Geometry of Voting

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Introduction

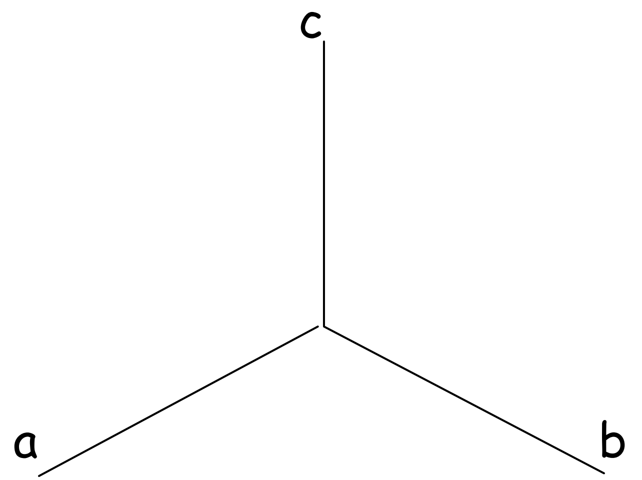
- We have seen what paradoxical situations could occur
- Now it's time to provide some explanation for it
- Don Saari's **"Basic Geometry of Voting (1995)"**
  - More than 50 papers and 6 books about this topic
- We will focus on 3 things:
  - Use geometry to determine possible voting outcomes for scoring rules
  - Decompose profiles to explain paradoxes between scoring rules and simple majority rule
  - Representation polytope and some applications

Mathematical prerequisites

- Convex sets
- Linear mappings
- Convexity property of linear functions
  - If *A* is a convex set in the domain and if f is a linear function, then the image of *A* is a convex set in the image space.
  - If f is a linear mapping with a convex domain if *D* is a convex set in the image set, then $f^{-1}(D)$ is convex set.
- Convex hull
  - The convex hull of the vertices $\{v_i\}_1^m$ is the smallest convex set containing all vertices.
- Linear functions now map convex hulls into convex hulls

Saari Triangle

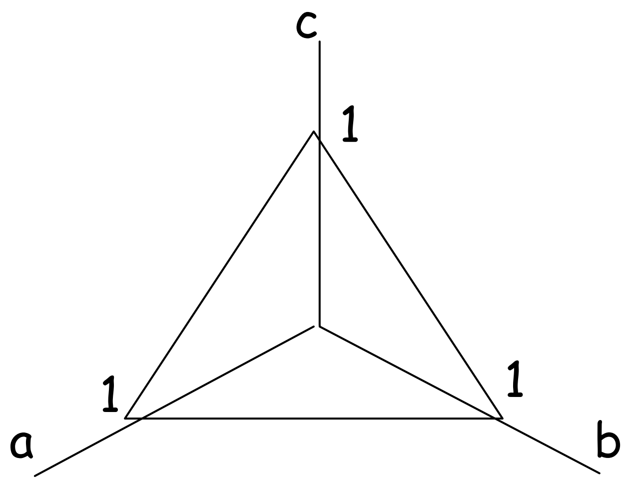
- Consider any voting rule that ranks the alternatives according to some assignment of points to them
  - E.g. scoring rules such as the Borda rule or Plurality rule
  - We could try to see each point score for an alternative as a point on the axis for that alternative
  - What happens with 2 alternatives?
  - … and for 3 alternatives?



  - We get a point in 3-dimensional space

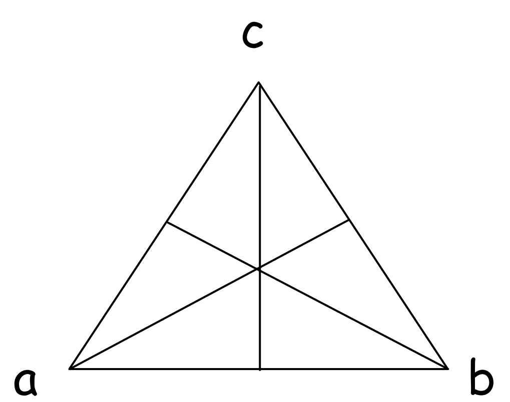
Saari Triangle

- How can we make this look simpler?
- Whatever the election tally, we could try to normalize it
- E.g. plurality vector (9, 6, 5) could be normalized to (9/20, 6/20, 5/20)
- This can be plotted in the simplex



Saari Triangle

A projection of this leads to the following triangle:



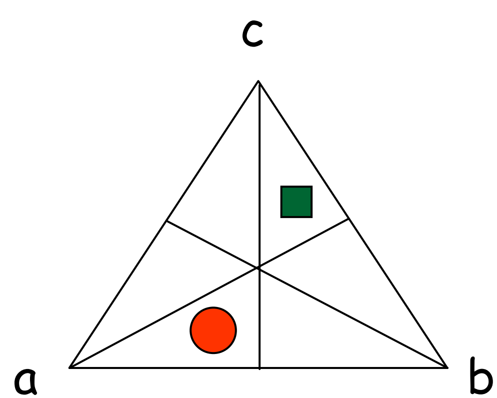
Which we could divide into proximity (or ranking) regions.

- The closer to one of the edges the "better"
- Leads to 6 ranking regions

Saari Triangle

Let X = {a,b,c}. There are 6 types of strict rankings:

$$\begin{array}{cccccc} a & a & c & c & b & b \\ b & c & a & b & c & a \\ c & b & b & a & a & c \end{array}$$



Can use this to determine the outcomes. How?

However, we have a "slight" dimensionality problem when we increase the number of alternatives!

Scoring Rules

For $|X| = 3$ we know that the scoring vectors are:

Plurality: $W_{PR} = (1,0,0)$ or $w_{PR} = (1,0,0)$

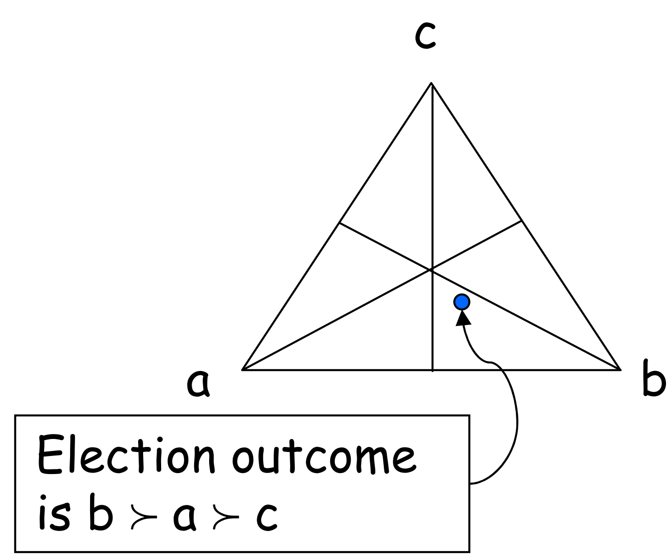
Antiplurality: $W_{AP} = (1,1,0)$ or $w_{AP} = (\frac{1}{2},\frac{1}{2},0)$

Borda: $W_B = (2,1,0)$ or $w_B = (2/3,1/3, 0)$

In general $w_s = (1-s,s,0)$; $s \in [0, \frac{1}{2}]$

Now, use the scoring vector to get a vector of scores for the alternatives (e.g. (10,14,9)).

Normalize this vector to get a point in the simplex (e.g. $(\frac{10}{33}, \frac{14}{33}, \frac{9}{33})$).



Example

6	5	4
M	B	W
W	W	B
B	M	M

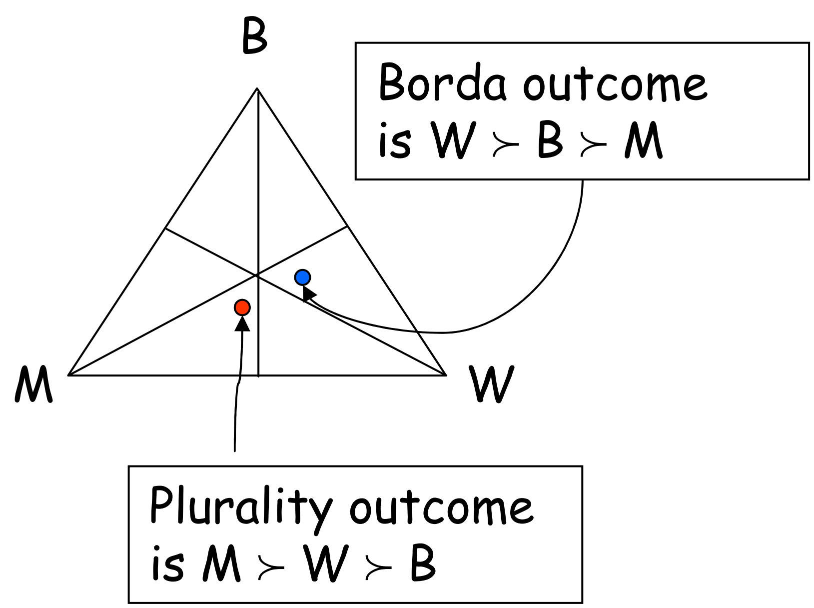
scoring vector $w_B = (\frac{2}{3}, \frac{1}{3}, 0)$

$Borda$	$scores$
M	12
W	19
B	14

$Borda$	$norm.scores$
M	$12/45$
W	$19/45$
B	$14/45$

scoring vector $w_{PR} = (1, 0, 0)$

$Plur$	$scores$	$norm.scores$
M	6	$6/15$
W	5	$5/15$
B	4	$4/15$

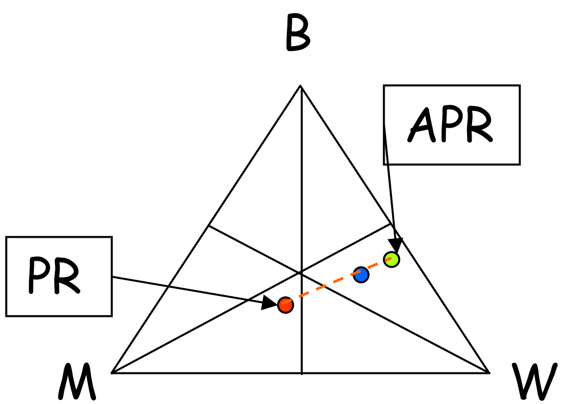


Example

6	5	4
M	B	W
W	W	B
B	M	M

scoring vector $w_{AP} = (\frac{1}{2}, \frac{1}{2}, 0)$

$APlur$	$scores$	$norm.scores$
M	6	$6/30$
W	15	$15/30$
B	9	$9/30$



Because of linearity the Borda outcome lies on the line between the PR and the APR outcome.

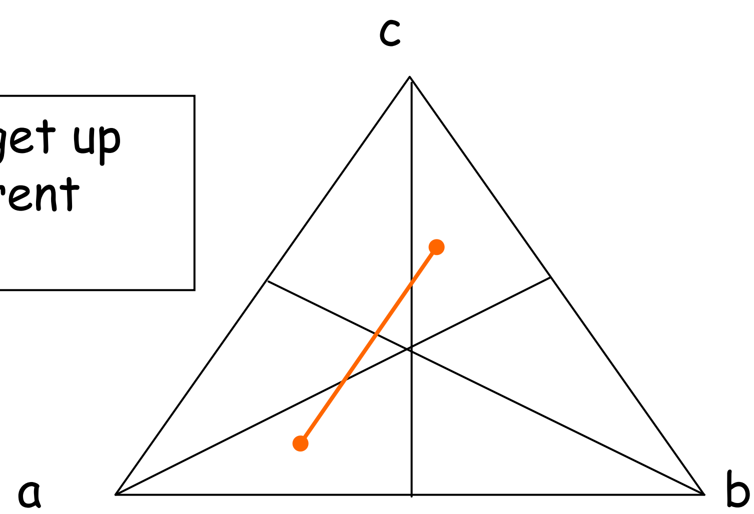
$$(\tfrac{2}{3}, \tfrac{1}{3}, 0) = \tfrac{1}{3}(1,0,0) + \tfrac{2}{3}(\tfrac{1}{2}, \tfrac{1}{2}, 0)$$

The line connecting PR with APR is the *Procedure Line* and contains all possible outcomes of scoring rules for the given profile.

Procedure Line

Using the procedure line we can see how bad it can get among different scoring rules.

We could get up to 7 different outcomes!



Whenever PR and APR are in the same ranking region, ALL scoring rules give the same voting outcome!

With more alternatives we get spaces of possible scoring rule outcomes.

More alternatives lead to more problems! With 10 candidates there exists a profile that gives 84,830,767 different election rankings for different scoring rules (Saari, 1995).

Representing profiles

- For |X|=3 we do have how many linear orders?
- So we could think of our profile as a vector in 6 dimensional space

$$\begin{array}{cccccc} a & a & b & b & c & c \\ b & c & a & c & a & b \\ c & b & c & a & b & a \end{array}$$

- E.g.: p = (32,0,10,22,20,16)
- So any profile is a 6-dimensional vector
- Dimensions increase massively with candidates
- Now Saari (1995) thinks of certain subspaces, which have a specific impact on certain voting rules
  - This should help us understand certain voting paradoxes and differences in voting outcomes
- → **Saari's profile decomposition**

Profile Decomposition

What is the driving force behind the different outcomes?

Or, how can we create profiles that lead to differences between scoring rules and majority rule?

Profile Decomposition (Saari, 1995)

What would we expect for the following combinations of preferences?

$$\boxed{\begin{array}{cc} a & c \\ b & b \\ c & a \end{array}}$$

REVERSAL PORTION

Those two preferences should cancel out!

But only true for majority rule and Borda rule, not for any other scoring rule!

Profile Decomposition

$$\begin{array}{ccc} a & c & b \\ b & a & c \\ c & b & a \end{array}$$

Every alternative is once in each position. Should not influence the voting outcome.

But only true for scoring rules, not for majority rule or any Condorcet extension as it creates or strengthens a cycle!

CONDORCET PORTION

Every alternative is in each position twice.

Gives indifference for ALL voting procedures!

$$\begin{array}{cccccc} a & a & c & c & b & b \\ b & c & a & b & c & a \\ c & b & b & a & a & c \end{array}$$

KERNEL PORTION

Profile Decomposition

Finally there is a portion that gives the same outcome for every scoring rule AND majority rule!

BASIC PORTION

Example:

$$\begin{array}{ccc} 4 & 2 & 1 \\ \hline a & b & c \\ b & a & a \\ c & c & b \end{array}$$

Both, PR and APR give the same a $\succ$ b $\succ$ c ranking and hence all scoring rules give this ranking.

Also majority rule gives this ranking!

If we now add 6 Condorcet portions we get:

$$\begin{array}{ccc} 4 & 2 & 1 \\ \hline a & b & c \\ b & a & a \\ c & c & b \end{array} + 6\times \begin{array}{ccc} a & c & b \\ b & a & c \\ c & b & a \end{array} = \begin{array}{cccc} 10 & 2 & 7 & 6 \\ \hline a & b & c & b \\ b & a & a & c \\ c & c & b & a \end{array}$$

The scoring rule outcomes don't change but we now get the majority cycle a $\succ$ b $\succ$ c $\succ$ a.

Profile Decomposition

Now, add the following reversal portions:

5	5	2	2
b	c	a	c
a	a	b	b
c	b	c	a

This leads to the following new profile:

12	7	12	6	2
a	b	c	b	c
b	a	a	c	b
c	c	b	a	a

Now, the Borda ranking is still a $\succ$ b $\succ$ c , there is still a majority cycle, but the new Plurality ranking is c $\succ$ b $\succ$ a.

Profile Decomposition

Actually the real portions look as follows (and consider now w=(1,s,0)):

Basic Portion for "a"

1	1	0	−1	0	−1
a	a	b	b	c	c
b	c	a	c	a	b
c	b	c	a	b	a

For any scoring rule:

a receives 2 points

b and c receive 0 points

And SMR?

What do negative voters mean? We need them to create an orthogonal coordinate system, to separate their effects from other effects.

Basic Portion for "b"

0	−1	1	1	−1	0
a	a	b	b	c	c
b	c	a	c	a	b
c	b	c	a	b	a

Profile Decomposition

a-reversal portion

1	1	-2	1	-2	1
a	a	b	b	c	c
b	c	a	c	a	b
c	b	c	a	b	a

For any scoring rule:

a receives 2-4s points

b and c receive 2s-1 points

And SMR?

b-reversal portion

-2	1	1	1	1	-2
a	a	b	b	c	c
b	c	a	c	a	b
c	b	c	a	b	a

ALL possible differences in 3-alternatives elections are caused by reversal portions (Saari, 1999)

Profile Decomposition

Condorcet portion

1	−1	−1	1	1	−1
a	a	b	b	c	c
b	c	a	c	a	b
c	b	c	a	b	a

For any scoring rule:

a,b,c receive 0 points

For |X|=3 we have now all our coordinate directions, i.e. our 4 portions span the six-dimensional profile space.

- These profile coordinates account for every problem that might occur.
- Any other configuration of profiles that impacts on election outcomes must be a combination of these.
- However, other profile coordinate systems are possible.

So any profile can be represented by those portions

- E.g.: $p = 3B_a + 2B_b - 5R_a + 1R_b - 3C + 14K$

Representing profiles

- Let us think more about pairwise voting now
- Start with a profile

a	a	b	b	c	c
b	c	a	c	a	b
c	b	c	a	b	a

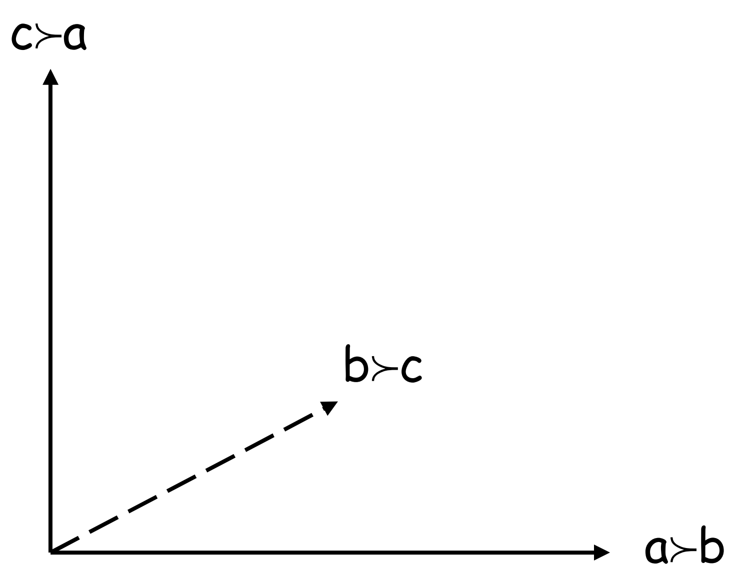
- E.g.: p = (32,0,10,22,20,16)
- And we could normalize it to (.32,0,.1,.22,.2,.16) by dividing through |N|
- With pairwise votes this maps into a 3 dimensional space
  - One dimension for each pair
  - Use the majority margins: $k_{xy} = |\{i \in N : xR_iy\}| - |\{i \in N : yR_ix\}|$ and normalize them

$$f : Si(m!) \to [-1,1]^{\binom{m}{2}}$$

Pairwise voting

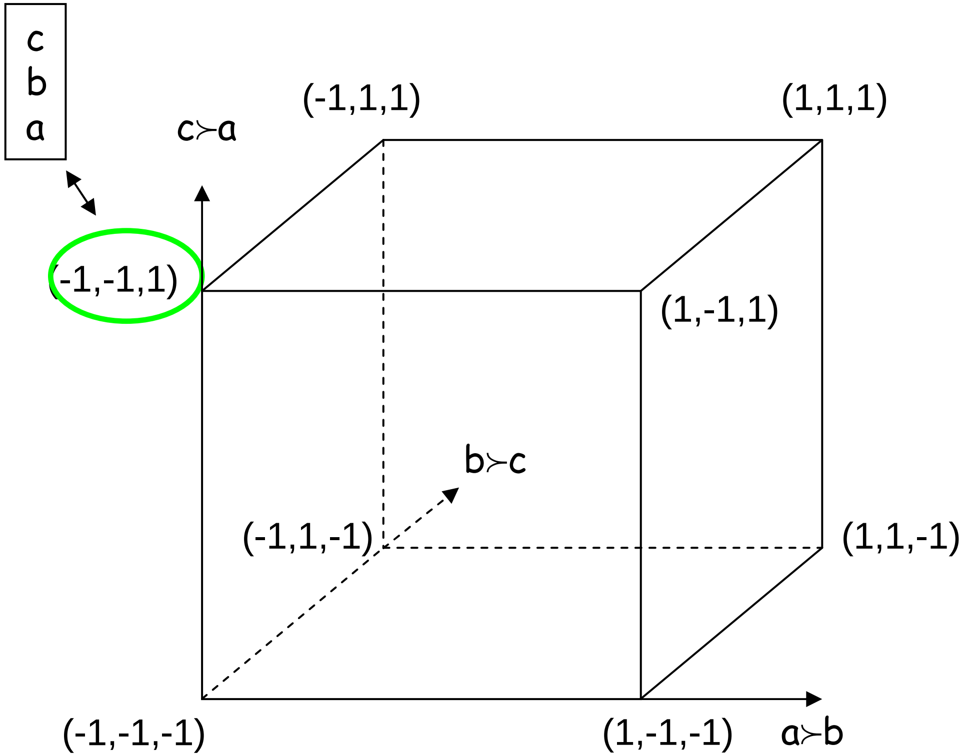
Consider set of alternatives: $X = \{a,b,c\}$

Family of pairwise comparisons: $\{a \succ b,\ b \succ c,\ c \succ a\}$



Any preference (outcome) can be represented by a point in this 3-dimensional space.

Representation cube

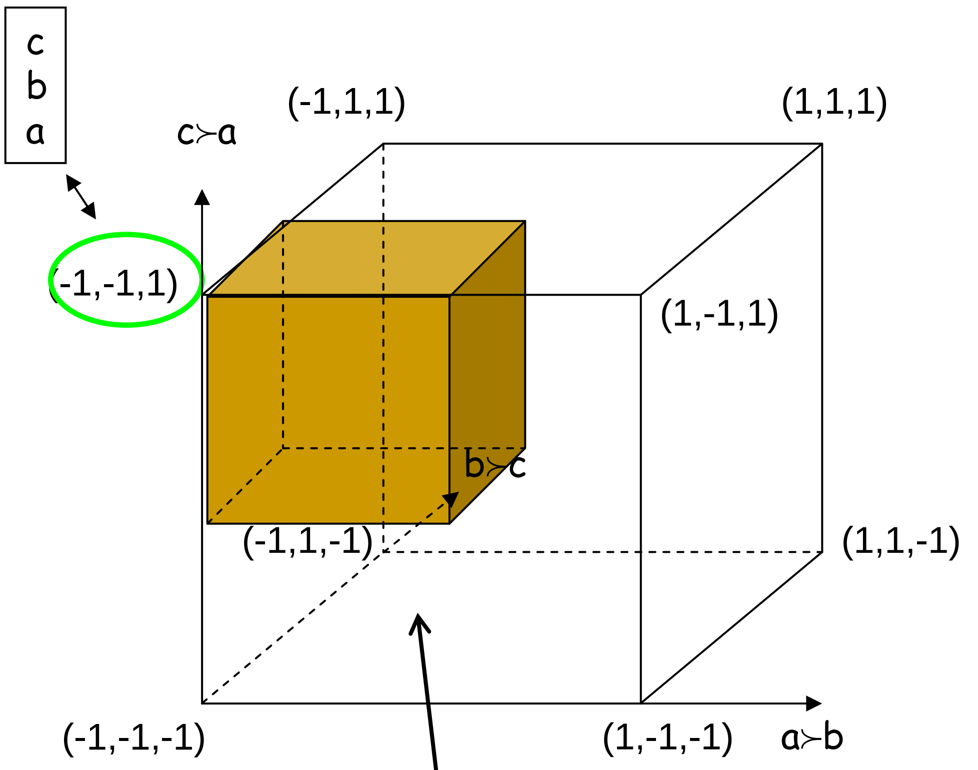


Family of pairwise comparisons: {a≻b, b≻c, c≻a}

Vectors representing cyclic voters are:

(1,1,1) and (-1,-1,-1)

Representation cube



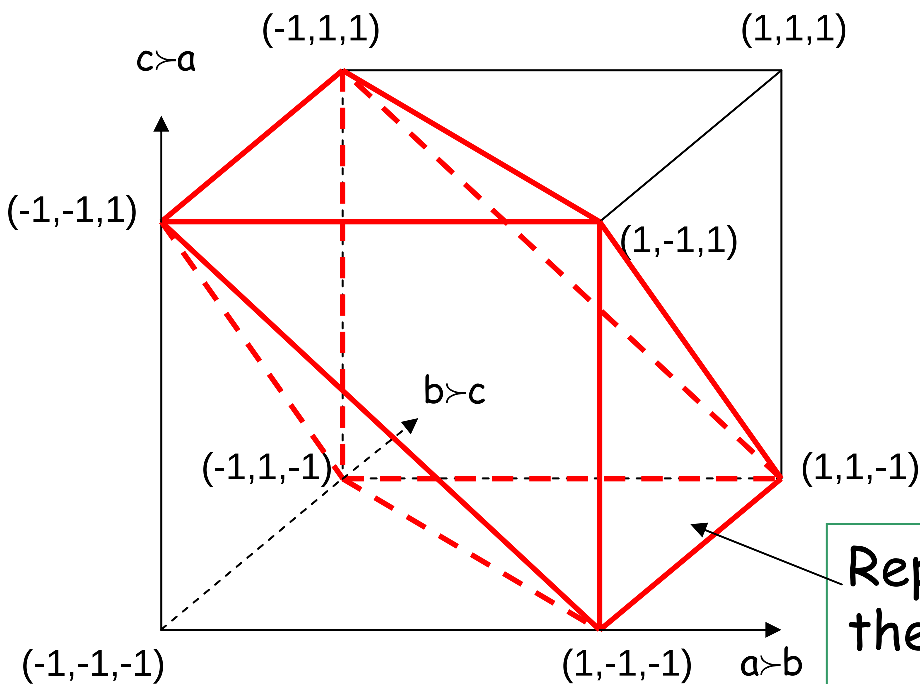
Majority subcube for vertex (-1,-1,1)

Its Euclidean distance from the vertex determines the majority outcome.

- Hence we have 8 subcubes
- Two of them lead to cycling SMR outcome

Can use this representation cube to prove Sen's theorem. How?

Representation cube



Family of pairwise valuations: {a≻b, b≻c, c≻a}

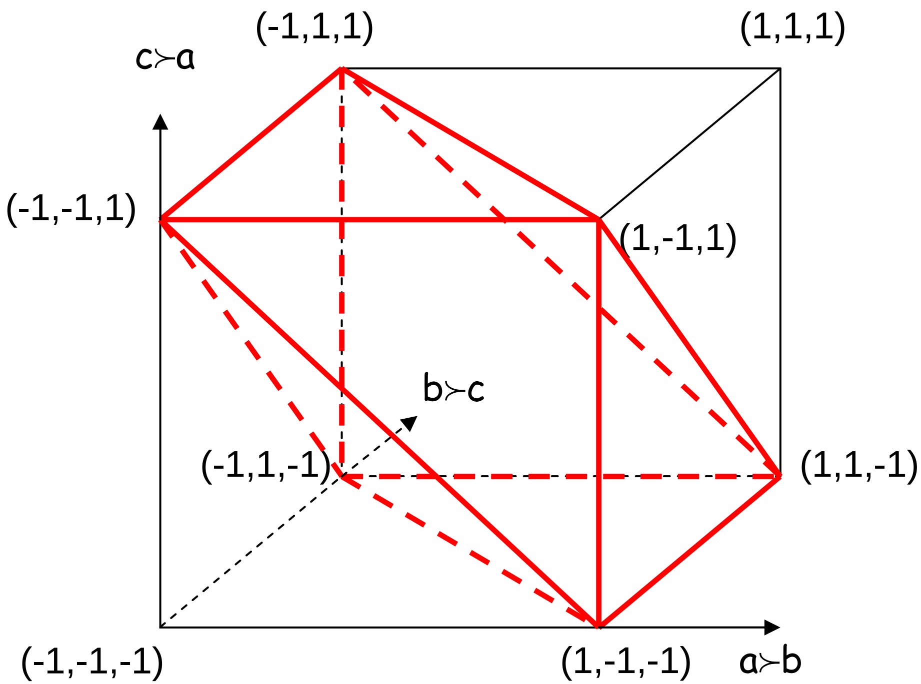
Vectors representing cyclic voters are:

(1,1,1) and (-1,-1,-1)

Representation polytope being the convex hull of all feasible vertices, which are all unanimity profiles.

Those are all the points a profile can be mapped into by SMR.

Representation cube



- As SMR outcome is the vertex closest to the point that the profile maps into, we see that it cuts through the two subcubes of cyclic outcomes

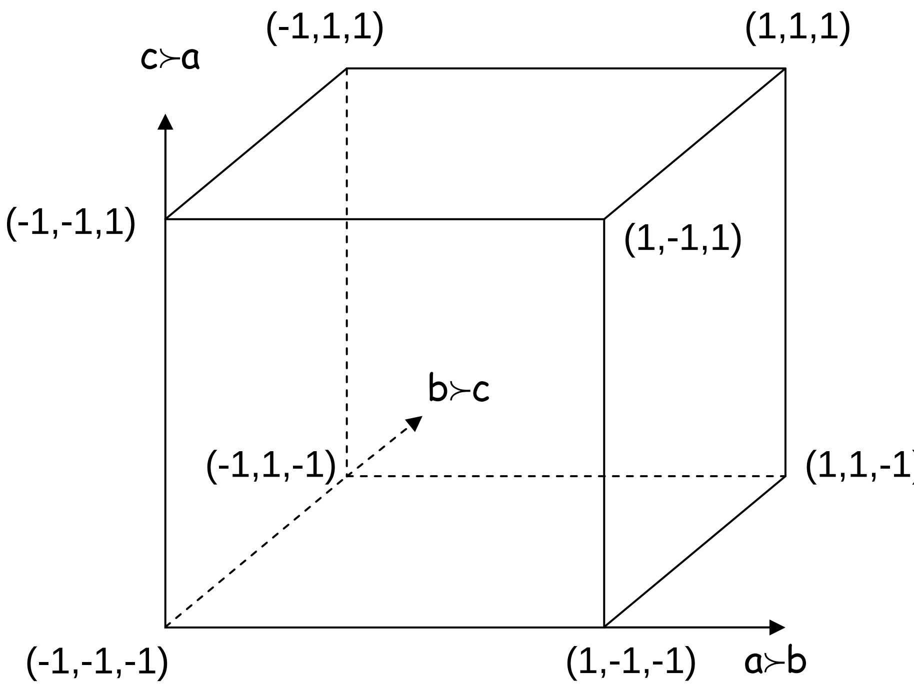
Reduced profiles

- Can also reduce the profile to see more clearly when problems occur with SMR
  - eliminate reversal portions

p_1	p_2	p_3	p_4	p_5	p_6
a	a	b	b	c	c
b	c	a	c	a	b
c	b	c	a	b	a

  - p = (10,12,3,8,6,5) can be reduced to what?
  - Do we get problems? Check the cube!

Representation cube

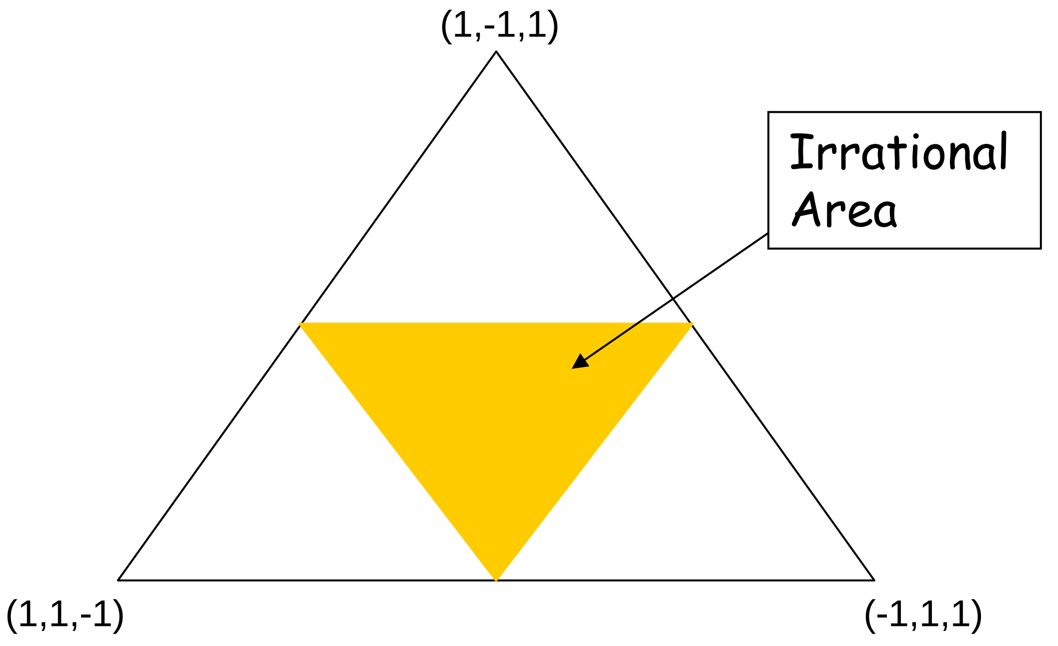


- And for p = (10,3,6,9,7,3)?
- Problems!

p_1	p_2	p_3	p_4	p_5	p_6
a	a	b	b	c	c
b	c	a	c	a	b
c	b	c	a	b	a

Profile Decomposition

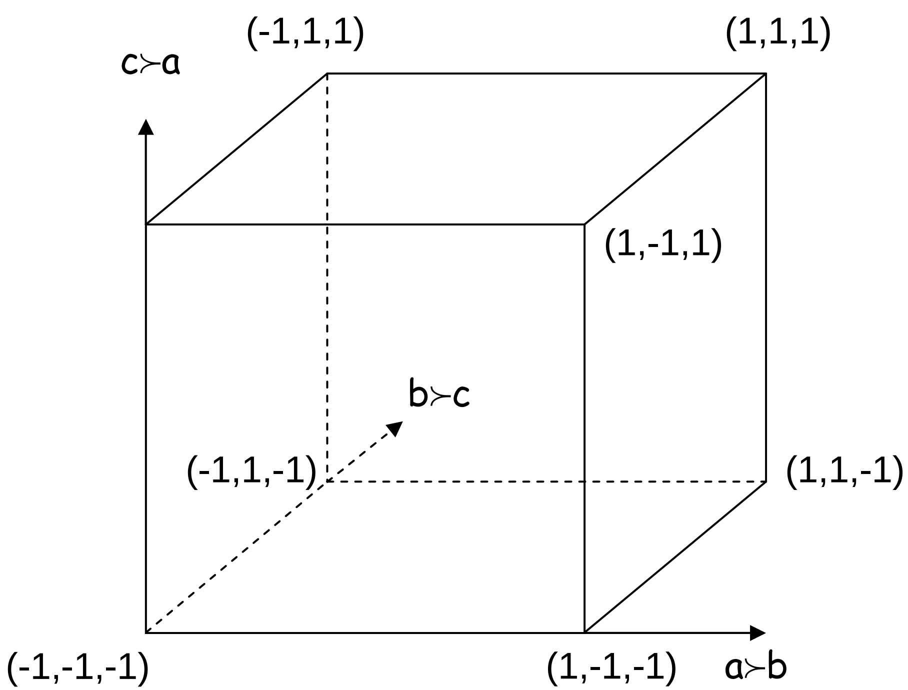
We can see the problems in the following triangle



In principle we can now create domain restrictions in the form of single-peakedness condition by Black to make sure that no profile can be plotted in any of those irrational areas.

Domain restrictions

- What does single-peakedness mean?
- Certain combinations of individual proferences are note allowed.
- What does this imply for our cube?



p_1	p_2	p_3	p_4	p_5	p_6
a	a	b	b	c	c
b	c	a	c	a	b
c	b	c	a	b	a

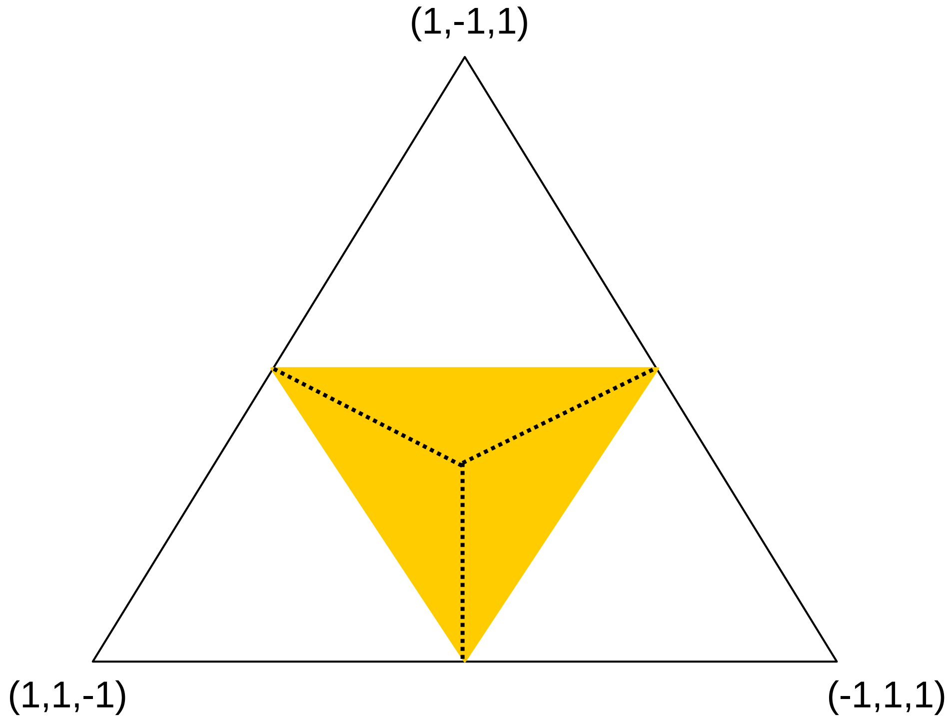
Distance Based Rules

Besides domain restrictions, there is an alternative way to guarantee collective rationality, via distance-based rules (Kemeny, 1959).

Determine the social outcome as the ranking that *minimizes* the distance to the individual rankings.

Distance Based Rules

Geometrically, this means dividing the yellow triangle into three areas based on their distance to the vertices.



This guarantees a consistent social outcome and is equivalent to switching the pair of alternatives which is closest to the 50-50 threshold (see Merlin and Saari, 2000).

Conclusion

- Geometry as a tool to understand voting results and differences in outcomes
- Saari triangles
- Profile decomposition
- Representation cubes
- Also useful for other aggregation frameworks such as in judgment aggregation
- Other geometric approaches possible (e.g. Zwicker, 2008)

Literature

Some interesting literature on this topic:

- Saari, D.G. (1995): Basic Geometry of Voting. Springer, Berlin.
- Saari, D.G. (1999): Explaining all three-alternative voting outcomes. Journal of Economic Theory, 87, 313-355.
- Saari, D.G. (2000): Mathematical structure of voting paradoxes I: Pairwise Vote. Economic Theory, 15, 1-53.
- Saari, D.G. (2000): Mathematical structure of voting paradoxes II: Positional Voting. Economic Theory, 15, 55-101.
- Saari, D.G. and V.R. Merlin (2000): A geometric examination of Kemeny's rule. Social Choice and Welfare, 17, 403-438.
- Saari, D.G. (2008): Disposing Dictators, Demystifying Voting Paradoxes. Cambridge University Press, New York.
- Zwicker, W.S. (2008): Consistency without neutrality in voting rules: when is a vote an average? Mathematical and Computer Modelling, 48, 1357-1373.