A Computational Analysis of Minimal Unidirectional Covering Sets

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Outline

Solution Concepts

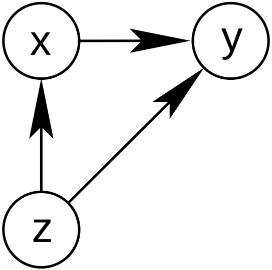
Binary dominance relations

Identify the "most desirable" elements in a pairwise majority relation:

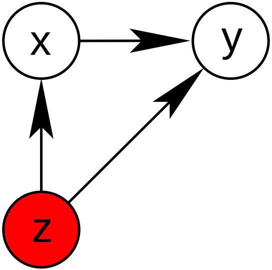
- game theory
- social choice theory
- argumentation theory
- sports tournaments
- ...

Natural concept: Choose the maximal element.

Example



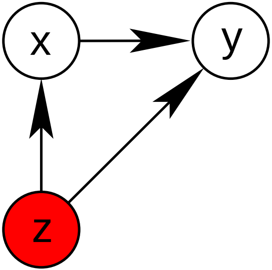
Example

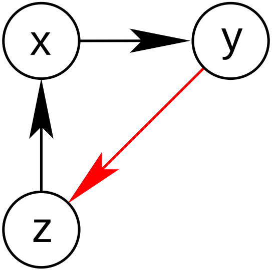


Maximal element

z is the winner.

Example





Maximal element

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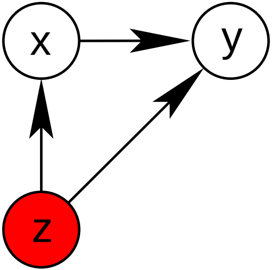
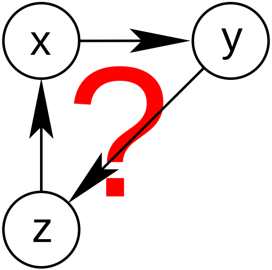
Example





Maximal element

z is the winner.

Maximal element

There is no winner!

Condorcet's Paradox renders maximality useless
$\Rightarrow$ solution concepts

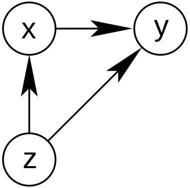
Solution Concept: Minimal Unidirectional Covering Sets

Unidirectional Covering

Let A be a finite set of alternatives, $B \subseteq A$, $\succ \subseteq A \times A$ a dominance relation, and let $x, y \in B$.

- x upward covers y (xC_uy) if $x \succ y$ and for all $z \in B$, $z \succ x$ implies $z \succ y$.

xC_uy, zC_ux, and zC_uy



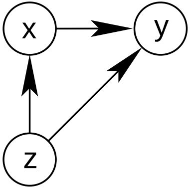
Solution Concept: Minimal Unidirectional Covering Sets

Unidirectional Covering

Let A be a finite set of alternatives, $B \subseteq A$, $\succ \subseteq A \times A$ a dominance relation, and let $x, y \in B$.

- x **upward covers** y (xC_uy) if $x \succ y$ and for all $z \in B$, $z \succ x$ implies $z \succ y$.
- x **downward covers** y (xC_dy) if $x \succ y$ and for all $z \in B$, $y \succ z$ implies $x \succ z$.

xC_uy, zC_ux, and zC_uy
zC_dx, zC_dy, and xC_dy



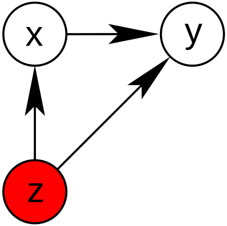
Solution Concept: Minimal Unidirectional Covering Sets

Uncovered Set

Let A be a finite set of alternatives, $B \subseteq A$, $\succ \subseteq A \times A$ a dominance relation, and let C be a covering relation on A. The uncovered set of B with respect to C is:

$$UC_C(B) = \{x \in B \mid yCx \text{ for no } y \in B\}.$$

$$UC_u(\{x, y, z\}) = \{z\}$$
$$UC_d(\{x, y, z\}) = \{z\}$$



Solution Concept: Minimal Unidirectional Covering Sets

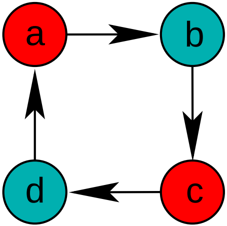
Minimal Covering Set

Let A be a finite set of alternatives, $\succ \subseteq A \times A$ a dominance relation, and C a covering relation. $B \subseteq A$ is a covering set for A under C, if:

- $UC_C(B) = B$ (internal stability), and
- for all $x \in A - B, x \notin UC_C(B \cup \{x\})$ (external stability).

Such a B is minimal if no $B' \subset B$ is a covering set for A under C.

Minimal upward covering sets:
$B_1 = \{a, c\}$ and $B_2 = \{b, d\}$
Minimal downward covering set:
$B_3 = \{a, b, c, d\}$



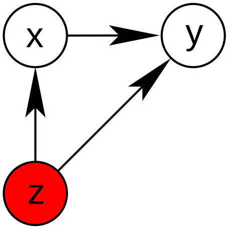
Minimal Upward Covering Set Member

Definition

Name: Minimal Upward Covering Set Member (MC_u-Member).
Instance: A set A of alternatives, a dominance relation $\succ$ on A, and a distinguished element $d \in A$.
Question: Is d contained in some minimal upward covering set for A?

$$\begin{aligned} A &= \{x, y, z\} \\ \succ &= \{(z, x), (z, y), (x, y)\} \end{aligned}$$

$$\begin{aligned} (A, \succ, z) &\in MC_u\text{-Member} \\ (A, \succ, x) &\notin MC_u\text{-Member} \end{aligned}$$



Unidirectional Covering Set Problems

- MC_{u}-Size: Given a set A of alternatives, a dominance relation $\succ$ on A, and a positive integer k, does there exist some minimal upward covering set for A containing at most k alternatives?
- MC_{u}-Member-All: Given a set A of alternatives, a dominance relation $\succ$ on A, and a distinguished element $d \in A$, is d contained in all minimal upward covering sets for A?
- MC_{u}-Unique: Given a set A of alternatives and a dominance relation $\succ$ on A, does there exist a unique minimal upward covering set for A?
- MC_{u}-Test: Given a set A of alternatives, a dominance relation $\succ$ on A, and a subset $M \subseteq A$, is M a minimal upward covering set for A?
- MC_{u}-Find: Given a set A of alternatives and a dominance relation $\succ$ on A, find a minimal upward covering set for A.

Minimality versus Minimum Size

Set-inclusion Minimality versus Minimum Cardinality

- **cardinality**: classical problems (maximum-size independent set, minimum-size dominating set, etc.)
- **set inclusion**: minimal upward covering set member.

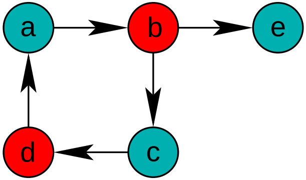
$\Rightarrow$ Standard techniques are not directly applicable.

Upward covering sets:

$$S = \{a, c, e\}$$
$$T = \{b, d\}$$

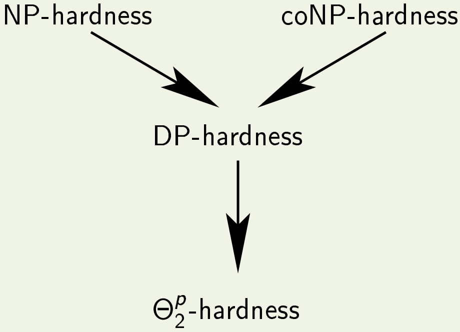
set inclusion minimal: S and T

cardinality minimal: only T



Lower Bound

Approach for proving Θ_2^p-hardness



NP-Hardness

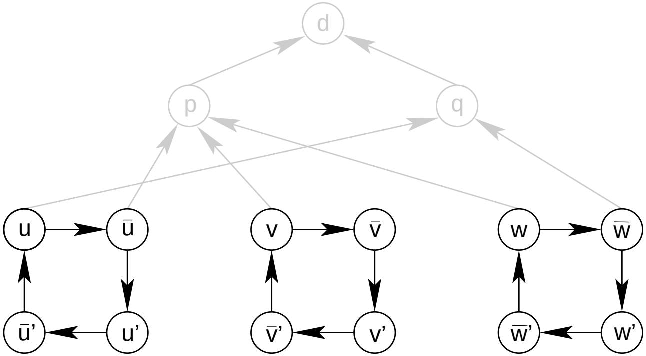
Reduction from SAT to MC_u-Member

There is a satisfying assignment for φ

$\Leftrightarrow$

there is a minimal upward covering set that contains d.

$\varphi = (\overline{u} \vee v \vee w) \wedge (u \vee \overline{w})$



Example: NP-Hardness

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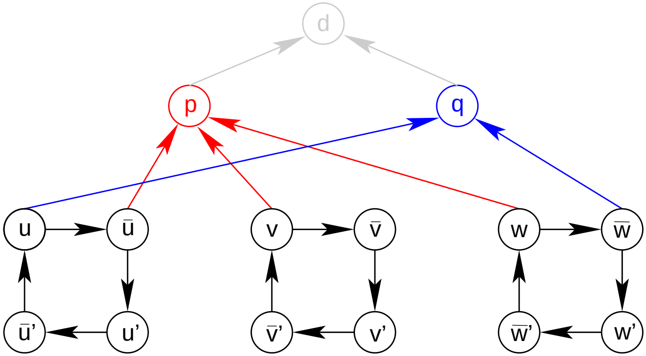

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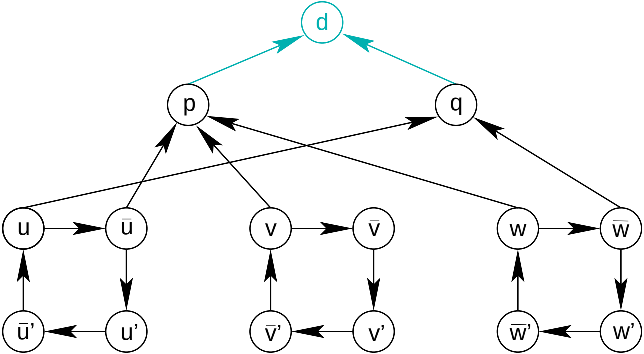


Example: NP-Hardness

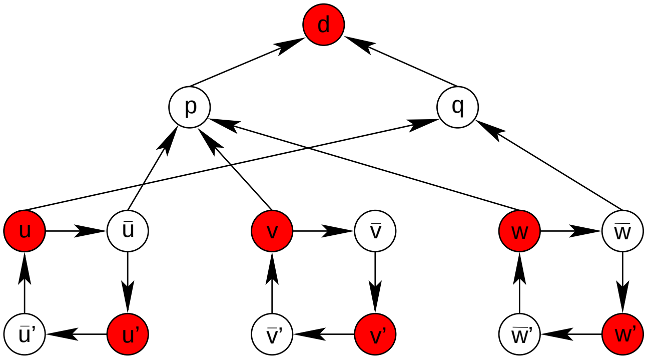
Reduction from SAT to MC_u-Member

There is a satisfying assignment for φ

$\Leftrightarrow$

there is a minimal upward covering set that contains d.

$\varphi = (\overline{u} \vee v \vee w) \wedge (u \vee \overline{w})$, satisfying assignment: $u = v = w = 1$



coNP-Hardness

The class coNP

Class of sets whose complements are in NP.

Reduction from SAT to the complement of MC_u-Member

There is a satisfying assignment for ψ

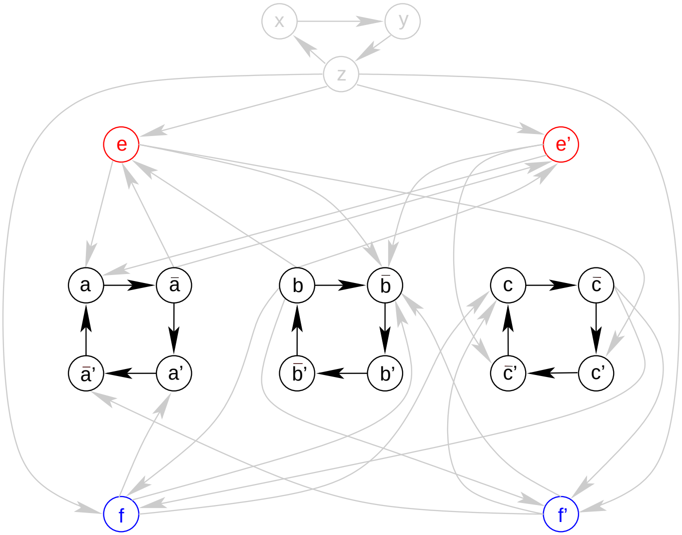
$$\Leftrightarrow$$

there is no minimal upward covering set that contains e.

Additionally: e is contained in all minimal upward covering sets if and only if there is no satisfying assignment for ψ.

Example: coNP-Hardness

$\psi = (\overline{a} \vee b) \wedge (b \vee \overline{c})$



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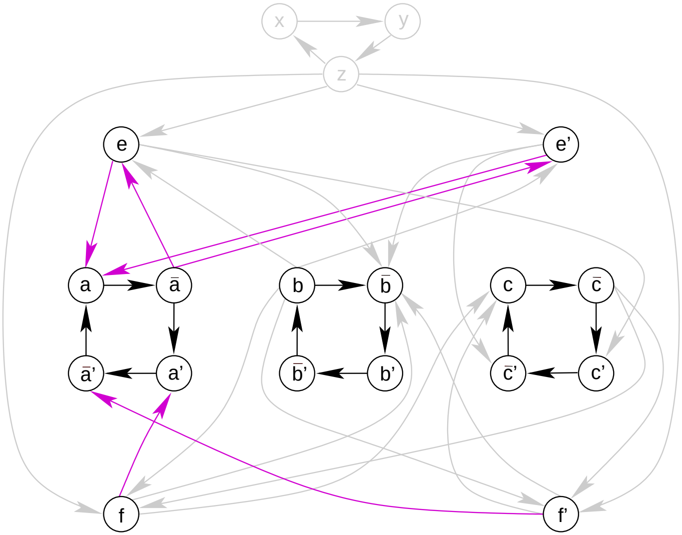

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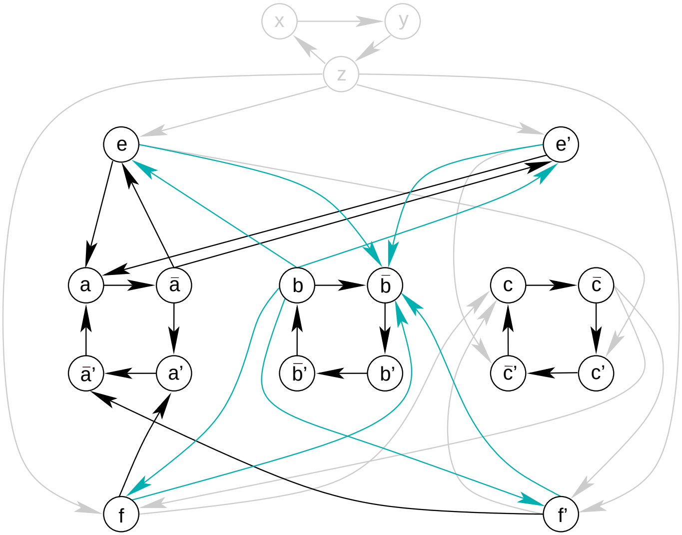

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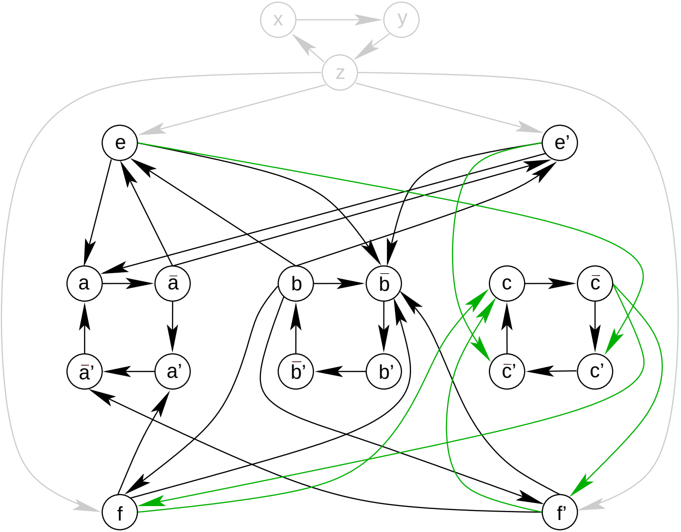

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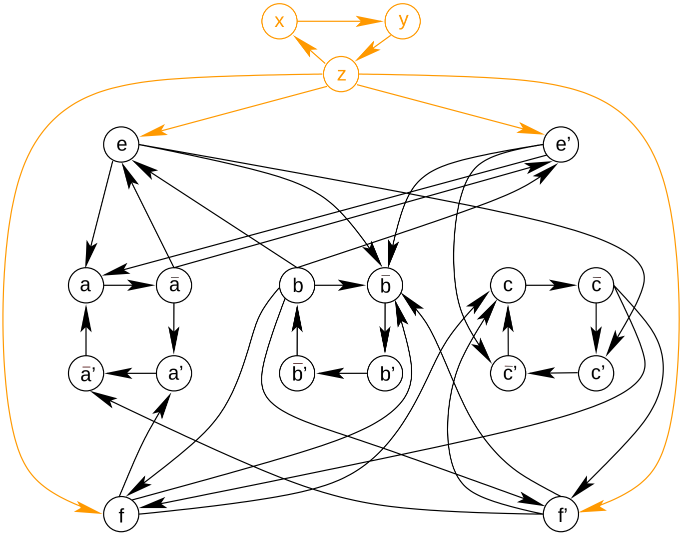
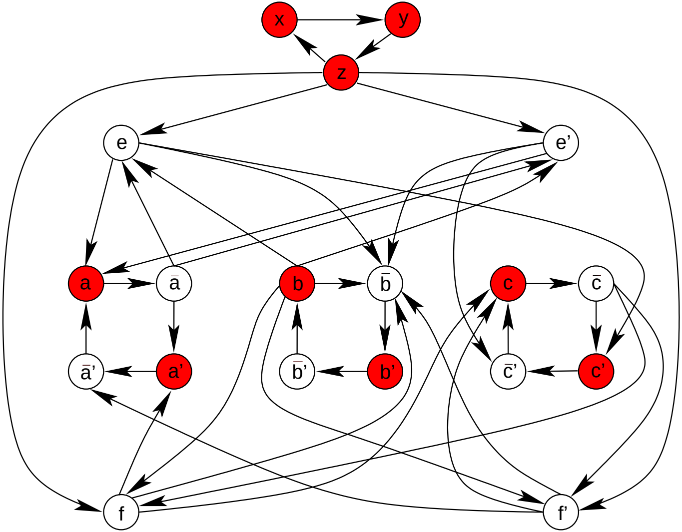


Example: coNP-Hardness

$\psi = (\overline{a} \vee b) \wedge (b \vee \overline{c})$, satisfying assignment: $a = b = c = 1$



DP-Hardness

The class DP

The class of differences of two NP sets: $\text{DP} = \{A - B \mid A, B \in \text{NP}\}$.

$$\text{NP} \cup \text{coNP} \subseteq \text{DP}.$$

Wagner's Lemma for DP-Hardness

Let A be some NP-complete problem, let B be an arbitrary problem. If there exists a polynomial-time computable function f such that, for all strings x_1, x_2 satsifying that if $x_2 \in A$ then $x_1 \in A$, it holds:

$$(x_1 \in A \text{ and } x_2 \notin A) \Leftrightarrow f(x_1, x_2) \in B,$$

then B is DP-hard.

Construction

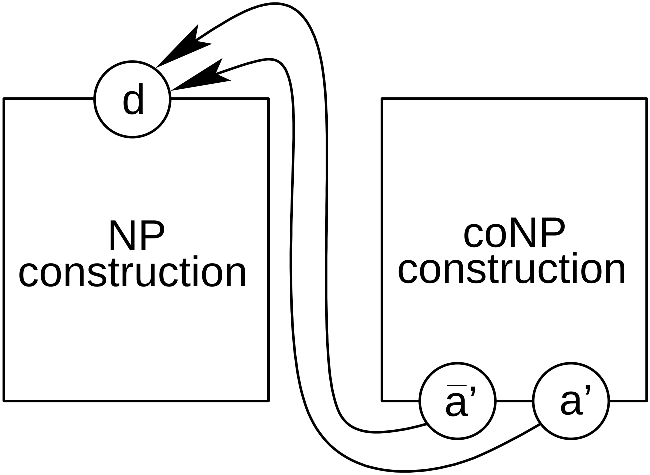
There is a satisfying assignment for φ, and none for ψ

$$\Leftrightarrow$$

there is a minimal upward covering set that contains d.

Proof Sketch: DP-Hardness

Combination of the previously presented NP and coNP reductions.



Θ_2^p-Hardness

The class Θ_2^p

Θ_2^p (also known as $P_{||}^{NP}$) is the class of problems solvable by a polynomial-time algorithm having parallel access to an NP oracle.

$$\mathrm{NP} \cup \mathrm{coNP} \subseteq \mathrm{DP} \subseteq \Theta_2^p.$$

Wagner's Lemma for Θ_2^p-Hardness

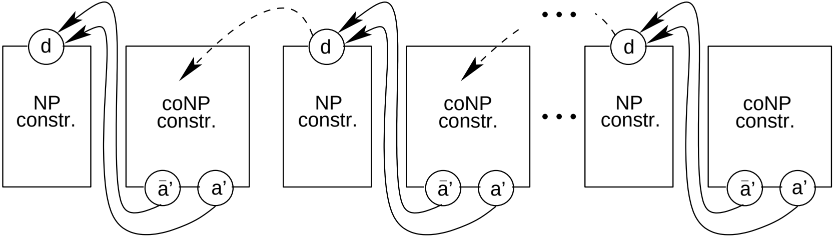
Let A be some NP-complete problem, and let B be an arbitrary problem. If there exists a polynomial-time computable function f such that, for all $m \geq 1$ and all strings $x_1, x_2, \ldots, x_{2m}$ satisfying that if $x_j \in A$ then $x_{j-1} \in A$, $1 < j \leq 2m$, it holds that

$$||\{i \mid x_i \in A\}|| \text{ is odd } \Leftrightarrow f(x_1, x_2, \ldots, x_{2m}) \in B,$$

then B is Θ_2^p-hard.

Proof Sketch: Θ_2^p-Hardness

Concatenation of the construction used to show DP-hardness.



There is some odd i such that $\varphi_i \in \text{SAT}$ and $\varphi_{i+1} \notin \text{SAT}$

$$\Leftrightarrow$$

there is a minimal upward covering set that contains d.

Summary of Results

Problem	MC_u, MC_d	MSC_u	MSC_d
Size	NP-complete	NP-complete	NP-complete
Member	Θ_2^p-hard, in Σ_2^p	Θ_2^p-complete	coNP-hard, in Θ_2^p
Member-All	coNP-complete	Θ_2^p-complete	coNP-hard, in Θ_2^p
Unique	coNP-hard, in Σ_2^p	coNP-hard, in Θ_2^p	coNP-hard, in Θ_2^p
Test	coNP-complete	coNP-complete	coNP-complete
Find	not in polynomial time unless P $=$ NP		

Thank you for your attention!

The Complexity of Computing Minimal Unidirectional Covering Sets, D. Baumeister, F. Brandt, F. Fischer, J. Hoffmann, and J. Rothe, *to appear in the Proceedings of CIAC 2010*.