Towards a Dichotomy for the POSSIBLE WINNER Problem in Elections Based on Scoring Rules

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joint work with

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Motivation

Typical voting scenario for joint decision making:

Voters give preferences over a set of candidates as linear orders.

Example: candidates: $C = \{a, b, c, d\}$

profile:

vote 1:	a	>	b	>	c	>	d
vote 2:	a	>	d	>	c	>	b
vote 3:	b	>	d	>	c	>	a

Aggregate preferences according to a voting rule

Kind of voting rules considered in this work: **Scoring rules**

Scoring rules

Preferences as linear orders, scoring rules. Reminder:
Examples:

- plurality: $(1, 0, \dots, 0)$
- 2-approval: $(1, 1, 0, \dots, 0)$
- veto: $(1, \dots, 1, 0)$
- Borda: $(m-1, m-2, \dots, 0)$ (m = number of candidates)
- Formula 1 scoring: $(25, 18, 15, 12, 10, 8, 6, 4, 2, 1, 0, \dots, 0)$

Scoring rules

m candidates: scoring vector $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m$ and $\alpha_m = 0$

Scoring rule

provides a scoring vector for every number of candidates.

- non-trivial: $\alpha_1 \neq 0$
- pure: the scoring vector for i candidates can be obtained from the scoring vector for $i-1$ candidates by inserting an additional score value at an arbitrary position

Example:

3 candidates:	$(6,3,0)$
4 candidates:	pure: $(6,3,3,0)$, $(6,5,3,0)$, $(8,6,3,0)$, $\ldots$
	not pure: $(6,6,0,0)$, $(6,3,2,1)$, $\ldots$

Partial information

Recall: In the typical model, votes need to be presented as linear orders.

Realistic settings: voters may only provide partial information.

For example:

- not all voters have given their preferences yet
- new candidates are introduced
- a voter cannot compare several candidates because of lack of information/because he doesn't want to

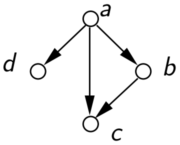
How to deal with partial information?

We consider the question if a distinguished candidate can still win.

Partial vote

A **partial vote** is a transitive and antisymmetric relation.

Example: $C = \{a, b, c, d\}$
partial vote: $a \succ b \succ c, a \succ d$



possible **extensions**:

1. $a > d > b > c$
2. $a > b > d > c$
3. $a > b > c > d$

An extension of a profile of partial votes extends every partial vote.

Computational Problem

Possible Winner

Input: A voting rule r, a set of candidates C, a profile of partial votes, and a distinguished candidate c.
Question: Is there an extension profile where c wins according to r?

Known results for scoring rules

Two studied scenarios for POSSIBLE WINNER:

1. weighted voters:
   NP-completeness for all scoring rules except plurality (holds even for a constant number of candidates)
   (follows by dichotomy for the special case of MANIPULATION [HEMASPAANDRA AND HEMASPAANDRA, JCSS 2007])
2. unweighted voters:
   a) constant number of candidates: always polynomial time [CONITZER, SANDHOLM, AND LANG, JACM 2007]
   b) unbounded number of candidates:

Known results for scoring rules

- unweighted voters

  b) unbounded number of candidates:

  - NP-complete for scoring rules that fulfill the following:
    [Xia and Conitzer, AAAI 2008]
    there is a position b with

    $$\alpha_b - \alpha_{b+1} = \alpha_{b+1} - \alpha_{b+2} = \alpha_{b+2} - \alpha_{b+3}$$

    and

    $$\alpha_{b+3} > \alpha_{b+4}$$

    Examples: $(\ldots, 6, 5, 4, 3, 0, \ldots)$, $(\ldots, 17, 14, 11, 8, 7, \ldots)$
  - Parameterized complexity study for some scoring rules:
    [Betzler, Hemmann, and Niedermeier, IJCAI 2009]
    k-approval is NP-hard for two partial votes when k is part of the input

Main Theorem

Theorem

For non-trivial pure scoring rules, POSSIBLE WINNER is

- polynomial-time solvable for plurality and veto,
- open for $(2, 1, \dots, 1, 0)$, and
- NP-complete for all other cases.

Recently,the case $(2, 1, \dots, 1, 0)$ has been shown to be NP-complete as well! [BAUMEISTER, ROTHE, 2010]

Examples for new results:

- 2-approval: $(1, 1, 0, \dots)$
- voting systems in which one can specify a small group of favorites and a small group of disliked candidates, like $(2, 2, 2, 1, \dots, 1, 0, 0)$ or $(3, 1, \dots, 1, 0)$

Plurality

Example: $C = \{a, b, c, d\}$, distinguished candidate c

$v_1 : a \succ c \succ d, b \succ c$

$v_2 : c \succ a \succ b$

$v_3 : a \succ d \succ b$

$v_4 : a \succ b \succ c$

$v_5 : a \succ c, b \succ d$

Plurality

Example: $C = \{a, b, c, d\}$, distinguished candidate c

$v_1 : a \succ c \succ d, b \succ c$

$v_2 : c \succ a \succ b$ $\Rightarrow c > a > b > d$

$v_3 : a \succ d \succ b$ $\Rightarrow c > a > d > b$

$v_4 : a \succ b \succ c$

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Step I: Maximize score of c

Plurality

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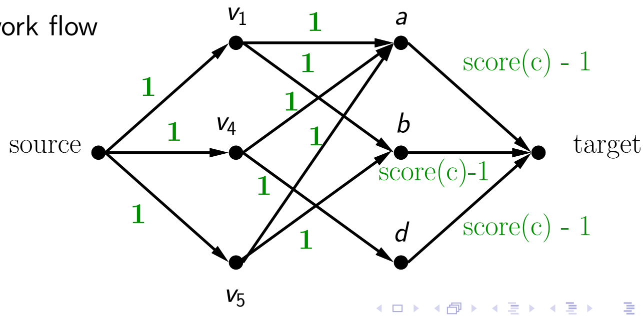
Step I: Maximize score of c

Step II: Network flow



Plurality

Example: $C = \{a, b, c, d\}$, distinguished candidate c

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$v_4 : a \succ b \succ c \quad \Rightarrow d > a > b > c$

$v_5 : a \succ c, b \succ d \quad \Rightarrow b > a > c > d$

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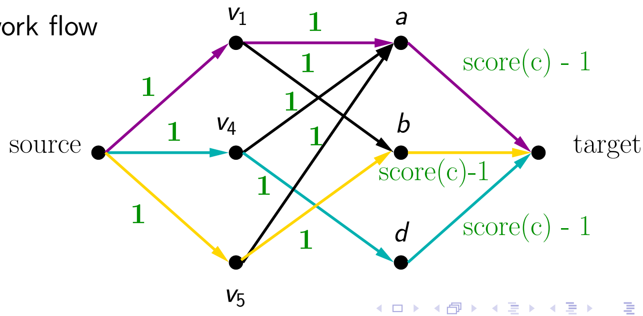

What about non-pure scoring rules?

Theorem

For non-trivial pure scoring rules, POSSIBLE WINNER is

- polynomial-time solvable for plurality and veto,
- open for $(2, 1, \ldots, 1, 0)$, and
- NP-complete for all other cases.

Problem: scoring rules which have “easy” scoring vectors for nearly all number of candidates and still “hard” scoring vectors for some unbounded numbers of candidates

Property of pure scoring rules: can never go back to an easy vector
Examples: $(1, 0, 0)$, $(1, 1, 0, 0) \rightarrow$ not $(1, 0, 0, 0, 0)$ or $(1, 1, 1, 1, 0)$
$(1, 1, 1, 0)$, $(2, 1, 1, 1, 0), \ldots$

Open questions

- How to compare candidates in partial votes?
  Counting version: In how many extensions does a distinguished candidate win?
- NP-complete problems: Find approximation/exact exponential algorithm
- Parameter number of candidates: combinatorial algorithm?