# The Approximability of Dodgson Elections

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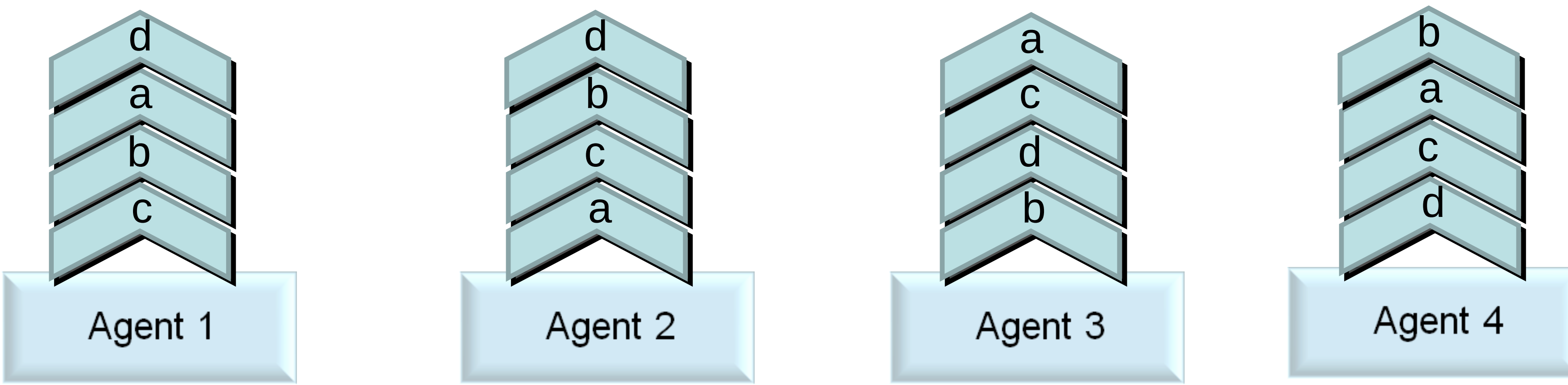
# Voting rules

- Input: Agents' preferences (preference profile)



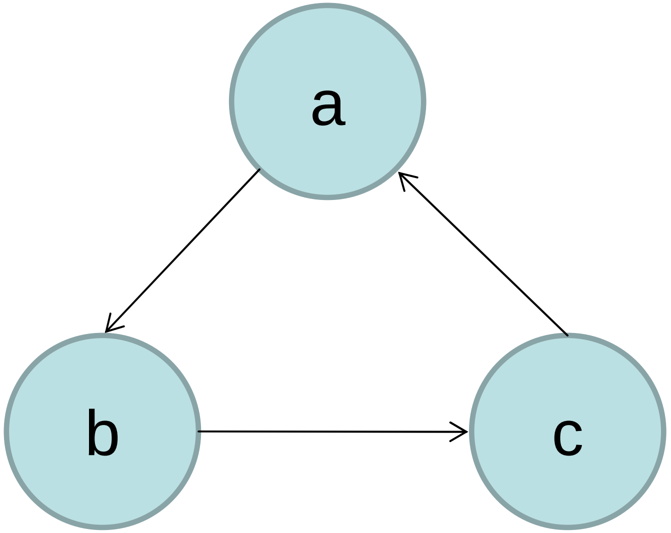

- Output: Winner(s) of the election or a ranking of the alternatives

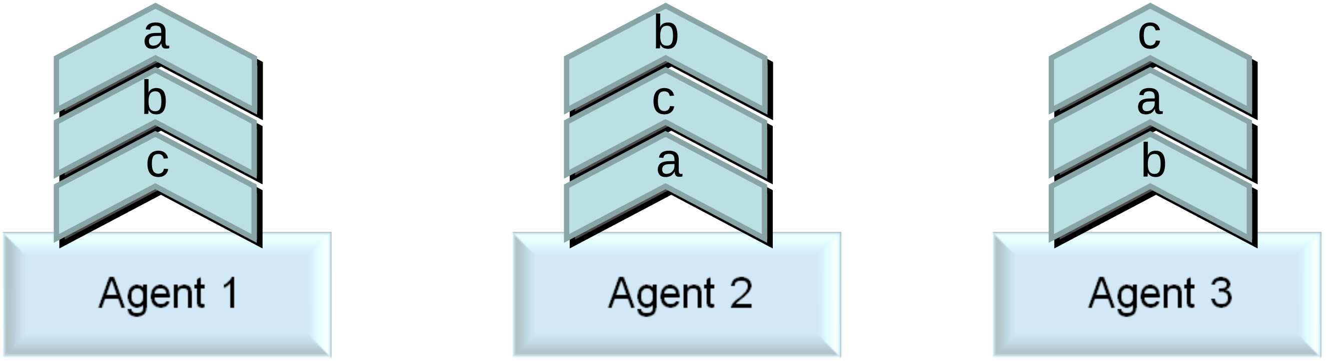
# Condorcet criterion

- Alternative x beats y in a pairwise election if the majority of agents prefers x to y
- Alternative x is a Condorcet winner if x beats any other alternative in a pairwise election
- Condorcet paradox: A Condorcet winner may not exist
- Choose an alternative as close as possible to a Condorcet winner according to some proximity measure
  - Dodgson's rule

# Condorcet paradox

- a beats b
- b beats c
- c beats a

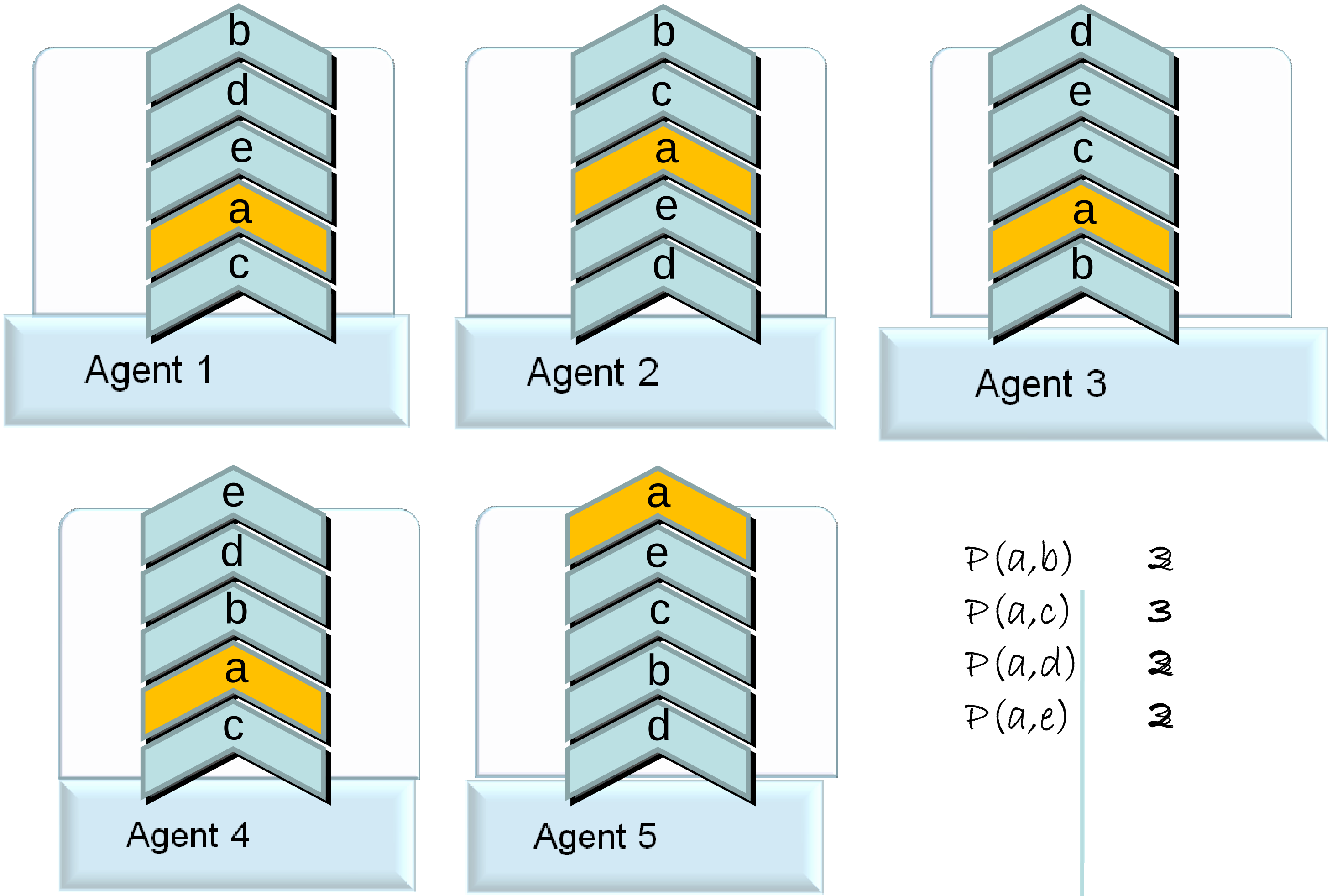


# Dodgson's method

- Dodgson score of x:
  - the minimum number of exchanges between adjacent alternatives needed to make x a Condorcet winner
- Dodgson ranking:
  - the alternatives are ranked in non-decreasing order of their Dodgson score
- Dodgson winner:
  - an alternative with the minimum Dodgson score

# An example of Dodgson




$P(a,b)$ 2
$P(a,c)$ 3
$P(a,d)$ 2
$P(a,e)$ 2

# Related combinatorial problems

- Dodgson score:
  - Given a preference profile, a particular alternative x, and an integer K, is the Dodgson score of x at most K?
  - NP-complete : Bartholdi, Tovey, and Trick (Social Choice & Welfare, 1989)
- Dodgson winner:
  - Given a preference profile and a particular alternative x, is x a Dodgson winner?
  - NP-hard: Bartholdi, Tovey, and Trick (Social Choice & Welfare, 1989) and Hemaspaandra, Hemaspaandra, and Rothe (J. ACM, 1997)
- Dodgson ranking:
  - Given a preference profile, compute a Dodgson ranking

# Approximation algorithms

- Can we approximate the Dodgson score and ranking?
- i.e., is there an algorithm which, on input a preference profile and a particular alternative x, computes a score which is at most a multiplicative factor away the Dodgson score of x?
- A $\rho$-approximation algorithm guarantees that Dodgson score of x $\leq$ score returned by the algorithm for x $\leq \rho$ times Dodgson score of x
- An approximation algorithm naturally defines an alternative voting rule

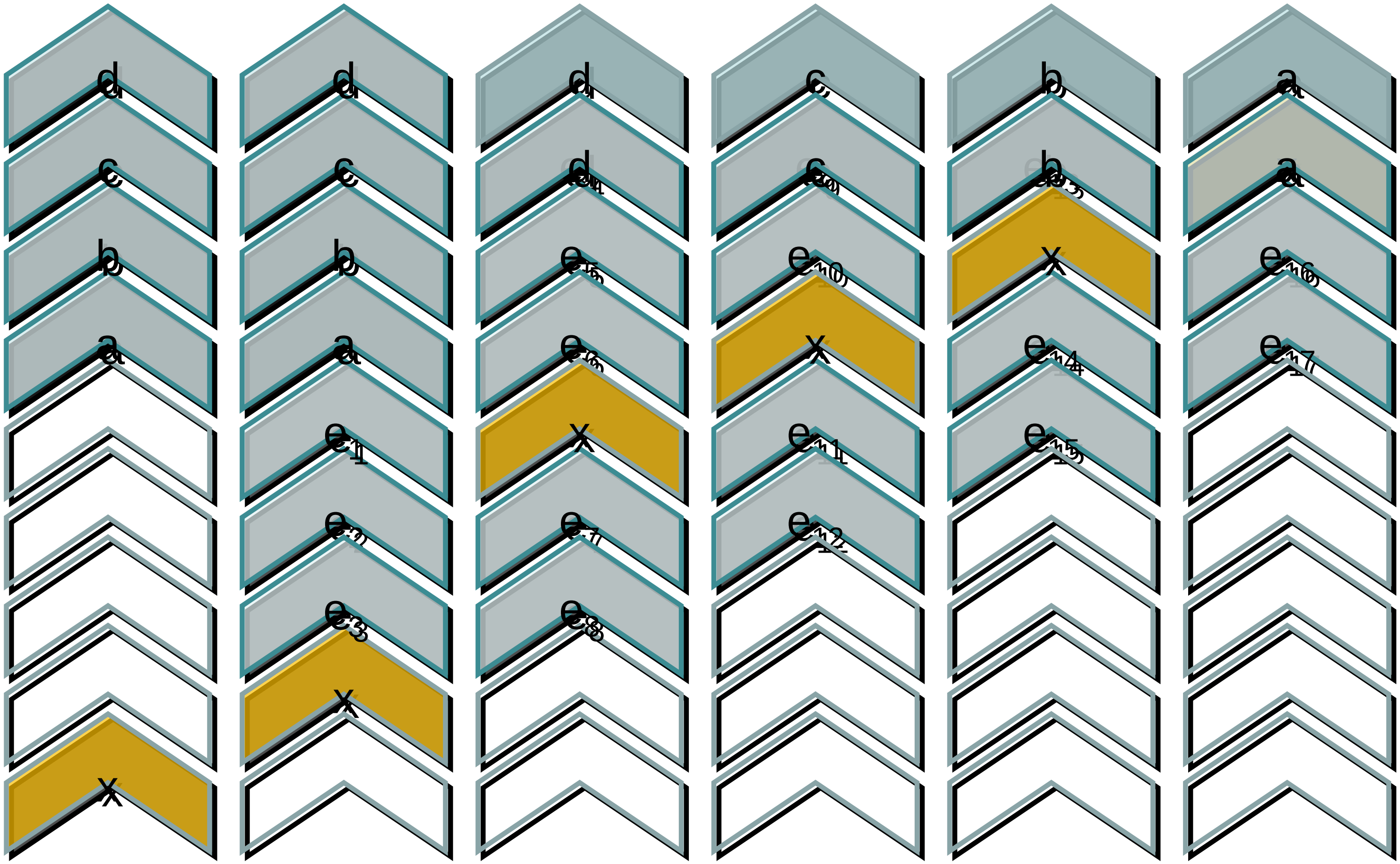
# Our results

- Approximation of Dodgson’s rule
  - Greedy algorithm
  - An algorithm based on linear programming
- Inapproximability results for the Dodgson ranking and Dodgson score

# The greedy algorithm

- Input:
  - A preference profile and a specific alternative x
- Notions:
  - def(x,c) = number of additional agents that must rank x above c in order for x to beat c in a pairwise election
  - c is alive iff def(x,c)>0, otherwise dead
  - Cost-effectiveness of pushing alternative x upwards at the preference of an agent = ratio between the number of alive alternatives overtaken by x over number of positions pushed
- Greedy algorithm: While there are alive alternatives, perform the most cost-effective push

# The greedy algorithm: an example



# Greedy algorithm performance

- **Theorem :** The greedy algorithm has approximation ratio at most $H_{m-1}$
- The proof uses the equivalent ILP for the computation of Dodgson score and its LP relaxation as analysis tools

# ILP for Dodgson score

- Variables $y_{ij}$:
  - 1 if agent i pushes x j positions, 0 otherwise
- Constants $a_{ij}^{c}$ :
  - 1 if pushing x j positions in agent i gives x an additional vote against c, 0 otherwise

$$
\begin{array}{lll}
\text{minimize} & \sum_{i,j} j \cdot y_{ij} & \\
\text{subject to:} & \sum_{j} y_{ij} = 1 & \forall i \\
& \sum_{i,j} a_{ij}^{c} \cdot y_{ij} \geq \mathrm{def}(x,c) & \forall c \neq x \\
& y_{ij} \in \{0,1\} & \forall i, j
\end{array}
$$

# LP relaxation for Dodgson score

- Variables $y_{ij}$ are fractional
- Constants $a_{ij}^{c}$:
  - 1 if pushing x j positions in agent i gives x an additional vote against c, 0 otherwise

$$
\begin{array}{lll}
\text{minimize} & \sum_{i,j} j \cdot y_{ij} & \\
\text{subject to:} & \sum_{j} y_{ij} = 1 & \forall i \\
& \sum_{i,j} a_{ij}^{c} \cdot y_{ij} \geq \mathrm{def}(x,c) & \forall c \neq x \\
& 0 \leq y_{ij} \leq 1 & \forall i, j
\end{array}
$$

# Greedy algorithm performance

- **Theorem:** The greedy algorithm has approximation ratio at most $H_{m-1}$
- The proof uses the equivalent ILP for the computation of Dodgson score and its LP relaxation as analysis tools
  - We know that LP $\leq$ ILP = Dodgson score
  - We use a technique known as dual fitting to show that the score computed by the algorithm is upper bounded by the solution of LP times $H_{m-1}$
  - This means that the greedy algorithm approximates the Dodgson score within $H_{m-1}$

# An LP-based algorithm

- Solve the LP and multiply its solution by $H_{m-1}$
- **Theorem:** The LP-based algorithm computes an $H_{m-1}$ – approximation of the Dodgson score
- Why?
  - We know that LP ≤ Dodgson score ≤ LP $H_{m-1}$
  - Hence, Dodgson score ≤ LP $H_{m-1}$ ≤ Dodgson score times $H_{m-1}$

# Inapproximability of Dodgson's ranking

- **Theorem:** It is NP-hard to decide whether a given alternative is a Dodgson winner or in the last $6\sqrt{m}$ positions in the Dodgson ranking
  - The proof uses a reduction from vertex cover in 3-regular graphs
- Complexity-theoretic explanation of sharp discrepancies observed in the Social Choice literature when comparing Dodgson voting rule to other (polynomial-time computable) voting rules (e.g., Copeland or Borda)
  - Klamer (Math. Social Sciences, 2004)
  - Ratliff (Economic Theory, 2002)

# Inapproximability of Dodgson's score

- **Theorem:** No polynomial-time algorithm can approximate the Dodgson score of a particular alternative within (1/2-ε) lnm unless problems in NP have superpolynomial-time algorithms
  - The proof uses a reduction from Set Cover

# A socially desirable property

- A voting rule is weakly monotonic if pushing an alternative upwards in the preferences of some agents cannot worsen its score
- Greedy is not weakly monotonic
- The LP-based algorithm is weakly monotonic

# More socially desirable approximations for Dodgson

- In the forthcoming paper:
  - Caragiannis, Kaklamanis, K, & Procaccia (EC 10)

# Thank you!