General Aspects of Social Choice Theory

Christian Klamler
University of Graz

10. April 2010

Overview

Overview

- Aggregation not only important for voting theory but also for welfare economics

Overview

- Aggregation not only important for voting theory but also for welfare economics
- Decision between different policies that have different impact on different people

Overview

- Aggregation not only important for voting theory but also for welfare economics
- Decision between different policies that have different impact on different people
- Some historical aspects

Overview

- Aggregation not only important for voting theory but also for welfare economics
- Decision between different policies that have different impact on different people
- Some historical aspects
  - Bentham - utilitarianism

Overview

- Aggregation not only important for voting theory but also for welfare economics
- Decision between different policies that have different impact on different people
- Some historical aspects
  - Bentham - utilitarianism
  - challenged in the 1930s

Overview

- Aggregation not only important for voting theory but also for welfare economics
- Decision between different policies that have different impact on different people
- Some historical aspects
  - Bentham - utilitarianism
  - challenged in the 1930s
  - ordinal vs. cardinal

Overview

Overview

- Main goal: Formal introduction to Social Choice Theory

Overview

- Main goal: Formal introduction to Social Choice Theory
- Elaborate the formal framework

Overview

- Main goal: Formal introduction to Social Choice Theory
- Elaborate the formal framework
- State and "prove" 3 most famous social choice results:

Overview

- Main goal: Formal introduction to Social Choice Theory
- Elaborate the formal framework
- State and "prove" 3 most famous social choice results:
    - Arrow's theorem - general aspects (1951)

Overview

- Main goal: Formal introduction to Social Choice Theory
- Elaborate the formal framework
- State and "prove" 3 most famous social choice results:
    - Arrow's theorem - general aspects (1951)
    - Sen's theorem - freedom aspects (1970)

Overview

- Main goal: Formal introduction to Social Choice Theory
- Elaborate the formal framework
- State and "prove" 3 most famous social choice results:
  - Arrow's theorem - general aspects (1951)
  - Sen's theorem - freedom aspects (1970)
  - Gibbard-Satterthwaite theorem - strategic aspects (1973/75)

Collective Decision Rule

Collective Decision Rule

- What are we doing when we look for a collective decision?

Collective Decision Rule

- What are we doing when we look for a collective decision?
- Use a function (collective decision rule) that assigns to any input of individual preferences a social outcome.

Collective Decision Rule

- What are we doing when we look for a collective decision?
- Use a function (collective decision rule) that assigns to any input of individual preferences a social outcome.
  - What is the input?

Collective Decision Rule

- What are we doing when we look for a collective decision?
- Use a function (collective decision rule) that assigns to any input of individual preferences a social outcome.
    - What is the input?
    - What is the output?

Collective Decision Rule

- What are we doing when we look for a collective decision?
- Use a function (collective decision rule) that assigns to any input of individual preferences a social outcome.
  - What is the input?
  - What is the output?
  - What does the collective decision rule look like?

What is the input?

What is the input?

- Finite set X of alternatives/candidates or social states with certain characteristics.

What is the input?

- Finite set X of alternatives/candidates or social states with certain characteristics.
- Finite set N of voters.

What is the input?

- Finite set X of alternatives/candidates or social states with certain characteristics.
- Finite set N of voters.
- Individual preferences over X by individual i are given as binary relation $R_i \subseteq X \times X$, and we write xR_iy to denote x at least as good as y in i's terms.

What is the input?

- Finite set X of alternatives/candidates or social states with certain characteristics.
- Finite set N of voters.
- Individual preferences over X by individual i are given as binary relation $R_i \subseteq X \times X$, and we write xR_iy to denote x at least as good as y in i's terms.
- Given R we can construct two related preferences P (strict preference) and I (indifference):

What is the input?

- Finite set X of alternatives/candidates or social states with certain characteristics.
- Finite set N of voters.
- Individual preferences over X by individual i are given as binary relation $R_i \subseteq X \times X$, and we write xR_iy to denote x at least as good as y in i's terms.
- Given R we can construct two related preferences P (strict preference) and I (indifference):
  - $xPy \Leftrightarrow xRy \wedge \neg yRx$

What is the input?

- Finite set X of alternatives/candidates or social states with certain characteristics.
- Finite set N of voters.
- Individual preferences over X by individual i are given as binary relation $R_i \subseteq X \times X$, and we write xR_iy to denote x at least as good as y in i's terms.
- Given R we can construct two related preferences P (strict preference) and I (indifference):
  - $xPy \Leftrightarrow xRy \wedge \neg yRx$
  - $xIy \Leftrightarrow xRy \wedge yRx$

What is the input?

- Finite set X of alternatives/candidates or social states with certain characteristics.
- Finite set N of voters.
- Individual preferences over X by individual i are given as binary relation $R_i \subseteq X \times X$, and we write xR_iy to denote x at least as good as y in i's terms.
- Given R we can construct two related preferences P (strict preference) and I (indifference):
  - $xPy \Leftrightarrow xRy \wedge \neg yRx$
  - $xIy \Leftrightarrow xRy \wedge yRx$

Definition

A binary relation R on X is

What is the input?

- Finite set X of alternatives/candidates or social states with certain characteristics.
- Finite set N of voters.
- Individual preferences over X by individual i are given as binary relation $R_i \subseteq X \times X$, and we write xR_iy to denote x at least as good as y in i's terms.
- Given R we can construct two related preferences P (strict preference) and I (indifference):
  - $xPy \Leftrightarrow xRy \wedge \neg yRx$
  - $xIy \Leftrightarrow xRy \wedge yRx$

Definition

A binary relation R on X is

- complete

What is the input?

- Finite set X of alternatives/candidates or social states with certain characteristics.
- Finite set N of voters.
- Individual preferences over X by individual i are given as binary relation $R_i \subseteq X \times X$, and we write xR_iy to denote x at least as good as y in i's terms.
- Given R we can construct two related preferences P (strict preference) and I (indifference):
  - $xPy \Leftrightarrow xRy \wedge \neg yRx$
  - $xIy \Leftrightarrow xRy \wedge yRx$

Definition

A binary relation R on X is

- complete if $\forall x, y \in X$, either xRy or yRx

What is the input?

- Finite set X of alternatives/candidates or social states with certain characteristics.
- Finite set N of voters.
- Individual preferences over X by individual i are given as binary relation $R_i \subseteq X \times X$, and we write xR_iy to denote x at least as good as y in i's terms.
- Given R we can construct two related preferences P (strict preference) and I (indifference):
  - $xPy \Leftrightarrow xRy \wedge \neg yRx$
  - $xIy \Leftrightarrow xRy \wedge yRx$

Definition

A binary relation R on X is

- complete if $\forall x, y \in X$, either xRy or yRx
- reflexive

What is the input?

- Finite set X

Preferences and Properties

Definition

A binary relation R on X is

Preferences and Properties

Definition

A binary relation R on X is

- transitive

Preferences and Properties

Definition

A binary relation R on X is

- transitive if $\forall x, y, z \in X$, xRy and yRz implies xRz

Preferences and Properties

Definition

A binary relation R on X is

- transitive if $\forall x, y, z \in X$, xRy and yRz implies xRz
- quasi-transitive

Preferences and Properties

Definition

A binary relation R on X is

- transitive if $\forall x, y, z \in X$, xRy and yRz implies xRz
- quasi-transitive if $\forall x, y, z \in X$, xPy and yPz implies xPz

Preferences and Properties

Definition

A binary relation R on X is

- transitive if $\forall x, y, z \in X$, xRy and yRz implies xRz
- quasi-transitive if $\forall x, y, z \in X$, xPy and yPz implies xPz
- acyclic

Preferences and Properties

Definition

A binary relation R on X is

- transitive if $\forall x, y, z \in X$, xRy and yRz implies xRz
- quasi-transitive if $\forall x, y, z \in X$, xPy and yPz implies xPz
- acyclic if $\forall x, y, z_1, ..., z_k \in X$, xPz_1, z_1Pz_2, ..., z_kPy implies xRy

Preferences and Properties

Definition

A binary relation R on X is

- transitive if $\forall x, y, z \in X$, xRy and yRz implies xRz
- quasi-transitive if $\forall x, y, z \in X$, xPy and yPz implies xPz
- acyclic if $\forall x, y, z_1, ..., z_k \in X$, xPz_1, z_1Pz_2, ..., z_kPy implies xRy

Definition

R is called a *weak order* if it is complete, reflexive and transitive.

Preferences and Properties

Definition

A binary relation R on X is

- transitive if $\forall x, y, z \in X$, xRy and yRz implies xRz
- quasi-transitive if $\forall x, y, z \in X$, xPy and yPz implies xPz
- acyclic if $\forall x, y, z_1, ..., z_k \in X$, xPz_1, z_1Pz_2, ..., z_kPy implies xRy

Definition

R is called a *weak order* if it is complete, reflexive and transitive.

Example

Let $X = \{x, y, z\}$ and xPy,yIz and xIz. What properties does this relation satisfy?

Preference profile

Definition

A preference profile is an n-tuple of weak orders $p = (R_1, ..., R_n)$.

Preference profile

Definition

A preference profile is an n-tuple of weak orders $p = (R_1, \ldots, R_n)$.

Usually in social choice theory we work with *linear orders*, i.e. strict rankings of the alternatives.

What is the output?

What is the output?

What is it that we want to get as social output?

What is the output?

What is it that we want to get as social output?
There are various possibilities:

What is the output?

What is it that we want to get as social output?
There are various possibilities:

- singletons from X

What is the output?

What is it that we want to get as social output?
There are various possibilities:

- singletons from X
- subsets from X

What is the output?

What is it that we want to get as social output?
There are various possibilities:

- singletons from X
- subsets from X
- binary relations on X

What is the output?

What is it that we want to get as social output?
There are various possibilities:

- singletons from X
- subsets from X
- binary relations on X
- choice functions on X

What is the output?

What is it that we want to get as social output?
There are various possibilities:

- singletons from X
- subsets from X
- binary relations on X
- choice functions on X

What is a choice function?

What is the output?

What is it that we want to get as social output?
There are various possibilities:

- singletons from X
- subsets from X
- binary relations on X
- choice functions on X

What is a choice function?

Definition (Choice function)

Let $\mathcal{X}$ be the set of all non-empty subsets of X. A choice function is a function $C : \mathcal{X} \to \mathcal{X}$ s.t. $\forall S \in \mathcal{X}$, $C(S) \subseteq S$.

Choice and preferences

Is there a relationship between choices and preferences?

Choice and preferences

Is there a relationship between choices and preferences?

Definition (Rationalizability)

A choice function C is rationalizable if there exists a preference R s.t. $\forall S \in \mathcal{X}$, $C(S) = \{x \in S : \forall y \in S, xRy\}$.

Choice and preferences

Is there a relationship between choices and preferences?

Definition (Rationalizability)

A choice function C is rationalizable if there exists a preference R s.t. $\forall S \in \mathcal{X}$, $C(S) = \{x \in S : \forall y \in S, xRy\}$.

Example

Which choice function is rationalized by xPy, yIz and xIz?

Choice and preferences

Is there a relationship between choices and preferences?

Definition (Rationalizability)

A choice function C is rationalizable if there exists a preference R s.t. $\forall S \in \mathcal{X}$, $C(S) = \{x \in S : \forall y \in S, xRy\}$.

Example

Which choice function is rationalized by xPy, yIz and xIz?

Is every choice function rationalizable by a preference R?

Choice and preferences

Is there a relationship between choices and preferences?

Definition (Rationalizability)

A choice function C is rationalizable if there exists a preference R s.t. $\forall S \in \mathcal{X}$, $C(S) = \{x \in S : \forall y \in S, xRy\}$.

Example

Which choice function is rationalized by xPy, yIz and xIz?

Is every choice function rationalizable by a preference R?

Example

Let $X = \{x, y, z\}$ and the choice function be s.t.
$C(\{x, y, z\} = C(\{x, y\}) = y$ and $C(\{x, z\}) = C(\{y, z\}) = z$.

Collective decision rules

Now the type of output determines what type of collective decision rule we consider.

Collective decision rules

Now the type of output determines what type of collective decision rule we consider.

Definition (Preference aggregation rule)

Let $\mathcal{B}$ denote the set of all complete and reflexive binary relations on X and $\mathcal{R} \subseteq \mathcal{B}$ the set of all weak orders.
A preference aggregation rule is a mapping $f : \mathcal{R}^n \to \mathcal{B}$

Collective decision rules

Now the type of output determines what type of collective decision rule we consider.

Definition (Preference aggregation rule)

Let $\mathcal{B}$ denote the set of all complete and reflexive binary relations on X and $\mathcal{R} \subseteq \mathcal{B}$ the set of all weak orders.
A preference aggregation rule is a mapping $f : \mathcal{R}^n \to \mathcal{B}$

Other types of collective decision rules:

Collective decision rules

Now the type of output determines what type of collective decision rule we consider.

Definition (Preference aggregation rule)

Let $\mathcal{B}$ denote the set of all complete and reflexive binary relations on X and $\mathcal{R} \subseteq \mathcal{B}$ the set of all weak orders.
A preference aggregation rule is a mapping $f : \mathcal{R}^n \to \mathcal{B}$

Other types of collective decision rules:

- Social Welfare Function: $f : \mathcal{R}^n \to \mathcal{R}$

Collective decision rules

Now the type of output determines what type of collective decision rule we consider.

Definition (Preference aggregation rule)

Let $\mathcal{B}$ denote the set of all complete and reflexive binary relations on X and $\mathcal{R} \subseteq \mathcal{B}$ the set of all weak orders.
A preference aggregation rule is a mapping $f : \mathcal{R}^n \to \mathcal{B}$

Other types of collective decision rules:

- Social Welfare Function: $f : \mathcal{R}^n \to \mathcal{R}$
- Social Decision Function: $f : \mathcal{R}^n \to \mathcal{A}$, where $\mathcal{A}$ is the set of all complete, reflexive and acyclic binary relations on X.

Collective decision rules

Now the type of output determines what type of collective decision rule we consider.

Definition (Preference aggregation rule)

Let $\mathcal{B}$ denote the set of all complete and reflexive binary relations on X and $\mathcal{R} \subseteq \mathcal{B}$ the set of all weak orders.

Examples of collective decision rules

Examples of collective decision rules

Example

$f : \mathcal{R}^n \to \mathcal{B}$ is called simple majority rule if $\forall p \in \mathcal{R}^n$ and all $x, y \in X$, xRy if and only if $|\{i \in N : xR_iy\}| \geq |\{i \in N : yR_ix\}|$.

Examples of collective decision rules

Example

$f : \mathcal{R}^n \to \mathcal{B}$ is called simple majority rule if $\forall p \in \mathcal{R}^n$ and all $x, y \in X$, xRy if and only if $|\{i \in N : xR_i y\}| \geq |\{i \in N : yR_i x\}|$.

For the following example let for all $B \in \mathcal{B}$ and $S \in \mathcal{X}$, $M(S, R) = \{x \in S | \nexists y \in S : yPx\}$.

Examples of collective decision rules

Example

$f : \mathcal{R}^n \to \mathcal{B}$ is called simple majority rule if $\forall p \in \mathcal{R}^n$ and all $x, y \in X$, xRy if and only if $|\{i \in N : xR_i y\}| \geq |\{i \in N : yR_i x\}|$.

For the following example let for all $B \in \mathcal{B}$ and $S \in \mathcal{X}$, $M(S, R) = \{x \in S | \nexists y \in S : yPx\}$.
Also, let for all $B \in \mathcal{B}$, B^* denote its transitive closure, i.e. xB^*y if and only if there exists a sequence $z_1, z_2, ..., z_k \in X$ s.t. xBz_1, z_1Bz_2, ... ,z_kBy.

Examples of collective decision rules

Example

$f : \mathcal{R}^n \to \mathcal{B}$ is called simple majority rule if $\forall p \in \mathcal{R}^n$ and all $x, y \in X$, xRy if and only if $|\{i \in N : xR_i y\}| \geq |\{i \in N : yR_i x\}|$.

For the following example let for all $B \in \mathcal{B}$ and $S \in \mathcal{X}$, $M(S, R) = \{x \in S | \nexists y \in S : yPx\}$.
Also, let for all $B \in \mathcal{B}$, B^* denote its transitive closure, i.e. xB^*y if and only if there exists a sequence $z_1, z_2, ..., z_k \in X$ s.t. xBz_1, $z_1 Bz_2$, ... ,$z_k By$.

Example

The transitive closure rule assigns to all $p \in \mathcal{R}^n$ a choice function on X s.t. $\forall S \in \mathcal{X}$, $C(S) = M(S, B^*)$, where B^* is the transitive closure of the simple majority relation B for p.

Examples of collective decision rules

Example

$f : \mathcal{R}^n \to \mathcal{B}$ is called simple majority rule if $\forall p \in \mathcal{R}^n$ and all $x, y \in X$, xRy if and only if $|\{i \in N : xR_iy\}| \geq |\{i \in N : yR_ix\}|$.

For the following example let for all $B \in \mathcal{B}$ and $S \in \mathcal{X}$, $M(S, R) = \{x \in S | \nexists y \in S : yPx\}$.
Also, let for all $B \in \mathcal{B}$, B^* denote its transitive closure, i.e. xB^*y if and only if there exists a sequence $z_1, z_2, ..., z_k \in X$ s.t. xBz_1, z_1Bz_2, ... ,z_kBy.

Example

The transitive closure rule assigns to all $p \in \mathcal{R}^n$ a choice function on X s.t. $\forall S \in \mathcal{X}$, $C(S) = M(S, B^*)$, where B^* is the transitive closure of the simple majority relation B for p.
Let $X = \{x, y, z\}$, what does the transitive closure rule give for the Condorcet paradox?

Properties of social welfare functions

Properties of social welfare functions

Definition (Unrestricted Domain)

The domain of f includes all logically possible n-tuples of individual weak orders over X.

Properties of social welfare functions

Definition (Unrestricted Domain)

The domain of f includes all logically possible n-tuples of individual weak orders over X.

Definition (Weak Pareto)

For all $p \in \mathcal{R}^n$ and all $x, y \in X$; $\forall i \in N$, xP_iy implies xPy.

Properties of social welfare functions

Definition (Unrestricted Domain)

The domain of f includes all logically possible n-tuples of individual weak orders over X.

Definition (Weak Pareto)

For all $p \in \mathcal{R}^n$ and all $x, y \in X$; $\forall i \in N$, xP_iy implies xPy.

Definition (Independence of Irrelevant Alternatives)

For all $p, p' \in \mathcal{R}^n$ and all $x, y \in X$; $\forall i \in N$, $xR_iy \Leftrightarrow xR'_iy$ implies $xRy \Leftrightarrow xR'y$.

Properties of social welfare functions

Definition (Unrestricted Domain)

The domain of f includes all logically possible n-tuples of individual weak orders over X.

Definition (Weak Pareto)

For all $p \in \mathcal{R}^n$ and all $x, y \in X$; $\forall i \in N$, xP_iy implies xPy.

Definition (Independence of Irrelevant Alternatives)

For all $p, p' \in \mathcal{R}^n$ and all $x, y \in X$; $\forall i \in N$, $xR_iy \Leftrightarrow xR'_iy$ implies $xRy \Leftrightarrow xR'y$.

Which social welfare functions satisfy those three conditions?

Properties of social welfare functions

Definition (Unrestricted Domain)

The domain of f includes all logically possible n-tuples of individual weak orders over X.

Definition (Weak Pareto)

For all $p \in \mathcal{R}^n$ and all $x, y \in X$; $\forall i \in N$, $xP_i y$ implies xPy.

Definition (Independence of Irrelevant Alternatives)

For all $p, p' \in \mathcal{R}^n$ and all $x, y \in X$; $\forall i \in N$, $xR_i y \Leftrightarrow xR'_i y$ implies $xRy \Leftrightarrow xR'y$.

Which social welfare functions satisfy those three conditions?
Dictatorship

Arrow's impossibility theorem

Arrow's impossibility theorem

Definition (Nondictatorship)

$\nexists i \in N$ s.t. $\forall p \in \mathcal{R}^n$ and $x, y \in X$, xP_iy implies xPy.

Theorem (Arrow's theorem)

Let $|N| \geq 2$ and $|X| \geq 3$. There exists no SWF that satisfies UD, WP, IIA and ND.

Rules and those properties

Before proving Arrow's theorem, which of the properties do certain rules violate?

Rules and those properties

Before proving Arrow's theorem, which of the properties do certain rules violate?

- Dictatorship

Rules and those properties

Before proving Arrow's theorem, which of the properties do certain rules violate?

- Dictatorship satisfies UD, WP, IIA

Rules and those properties

Before proving Arrow's theorem, which of the properties do certain rules violate?

- Dictatorship satisfies UD, WP, IIA but violates ND

Rules and those properties

Before proving Arrow's theorem, which of the properties do certain rules violate?

- Dictatorship satisfies UD, WP, IIA but violates ND
- constant rule

Rules and those properties

Before proving Arrow's theorem, which of the properties do certain rules violate?

- Dictatorship satisfies UD, WP, IIA but violates ND
- constant rule satisfies UD, IIA, ND

Rules and those properties

Before proving Arrow's theorem, which of the properties do certain rules violate?

- Dictatorship satisfies UD, WP, IIA but violates ND
- constant rule satisfies UD, IIA, ND but violates WP

Rules and those properties

Before proving Arrow's theorem, which of the properties do certain rules violate?

- Dictatorship satisfies UD, WP, IIA but violates ND
- constant rule satisfies UD, IIA, ND but violates WP
- Borda rule

Rules and those properties

Before proving Arrow's theorem, which of the properties do certain rules violate?

- Dictatorship satisfies UD, WP, IIA but violates ND
- constant rule satisfies UD, IIA, ND but violates WP
- Borda rule satisfies UD, WP, ND

Rules and those properties

Before proving Arrow's theorem, which of the properties do certain rules violate?

- Dictatorship satisfies UD, WP, IIA but violates ND
- constant rule satisfies UD, IIA, ND but violates WP
- Borda rule satisfies UD, WP, ND but violates IIA

Rules and those properties

Before proving Arrow's theorem, which of the properties do certain rules violate?

- Dictatorship satisfies UD, WP, IIA but violates ND
- constant rule satisfies UD, IIA, ND but violates WP
- Borda rule satisfies UD, WP, ND but violates IIA

Example (Violation of IIA by Borda rule)

R_1	R_2	R_3	R'_1	R'_2	R'_3
a	d	d	a	d	d
c	c	c	b	c	c
b	a	a	d	a	a
d	b	b	c	b	b

Proof of Arrow's theorem

For the proof we need the following definitions:

Proof of Arrow's theorem

For the proof we need the following definitions:

Definition (Decisiveness)

$G \subseteq N$ is decisive over the ordered pair $\{x, y\}$, $\bar{D}_G(x, y)$ iff xP_iy, $\forall i \in G$ implies xPy.

Proof of Arrow's theorem

For the proof we need the following definitions:

Definition (Decisiveness)

$G \subseteq N$ is decisive over the ordered pair $\{x, y\}$, $\bar{D}_G(x, y)$ iff xP_iy, $\forall i \in G$ implies xPy.

Definition (Almost decisiveness)

$G \subseteq N$ is almost decisive over ordered pair $\{x, y\}$, $D_G(x, y)$ iff xP_iy, $\forall i \in G$ and yP_ix, $\forall i \in N \backslash G$ implies xPy.

Two lemmata (Sen)

Two lemmata (Sen)

The proof of Arrow's theorem is achieved in different forms. One is via the following two lemmata:

Two lemmata (Sen)

The proof of Arrow's theorem is achieved in different forms. One is via the following two lemmata:

Lemma (Field expansion lemma)

For any SWF satisfying UD, WP and IIA and $|X| \geq 3$*, if a group* G *is almost decisive over some ordered pair* $\{x, y\}$*, then it is decisive over every ordered pair, i.e.*
$[\exists x, y \in X : D_G(x, y)] \Rightarrow \left[\forall a, b \in X : \bar{D}_G(a, b)\right]$

Two lemmata (Sen)

The proof of Arrow's theorem is achieved in different forms. One is via the following two lemmata:

Lemma (Field expansion lemma)

For any SWF satisfying UD, WP and IIA and $|X| \geq 3$, if a group G is almost decisive over some ordered pair $\{x, y\}$, then it is decisive over every ordered pair, i.e.
$[\exists x, y \in X : D_G(x, y)] \Rightarrow \left[\forall a, b \in X : \bar{D}_G(a, b)\right]$

Lemma (Group contraction lemma)

For any SWF satisfying UD, WP and IIA and $|X| \geq 3$, if any group G with $|G| > 1$ is decisive, then so is some proper subgroup of G.

Field expansion lemma

Consider $X = \{x, y, a, b\}$ and the following profile where $D_G(x, y)$:

$i \in G$	$rest(k \notin G)$
a	aP_kx
x	yP_kb
y	yP_kx
b	

Field expansion lemma

Consider $X = \{x, y, a, b\}$ and the following profile where $D_G(x, y)$:

$i \in G$	$rest(k \notin G)$
a	$aP_k x$
x	$yP_k b$
y	$yP_k x$
b	

- aPx and yPb

Field expansion lemma

Consider $X = \{x, y, a, b\}$ and the following profile where $D_G(x, y)$:

$i \in G$	$rest(k \notin G)$
a	aP_kx
x	yP_kb
y	yP_kx
b	

- aPx and yPb because of WP

Field expansion lemma

Consider $X = \{x, y, a, b\}$ and the following profile where $D_G(x, y)$:

$i \in G$	$rest(k \notin G)$
a	aP_kx
x	yP_kb
y	yP_kx
b	

- aPx and yPb because of WP
- xPy

Field expansion lemma

Consider $X = \{x, y, a, b\}$ and the following profile where $D_G(x, y)$:

$i \in G$	$rest(k \notin G)$
a	aP_kx
x	yP_kb
y	yP_kx
b	

- aPx and yPb because of WP
- xPy because of $D_G(x, y)$

Field expansion lemma

Consider $X = \{x, y, a, b\}$ and the following profile where $D_G(x, y)$:

$i \in G$	$rest(k \notin G)$
a	$aP_k x$
x	$yP_k b$
y	$yP_k x$
b	

- aPx and yPb because of WP
- xPy because of $D_G(x, y)$
- aPb

Field expansion lemma

Consider $X = \{x, y, a, b\}$ and the following profile where $D_G(x, y)$:

$i \in G$	$rest(k \notin G)$
a	aP_kx
x	yP_kb
y	yP_kx
b	

- aPx and yPb because of WP
- xPy because of $D_G(x, y)$
- aPb because of (quasi) transitivity of f

Field expansion lemma

Consider $X = \{x, y, a, b\}$ and the following profile where $D_G(x, y)$:

$i \in G$	$rest(k \notin G)$
a	$aP_k x$
x	$yP_k b$
y	$yP_k x$
b	

- aPx and yPb because of WP
- xPy because of $D_G(x, y)$
- aPb because of (quasi) transitivity of f
- by IIA this only depends on orderings of a and b

Field expansion lemma

Consider $X = \{x, y, a, b\}$ and the following profile where $D_G(x, y)$:

$i \in G$	$rest(k \notin G)$
a	aP_kx
x	yP_kb
y	yP_kx
b	

- aPx and yPb because of WP
- xPy because of $D_G(x, y)$
- aPb because of (quasi) transitivity of f
- by IIA this only depends on orderings of a and b of which only those in group G have been specified

Field expansion lemma

Consider $X = \{x, y, a, b\}$ and the following profile where $D_G(x, y)$:

$i \in G$	$rest(k \notin G)$
a	aP_kx
x	yP_kb
y	yP_kx
b	

- aPx and yPb because of WP
- xPy because of $D_G(x, y)$
- aPb because of (quasi) transitivity of f
- by IIA this only depends on orderings of a and b of which only those in group G have been specified
- Hence: $\bar{D}_G(a, b)$

Group contraction lemma

Partition G into G_1 and G_2

Group contraction lemma

Partition G into G_1 and G_2

G_1	G_2	$rest(k \notin G)$
x	y	z
y	z	x
z	x	y

Group contraction lemma

Partition G into G_1 and G_2

G_1	G_2	$rest(k \notin G)$
x	y	z
y	z	x
z	x	y

- yPz

Group contraction lemma

Partition G into G_1 and G_2

G_1	G_2	$rest(k \notin G)$
x	y	z
y	z	x
z	x	y

- yPz by decisiveness of G

Group contraction lemma

Partition G into G_1 and G_2

G_1	G_2	$rest(k \notin G)$
x	y	z
y	z	x
z	x	y

- yPz by decisiveness of G
- xPz or zRx by completeness of R

Group contraction lemma

Partition G into G_1 and G_2

G_1	G_2	$rest(k \notin G)$
x	y	z
y	z	x
z	x	y

- yPz by decisiveness of G
- xPz or zRx by completeness of R
- xPz or yPx

Group contraction lemma

Partition G into G_1 and G_2

G_1	G_2	$rest(k \notin G)$
x	y	z
y	z	x
z	x	y

- yPz by decisiveness of G
- xPz or zRx by completeness of R
- xPz or yPx by yPz and transitivity of R

Group contraction lemma

Partition G into G_1 and G_2

G_1	G_2	$rest(k \notin G)$
x	y	z
y	z	x
z	x	y

- yPz by decisiveness of G
- xPz or zRx by completeness of R
- xPz or yPx by yPz and transitivity of R
- hence either G_1 is almost decisive over $\{x, z\}$

Group contraction lemma

Partition G into G_1 and G_2

G_1	G_2	$rest(k \notin G)$
x	y	z
y	z	x
z	x	y

- yPz by decisiveness of G
- xPz or zRx by completeness of R
- xPz or yPx by yPz and transitivity of R
- hence either G_1 is almost decisive over $\{x, z\}$ or G_2 is almost decisive over $\{y, x\}$

Group contraction lemma

Partition G into G_1 and G_2

G_1	G_2	$rest(k \notin G)$
x	y	z
y	z	x
z	x	y

- yPz by decisiveness of G
- xPz or zRx by completeness of R
- xPz or yPx by yPz and transitivity of R
- hence either G_1 is almost decisive over $\{x, z\}$ or G_2 is almost decisive over $\{y, x\}$
- from field expansion lemma either G_1 or G_2 is decisive

Proof of Arrow's theorem

Proof.

Proof of Arrow's theorem

Proof.

- WP and field expansion lemma implies that N is decisive

Proof of Arrow's theorem

Proof.

- WP and field expansion lemma implies that N is decisive
- by the group contraction lemma we can eliminate members of N until we are left with a dictator.

$\square$

Proofs and resolutions

Proofs and resolutions

- Other proof techniques have been used by e.g. Saari or Austen-Smith and Banks.

Proofs and resolutions

- Other proof techniques have been used by e.g. Saari or Austen-Smith and Banks.
-

Proofs and resolutions

- Other proof techniques have been used by e.g. Saari or Austen-Smith and Banks.
- Ways to overcome the negative results?
  - Domain restrictions (single-peaked preferences)

Proofs and resolutions

- Other proof techniques have been used by e.g. Saari or Austen-Smith and Banks.
- Ways to overcome the negative results?
  - Domain restrictions (single-peaked preferences)
  - Relaxing the consistency conditions of the social outcome to quasi-transitivity or acyclicity.

Proofs and resolutions

- Other proof techniques have been used by e.g. Saari or Austen-Smith and Banks.
- Ways to overcome the negative results?
  - Domain restrictions (single-peaked preferences)
  - Relaxing the consistency conditions of the social outcome to quasi-transitivity or acyclicity.
  - Use of broader informational basis, i.e. interpersonal comparisons

Proofs and resolutions

- Other proof techniques have been used by e.g. Saari or Austen-Smith and Banks.
- Ways to overcome the negative results?
  - Domain restrictions (single-peaked preferences)
  - Relaxing the consistency conditions of the social outcome to quasi-transitivity or acyclicity.
  - Use of broader informational basis, i.e. interpersonal comparisons
- but many resolutions lead to other "dictator-like" results with veto rights or oligarchies

Sen's Liberal Paradox

Sen's Liberal Paradox

We have not considered any aspects of choices among alternatives that lie in one's private domain.

Sen's Liberal Paradox

We have not considered any aspects of choices among alternatives that lie in one's private domain.

> *[Sen, 1970] If you prefer to have pink walls rather then white, the society should permit you to have this even if a majoritiy of the community would like to see your walls white.*

Sen's liberal paradox

Let $f : \mathcal{R}^n \rightarrow \mathcal{A}$ be a social decision function and consider the following property:

Sen's liberal paradox

Let $f : \mathcal{R}^n \to \mathcal{A}$ be a social decision function and consider the following property:

Definition (Minimal Liberalism)

There exist at least 2 individuals s.t. each of them is decisive over at least one pair of alternatives, i.e. if i is decisive over (x, y), then $xP_iy \Rightarrow xPy$.

Sen's liberal paradox

Let $f : \mathcal{R}^n \to \mathcal{A}$ be a social decision function and consider the following property:

Definition (Minimal Liberalism)

There exist at least 2 individuals s.t. each of them is decisive over at least one pair of alternatives, i.e. if i is decisive over (x, y), then $xP_iy \Rightarrow xPy$.

Theorem (Sen, 1970)

There exists no social decision function satisfying UD, WP and ML.

Proof

Proof

Proof.

Let $X = \{x, y, z\}$ and $i, j \in N$ be such that $\bar{D}_i(x, y)$ and $\bar{D}_j(x, z)$. The preferences are considered as follows:

Proof

Proof.

Let $X = \{x, y, z\}$ and $i, j \in N$ be such that $\bar{D}_i(x, y)$ and $\bar{D}_j(x, z)$. The preferences are considered as follows:

R_i	R_j	$rest(k \neq i, j)$
x	y	yP_kz
y	z	
z	x	

Proof

Proof.

Let $X = \{x, y, z\}$ and $i, j \in N$ be such that $\bar{D}_i(x, y)$ and $\bar{D}_j(x, z)$. The preferences are considered as follows:

R_i	R_j	$rest(k \neq i, j)$
x	y	yP_kz
y	z	
z	x	

- xPy because of ML of i

Proof

Proof.

Let $X = \{x, y, z\}$ and $i, j \in N$ be such that $\bar{D}_i(x, y)$ and $\bar{D}_j(x, z)$. The preferences are considered as follows:

R_i	R_j	$rest(k \neq i, j)$
x	y	yP_kz
y	z	
z	x	

- xPy because of ML of i
- yPz because of WP

Proof

Proof.

Let $X = \{x, y, z\}$ and $i, j \in N$ be such that $\bar{D}_i(x, y)$ and $\bar{D}_j(x, z)$. The preferences are considered as follows:

R_i	R_j	$rest(k \neq i, j)$
x	y	yP_kz
y	z	
z	x	

- xPy because of ML of i
- yPz because of WP
- zPx because of ML of j

Proof

Proof.

Let $X = \{x, y, z\}$ and $i, j \in N$ be such that $\bar{D}_i(x, y)$ and $\bar{D}_j(x, z)$. The preferences are considered as follows:

R_i	R_j	$rest(k \neq i, j)$
x	y	yP_kz
y	z	
z	x	

- xPy because of ML of i
- yPz because of WP
- zPx because of ML of j
- Leads to a cycle!

$\square$

Relevance

Relevance

- liberal values conflict with the Pareto principle in a basic sense

Relevance

- liberal values conflict with the Pareto principle in a basic sense
- Compared to Arrow's theorem

Relevance

- liberal values conflict with the Pareto principle in a basic sense
- Compared to Arrow's theorem
  - it also works if we just consider the possibility of choices, i.e. acyclic social preferences

Relevance

- liberal values conflict with the Pareto principle in a basic sense
- Compared to Arrow's theorem
    - it also works if we just consider the possibility of choices, i.e. acyclic social preferences
    - it does not use the rather criticized IIA condition

Relevance

- liberal values conflict with the Pareto principle in a basic sense
- Compared to Arrow's theorem
  - it also works if we just consider the possibility of choices, i.e. acyclic social preferences
  - it does not use the rather criticized IIA condition
  - there is no satisfactory resolution via a broadening of the informational basis through interpersonal comparisons

Strategic aspects in voting

Strategic aspects in voting

Strategic aspects in voting have been known for a long time:

Strategic aspects in voting

Strategic aspects in voting have been known for a long time:

My scheme is only intended for honest men! [Borda]

Strategic aspects in voting

Strategic aspects in voting have been known for a long time:

> *My scheme is only intended for honest men! [Borda]*
>
> *Voters adopt a principle of voting which makes it more of a game of skill than a real test of the wishes of the electors. [Dodgson]*

Strategic aspects in voting

Strategic aspects in voting have been known for a long time:

My scheme is only intended for honest men! [Borda]

Voters adopt a principle of voting which makes it more of a game of skill than a real test of the wishes of the electors. [Dodgson]

Politicians are continually poking and pushing the world to get the results they want. The reason they do this is they believe (and rightly so) that they can change outcomes by their efforts. It is often the case that voting need not have turned out the way it did. [Riker]

Manipulability

Manipulability

Let $p = (R_1, ..., R_n) \in \mathcal{R}^n$ and let (p_{-i}, p_i') denote the profile $p' = (R_1, ..., R_i^{'}, ..., R_n)$. Now:

Manipulability

Let $p = (R_1, ..., R_n) \in \mathcal{R}^n$ and let (p_{-i}, p_i') denote the profile $p' = (R_1, ..., R_i', ..., R_n)$. Now:

Definition (Manipulability)

Social choice rule $f : \mathcal{R}^n \to X$ is manipulable by i at profile p via R_i' if $f(p')P_i f(p)$.

Manipulability

Let $p = (R_1, ..., R_n) \in \mathcal{R}^n$ and let (p_{-i}, p'_i) denote the profile $p' = (R_1, ..., R'_i, ..., R_n)$. Now:

Definition (Manipulability)

Social choice rule $f : \mathcal{R}^n \to X$ is manipulable by i at profile p via R'_i if $f(p')P_i f(p)$.

Theorem (Gibbard-Satterthwaite)

Let $|N| \geq 2$ and $|X| \geq 3$. If f is non-manipulable and satisfies WP, it is a dictatorship.

Conclusion

Conclusion

We have discussed 3 of the major impossibility results in Social Choice Theory

Conclusion

We have discussed 3 of the major impossibility results in Social Choice Theory

- There is an inconsistency between basic reasonable properties. [Arrow]

Conclusion

We have discussed 3 of the major impossibility results in Social Choice Theory

- There is an inconsistency between basic reasonable properties. [Arrow]
- There is an inconsistency between basic liberal aspects and the Pareto principle. [Sen]

Conclusion

We have discussed 3 of the major impossibility results in Social Choice Theory

- There is an inconsistency between basic reasonable properties. [Arrow]
- There is an inconsistency between basic liberal aspects and the Pareto principle. [Sen]
- There is an inconsistency between basic strategic aspects and the Pareto principle. [Gibbard-Satterthwaite]

Literature

Literature

Some basic literature:

Literature

Some basic literature:

- Arrow, K.J. (1963): Social choice and Individual Values (2nd ed.). Yale University Press, New Haven.
- Sen, A.K. (1970): Impossibility of a Paretian Liberal. *Journal of Political Economy*, **78**, 152–157.
- Gibbard, A. (1973): Manipulation of voting schemes: a general result. *Econometrica*, **41**, 587–601.
- Satterthwaite, M. (1975): Strategy-proofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions. *Journal of Economic Theory*, **10**, 187–217.
- Sen, A.K. (1970): Collective Choice and Social Welfare. North Holland, Amsterdam.

Literature

- Riker, W.H. (1982): Liberalsism against populism. W.H. Freeman and Company.
- Sen, A.K. (1986): Social choice theory. in: Arrow, K.J. and M.D. Intriligator, Handbook of Mathematical Economics. Elsevier.
- Austen-Smith, D. and J.S. Banks (2000): Positive Political Theory I. University of Michigan Press, Ann Arbor.
- Taylor, A.D. (2005): Social choice and the mathematics of manipulation. Cambridge University Press, New York.