},\frac{1}{3},\frac{1}{3},0\right), \\ \pi(u_{\{1,2,3\}}) &= \left(\frac{4}{12},\frac{3}{12},\frac{3}{12},\frac{2}{12}\right), \\ \text{and} \quad \phi^{\widetilde{G}}(u_{\{1,2,3\}}) &= \left(\frac{23}{70},\frac{21}{70},\frac{21}{70},\frac{5}{70}\right). \end{aligned}$$

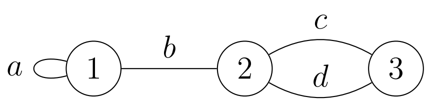
Note that $\phi^{\widetilde{G}}(u_{\{1,2,3\}})$ is not a convex combination of $\mu(u_{\{1,2,3\}})$ and $\pi(u_{\{1,2,3\}})$. △

In general, a network control value is not necessarily a convex combination of the Myerson value and the position value, even if the underlying game is a unanimity game and the underlying communication network is cycle-free.

9.5 Future extensions

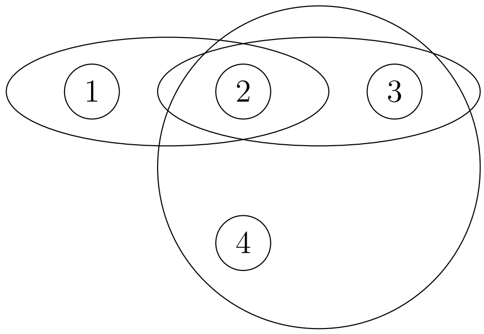
We conclude this chapter with two examples of possible extensions of the decomposition theory to more general communication networks: undirected multigraphs and hypergraphs. Hypergraph communication structures were introduced by Myerson (1980) and further studied by Van den Nouweland, Borm, and Tijs (1992). For convenience, we restrict ourselves to an outline of the link game and the corresponding position value in these examples.

Example 9.8
Let $\{1,2,3\}$ be the set of nodes and let $\{a,b,c,d\}$ be the set of links of the multigraph illustrated as follows.



Consider the communication structure with underlying game $u_{\{1,3\}}$. The collection of coalitions of links which induce a minimal $\{1,3\}$-connecting link-induced subgraph is given by $\Big\{\{b,c\},\{b,d\}\Big\}$. The corresponding link game can be written as $u_{\{b,c\}} + u_{\{b,d\}} - u_{\{b,c,d\}}$. The position value of this communication structure is given by $(\frac{2}{6},\frac{3}{6},\frac{1}{6})$. △

Example 9.9
Let $\{1,2,3,4\}$ be the set of nodes and let $\Big\{\{1,2\},\{2,3\},\{2,3,4\}\Big\}$ be the set of (hyper)links of the hypergraph illustrated as follows.



Consider the communication structure with underlying game $u_{\{1,3\}}$. The collection of coalitions of links which induce a minimal $\{1,3\}$-connecting link-induced subgraph is given by $\Big\{\big\{\{1,2\},\{2,3\}\big\},\big\{\{1,2\},\{2,3,4\}\big\}\Big\}$. The corresponding link game can be written as $u_{\{\{1,2\},\{2,3\}\}}+u_{\{\{1,2\},\{2,3,4\}\}}-u_{\{\{1,2\},\{2,3\},\{2,3,4\}\}}$. The position value of this communication structure is given by $(\frac{12}{36},\frac{17}{36},\frac{5}{36},\frac{2}{36})$. $\triangle$

Future research could formalize these or other extensions to more general communication structures (cf. Bilbao (2000)). Moreover, one could aim to axiomatically characterize the class of network control values or a specific network control value for communication situations.

References

Arin, J. and E. Iñarra (2001). Egalitarian solutions in the core. *International Journal of Game Theory*, **30**(2), 187–193.

Arin, J., J. Kuipers, and D. Vermeulen (2003). Some characterizations of egalitarian solutions on classes of TU-games. *Mathematical Social Sciences*, **46**(3), 327–345.

Aumann, R. (1985). On the non-transferable utility value: a comment on the Roth-Shafer examples. *Econometrica*, **53**(3), 667–678.

Aumann, R. (1986). On the non-transferable utility value: rejoinder. *Econometrica*, **54**(4), 985–989.

Aumann, R. and M. Maschler (1985). Game theoretic analysis of a bankruptcy problem from the Talmud. *Journal of Economic Theory*, **36**(2), 195–213.

Aumann, R. and B. Peleg (1960). Von Neumann-Morgenstern solutions to cooperative games without side payments. *Bulletin of the American Mathematical Society*, **66**(3), 173–179.

Bilbao, J. (2000). *Cooperative Games on Combinatorial Structures.* Springer.

Bondareva, O. (1963). Some applications of linear programming methods to the theory of cooperative games. *Problemy Kibernetiki*, **10**, 119–139.

Borm, P., H. Keiding, R. McLean, S. Oortwijn, and S. Tijs (1992). The compromise value for NTU-games. *International Journal of Game Theory*, **21**(2), 175–189.

Borm, P., A. van den Nouweland, and S. Tijs (1994). Cooperation and communication restrictions: a survey. In: R. Gilles and P. Ruys (Eds.), *Imperfections and Behavior in Economic Situations*, pp. 195–227. Springer.

Borm, P., G. Owen, and S. Tijs (1992). On the position value for communication situations. *SIAM Journal on Discrete Mathematics*, **5**(3), 305–320.

Branzei, R., D. Dimitrov, and S. Tijs (2006). The equal split-off set for cooperative games. *Banach Center Publications*, **71**, 39–46.

Chun, Y. (1988). The proportional solution for rights problems. *Mathematical Social Sciences*, **15**(3), 231–246.

Chun, Y. and W. Thomson (1990). Nash solution and uncertain disagreement points. *Games and Economic Behavior*, **2**(3), 213–223.

Chun, Y. and W. Thomson (1992). Bargaining problems with claims. *Mathematical Social Sciences*, **24**(1), 19–33.

Curiel, I., M. Maschler, and S. Tijs (1987). Bankruptcy games. *Zeitschrift für Operations Research*, **31**(5), 143–159.

Dagan, N. (1996). New characterizations of old bankruptcy rules. *Social Choice and Welfare*, **13**(1), 51–59.

Dagan, N. and O. Volij (1993). The bankruptcy problem: a cooperative bargaining approach. *Mathematical Social Sciences*, **26**(3), 287–297.

Davis, M. and M. Maschler (1965). The kernel of a cooperative game. *Naval Research Logistics Quarterly*, **12**(3), 223–259.

Dietzenbacher, B. (2018). Bankruptcy games with nontransferable utility. *Mathematical Social Sciences*, **92**, 16–21.

Dietzenbacher, B., P. Borm, and A. Estévez-Fernández (2017). NTU-bankruptcy problems: consistency and the relative adjustment principle. *CentER Discussion Paper*, 2017–044.

Dietzenbacher, B., P. Borm, and R. Hendrickx (2017a). Decomposition of network communication games. *Mathematical Methods of Operations Research*, **85**(3), 407–423.

Dietzenbacher, B., P. Borm, and R. Hendrickx (2017b). Egalitarianism in nontransferable utility games. *CentER Discussion Paper*, 2017–023.

Dietzenbacher, B., P. Borm, and R. Hendrickx (2017c). The procedural egalitarian solution. *Games and Economic Behavior*, **106**, 179–187.

Dietzenbacher, B., A. Estévez-Fernández, P. Borm, and R. Hendrickx (2016). Proportionality, equality, and duality in bankruptcy problems with nontransferable utility. *CentER Discussion Paper*, 2016–026.

Dietzenbacher, B. and H. Peters (2018). Characterizing NTU-bankruptcy rules using bargaining axioms. *CentER Discussion Paper*, 2018–005.

Dutta, B. (1990). The egalitarian solution and reduced game properties in convex games. *International Journal of Game Theory*, **19**(2), 153–169.

Dutta, B. and D. Ray (1989). A concept of egalitarianism under participation constraints. *Econometrica*, **57**(3), 615–635.

Estévez-Fernández, A., P. Borm, and M. Fiestras-Janeiro (2014). Nontransferable utility bankruptcy games. *Tinbergen Institute Discussion Paper*, TI 2014–30/II.

Gerard-Varet, L. and S. Zamir (1987). Remarks on the reasonable set of outcomes in a general coalition function form game. *International Journal of Game Theory*, **16**(2), 123–143.

Gillies, D. (1959). Solutions to general non-zero-sum games. In: A. Tucker and R. Luce (Eds.), *Contributions to the Theory of Games IV*, pp. 47–85. Princeton University Press.

Greenberg, J. (1985). Cores of convex games without side payments. *Mathematics of Operations Research*, **10**(3), 523–525.

Harsanyi, J. (1959). A bargaining model for the cooperative n-person game. In: A. Tucker and R. Luce (Eds.), *Contributions to the Theory of Games IV*, pp. 325–355. Princeton University Press.

Harsanyi, J. (1963). A simplified bargaining model for the n-person cooperative game. *International Economic Review*, **4**(2), 194–220.

Harsanyi, J. (1980). Comments on Roth's paper, 'Values for games without side payments'. *Econometrica*, **48**(2), 477.

Hart, S. (1985). Nontransferable utility games and markets: some examples and the Harsanyi solution. *Econometrica*, **53**(6), 1445–1450.

Hart, S. and M. Kurz (1983). Endogenous formation of coalitions. *Econometrica*, **51**(4), 1047–1064.

Hart, S. and A. Mas-Colell (1989). Potential, value, and consistency. *Econometrica*, **57**(3), 589–614.

Hendrickx, R., P. Borm, and J. Timmer (2002). A note on NTU convexity. *International Journal of Game Theory*, **31**(1), 29–37.

Herrero, C. (1997). Endogenous reference points and the adjusted proportional solution for bargaining problems with claims. *Social Choice and Welfare*, **15**(1), 113–119.

Herrero, C. and A. Villar (2001). The three musketeers: four classical solutions to bankruptcy problems. *Mathematical Social Sciences*, **42**(3), 307–328.

Herrero, C. and A. Villar (2002). Sustainability in bankruptcy problems. *TOP*, **10**(2), 261–273.

Ichiishi, T. (1981). Super-modularity: applications to convex games and to the greedy algorithm for LP. *Journal of Economic Theory*, **25**(2), 283–286.

Kalai, E. (1977). Proportional solutions to bargaining situations: interpersonal utility comparisons. *Econometrica*, **45**(7), 1623–1630.

Kalai, E. and R. Rosenthal (1978). Arbitration of two-party disputes under ignorance. *International Journal of Game Theory*, **7**(2), 65–72.

Kalai, E. and D. Samet (1985). Monotonic solutions to general cooperative games. *Econometrica*, **53**(2), 307–327.

Kalai, E. and M. Smorodinsky (1975). Other solutions to Nash's bargaining problem. *Econometrica*, **43**(3), 513–518.

Klijn, F., M. Slikker, S. Tijs, and J. Zarzuelo (2000). The egalitarian solution for convex games: some characterizations. *Mathematical Social Sciences*, **40**(1), 111–121.

Livne, Z. (1988). The bargaining problem with an uncertain conflict outcome. *Mathematical Social Sciences*, **15**(3), 287–302.

Mariotti, M. and A. Villar (2005). The Nash rationing problem. *International Journal of Game Theory*, **33**(3), 367–377.

Megiddo, N. (1974). On the nonmonotonicity of the bargaining set, the kernel and the nucleolus of a game. *SIAM Journal on Applied Mathematics*, **27**(2), 355–358.

Moulin, H. and S. Shenker (1992). Serial cost sharing. *Econometrica*, **60**(5), 1009–1037.

Musegaas, M., B. Dietzenbacher, and P. Borm (2016). On Shapley ratings in brain networks. *Frontiers in Neuroinformatics*, **10**(51).

Myerson, R. (1977). Graphs and cooperation in games. *Mathematics of Operations Research*, **2**(3), 225–229.

Myerson, R. (1980). Conference structures and fair allocation rules. *International Journal of Game Theory*, **9**(3), 169–182.

Nash, J. (1950). The bargaining problem. *Econometrica*, **18**(2), 155–162.

Neumann, J. von and O. Morgenstern (1944). *Theory of Games and Economic Behavior*. Princeton University Press.

Nouweland, A. van den, P. Borm, and S. Tijs (1992). Allocation rules for hypergraph communication situations. *International Journal of Game Theory*, **20**(3), 255–268.

O'Neill, B. (1982). A problem of rights arbitration from the Talmud. *Mathematical Social Sciences*, **2**(4), 345–371.

Orshan, G., F. Valenciano, and J. Zarzuelo (2003). The bilateral consistent prekernel, the core, and NTU bankruptcy problems. *Mathematics of Operations Research*, **28**(2), 268–282.

Otten, G., P. Borm, B. Peleg, and S. Tijs (1998). The MC-value for monotonic NTU-games. *International Journal of Game Theory*, **27**(1), 37–47.

Owen, G. (1986). Values of graph-restricted games. *SIAM Journal on Algebraic Discrete Methods*, **7**(2), 210–220.

Peters, H. (2010). Characterizations of bargaining solutions by properties of their status quo sets. In: A. van Deemen and A. Rusinowska (Eds.), *Collective Decision Making*, pp. 231–247. Springer.

Peters, H. and E. van Damme (1991). Characterizing the Nash and Raiffa bargaining solutions by disagreement point axioms. *Mathematics of Operations Research*, **16**(3), 447–461.

Peters, H., S. Tijs, and J. Zarzuelo (1994). A reduced game property for the Kalai-Smorodinsky and egalitarian bargaining solutions. *Mathematical Social Sciences*, **27**(1), 11–18.

Quant, M., P. Borm, H. Reijnierse, and B. van Velzen (2005). The core cover in relation to the nucleolus and the Weber set. *International Journal of Game Theory*, **33**(4), 491–503.

Raiffa, H. (1953). Arbitration schemes for generalized two-person games. In: H. Kuhn and A. Tucker (Eds.), *Contributions to the Theory of Games II*, pp. 361–387. Princeton University Press.

Rawls, J. (1971). *A Theory of Justice*. Harvard University Press.

Roth, A. (1977). Independence of irrelevant alternatives, and solutions to Nash's bargaining problem. *Journal of Economic Theory*, **16**(2), 247–251.

Roth, A. (1979). Proportional solutions to the bargaining problem. *Econometrica*, **47**(3), 775–778.

Roth, A. (1980). Values for games without sidepayments: some difficulties with current concepts. *Econometrica*, **48**(2), 457–465.

Roth, A. (1986). On the non-transferable utility value: a reply to Aumann. *Econometrica*, **54**(4), 981–984.

Scarf, H. (1967). The core of an n-person game. *Econometrica*, **35**(1), 50–69.

Schmeidler, D. (1969). The nucleolus of a characteristic function game. *SIAM Journal on Applied Mathematics*, **17**(6), 1163–1170.

Schouten, J., B. Dietzenbacher, and P. Borm (2018). The nucleolus and inheritance of properties in communication situations. Tilburg University.

Selten, R. (1972). Equal share analysis of characteristic function experiments. In: H. Sauermann (Ed.), *Contributions to Experimental Economics III*, pp. 130–165. Mohr Siebeck.

Selten, R. (1987). Equity and coalition bargaining in experimental three-person games. In: A. Roth (Ed.), *Laboratory Experimentation in Economics*, pp. 42–98. Cambridge University Press.

Shafer, W. (1980). On the existence and interpretation of value allocation. *Econometrica*, **48**(2), 467–476.

Shapley, L. (1953). A value for n-person games. In: H. Kuhn and A. Tucker (Eds.), *Contributions to the Theory of Games II*, pp. 307–317. Princeton University Press.

Shapley, L. (1967). On balanced sets and cores. *Naval Research Logistics Quarterly*, **14**(4), 453–460.

Shapley, L. (1969). Utility comparison and the theory of games. In: *La Décision: Agrégation et Dynamique des Ordres de Préférence*, pp. 251–263. Editions du Centre National de la Recherche Scientifique.

Shapley, L. (1971). Cores of convex games. *International Journal of Game Theory*, **1**(1), 11–26.

Shapley, L. and M. Shubik (1953). Solutions of n-person games with ordinal utilities. *Econometrica*, **21**(2), 348–349.

Sugumaran, A., V. Thangaraj, and G. Ravindran (2013). Average rules for cooperative TU games. *International Game Theory Review*, **15**(4), 1340027.

Thomson, W. (2003). Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey. *Mathematical Social Sciences*, **45**(3), 249–297.

Thomson, W. (2011). Consistency and its converse: an introduction. *Review of Economic Design*, **15**(4), 257–291.

Thomson, W. (2013). Game-theoretic analysis of bankruptcy and taxation problems: recent advances. *International Game Theory Review*, **15**(3), 1340018.

Thomson, W. (2015). Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: an update. *Mathematical Social Sciences*, **74**, 41–59.

Thomson, W. (2016). A new characterization of the proportional rule for claims problems. *Economics Letters*, **145**, 255–257.

Thomson, W. and R. Myerson (1980). Monotonicity and independence axioms. *International Journal of Game Theory*, **9**(1), 37–49.

Thomson, W. and C. Yeh (2008). Operators for the adjudication of conflicting claims. *Journal of Economic Theory*, **143**(1), 177–198.

Tijs, S. and F. Lipperts (1982). The hypercube and the core cover of n-person cooperative games. *Cahiers du CERO*, **24**, 27–37.

Vilkov, V. (1977). Convex games without side payments. *Vestnik Leningrad University*, **7**, 21–24.

Weber, R. (1988). Probabilistic values for games. In: A. Roth (Ed.), *The Shapley Value: Essays in Honor of Lloyd S. Shapley*, pp. 101–119. Cambridge University Press.

Yeh, C. (2004). Sustainability, exemption, and the constrained equal awards rule: a note. *Mathematical Social Sciences*, **47**(1), 103–110.

Yeh, C. (2006). Protective properties and the constrained equal awards rule for claims problems: a note. *Social Choice and Welfare*, **27**(2), 221–230.

Young, H. (1985). Monotonic solutions of cooperative games. *International Journal of Game Theory*, **14**(2), 65–72.

Young, H. (1987). On dividing an amount according to individual claims or liabilities. *Mathematics of Operations Research*, **12**(3), 398–414.

Young, H. (1988). Distributive justice in taxation. *Journal of Economic Theory*, **44**(2), 321–335.

Young, H. (1995). *Equity*. Princeton University Press.

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