On fair allocation of indivisible goods to submodular agents

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Abstract

We consider the problem of fair allocation of indivisible goods to agents with submodular valuation functions, where agents may have either equal entitlements or arbitrary (possibly unequal) entitlements. We focus on share-based fairness notions, specifically, the maximin share (MMS) for equal entitlements and the anyprice share (APS) for arbitrary entitlements, and design allocation algorithms that give each agent a bundle of value at least some constant fraction of her share value. For the equal entitlement case (and submodular valuations), Ghodsi, Hajiaghayi, Seddighin, Seddighin, and Yami [EC 2018] designed a polynomial-time algorithm for $\frac{1}{3}$-maximin-fair allocation. We improve this result in two different ways. We consider the general case of arbitrary entitlements, and present a polynomial time algorithm that guarantees submodular agents $\frac{1}{3}$ of their APS. For the equal entitlement case, we improve the approximation ratio and obtain $\frac{10}{27}$-maximin-fair allocations. Our algorithms are based on designing strategies for a certain bidding game that was previously introduced by Babaioff, Ezra and Feige [EC 2021].

1 Introduction

We study the problem of allocating a set $\mathcal{M}$ of m indivisible items *fairly* to a set $\mathcal{N}$ of n agents, where each agent i has an individual non-negative submodular valuation function $v_i : 2^{\mathcal{M}} \to \mathbb{R}_{\geq 0}$ (the definition of submodularity appears in Section 1.3). In fair allocation settings, agents do not pay for the items. Instead, agents have arbitrary, possibly unequal entitlements to the items. Specifically, each agent i has an individual entitlement $0 < b_i \leq 1$, and the entitlements sum up to 1 ($\sum_{i=1}^{n} b_i = 1$). We focus on share-based fairness notions, specifically, the maximin share (MMS) for equal entitlements [Bud11] and the anyprice share (APS) for arbitrary entitlements [BEF21] (definitions of these notions appear in Section 1.5). We design allocation algorithms that give each agent a bundle of value at least some constant fraction of her share value.

The problem of fairly allocating indivisible items has been extensively studied, with various settings of the problem considered (items might be goods or chores, agents may have equal or unequal entitlements), and different fairness criteria explored, such as envy-based notions and share-based principles. See for example [ALMW22, ABFV22] and references therein.

The problem of allocating indivisible items to agents arises naturally in the real-world. We present one such example that illustrates aspects addressed in our work (unequal entitlement, no payments, non-additive valuation functions). The NBA draft is an annual event of the National Basketball Association (NBA) in which new eligible basketball players (typically graduate college players) are allocated to NBA teams. The allocation mechanism is a picking sequence composed of two rounds. In each round, each team in its turn picks a player among the eligible players. In this example, teams correspond to the agents, and basketball players correspond to the items. The teams do not have equal entitlement to the players. Teams of poorer performance in the previous season have higher entitlement than those of better performance (a policy that tries to maintain the competitiveness of the teams). This unequality of entitlement is reflected in the allocation mechanism, by having teams with higher entitlement pick earlier than teams of lower entitlement, in each of the rounds of the picking sequence. (In practice, the allocation mechanism is somewhat more complicated than described above, but these additional complications are not relevant to our presentation, and hence omitted.) Teams do not pay for the right to pick a player. (They will of course later pay the salary of the player, but these monetary aspects are only an aspect that determines how desirable the player is for the team, and are not part of the allocation processes.) The interests of teams do not seem to fit a model of an additive valuation function. For example, it is likely that the combined value for a team of two players that play in the "center" position is smaller than the sum of values of the individual players.

1.1 Fairness notions

In our paper, we focus on notions of *fairness* known as *share-based* notions. In share-based fairness, each agent cares only about her own bundle in the allocation, and expects its value to reach at least a certain target value. One such fairness notion is the *maximin share*, which was introduced by Budish [Bud11]. The *maximin share* (abbreviated as *MMS*) of an agent is defined to be the maximum value she can ensure for herself if she were to partition the goods into n bundles and then receive a minimum valued bundle. A *maximin fair* allocation is an allocation in which each agent gets a bundle that she values at least as her maximin share.

The maximin share notion is applicable when agents have equal entitlement. A notion of fairness for the case of arbitrary entitlements was presented by Babaioff, Ezra, and Feige [BEF21], and is referred to as the *AnyPrice share* (abbreviated as APS). See Definition 9. In the special case of equal entitlements, the AnyPrice share of an agent is at least as large as her *Maximin share*, and sometimes strictly larger.

1.2 Notions of approximation

In the context of the notions of the MMS and APS, there are two different tasks that involve approximations.

- Approximating the value of the MMS (or APS) of an agent. Both the MMS and APS of an agent are NP-hard to compute even if the valuation function is additive, in which case computing the exact value of the MMS is strongly NP-hard [Woe97], computing the exact value of the APS is weakly NP-hard, and the APS can be computed by a pseudo-polynomial time algorithm [BEF21]. For submodular valuation functions (a class that is considered in this paper), computing the MMS and the APS is APX-hard. See Section A for more details.

- Approximating a fair allocation (maximin-fair allocation, AnyPrice-fair allocation, etc.). For $\alpha \in (0, 1)$, we say that an allocation is α-*maximin-fair* (resp. α-*AnyPrice-fair)* if it gives every agent at least an α fraction of her MMS (resp. APS). Kurokawa, Procaccia, and Wang [KPW18] showed that for every $n \geq 3$, there exists an instance with n additive agents for which no *maximin-fair* allocation exists (in every allocation some agent gets a bundle she values strictly less than her MMS). As the APS of an agent is at least as large as her MMS, there are instances with additive valuations and no APS-allocation.

In our paper we will focus on the latter task, approximating MMS-fair (APS-fair) allocations.

1.3 Classes of valuation functions

Throughout this paper we assume that valuation functions are *normalized* (the value of the empty set is 0) and *monotone* ($v(S) \leq v(T)$ for $S \subset T$).

Lehman, Lehman, and Nissan [LLN06]introduce a hierarchy of families of valuation functions, and two prominent members of this hierarchy are *Submodular* and *XOS* valuations, as defined below.

Definition 1. ***(Submodular valuation)*** *A valuation function $v \colon 2^{\mathcal{M}} \to \mathbb{R}_{\geq 0}$ is submodular if the following (equivalent) conditions hold:*

- $\forall S, T \subseteq \mathcal{M}$ *we have* $v(S) + v(T) \geq v(S \cup T) + v(S \cap T)$
- $\forall S, T \subseteq \mathcal{M}$ *with* $S \subseteq T$*, and for any* $j \in \mathcal{M} \backslash T$ *we have* $v(S \cup \{j\}) - v(S) \geq v(T \cup \{j\}) - v(T)$

Definition 2. ***(XOS valuation)*** *A valuation function $v \colon 2^{\mathcal{M}} \to \mathbb{R}_{\geq 0}$ is XOS (also referred to as* fractionally subadditive*) if there exist a finite set of* ***additive*** *valuations $\{v_1, v_2, \dots, v_k\}$ such that*

$$\forall T \in \mathcal{M}, \quad v(T) = \max_{j \in [k]} v_j(T)$$

As shown in [LLN06], the hierarchy of these classes is as follows:

$$\textit{Additive} \subsetneq \textit{Submodular} \subsetneq \textit{XOS}$$

Let us briefly discuss the representation of valuation functions. The explicit representation of a valuation function requires exponential space in m (its domain size is 2^m). Consequently, as described in [LLN06], one typically assumes query access to valuation functions, rather than having an explicit representation for them. Our paper will focus on the value queries model (i.e., the function is implicitly given through a value oracle). In this model, a query is a set of items, and the answer is the value of the function on this set of items. We assume that each such query takes unit time. Consequently, polynomial allocation algorithms may make only polynomially many value queries to the underlying valuation functions.

1.4 Our main results

Our main results concern the existence (and polynomial time computability) of approximate MMS-fair (APS-fair) allocations in the equal entitlements (arbitrary entitlements) case for submodular agents. Previously, for the equal entitlement case, Ghodsi, Hajiaghayi, Seddighin, Seddighin and Yami [GHS$^+$18] designed a polynomial-time algorithm for $\frac{1}{3}$-*maximin-fair* allocations, and designed instances in which in every allocation, at least one agent gets a bundle of value not larger than a $\frac{3}{4}$ fraction of her MMS.

Our results are based on a bidding game mechanism introduced by [BEF21]. Each agent gets an initial budget equal to her entitlement. In every round, the highest bidder gets to choose an item and pays her bid (see Section 2.2.) We show here that in this bidding game, a submodular agent has a bidding strategy that guarantees at least a $\frac{1}{3}$ fraction of her APS in the arbitrary entitlements case. In the case of equal entitlements, we consider a slight modification of the bidding game in which agents who spent a substantial fraction of their budget drop out of the game. We refer to this version as the *altruistic version* of the bidding game (see Section 2.3). For the altruistic version of the bidding game (and equal entitlements), we present a bidding strategy that guarantees a submodular agent at least a $\frac{10}{27}$ fraction of her MMS.

Theorem 3. *Consider the bidding game described above, and an agent p with a submodular valuation function and entitlement b_p. Setting $\rho = \frac{1}{3-2b_p} > \frac{1}{3}$, a bidding strategy referred to as proportional(ρ) guarantees agent p a value of at least $\rho \cdot APS_p$. (In the case of equal entitlements, this gives $\rho = \frac{n}{3n-2}$.)*

Theorem 4. *Consider the altruistic version of the bidding game in the equal entitlement case. Every agent with a submodular valuation that uses a bidding strategy referred to as the proportional bidding strategy is guaranteed to get a bundle of value at least a $\rho = \frac{10}{27} + \Omega(\frac{1}{n}) >$ 0.37037 fraction of her MMS.*

The bidding strategies used in our two main theorems require computing the APS (or MMS) of the agents, tasks which are APX-hard. Nevertheless, known techniques [GHS+18] allow us to deduce the following corollary:

Corollary 5. *For agents with submodular valuations, there are polynomial time algorithms offering the following guarantees. In the case of arbitrary entitlements, each agent gets at least $\frac{1}{3}$-APS. For the case of equal entitlements, each agents gets at least $\frac{10}{27}$-MMS.*

Our results concerning submodular valuations can be combined with previous results of [BEF21] that concern bidding strategies for agents with subclasses of submodular valuations, namely additive valuations and unit demand valuations. This results in allocation algorithms for setting with submodular valuations, in which agents that have valuations coming from simple sub-classes of submodular valuations get improved guarantees.

Corollary 6. *There is a polynomial time allocation algorithm, which simultaneously guarantees for submodular agents $\frac{1}{3}$-APS, for additive agents $\frac{3}{5}$-APS, and for Unit-demand agents 1-APS.*

A class of valuations that is more general than submodular valuations is XOS valuations. A natural question concerning the bidding game is whether agents with XOS valuations have strategies that guarantee a constant fraction of their APS. The answer for this question is negative.

Proposition 7. *There is no bidding strategy that guarantees a constant fraction of the MMS to an agent with an XOS valuation function (not even in the case of equal entitlements).*

1.5 Definitions of shares

In this section we provide formal definitions for the share notions that are used in this paper. For equal entitlements, we use the well established notion of the maximin share.

Definition 8. ***(Maximin share (MMS))*** *Consider an allocation instance with a set $\mathcal{M} = \{e_1, \ldots, e_m\}$ of m items and a set $\mathcal{N} = \{1, \ldots, n\}$ of n agents, where each agent i has an individual non-negative valuation function $v_i \colon 2^{\mathcal{M}} \to \mathbb{R}_{\geq 0}$. Then the maximin share of agent i, denoted by MMS_i, is the maximum over all n-partitions of $\mathcal{M}$, of the minimum value under v_i of a bundle in the n-partition*

$$MMS_i = \max_{A_1, A_2, \ldots, A_n \in P_n(\mathcal{M})} \min_j v_i(A_j)$$

(where $P_n(\mathcal{M})$ is the set of all partitions of $\mathcal{M}$ to n pairwise disjoint sets)

For arbitrary (unequal) entitlements, we use a relatively new notion of a share, the anyprice share (APS). The APS of agent i is defined via the following *price and choose game*. First, an adversary assigns nonnegative prices to the items in $\mathcal{M}$, and agent i's entitlement b_i translates to a budget that is a b_i fraction of the sum of prices of all items. Then, agent i may choose any bundle of items of total price not exceeding this budget. The value that she can guarantee for herself as a chooser, no matter how items are priced, is her anyprice share APS_i. Note that the APS of an agent depends only on her valuation and entitlement, and is independent of valuations and entitlements of other agents. Comparisons of the APS with other share notions appear in [BEF21].

Definition 9. ***(AnyPrice share)*** *Consider a setting in which agent i with valuation v_i has entitlement b_i to a set of indivisible items $\mathcal{M}$. The AnyPrice share (APS) of agent i, denoted by $AnyPrice(b_i, v_i, \mathcal{M})$, is the value she can guarantee herself whenever the items in $\mathcal{M}$ are adversarially priced with non-negative prices that sum up to 1, and she picks her favorite affordable bundle. More formally, if $P = \{(p_1, \ldots, p_m) | \sum p_j = 1, \text{ and } \forall j, p_j \geq 0\}$ is the set of all possible pricing of $\mathcal{M}$, then the definition of the APS is:*

$$AnyPrice(b_i, v_i, \mathcal{M}) = \min_{(p_1, p_2, \ldots, p_m) \in P} \max_{S \subseteq \mathcal{M}} \left\{ v_i(S) \mid \sum_{j \in S} p_j \leq b_i \right\}$$

When $\mathcal{M}$ and v_i are clear from context we denote the APS share of an agent i with entitlement b_i by $AnyPrice(b_i)$, instead of $AnyPrice(b_i, v_i, \mathcal{M})$.

As shown in [BEF21] (for all classes of valuation functions), the AnyPrice share has the following equivalent definition.

Definition 10. ***(AnyPrice share dual definition)*** *Consider a setting in which agent i with valuation v_i has entitlement b_i to a set of indivisible items $\mathcal{M}$. The AnyPrice share of i, denoted by $AnyPrice(b_i, v_i, \mathcal{M})$, is the maximum value z she can get by coming up with nonnegative weights $\{\lambda_T\}_{T \subseteq \mathcal{M}}$ that total to 1 (a distribution over sets), such that any set T of value below z has a weight of zero, and any item appears in sets of a total weight at most b_i:*

$$AnyPrice(b_i, v_i, \mathcal{M}) = \max z$$

subject to the following set of constraints being feasible for z:

- $\sum_{T \subseteq \mathcal{M}} \lambda_T = 1$
- $\lambda_T \geq 0, \forall T \subseteq \mathcal{M}$
- $\lambda_T = 0, \forall T \subseteq \mathcal{M}$ *s.t* $v_i(T) < z$
- $\sum_{T : j \in T} \lambda_T \leq b_i, \forall j \in \mathcal{M}$

1.6 Related work

There are several different approaches trying to define fairness criteria for allocation of items. One approach concerns elimination (or minimization) of envy among agents. An allocation is *envy-free* if no agent strictly prefers a bundle of another agent over her own bundle [Fol67]. Envy free allocations exist in setting with divisible items, but need not exist in settings with indivisible items (e.g., when there are fewer agents than items). Consequently, various relaxations of the envy-free property have been introduced, among them EF1 ([LMMS04],[Bud11]) and EFX [CKM$^+$19]. In this work we do not consider envy-based fairness notions.

Perhaps the first share-based fairness notion to have been introduced is the proportional share. For agent i with entitlement b_i and valuation function v_i, it equals $b_i \cdot v_i(\mathcal{M})$. This notion and various relaxations of it (Prop1) may be appropriate when valuations functions are additive, but is hard to justify for other classes of valuation functions. In this work we are concerned with submodular valuations. For the case of equal entitlements, we consider the maximin share (MMS) [Bud11], which is the share notion that is most commonly used for allocation of indivisible items to agents with equal entitlements. For the case of arbitrary entitlements, we use the anyprice share (APS) [BEF21]. We remark that there are other notions of shares that have been proposed for settings with unequal entitlements and are not considered in our work. These include the *weighted maximin share* (WMMS) of [FGH+19], and the *l-out-of-d* share [BNT21]. Arguments for preferring the APS over these other notions are presented in [BEF21], and are omitted here for lack of space.

We present here some known approximation results for *MMS-fair* and *APS-fair* allocations:

- Additive valuations with equal entitlements (approximate MMS-allocations).
  - Impossibility results. As mentioned above, Kurokawa, Procaccia, and Wang [KPW18] were the first to show that for each $n \geq 3$, there exists an instance with n additive agents, such that no *maximin-fair* allocation exists. Later, Feige, Sapir, and Tauber [FST21] showed an example of an instance with $n = 3$ agents with additive valuations, where for any allocation, at least one of the agents gets a bundle she values at most $\frac{39}{40}$ of her MMS).
  - Existence results. [KPW18] showed existence of $\approx \frac{2}{3}$-*maximin-fair* allocation (up to $O(\frac{1}{n})$). Ghodsi et al [GHS+18] showed existence of $\frac{3}{4}$-*maximin-fair* allocations.
  - Algorithmic results. Garg and Taki [GT21], presented a polynomial time algorithm that finds a $\frac{3}{4}$-*maximin-fair* allocation, assuming value queries.
- Submodular valuations with equal entitlements. [BK20] showed existence and polynomial time computability of ≈ 0.21-MMS fair allocations. Their algorithm is based on a simple round-robin algorithm. Ghodsi et al [GHS+18] showed a polynomial-time algorithm for $\frac{1}{3}$-*maximin-fair* allocations, and examples in which ρ-*maximin-fair* allocations do not exist, for any $\rho > \frac{3}{4}$ and any $n \geq 2$ (number of agents).
- For XOS valuations with equal entitlements, Ghodsi et al [GHS+18] show the existence of $\frac{1}{5}$-MMS allocations, and presented examples in which ρ-*maximin-fair* allocations do not exist, for any $\rho > \frac{1}{2}$ and any $n \geq 2$.
- Additive valuations with arbitrary entitlements (approximate *AnyPrice-fair* allocations).
  - Impossibility result. The negative results stated for MMS allocations (such as upper bound of $\frac{39}{40}$ for $n = 3$ agents) extend to APS allocations, as the APS is at least as large as the MMS.
  - There exists a polynomial-time algorithm for computing a $\frac{3}{5}$-APS allocation, i.e., an algorithm which returns an allocation where each agent gets a bundle she values at least $\frac{3}{5}$ of her AnyPrice share [BEF21].

We are not aware of previous work concerning approximate APS-allocations for valuation functions that are submodular.

2 Proofs of our results

2.1 Preliminaries

Definition 11. *Let v_p be the valuation function of agent p and let $t \geq 0$ be a scalar, then we define v_p^t, the valuation v_p truncated at t, as follows:*

$$v_p^t(B) \coloneqq \min\{v_p(B), t\}$$

Observe that if v_p is submodular, then also v_p^t is submodular.

Claim 12. *Let v_p be the valuation function of an agent p, and set $t \leq MMS_p$ (respectively $t \leq APS_p$). Then the new MMS (APS) of agent p is t if we consider her valuation function to be v_p^t (Definition 11). In the special case of $t = MMS_p$ ($t = APS_p$), this implies that the MMS (APS) of the agent remains unchanged.*

Proof. The partition (fractional partition) that certifies that the MMS (APS) with respect to v_p is at least t, certifies the same with respect to v_p^t (because every bundle that has value at least t with respect to v_p has value t with respect to v_p^t). The MMS (APS) with respect to v_p^t cannot be larger than t, as no bundle has v_p^t value larger than t. ■

2.2 Approximate APS-fair allocations for submodular agents

We now describe an allocation game (introduced in [BEF21]), that we refer to as the bidding game. Initially, every agent i is *active*, is given a budget of $b_i^0 = b_i$ (in particular, in the equal entitlement case $b_i^0 = \frac{1}{n}$), and has an empty bundle S_i^0 of items. The set of initially unallocated items is denoted by M^0.

The game proceeds in rounds, and in every round, one item is allocated. In round $r \geq 1$, to decide which item is allocated, we do the following.

1. If there are no active agents, end the allocation algorithm. (The remaining items, if there are any, can be allocated arbitrarily.)
2. Every active agent i submits a nonnegative bid p_i^r of her choice, not exceeding her budget. Namely, $0 \leq p_i^r \leq b_i^{r-1}$.
3. The agent i with the highest bid (breaking ties arbitrarily) wins, and selects an arbitrary item of her choice. Denote this selected item as e^r. We update $M^r = M^{r-1} \setminus \{e^r\}$ and $S_i^r = S_i^{r-1} \cup \{e^r\}$. Her budget is updated to $b_i^r = b_i^{r-1} - p_i^r$. That is, the winner pays her bid. If $b_i^r = 0$, then agent i stops being active. In any case, for agents $j \neq i$, we have $S_j^r = S_j^{r-1}$ and $b_j^r = b_j^{r-1}$.

Remark 13. *One may consider a second-price version of the bidding game, in which the winner pays the second highest bid. All results of this paper hold without change also with respect to the second-price version.*

To help illustrate the key methods used in the proof of Theorem 3, we first present a sketch of proof for a weaker version. This will pave the way for the subsequent proof of Theorem 3.

Proposition 14. *Consider the bidding game described above and an agent p with a submodular valuation function in an equal entitlements setting. Employing a bidding-strategy referred to as proportional($\frac{1}{3}$) guarantees agent p a minimum value of $\frac{1}{3} \cdot MMS_p$.*

Proof. In the bidding game, each agent is initially assigned a budget equal to her entitlement. In the equal entitlement case, each agent receives a budget of $\frac{1}{n}$. The *proportional*$(\frac{1}{3})$ bidding strategy involves the following steps. Initially, agent p calculates MMS_p (her MMS value). At the beginning of round r, let $\mathcal{M}^r$ denote the set of items that are still available, let C^r denote the set of items that agent p won prior to round r, and let b_p^r denote the budget that the agent still holds. In round r, agent p bids $\frac{3}{2} \cdot \frac{1}{n} \cdot \frac{1}{MMS_p} \cdot \max_{e \in \mathcal{M}^r}[v_p(e \mid C^r)]$. In other words, the bid of the agent is equal to $\frac{3}{2}$ times the *scaled value* of the marginal value of the item of highest marginal value that still remains, where the scaling factor $\frac{1}{n} \cdot \frac{1}{MMS_p}$ is such that after this scaling, the MMS value of p equals the original budget of p. If this bid value exceeds b_p^r (the remaining budget of p), then p bids b_p^r. In any case, if p wins the bid, she selects the item of highest marginal value in $\mathcal{M}^r$, and pays her bid. We note here that the factor $\frac{3}{2}$ was chosen so as to equal $\frac{1}{2\rho}$, for our choice of $\rho = \frac{1}{3}$. The same type of expression, $\frac{1}{2\rho}$, will appear also in the proof of Theorem 3.

We now provide a sketch of proof that the above bidding strategy guarantees agent p a bundle of value at least $\frac{1}{3}MMS_p$. In this sketch, $\{B_i\}_{i=1}^n$ denotes an MMS partition for agent p. Namely, $v_p(B_i) \geq MMS_p$ for every i.

Call an item e *large* if $v_p(e) > \frac{2}{3}MMS_p$. We claim that, without loss of generality, we can assume that there are no large items. As long as a large item exists, p bids her entire budget. If agent p wins the round, we are done. If a different agent q wins the round, that agent spends her entire budget and leaves the bidding game after winning only a single item. Intuitively, q did not "hurt" p, since $n-1$ bundles of the MMS partition of p still contain all their items, whereas only $n-1$ agents remain to compete on them. Further details of this argument are omitted.

According to the bidding strategy, if agent p manages to spend at least half of her budget during the bidding game, she will receive a $\frac{1}{3}$-fraction of her MMS. Therefore, our goal is to show that p manages to spend at least half of her budget.

The main idea is that until p spent half of her budget, other agents cannot do much damage to p. The bidding strategy of p has two key properties that are easy to verify. First, the bidding sequence is non-increasing (this is a consequence of submodularity of v_p). Second, in the absence of large items (an assumption that we can make without loss of generality), as long as p has spent at most half of her budget, her remaining budget does not constrain her from providing a full bid according to the bidding strategy. Due to this latter property, if another agent q wins an item, q pays for the item that she takes at least $\frac{3}{2}$ times the scaled marginal value that p has for the item.

Next, we analyze how much harm the other agents can cause p up to the point when she spends half of her budget. We denote by C the bundle that p holds at the last point in time in which she has not spent half her budget. A sufficient condition for p to exceed $\frac{1}{3}MMS_p$ is if there exists a bundle B_i from p's MMS partition that has sufficiently high marginal value (relative to C, and after excluding from B_i those items won by other agents) so that together with C, the value exceeds $\frac{1}{3}MMS_p$. For every item e that another agent wins, that agent pays at least $\frac{3}{2}\frac{1}{n}\frac{1}{MMS_p}v_p(e \mid C)$. (The fact that we can compare to marginal value relative to C is a consequence of submodularity of v_p.) Hence, the ratio between the value taken from bundles of the MMS partition, and the payment done by the other agents is $\alpha = \frac{3}{2}\frac{1}{n}\frac{1}{MMS_p}$. Therefore, using the fact that the total budget of all the agents together is 1, we obtain an upper bound on the total value taken by the other agents, which is $\frac{2}{3} \cdot n \cdot MMS_p$. Hence, there must exist a bundle B_i in which the other agents took items of marginal value relative to C of at most $\frac{2}{3}MMS_p$. Hence, together with C, there is enough value left in B_i for p to surpass $\frac{1}{3}MMS_p$. (This last argument again uses submodularity of v_p.) ∎

Before presenting the proof of Theorem 3, we discuss some of the challenges we will face when extending the (sketch of) proof of Proposition 14 to the more general setting of Theorem 3.

- Proposition 14 considers the MMS, whereas Theorem 3 considers the APS, which is always at least as large as the MMS, and sometimes larger. The analysis needs to be extended to hold relative to this stronger notion, and in particular, can no longer assume the existence of an integral MMS partition.
- The setting considered in Theorem 3 allows for arbitrary entitlements. The proof of Proposition 14 uses the assumption that the setting is that of equal entitlement (for example, in its treatment of large items).
- Theorem 3 provides a guarantee that is somewhat better than $\frac{1}{3}$-fraction of the APS (that becomes significant if the entitlement is large).

The proof of Theorem 3 appears in Appendix C.

2.3 Approximate MMS-fair allocations for submodular agents

In the standard version of the bidding game, every agent has an initial budget, and in each round, the highest bidder picks an item. Each agent plays until she exhausts her budget, and the game ends when either there are no more items left, or all agents exhaust their budgets. Now we introduce a *ρ-altruistic version* of the bidding game, in which agents who spent a ρ-fraction of their budget leave the game, without exhausting their full budget. Consequently, the agents that remain in the game face less competition for the remaining items, making it easier for them to win additional items. For the equal entitlement case, this is helpful in the design of bidding strategies that guarantee agents a higher fraction of their MMS.

Definition 15. *The ρ-altruistic version of the bidding game is the same bidding game with the change that every agent becomes inactive after spending a ρ-fraction of her budget.*

We shall consider a bidding strategy for the ρ-altruistic bidding game, that we shall refer to as the *proportional bidding strategy*. We make two assumptions that simplify our presentation. These assumptions have no effect on the correctness of Theorem 4 that follows. These assumptions concern the submodular valuation function v_p of agent p that uses the proportional bidding strategy.

1. The valuation function is scaled so that $MMS_p = b_p$ (the MMS of agent p equals her entitlement).
2. The valuation function is truncated at MMS_p (as in Definition 11). By Claim 12, this truncation does not affect the MMS value.

We now present the proportional bidding strategy, as used by agent p. At the beginning of round r, let $\mathcal{M}^r$ denote the set of items not yet allocated, let C^r denote the set of items already allocated to p, and let b_p^r denote the budget remaining for agent p. Then the agent bids $\max_{e \in \mathcal{M}^r}[v_p(e \mid C^r)]$ (the highest marginal value that a yet unallocated item has) if this bid is not larger than b_p^r, and bids her remaining budget b_p^r otherwise.

Recall Theorem 4:

Theorem 4. *Consider the altruistic version of the bidding game in the equal entitlement case. Every agent with a submodular valuation that uses a bidding strategy referred to as the proportional bidding strategy is guaranteed to get a bundle of value at least a* $\rho = \frac{10}{27} + \Omega(\frac{1}{n}) > 0.37037$ *fraction of her MMS.*

Before presenting the proof of Theorem 4, we sketch the proof for a weaker version of the theorem, setting $\rho = \frac{4}{11}$ instead of $\rho = \frac{10}{27} > \frac{4}{11}$. Already this weaker version improves over the ratio of $\rho = \frac{n}{3n-2}$ of Theorem 3 (when $n > 8$). Moreover, the proof of this weaker version of Theorem 4 conveys some intuition that may be helpful for following the proof of Theorem 4.

Proposition 16. *In the equal entitlement case with submodular valuations, for $\rho = \frac{4}{11}$, every agent that uses the proportional strategy in the altruistic version of the bidding game is guaranteed to get a bundle of value at least a $\rho = \frac{4}{11}$ fraction of her MMS.*

Proof. We only sketch the proof, as we shall later present a full proof for Theorem 4.

If agent p that uses the proportional bidding strategy manages to spend $\rho \cdot b_p$, then she also wins items of total value at least $\rho \cdot MMS_p$, and we are done. Hence it remains to exclude the case that agent p failed to spend ρb_p. In this case, partition the agents other than p into three classes, X_0, X_1, Y.

Class X_0 contains those agents that by the end of the bidding game take only one item. Intuitively, an agent i of class X_0 who took one item does not "hurt" p, because there are $n-1$ bundles in the MMS partition of p from which agent i does not take any item, and only $n-1$ agents (including p) compete for items in these bundles. Hence we may assume that no agent is in class X_0. See further details in Claim 29.

Class X_1 contains those agents that by the end of the bidding game take two items. Being in the ρ-altruistic bidding game implies that for the first item that such an agent i took she paid at most ρb_p (here we use the fact that agents have equal entitlements), implying that the bid of p at the time was at most $\rho b_p = \rho MMS_p$. By the proportional strategy, this bid was equal to the highest marginal value for p for any of the remaining items at the time, and as the sequence of highest marginal values cannot increase as rounds progress, this further implies that the marginal v_p values of items taken by agent i is at most $2\rho MMS_p$. A key observation is that if there are more than $\frac{n}{2}$ agents in class X_1, then there must be a bundle B in the MMS partition of p from which they took at least two items. As it does not matter for p which of the agents in X_1 takes which of the items that the agents in X_1 collectively take (as long as each agent in X_1 takes two items, and each such item has marginal value at most ρb_p), we may pretend that there is an agent i in X_1 for which the two items that she takes are from this bundle B. This agent i does not "hurt" p, because there are $n-1$ bundles in the MMS partition of p from which agent i does not take any item, and only $n-1$ agents (including p) compete for items in these bundles. Hence, we may assume that at most $n/2$ agents are in class X_1. See further details in Claim 30.

Class Y contains all remaining agents. Such an agent i either takes no items (and then of course she does not hurt p), or takes only one item with marginal value (to p) of at most ρAPS_p, or takes $k \geq 3$ items. In the latter case, being in the ρ-altruistic bidding game implies that for her first $k-1$ items agent i spent at most ρb_p, and hence the bid of p for the last item taken by i was at most $\frac{1}{k-1}\rho b_p$. This implies that the total marginal v_p values (with respect to items held by p) of items taken by i is at most $\frac{k}{k-1}\rho MMS_p$. For $k \geq 3$, this is maximized when $k = 3$, giving $\frac{3}{2}\rho MMS_p$.

Summing up, agents other than p take a total marginal v_p value of at most $\frac{n}{2} \cdot 2\rho MMS_p + (\frac{n}{2}-1)\cdot\frac{3}{2}\rho MMS_p = (\frac{7n}{4}-\frac{3}{2})\rho MMS_p$. Hence from at least one of the bundles B of the MMS partition of p, the total marginal value taken by other agents is at most $(\frac{7}{4}-\frac{3}{2n})\rho MMS_p$. As the bidding game ended with p spending strictly less than $\rho b_p = \rho MMS_p$, no items left in B have any marginal value for p. Denoting the bundle that p receives by C, this implies that $v_p(B \mid C) \leq (\frac{7}{4}-\frac{3}{2n})\rho MMS_p$. Hence. we have that $MMS_p \leq v_p(B) \leq v_p(B \mid C) + v_p(C) \leq (\frac{7}{4}-\frac{3}{2n})\rho MMS_p + v_p(C)$. For $\rho = \frac{4}{11}$, the assumption that $v_p(C) < \rho MMS_p$ leads to a contradiction in the above inequality, thus proving that $v_p(C) \geq \rho MMS_p$, as desired. ∎

Observe that due to the slackness factor of $\frac{3}{2n}$ in the last paragraph of the proof of Proposition 16, we can adapt the proof to get a slightly better bound for ρ, of the order of $\frac{4}{11} + \Theta(\frac{1}{n})$. Though this slackness term does not substantially change the approximation ratio, it does prove useful for designing a polynomial time algorithm that outputs an allocation in which each agent receives at least a $\frac{4}{11}$ fraction of her APS. See more details in Section 2.4.

Having seen the proof of Proposition 16, let us explain the source of improvement that leads to the proof of Theorem 4. The $\frac{4}{11}$ approximation ratio (rather than a better one) comes from the possibility that agents other than p take $2 \cdot \frac{n}{2}$ items of value ρMMS_p, and $3 \cdot (\frac{n}{2} - 1)$ items of value $\frac{1}{2}\rho MMS_p$, for a total value of nearly $\frac{7}{4}\rho MMS_p$. However, in this case the other agents take fewer than $3n$ items, implying that in at least one of the bundles of the MMS partition of p, they take at most two items, of values ρMMS_p and $\frac{1}{2}\rho MMS_p$ (recall that we may assume that no two items of value ρMMS_p are in the same bundle of p's MMS partition). Hence one of these bundles still has value of $MMS_p - \frac{3}{2}\rho MMS_p$. The assumption that p gets a value of at most ρMMS_p then implies that $MMS_p \leq \frac{5}{2}\rho MMS_p$, implying that $\rho \geq \frac{2}{5}$.

The other agents may do damage to p in a different way. $\frac{n}{2}$ agents might each take two items of value ρMMS_p, $\frac{n}{3}$ agents might each take three items of value $\frac{1}{2}\rho MMS_p$, and $\frac{n}{6} - 1$ agents might each take six items of value $\frac{1}{5}\rho MMS_p$. Ignoring the missing one agent (that is important, but is ignored only for the sake of the argument), this allows the other agents to take three items from each MMS bundle, of values ρMMS_p, $\frac{1}{2}\rho MMS_p$ and $\frac{1}{5}\rho MMS_p$. This leads to the inequality $MMS_p \leq \frac{27}{10}\rho MMS_p$, implying that $\rho \geq \frac{10}{27}$. The proof of Theorem 4 shows that this is the most damage that the other agents can do.

The proof of Theorem 4 appears in Appendix D

2.4 Polynomial time algorithms

Theorems 3 and 4 imply (among other things) the existence of allocations that give each agent with a submodular valuation a certain fraction of her APS (or MMS). However, they do not provide polynomial time algorithms to find such allocations because they assume that the APS (or MMS) value is known (or can be computed by the agent), whereas computing this value is NP-hard. Nevertheless, by using a technique presented in [GHS+18], we obtain a polynomial time implementation, proving Corollary 5. The basic idea is as follows. One runs the bidding game with all agents using our proposed proportional strategy, but each agent starts with an estimate for her MMS (or APS) that is higher than the true value. If all agents get the desired fraction of their estimated MMS, we are done. If not, then for those agents that get a fraction that is too small, we lower their estimate for their MMS by a factor of $(1 - \epsilon)$, and repeat the whole process. No agent will ever need to lower her estimate to below a $(1 - \epsilon)$ fraction of her true MMS. For the full proof, see Appendix E.

Corollary 6 states that if valuation functions of agents come from different classes, then there is an allocation that simultaneously gives each agent a bundle of value that is a certain fraction of her APS, where this fraction depends on the class (1 for unit demand, $\frac{3}{5}$ for additive, $\frac{1}{3}$ for submudular). This is a consequence of the fact that all these ratios can be achieved by bidding strategies for a certain bidding game. The bidding game that is used is a variation on the bidding games considered in the current paper, in which agents who win a bid may pick more than a single item (as long as they can afford to pay for the items that they pick). This variation is used in [BEF21] in their proof of the $\frac{3}{5}$ ratio for additive valuations. Our $\frac{1}{3}$ ratio for submudular valuations extends also to this version of the bidding game (and unit demand bidders have trivial bidding strategies), establishing Corollary 6. For more details, see Appendix E.

2.5 Negative examples

The following proposition shows that our analysis in Theorem 4, showing a ratio of $\frac{10}{27} \simeq 0.37$, is nearly tight (for the particular bidding game and bidding strategy used in that theorem). Its proof is presented in Section F.

Proposition 17. *For every constant $\rho > \lim_{k\to\infty} \rho_k \simeq 0.3716$ (where for each $k \in \mathbb{N}$ we will define ρ_k in the proof), there is an allocation instance with equal entitlements and an adversarial run of the* altruistic *version of bidding game, in which an agent p that has a submodular valuation function and uses the proportional bidding strategy gets a bundle of value smaller than ρMMS_p.*

The following proposition shows that in Theorem 3, the value of ρ cannot be improved to a constant (independent of b_p) larger than $\frac{1}{3}$. Its proof is presented in Section F.

Proposition 18. *For every constant $\rho > \frac{1}{3}$, there is an allocation instance with equal entitlements and an adversarial run of the bidding game, in which an agent p that has a submodular valuation function and uses the proportional(ρ) bidding strategy gets a bundle of value smaller than ρMMS_p.*

We now restate and prove Proposition 7, showing that our proof techniques do not extend to XOS valuations.

Proposition 7. *There is no bidding strategy that guarantees a constant fraction of the MMS to an agent with an XOS valuation function (not even in the case of equal entitlements).*

Proof. For parameters n, k, define the instance $I(n, k)$ as follows. There are n agents with equal entitlements. The set $\mathcal{M}$ of items consists of nk items e_{ij} for $1 \le i \le k$ and $1 \le j \le n$. We think of $\mathcal{M}$ as arranged in an $k \times n$ matrix, with e_{ij} in the ij entry. For every column j, let c_j be the additive valuation function defined by giving value 1 to items in column j, and 0 to all other items. Let v be the pointwise maximum of the functions c_j, that is, $v(S) = \max_j c_j(S)$ for every $S \subseteq \mathcal{M}$. Then the valuation function v is an XOS function by its definition. We focus on a specific agent p whose valuation function is $v_p = v$. (The other agents may have arbitrary valuations.)

Claim 19. *If $n \ge 4k^2$, no bidding strategy can guarantee p more than a $1/k$-fraction of MMS_p.*

Proof. For convenience, assume all agents are given a budget of k. We give the other agents adversarial bidding strategies, as follows. There are two types of agents.

- Type 1 agents consist of $n/2$ of the agents that always bid $1/2$, and take an arbitrary available item upon winning.
- Type 2 agents consist of the rest $n/2 - 1$ agents, which operate as follows. Once agent p wins an item e_{ij}, an agent of this type bids all of her budget in the next $k-1$ rounds, and upon winning, chooses an available item from column j (and becomes inactive).

An agent of type 1 becomes inactive after winning exactly $2k$ items. As the number of items is nk, it follows that there exists an active agent of type 1 in every round. Thus, agent p must pay $1/2$ for every won item, so she can win at most $2k$ items overall. Once p wins her first item from some column j, if there exist at least $k - 1$ active agents of type 2, all other items in column j will be taken by them in the next $k - 1$ rounds. So, if we start with at least $(2k) \cdot (k - 1)$ agents of type 2, p will not win more than one item from every column. As $(2k) \cdot (k - 1) \le 2k^2 - 1 \le n/2 - 1$, this indeed holds, so agent p cannot win a bundle of value more than 1. Observe that $MMS_p = k$, so the claim follows. ■

Proposition 7 is an immediate consequence of Claim 19. ■

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A APX-hardness for computing MMS and APS for submodular valuations

The current section is presented in a somewhat sketchy way, and without detailed proofs, as it is not the main focus of the current paper.

In general, computing the MMS and the APS are both NP-hard tasks. For example, for the case of two agents with equal entitlements and additive valuations, weak NP-hardness is a straightforward consequence of the NP-hardness of the PARTITION problem (given a set of integers, is there a subset whose sum of values is exactly half of the total sum?). Computing the APS is in some sense an easier task than computing the MMS. In particular, for additive valuations there are pseudo-polynomial time algorithms for computing the APS [BEF21], whereas computing the MMS is strongly NP-hard. Also, as the value of the APS is a solution to a linear program with exponentially many constraints (Definition 9), the APS can be computed in polynomial time (using the ellipsoid algorithm) if there is a separation oracle for the linear program. This separation oracle corresponds to a computational problem that has a natural economic interpretation: given prices to the items and a budget for the agent, which is the highest value bundle that the agent can afford? For a given valuation function, if one can answer such queries in polynomial time, then the APS can be computed in polynomial time.

In the case of equal entitlement, the APS is at least as large as the MMS, and sometimes strictly larger. For submodular valuations, a ratio of $\frac{5}{6}$ between the MMS and the APS is demonstrated in [BEF21], for an allocation instance with six items and two agents with equal entitlements.

For submodular valuations, both the MMS and the APS are APX-hard to compute. We are not aware of a reference in which such a statement is proved explicitly. However,

$(1 - \frac{1}{e})$ approximation hardness can be proved using known techniques. (For simplicity, we omit here low order additive terms when stating approximation ratios.) Let us briefly explain how. A certain reduction template (starting from the APX-hard problem Max 3SAT) described in [Fei98] established hardness of approximation results for Min Set Cover (within a ratio of $\ln n$) and Max k-Coverage (within a ratio of $1 - \frac{1}{e}$). The same reduction template, but starting from a different APX-hard problem (Max 3-Coloring), was used in [FHKS02] to prove hardness of approximation of the Domatic Number (within a ratio of $\ln n$), and in [KLMM08] to prove that the maximum welfare problem with submodular valuations is hard to approximate within a ratio of $1 - \frac{1}{e}$. The reduction used to prove this last result implies that for submodular valuations it is NP-hard to approximate the MMS within a ratio better than $1 - \frac{1}{e}$. The reason for this is that in the maximum welfare instance constructed by the reduction, all agents have the same submodular valuation function (call it v). Moreover, on *yes* instances, the maximum welfare allocation gives all agents the same value (call it t, with the total welfare being $n \cdot t$). Hence on *yes* instances, the MMS of v is t (as the maximum welfare allocation serves as an MMS partition). On *no* instances, the fact that the welfare is at most $(1 - e)n \cdot t$ implies that in every partition to n bundles, at least one of the bundles has value at most $(1 - e)t$, showing that the MMS is at most $(1 - \frac{1}{e})t$. As it is NP-hard to distinguish between *yes* and *no* instances, we get a hardness of approximation for the MMS.

To derive $(1-\frac{1}{e})$ approximation hardness for the APS, one further observes the following property of *no* instances that result from using the reduction template: every set that contains a $\frac{1}{n}$ fraction of the items has value at most $(1 - \frac{1}{e})t$. This property is inherited from the fact that the same reduction template is used in the proof in [Fei98] that Max k-Coverage is hard to approximate within a ratio better than $1 - \frac{1}{e}$. When the entitlement is $\frac{1}{n}$, the APS fractional partition (of Definition 10) must contain at least one bundle with at most a $\frac{1}{n}$ fraction of the items, and hence the APS for *no* instances is at most $(1 - \frac{1}{e})t$.

B Simple technical claims

Claim 20. *Let I be an allocation instance with a set $\mathcal{M}$ of items, let i be an agent with entitlement $0 < b_i < 1$, let j be an agent with entitlement $b_j \leq b_i$, and let $e \in \mathcal{M}$ be an arbitrary item. Consider an allocation instance I' that differs from I only in that agent i is removed, the entitlements of all other agents are scaled by $\frac{1}{1-b_i}$ (so that they sum up to 1), and item e is removed. Then the APS of j in I' is at least as large as the APS of j in I. Moreover, if I was an instance with equal entitlement, then the same applies to the MMS of j*

Proof. The claim above states that for each agent j with $b_j \leq b_i$, if we define $b_j^* := \frac{1}{1-b_i} \cdot b_j$, then:

$$APS_j(\mathcal{M} \setminus \{e\}, b_j^*) \geq APS_j(\mathcal{M}, b_j)$$

Let $\{\lambda_S S\}$ be a fractional APS partition for j in the original instance I. Let $w = \sum_{S|e \in S} \lambda_S$ denote the total weight of bundles in which e participates in the fractional partition. Note that by definition of the APS, $w \leq b_j$, and since $b_j \leq b_i$, we also have $w \leq b_i$. Define the following new weights:

$$\lambda_S^* = \begin{cases} \frac{1}{(1-w)} \cdot \lambda_S & \text{if } e \notin S \\ 0 & \text{otherwise} \end{cases}$$

We show that the new weights induce a fractional APS partition for the new instance I' (with agent i and item e removed). Indeed, every item is in bundles of total weight at most

b_j^*:

$$\forall e' \in \mathcal{M} \setminus \{e\}, \quad \sum_{S|e' \in S} \lambda_S^* \leq \frac{1}{(1-w)} \sum_{S|e' \in S} \lambda_S \leq \frac{1}{(1-w)} b_j \leq \frac{1}{(1-b_i)} b_j = b_j^*$$

The sum of weights of all bundles is 1:

$$\sum_S \lambda_S^* = \sum_{S|e \notin S} \lambda_S^* = \sum_{S|e \notin S} \frac{1}{(1-w)} \lambda_S = \frac{1}{(1-w)} \cdot (1-w) = 1$$

Trivially all bundles in the support of $\{\lambda_S^* S\}$ are of value at least $APS_j(\mathcal{M}, b_j)$, proving the claim. ■

We recall notation used when describing the *proportional*(ρ) bidding strategy for the bidding game in Section 2.2. Let I_s denote the residual instance after round s, which is the first point in time after which no items satisfy $v_p(e) > 2\rho APS_p$. The total budget b^s remaining for all agents is denoted by γ. We consider a new instance $\hat{I}_s$ to be the following:

- The set of agents is those who are active in I_s.
- The entitlement of an agent in $\hat{I}_s$ is her budget in I_s, scaled by $\frac{1}{\gamma}$. Consequently, the entitlements are non-negative and sum up to 1.
- The items of $\hat{I}_s$ are those of I_s (denoted by $\mathcal{M}^s$), and v_p remains unchanged (over subsets of $\mathcal{M}^s$).

Claim 21. *Given that agent p did not win an item yet (in the first s rounds), then $APS(\mathcal{M}^s, v_p, \frac{1}{\gamma} b_p) \geq APS(\mathcal{M}, v_p, b_p)$. In other words, the APS of agent p in $\hat{I}_s$ is at least as her APS in I_0, the original instance. (Referring Definition 11, in the special case of $v_p = v_p^t$ with $t = APS_p$, then the claim holds with equality)*

Proof. Agent p did not win an item in the first s rounds, while there is an item with a value greater than 2ρ of her APS. Therefore by the proportional bidding strategy, she bids her entire budget, b_p, in each of these rounds. Thus, after these s rounds, $\gamma = b^s \geq 1 - s \cdot b_p$. Let $P_0 = \{\lambda_S S\}$ be a fractional partition associated with the APS of agent p. then we define a new fractional partition $P_s = \{\lambda'_S S\}$ to be the following

$$\lambda'_S = \begin{cases} \frac{1}{\gamma} \lambda_S & \text{if } S \subseteq \mathcal{M}^s \\ 0 & \text{otherwise} \end{cases}$$

We claim that P_s witness that indeed $APS(\mathcal{M}^s, v_p, \frac{1}{\gamma} b_p) \geq APS(\mathcal{M}, v_p, b_p)$.

-

$$\begin{aligned} \sum_{S \subseteq \mathcal{M}^s} \lambda'_S &= \frac{1}{\gamma} \sum_{S \subseteq \mathcal{M}^s} \lambda_S = \frac{1}{\gamma} \Big(\sum_{S \subseteq \mathcal{M}} \lambda_S - \sum_{e \in \mathcal{M} \setminus \mathcal{M}^s} \sum_{S|e \in S} \lambda_S \Big) \\ &\underset{*}{\geq} \frac{1}{\gamma} \Big(1 - \sum_{e \in \mathcal{M} \setminus \mathcal{M}^s} b_p \Big) \\ &= \frac{1}{\gamma} (1 - s \cdot b_p) \\ &\geq 1 \end{aligned}$$

By definition of P_o, for each $e \in \mathcal{M}$, $\sum_{S|e \in S} \lambda_S \leq b_p$, which justify inequality *.

- Each $e \in \mathcal{M}^s$ satisfies:
$$\sum_{S|e\in S} \lambda'_S = \frac{1}{\gamma} \sum_{S|e\in S} \lambda_S \leq \frac{1}{\gamma} b_p$$
Where $\frac{1}{\gamma} b_p$ is the entitlement of p in $\hat{I}_s$.

- Each $S \subseteq \mathcal{M}^s$ with strictly positive weight in P_s is of value $v_p(S) \geq APS_p$ (since a bundle with positive weight also has positive weight in P_0). In the special case of $v_p = v_p^t$ with $t = APS_p$, (no bundle equals more than APS_p) then clearly also $v_p(S) \leq APS_p$, and therefore $v_p(S) = APS_p$

■

C Proof of Theorem 3

In this section we present the proof of Theorem 3.

We describe a bidding strategy for the bidding game in the arbitrary entitlement case. It has a parameter $\rho > 0$, and we refer to it as the proportional bidding strategy $proportional(\rho)$. As we shall later prove, if ρ is chosen to satisfy $\rho \leq \frac{1}{3-2b_p}$, then the $proportional(\rho)$ bidding strategy will guarantee that agent p receives a bundle of value at least ρAPS_p. A player p (with valuation function v_p and entitlement b_p) that uses $proportional(\rho)$ first computes her $APS(\mathcal{M}, v_p, b_p)$ value, which we refer to as APS_p. Up to some minor technical details (that manifest themselves only if other agents spend their budgets at a rate that is higher than that dictated by the bids of p – these details do not affect the guarantees offered by the bidding strategy), $proportional(\rho)$ is equivalent to the following simple strategy. Scale the valuation function of p such that her APS equals her entitlement (and budget) b_p. In each round, bid $\frac{1}{2\rho}$ times the marginal value (with respect to the items that p already holds) of the item of highest marginal value that is not yet allocated, if p has sufficient budget to do so, and bid the total remaining budget otherwise. If p wins the bid, she selects the item of highest marginal value.

To simplify the proofs that follow later, we now present the $proportional(\rho)$ bidding strategy in more detail, and introduce terminology that will be used in the proofs. We first present $proportional(\rho)$ in the special case in which $v_p(e) \leq 2\rho APS_p$ holds for all items. We refer to the strategy in this special case as $proportional_1(\rho)$. At the beginning of round r, let $\mathcal{M}^r$ denote the set of items not yet allocated, let C^r denote the set of items already allocated to p, and let b_p^{r-1} denote the budget remaining for agent p. Then the agent bids $\frac{1}{2\rho} \cdot \frac{b_p}{APS_p} \cdot \max_{e\in\mathcal{M}^r}[v_p(e \mid C^r)]$ (the highest marginal value that a yet unallocated item has, scaled by $\frac{1}{2\rho}\frac{b_p}{APS_p}$) if this bid is not larger than b_p^{r-1}, and bids her remaining budget b_p^{r-1} otherwise. If the agent wins her bid, she selects the item with the highest marginal value with respect to C^r.

We now present the strategy for the remaining case, that in which there are large items that satisfy $v_p(e) > 2\rho APS_p$. We refer to the strategy in this case as $proportional_2(\rho)$.

Prior to the beginning of the bidding game, agent p truncates her valuation function, so that the value of each bundle S is $\min[v_p(S), APS_p]$. In the notation of Definition 11, this new valuation function is denoted as v_p^t, with $t = APS_p$. This does not affect APS_p, and preserves submodularity. To keep notation simple, we still use the notation v_p for this new valuation function.

In the bidding game itself, as long as there exists a large item that satisfies $v_p(e) > 2\rho APS_p$, p bids her entire budget. If the agent wins her bid, she selects the item with the highest value (which is at least $2\rho APS_p$) and leaves the game (as her budget is exhausted). If the agent does not win any of the large items, let s denote the last round in which there are large items, i.e., from round $s+1$ onward, all unallocated items satisfy $v_p(e) \leq 2\rho APS_p$. At this point, agent p basically switches to using *proportional*$_1(\rho)$. Below we introduce terminology that describes how this switch is done. This terminology will later be used in our proofs.

Let I_s denote the *residual instance* that remains after round s. It includes the set of items that were not yet allocated (which we denote by $\hat{\mathcal{M}}$), and each agent has whatever remains from her budget after the s rounds. View I_s as a new allocation instance, that we refer to as $\hat{I}_s$. The agents of $\hat{I}_s$ are the remaining active agents of I_s. The set of items of $\hat{I}_s$ is $\hat{\mathcal{M}}$ (those items remaining in I_s). Setting $\gamma = b^s$ to be the total remaining budget of agents, we update the entitlement of each agent to be $\hat{b_i} = \frac{1}{\gamma} b_i$. Agent p simulates *proportional*$_1(\rho)$ on $\hat{I}_s$. To do so, for every agent $i \neq p$, a bid x_i^r in I_s is viewed as a bid of $\frac{1}{\gamma}\hat{x}_i^{r-s}$ in $\hat{I}_s$, and any intended bid x^r of agent p in $\hat{I}_s$ is translated to a bid γx^{r+s} in I_s. (Since the ratio between a budget of an agent in I_s and in $\hat{I}_s$ is $\frac{1}{\gamma}$, by taking a bid of another agent i in I_s and scaling it by $\frac{1}{\gamma}$ we get a legal bid of the agent in $\hat{I}_s$. Similarly, scaling by γ a bid of agent p in $\hat{I}_s$ induces a legal bid in I_s.)

Observe that by Claim 21, the APS of p in $\hat{I}_s$ remains the same as the original value of APS_p.

Observation 22. *The sequence of bids of an agent that uses the proportional bidding strategy is weakly decreasing.*

Consider an APS fractional partition $\{\lambda_S\}_{S\subseteq\mathcal{M}}$ for agent p , where $\sum_S \lambda_S = 1$, $\sum_{S \mid e\in S} \lambda_S \leq b_p$ for every item e, and $v_p(S) \geq APS_p$ for every S with λ_S in the support. We trace three parameters throughout the execution of the algorithm. One is a *lower bound* on the total marginal value of the set of all remaining items to agent p, given her set of items at the time. Initially it is $L^0 = \sum_S \lambda_S v_p(S) \geq APS_p$. Another is the budget of agent i, which initially is $b_i^0 = b_i$. Another is the total remaining budget of all agents. Initially it is $b^0 = 1$.

Consider an agent p with valuation function v_p. We begin by proving Theorem 3 under the simplifying assumption that there are no large items i.e., every item satisfies $v_p(e) \leq 2\rho APS_p$.

Lemma 23. *For an agent p with a submodular valuation function, if $v_p(e) \leq 2\rho APS_p$ for every item e, then by setting $\rho = \frac{1}{3-2b_p} > \frac{1}{3}$, the proportional$(\rho)$ bidding strategy guarantees agent p a value of at least $\rho \cdot APS_p$.*

We shall present a sequence of claims that proves Lemma 23.

Claim 24. *In every round t of the algorithm, at least one of the following two conditions hold:*

1. *Agent p's bid is equal to $\frac{1}{2\rho}\cdot\frac{b_p}{APS_p} \cdot \max_{e\in\mathcal{M}^r}[v_p(e \mid C^r)]$ (the highest marginal value that a yet unallocated item has, scaled by $\frac{1}{2\rho}\frac{b_p}{APS_p}$)*
2. *Agent p already won a bundle with value at least ρAPS_p.*

Proof. Suppose that the second condition does not hold. If p did not win an item yet, then she still has her entire budget, and using the assumption that no item has a value greater than $2\rho APS_p$, condition 1 holds. Otherwise, agent p won items with a total value less than ρAPS_p by time t. Submodularity of v_p implies that the sequence of remaining maximal marginal values $\max_{e\in\mathcal{M}^r}[v_p(e \mid C^r)]$ is non-increasing in r. Hence $\max_{e\in\mathcal{M}^t}[v_p(e \mid C^t)] \leq \rho APS_p$. Therefore $\frac{1}{2\rho}\frac{b_p}{APS_p}\max_{e\in\mathcal{M}^t}[v_p(e \mid C^t)] \leq \frac{1}{2\rho}\rho b_p \leq \frac{1}{2}b_p \leq b_p^t$, and condition 1 holds. ∎

Let $\{\lambda_S\}_{S\subseteq\mathcal{M}}$ be the set of weights associated with the APS for v_p (i.e., for every $S \subseteq \mathcal{M}$, $\lambda_S > 0 \implies v_p(S) \geq APS_p$, also $\sum_S \lambda_S = 1$, and $\forall e \in \mathcal{M}$, $\sum_{S|e\in S}\lambda_S \leq b_p$).

Let $L^0 := \sum_S \lambda_S v(S)$. Observe that by the definition of APS, $L^0 \geq APS_p$. At the beginning of the algorithm (when no item has been allocated yet), L^0 is a lower bound on the marginal value that agent p has for the set of all items.

Let f denote the earliest round after which either all other agents become inactive, or all items have been allocated. Let $C \subseteq \mathcal{M}$ denote the set of items agent p has by the end of round f, and let O denote the set of items that the other agents have by the end of round f. Define $L^f = \sum_S \lambda_S \cdot v_p(S \setminus O\dot{\cup}C \mid C)$. Namely, L^f is the expected marginal value to agent p (who already holds the set C of items) of a bundle S selected at random according to the probability distribution over bundles implied by the coefficients λ_S, after one removes from S those items that were allocated by the end of round f.

Claim 25. *Let $\tilde{\mathcal{M}}$ be the set of items that remain unallocated after round f, i.e., $\tilde{\mathcal{M}} = \mathcal{M} \setminus (O\dot{\cup}C)$. Then $v_p(C \cup \tilde{\mathcal{M}}) = v_p(\mathcal{M} \setminus O) \geq v_p(C) + L^f$. In other words, $v_p(C) + L^f$ is a lower bound on the total value that agent p will have, if she receives all the remaining items ($\tilde{\mathcal{M}}$).*

Proof. If no items remain after round f (i.e., $C\dot{\cup}O = \mathcal{M}$), then for each $S \subseteq \mathcal{M}$, $S\setminus O\dot{\cup}C = \emptyset$ and $L^f = 0$. Hence, $v_p(C \cup \tilde{\mathcal{M}}) = v_p(C) = v_p(C) + L^f$, proving the claim.

If items do remain after round f, then every term $v_p(S \setminus O\dot{\cup}C \mid C)$ in the sum of $L^f = \sum_S \lambda_S \cdot v_p(S \setminus O\dot{\cup}C \mid C)$, is a marginal value of a partial set of the not-yet-allocated items (i.e., $(S \setminus O\dot{\cup}C) \subseteq \tilde{\mathcal{M}}$). Hence $v_p(S \setminus O\dot{\cup}C \mid C) \leq v_p(\tilde{\mathcal{M}} \mid C)$. Since the scalars $\{\lambda_S\}$ in the sum $L^f = \sum_S \lambda_S \cdot v_p(S \setminus O\dot{\cup}C \mid C)$ are non-negative and add up to 1, we obtain:

$$\begin{aligned} L^f =& \sum_S \lambda_S \cdot v_p(S \setminus O\dot{\cup}C \mid C) \\ \leq& \sum_S \lambda_S \cdot v_p(\tilde{\mathcal{M}} \mid C) \\ =& v_p(\tilde{\mathcal{M}} \mid C) \end{aligned}$$

Hence

$$v_p(C) + L^f \leq v_p(C) + v_p(\tilde{\mathcal{M}} \mid C) = v_p(C \cup \tilde{\mathcal{M}})$$

∎

Claim 26. $\min\{L^f + v_p(C), 2\rho APS_p\}$ *is a lower bound on the final total value of agent p.*

Proof. First, notice that if p is not active in time f, then p spent her entire budget, b_p. The bidding strategy of p (and submodularity of v_p) implies that in that case, p has a value of at least $2\rho APS_p$ in time f, i.e., $v_p(C) \geq 2\rho APS_p$ and the claim follows in this case.

Otherwise, p is an active agent after round f. We consider two cases. If some items remain after round f, then agent p is the only remaining active agent. Hence p is the only agent to win items from $\tilde{\mathcal{M}}$. Then, the agent will keep winning items until she becomes inactive or until she wins all remaining items. By Claim 25, $v_p(\mathcal{M} \setminus O) \geq v_p(C) + L^f$. The fact that the agent bids at most $\frac{1}{2\rho}\cdot\frac{b_p}{APS_p}$ times the marginal value of the item she wins in each round guarantees that agent p gets at least $\min\{L^f + v_p(C), 2\rho APS_p\}$. It remains to handle the case of agent p being active at time f while no items remain. In this case, $L^f = 0$, so the bound is trivial. ■

Claim 27. *The following holds:*

$$L^0 \leq L^f + v_p(C) + b_p \cdot \sum_{e \in O} v_p(e \mid C)$$

Proof.

$$L^f = \sum_S \lambda_S \cdot v_p(S \setminus O \dot{\cup} C \mid C) = \sum_S \lambda_S \cdot v_p(S \setminus O \mid C)$$

For every $S \subseteq \mathcal{M}$ we claim:

$$v_p(S) \underset{1.}{\leq} v_p(S \mid C) + v_p(C) \underset{2.}{\leq} v_p(S \setminus O \mid C) + v_p(C) + \sum_{e \in S \cap O} v_p(e \mid C)$$

proof of inequality 1:

$$v_p(S \mid C) + v_p(C) = v_p(S \cup C) - v_p(C) + v_p(C) = v_p(S \cup C) \geq v_p(S)$$

proof of inequality 2:

Consider an arbitrary order of the set $S \cap O = \{e_1, \ldots, e_k\}$. Then:

$$\begin{aligned} v_p(S \mid C) &= v_p\left(S \setminus O \mid C\right) + \sum_{i=1}^{k} v_p\left(e_i \mid (S \setminus O) \cup C \cup \left(\bigcup_{j=1}^{i-1} e_j\right)\right) \\ &\leq v_p\left(S \setminus O \mid C\right) + \sum_{i=1}^{k} v_p\left(e_i \mid C\right) \end{aligned}$$

Thus, using the last inequality, we obtain the following:

$$\begin{aligned} L^0 = \sum_{S \subseteq \mathcal{M}} \lambda_S v_p(S) &\leq \sum_{S \subseteq \mathcal{M}} \lambda_S \left(v_p(S \mid C) + v_p(C)\right) \\ &\leq \sum_{S \subseteq \mathcal{M}} \lambda_S \left(v_p(S \setminus O \mid C) + v_p(C) + \sum_{e \in S \cap O} v_p(e \mid C)\right) \\ &\underset{*}{\leq} v_p(C) + b_p \cdot \sum_{e \in O} v_p(e \mid C) + \sum_{S \subseteq \mathcal{M}} \lambda_S v_p(S \setminus O \mid C) \\ &= v_p(C) + b_p \cdot \sum_{e \in O} v_p(e \mid C) + L^f \end{aligned}$$

where inequality * is since each item $e \in \mathcal{M}$ has a total weight of at most b_p (by the APS definition). Overall, as we wanted to show, we obtained the following:

$$L^0 \leq L^f + v_p(C) + b_p \cdot \sum_{e \in O} v_p(e \mid C)$$

■

Claim 28. *Either* $v_p(C) \geq \rho APS_p$*, or*

$$\sum_{e \in O} v_p(e \mid C) \leq 2\rho(b^0 - b_p) \cdot \frac{APS_p}{b_p}$$

Proof. Suppose that $v_p(C) < \rho APS_p$. Let $C^e \subseteq C$ be the set of items agent p already won when another agent wins item e. Claim 24 implies that agent p bids $\frac{1}{2\rho} \cdot \frac{b_p}{APS_p}$ times the highest marginal value of an item w.r.t C^e. Hence, when the other agent i wins item e, agent p bid is at least $\frac{1}{2\rho} \frac{b_p}{APS_p} v_p(e \mid C^e) \geq \frac{1}{2\rho} \frac{b_p}{APS_p} v_p(e \mid C)$, and winning item e reduces the budget of agent i by at least $\frac{1}{2\rho} \frac{b_p}{APS_p} v_p(e \mid C)$. Since the budget of all other agents at the beginning is $b^0 - b_p$, we obtain

$$\sum_{e \in O} \frac{b_p}{2\rho APS_p} v_p(e \mid C) \leq \sum_{e \in O} \frac{b_p}{2\rho APS_p} v_p(e \mid C^e) \leq (b^0 - b_p)$$

The claim follows by rearranging (scaling both sides by $\frac{2\rho APS_p}{b_p}$). ∎

We are now ready to prove Lemma 23.

Proof. Considering Claim 27 and Claim 28, we have that either $v_p(C) \geq \rho b_p$, or the following holds:

$$APS_p \leq L^0 \leq v_p(C) + L^f + b_p \cdot \sum_{e \in O} v_p(e \mid C) \leq v_p(C) + L^f + b_p \cdot 2\rho \frac{APS_p}{b_p}(b^0 - b_p)$$

By rearranging the above, and plugging $b^0 = 1$ we obtain:

$$APS_p \cdot (1 - 2\rho b^0 + 2\rho b_p) = APS_p \cdot (1 - 2\rho + 2\rho b_p) \leq v_p(C) + L^f$$

By setting $\rho = \frac{1}{3 - 2b_p}$ the above gives $L^f + v_p(C) \geq \rho APS_p$. So far we obtained that either $v_p(C) \geq \rho APS_p$, or agent p is guaranteed to have a total final value of $\min\{L^f + v_p(C), 2\rho APS_p\}$ (Claim 26). Hence, in both cases, when setting $\rho = \frac{1}{3 - 2b_p}$, agent p is guaranteed to have at least ρ fraction of her APS. In the special case of equal entitlements (where $b_p = \frac{1}{n}$) it implies $\rho = \frac{n}{3n-2}$. This completes the proof of Lemma 23. ∎

We now restate and prove Theorem 3.

Theorem 3. *Consider the bidding game described above, and an agent p with a submodular valuation function and entitlement b_p. Setting $\rho = \frac{1}{3 - 2b_p} > \frac{1}{3}$, a bidding strategy referred to as proportional(ρ) guarantees agent p a value of at least $\rho \cdot APS_p$. (In the case of equal entitlements, this gives $\rho = \frac{n}{3n-2}$.)*

Proof. Lemma 23 handles the case that $v_p(e) \leq 2\rho APS_p$ for every item e. It remains to handle the case that there are items e of value $v_p(e) > 2\rho APS_p$.

Consider an input instance I_0. Run the bidding game with p using the proportional bidding strategy. As described in $proportional_2$, s denotes the last round in which there was an unallocated item e with $v_p(e) > 2\rho APS_p$. If agent p won some item by the end of round s, then she has a value of at least $2\rho APS_p$, and we are done. Hence, we may assume that agent p did not win any item in the first s rounds. Recall the definitions of residual instance I_s, $\hat{I}_s$ and γ (which is the scaling factor between bids in I_s and $\hat{I}_s$) presented in $proportional_2$.

By the definition of s, in each of the first s rounds, there is an available item with $v_p(e) > 2\rho APS_p$. Thus agent p bids her entire budget b_p in each such round, and the (other) agent who wins the round spends at least b_p. (In the special case of equal entitlement, this means that in each of the first s rounds, some agent wins a single item and becomes inactive. Hence, in the residual instance I_s there are $n - s$ active agents (including p), each active agent has her entire original budget, and no active agent has any items.)

Setting $\gamma = b^s \leq 1 - s \cdot b_p$, the entitlement of each agent in $\hat{I}_s$ is $\hat{b_i^s} = \frac{1}{\gamma} b_i^s$.

Recall that agent p simulates the bidding strategy ($proportional_1$) on $\hat{I}_s$. A bid p_i^r in I_s is interpreted as a bid of $\frac{1}{\gamma}\hat{p}_i^{r-s}$ in $\hat{I}_s$. By Claim 21, the APS of agent p at $\hat{I}_s$ stays the same as the APS of p in the original instance. Hence, in I_s, agent p will get the same bundle as she gets in the run on $\hat{I}_s$. By Lemma 23, this bundle is of value at least ρAPS_p, as desired. ∎

D Proof of Theorem 4

We now present rigorous proofs for two claims that were used in the proof of Proposition 16, and will also be used in the proof of Theorem 4. These claims imply that we can assume without loss of generality that there are no agents of type X_0, and at most $\frac{n}{2}$ agents of type X_1. For both claims, we present their proofs under a framework in which we prove by induction on n (the number of players) the statement that the proportional bidding strategy guarantees a ρ fraction of MMS_p to agent p. The statement is clearly true for the base case of $n = 1$. Hence for a given value of $n > 1$, we just prove the inductive step (assuming that the statement has already been proved for all $n' < n$). To simplify the presentation of the proofs of the claims, we make two assumptions. We stress that these assumptions do not affect the correctness of the claims. (Alternatively, these assumptions can be turned into facts, by simply incorporating them as part of the description of the proportional bidding strategy.)

- Agent p truncates her valuation function at $t = MMS_p$, (i.e., $v_p \leftarrow v_p^t$).
- Agent p fixes some order over $\mathcal{M}$ in order to break ties consistently. If she wins a round and there is more than one item with the highest marginal value, she will break the tie by picking the item appearing earlier in this order.

Claim 29. *In the inductive framework presented above, suppose that there is an allocation instance I with n agents, and that agent p has a submodular valuation function v_p and uses the proportional bidding strategy. If in the run of the bidding game there is at least one agent of type X_0, then agent p is guaranteed to receive a bundle of value at least ρMMS_p.*

Proof. Recall that by our assumption, the original valuation function v_p is truncated so that no bundle has value larger than MMS_p. Consider a run R of the bidding game, in which agent p uses the proportional bidding strategy with an order Π over the items $\mathcal{M}$. Suppose that in this run agent i is of type X_0. Let e denote the single item that agent i takes, and let r denote the round number in which i took item e. Suppose for the sake of contradiction that in this run R agent p receives a bundle of value strictly smaller than ρMMS_p. Then we show a new allocation instance I' with only $n-1$ agents, and a run R' in which an agent p' that uses the proportional bidding strategy gets value smaller than $\rho MMS_{p'}$. This contradicts our induction hypothesis.

The instance I' contains $n-1$ agents and the set $\mathcal{M}' = \mathcal{M} \setminus \{e\}$ of items. The valuation $v_{p'}$ is identical to v_p, though defined only over $\mathcal{M}'$ (for every $S \subseteq \mathcal{M}'$ we have that $v_{p'}(S) =$

$v_p(S)$). Note that even though there are $n-1$ agents, $MMS_{p'} = MMS_p$. (The MMS partition for v_p, after dropping the bundle containing e, can serve as an MMS partition for $v_{p'}$, showing that $MMS_{p'} \geq MMS_p$. Being truncated at MMS_p, we have that $MMS_{p'} \leq MMS_p$.) The order Π' used by p' over $\mathcal{M}'$ is identical to Π (but with item e removed). The run R' is identical to R, except for the following four changes:

- Agent i and her bids are removed.
- Agent p' plays in R' the role that agent p played in R.
- Round r (the one in which item e was taken) is removed.
- All bids are scaled by $\frac{n-1}{n}$ (as the budgets of agents are $\frac{1}{n-1}$ rather than $\frac{1}{n}$)

Consequently, the sequence of bids and item choices that p' makes in R', being derived from the bids of p in R, is consistent with p' using the proportional strategy. The bundle received by p' in R' is exactly the same bundle that p received in R, and hence of value below $\rho MMS_{p'}$. This contradicts our induction hypothesis. ■

Claim 30. *In the inductive framework presented above, suppose that there is an allocation instance I with n agents, and that agent p has a submodular valuation function v_p and uses the proportional bidding strategy. If in the run of the bidding game, more than $\frac{n}{2}$ agents are of type X_1, then agent p is guaranteed to receive a bundle of value at least ρMMS_p.*

Proof. Consider a run R of the bidding game, in which agent p uses the proportional bidding strategy with an order Π over the items $\mathcal{M}$. Suppose that in this run, more than $\frac{n}{2}$ agents are of type X_1. Suppose for the sake of contradiction that in this run R agent p receives a bundle of value strictly smaller than ρMMS_p. Then we show a new instance I' with only $n-1$ agents, and a run R' in which agent p' that uses the proportional bidding strategy gets a value strictly smaller than $\rho MMS_{p'}$. This contradicts our induction hypothesis.

Consider an MMS partition $\{B_i\}_{i=1}^n$ with respect to v_p. Since there are more than $\frac{n}{2}$ agents of type X_1, there are at least $n+1$ items taken by these agents. Hence, there exists a bundle B_i in the MMS partition that contains at least two of these items. Denote these items by e_1 and e_2, the agents who win them by a_1 and a_2, and the rounds in R in which these items were taken by r_1 and r_2.

The instance I' contains $n-1$ agents and the set $\mathcal{M}' = \mathcal{M} \setminus \{e_1, e_2\}$ of items. The valuation v'_p is identical to v_p, though defined only over $\mathcal{M}'$ (for every $S \subseteq \mathcal{M}'$ we have that $v'_p(S) = v_p(S)$). Note that even though there are $n-1$ agents, $MMS_{p'} = MMS_p$. ($\{B_i\}_{i=1}^n$, the MMS partition for p, can serve as an MMS partition for p', after dropping the bundle containing e_1 and e_2, showing that $MMS_{p'} \geq MMS_p$. Being truncated at MMS_p, we have that $MMS_{p'} \leq MMS_p$). The order Π' used by p' over $\mathcal{M}'$ is identical to Π (but with items e_1, e_2 removed). The run R' is identical to R, except for the following changes:

- Rounds r_1 and r_2 (the rounds in which items e_1, e_2 were taken) are removed.
- Agent p' plays in R' the role that agent p played in R.
- Agent a_1 and her bids are removed.
- If $a_1 \neq a_2$ (i.e e_1 and e_2 were taken by different agents) then let e_3 denote the other item taken by a_1 in R, and let r_3 denote the round in which it was taken. In R' agent a_2 takes item e_3 in round r_3 (instead of e_2 that a_2 took in the run R - recall that $e_2 \notin \mathcal{M}'$).
- All bids are scaled by $\frac{n-1}{n}$ (as the budgets of agents are $\frac{1}{n-1}$ rather than $\frac{1}{n}$)

Consequently, the sequence of bids and item choices that p' makes in R', being derived from the bids of p in R, is consistent with p' using the proportional strategy. The bundle received by p' in R' is exactly the same bundle that p received in R, and hence of value below $\rho MMS_{p'}$. This contradicts our induction hypothesis. ■

We are now ready to prove Theorem 4.

Proof. Consider an arbitrary allocation instance with equal entitlement, and an arbitrary agent p with a submodular valuation function v_p. We wish to show that by using the proportional strategy in the ρ-altruistic version of the bidding game (with a choice of $\rho = \frac{10}{27}$), p receives a bundle of value at least $\frac{10}{27} \cdot MMS_p$. Equivalently (by scaling v_p so that $MMS_p = \frac{27}{10}$), we wish to show that if $MMS_p = 2.7$, then p receives a bundle of value at least 1.

We begin by presenting a claim that will be used later. Let $\{B_i\}_{i=1}^n$ be an MMS partition of agent p, i.e., for each i $v_p(B_i) \geq MMS_p$. Let f be the earliest round after which no other agents are active. Let C denote the bundle p has by the end of round f, and let $O \subseteq \mathcal{M}$ be the set of items taken by other agents. The following claim is similar in nature to Claim 26

Claim 31. *In the altruistic version of the bidding game, for each $i \in [n]$, the final value that p has is at least*

$$\min\{\rho MMS_p, v_p(C) + v_p(B_i \setminus O \mid C)\} = \min\{\rho MMS_p, v_p(C \cup \{B_i \setminus O\})\}$$

Proof. If p is not active at the end of round f, then p has a value of at least ρMMS_p at that time, i.e., $v_p(C) \geq \rho APS_p$, as desired.

If p is active at the end of round f, we consider two cases. If some items remain after round f, then agent p is the only remaining active agent. Hence p is the only agent to win items from $\mathcal{M}$. Then, the agent will keep winning items until either she becomes inactive (and has value at least ρMMS_p) or until she wins all remaining items from $B_i \setminus O$. It remains to handle the case of agent p being active after time f while no items remain. In this case, $v_p(B_i \setminus O) = v_p(\emptyset) = 0$, so the bound is trivial. ■

Every negative example (i.e., an instance of the problem in which agent p does not get at least ρMMS_p) implies (by scaling the valuation function of the agent) the existence of another instance of the problem, in which $MMS_p = \frac{1}{\rho}$ and p gets a bundle of value less than 1. Denote the MMS of agent p by z. We set up a system of linear inequalities. The linear inequalities encode various constraints on the v_p value of items that agents other than p receive, given that p is using the proportional strategy and ends up with a value of at most 1. We show that the system of inequalities is feasible only if $z \leq 2.7$. This implies that $\rho \geq \frac{10}{27}$.

To simplify the presentation of the system of inequalities, we shall have a slight abuse of terminology. The term *payment* of an agent (say, for an item in round r) will correspond to the bid of agent p (in round r), even though the actual payment might be larger (if the bid of the winning agent was strictly higher than the bid of agent p). With this abuse of notation, the sequence of payments that an agent makes is nonincreasing. This implies that the total payment of an agent that wins $t > 1$ items is at most $\frac{t}{t-1}$, because otherwise the agent spent more than 1 on the first $t-1$ items that she picked, and would become inactive before winning its tth item. Observe that the total payment of p can be assumed to be not more than 1, as otherwise, the bundle that p receives according to the proportional strategy has v_p value at least 1, and we are done.

Consider a negative example for some ρ, i.e., an instance in which an agent does not get a ρ fraction of her MMS, and she gets a value of at most 1. Keeping this instance in mind, we will present our set of inequalities, and explain why the instance respects each of them.

The system of inequalities has nonnegative variables, $x_1, x_2, x_3, x_4, y, q, z$. For each $1 \leq i \leq 4$, x_i represents the fraction of agents who satisfy the following two conditions:

1. The agent takes $i+1$ items.
2. The total payments that the agent made is at least $\frac{6}{5}$.

The reason why we do not introduce a variable x_0 (for agents who take one item and pay at least $\frac{6}{5}$) is because Claim 29 implies that we can assume that there are no such agents. This also implies that a payment for an item is never larger than 1. Variable y represents the fraction of the rest of the agents, those that either take at least six items or paid at most $\frac{6}{5}$. Observe that in either case, any such agent paid a total of at most $\frac{6}{5}$. Variable z represents the MMS of agent p (and recall that we scale the valuation v_p so that $MMS_p = \frac{1}{\rho}$). The variable q needs a more detailed explanation. For every $0 \leq s \leq \frac{1}{2}$, let α_s denote the fraction of agents that satisfy the following conditions:

1. The agent takes at least three items.
2. The total payments made by the agent is larger than $\frac{6}{5}$.
3. Her payment for the first item that she takes is $\frac{1}{2} + s$.

Note that this implies that the number of items that such an agent takes is either three, in which case her total payments are at most $\frac{3}{2} - s$, or four, in which case her total payments are at most $\frac{5}{4} - \frac{s}{2}$, which is smaller than $\frac{4}{3} - s$ for $s < \frac{1}{6}$ (note that if $s \geq \frac{1}{6}$ then her total payments when taking four items are at most $\frac{6}{5}$, and hence this case is excluded). The variable q represents $\int_0^{\frac{1}{2}} s \cdot \alpha_s ds$.

We turn to present the linear inequalities, and explain why each of these inequalities must hold if p executes the proportional bidding strategy

1. $x_1 + x_2 + x_3 + x_4 + y \leq 1 - \frac{1}{n}$. The variables x_1, x_2, x_3, x_4, y represent fractions of the total number of agents, and agent p is not included in these fractions. As the total number of agents is n, the sum of the fractions is at most $1 - \frac{1}{n}$.
2. $z - 2x_1 - (3/2)x_2 - (4/3)x_3 - (5/4)x_4 - (6/5)y + q \leq 1$. Fix $\{B_i\}_{i=1}^n$, an MMS partition for agent p. Since we assume p is active at the end of the algorithm, Claim 31 implies that the final value of p is at least $v_p(C) + v_p\{B_i \setminus O \mid C\})$. Since the final value of p is at most 1, we obtain $1 \geq v_p(C) + v_p\{B_i \setminus O \mid C\})$

   Recall that $O \subseteq \mathcal{M}$ is the set of items taken by other agents. Consider the following partition of O , $\{O_i = B_i \cap O\}_{i=1}^n$. Then, for each i,

$$\begin{aligned}
1 &\geq v_p(C) + v_p\{B_i \setminus O \mid C\}) \\
&= v_p(C) + v_p(B_i \setminus O_i \mid C) \\
&\geq v_p(C) + v_p(B_i \mid C) - v_p(O_i \mid C) \\
&= v_p(C \cup B_i) - v_p(O_i \mid C) \\
&\geq v_p(B_i) - v_p(O_i \mid C) \\
&= z - v_p(O_i \mid C)
\end{aligned}$$

   Denote the total payments of agents other than p as P_O. Notice that $\sum_i v_p(O_i \mid C) \leq \sum_i \sum_{e \in O_i} v_p(e \mid C) \leq P_O$. Hence, there exists $i \in [n]$ for which $v_p(O_i \mid C) \leq \frac{1}{n} \cdot P_O$.

   We now upper bound $\frac{1}{n} \cdot P_O$. Recall that the total payment of an agent that takes $t > 1$ items is at most $\frac{t}{t-1}$. Moreover, if agents of type x_2 and x_3 have a payment strictly larger than $\frac{1}{2}$ for their first item, then they do not reach the maximum payment they can achieve ($\frac{3}{2}$ and $\frac{4}{3}$). By the definition of q, we obtain that their total payment is reduced by at least q. (Recall the discussion that follows the definition of q. It shows that if an agent of type x_2 pays $\frac{1}{2}+s$ for her first item, then the maximum payment she might reach after taking her three items is at most $\frac{3}{2}-s$. Likewise, it shows that if an agent of type x_3 pays $\frac{1}{2}+s$ for her first item, then the maximum payment she might reach after taking her four items is at most $\frac{4}{3}-s$.) Therefore we have

$$\frac{1}{n}\cdot P_O \leq 2x_1+(3/2)x_2+(4/3)x_3+(5/4)x_4+(6/5)y-q$$

The constraint follows by using this last inequality and the two previous inequalities $1 \geq z - v_p(O_i \mid C)$ and $v_p(O_i \mid C) \leq \frac{1}{n}P_0$.

3. $2x_1 \leq 1$. We can assume that at most half of the agents are of type x_1, by Claim 30.

4. $\frac{6}{5}y + q \geq (z-1-\frac{3}{2})\cdot(3-2x_1-3x_2-4x_3-5x_4)$. (This is not a linear inequality, but it will be linearized before it will be used.) We refer to items taken by agents represented by x_1, x_2, x_3, x_4 as *primary*, and to items taken by agents represented by y as *secondary*. How much payment do secondary items need to have so that payment of at least $z-1$ is paid for items in each bundle B_i of the MMS partition (which is a necessary condition for p having a total value of at most 1 Claim 31)? The total number of primary items is $(2x_1+3x_2+4x_3+5x_4)n$. We split the argument into two cases.

   In the first case, there are at most $3n$ *primary* items. The number of primary items missing to complete this number to $3n$ is $(3-2x_1-3x_2-4x_3-5x_4)n$. We refer to each such missing item as a *deficiency unit*. As noted above, we assume no two primary items of agents of type x_1 are taken from a single bundle. We analyze the distribution of *deficiency units* over the MMS bundles when a bundle with $\ell \leq 3$ *primary* items has $3-\ell$ *deficiencies*. Then, we will find properties of the distribution of *deficiencies* that minimizes the amount of payment to be paid by y agents to reach $z-1$ payment in each MMS-bundle.

   - A bundle B_j in the MMS partition with one *deficiency unit*, has a payment for *primary* items of at most $1+\frac{1}{2}+s$. At least $z-1-\frac{3}{2}-s$ needs to be paid by y agents in order to surpass a $z-1$ payment in B_j. I.e., a $z-1-\frac{3}{2}-s$ for the one unit of deficiency.
   - A bundle B_j in the MMS partition with two *deficiency units* has a payment for *primary* items of at most 1. At least $z-1-1$ needs to be paid by y agents to surpass a $z-1$ payment in B_j. I.e., a $\frac{z-2}{2}$ on average per *deficiency unit.*
   - A bundle with three *deficiency units*, needs a payment of at least $z-1$ by y agents. I.e., a $\frac{z-1}{3}$ on average for per deficiency unit.

   For $z \leq 3$, the minimum payment per deficiency occurs when every deficiency unit is in a different bundle (i.e., these bundles have one deficiency unit, the case of bullet two). Then, in each bundle with two primary items, one item can have payment arbitrarily close to 1, and the other $\frac{1}{2}+s$, with s as above. Hence, to reach $\sim (z-1)$, the bundle needs $\sim (z-1-\frac{3}{2})-s$. As integrating over *all* s we get q, the above considerations give the constraint $(6/5)y \geq (z-1-\frac{3}{2})\cdot(3-2x_1-3x_2-4x_3-5x_4)-q$ as desired.

   In the second case, the number of primary items is greater than $3n$. If $z > 2.5$, then the right-hand side of constraint 4 is a product of a positive scalar and a negative scalar, resulting in a negative scalar. By non-negativity of variables q and y, the left-hand side of the constraint is a non-negative scalar. Hence constraint 4 is valid for the range of values of z which will be considered in the proof, which only includes values larger than 2.5.

The fourth constraint is not a linear inequality. Nevertheless we may make use of the system of four constraints as if it is a system of linear inequalities. We do so by substituting candidate values for z (these values are larger than 2.5, so that the fourth constraint remains valid), simplifying the second and fourth constraints after making this substitution, and checking whether the resulting system of inequalities is feasible. If it is not feasible, this certifies that the substituted value for z was too high, and hence we obtain an upper bound on z. Substituting $z = 2.7$, the constraints become:

1. $x_1 + x_2 + x_3 + x_4 + y \leq 1 - \frac{1}{n}$
2. $-2x_1 - \frac{3}{2}x_2 - \frac{4}{3}x_3 - \frac{5}{4}x_4 - \frac{6}{5}y + q \leq -1.7$
3. $2x_1 \leq 1$
4. $-\frac{2}{5}x_1 - \frac{3}{5}x_2 - \frac{4}{5}x_3 - x_4 - \frac{6}{5}y - q \leq -\frac{3}{5}$.

Summing up the four constraints multiplied by coefficients $(1.8, 1, 0.2, 0.5)$ respectively, we obtain:

$$(\frac{2}{5} - \frac{1}{3})x_3 + \frac{x_4}{20} + \frac{q}{2} \leq -\frac{1.8}{n}$$

As x_3, x_4 and q are non-negative, this is a contradiction. Hence $z < 2.7$. In fact, the term $-\frac{1.8}{n}$ on the right hand side implies that $z \leq 2.7 - \Omega(\frac{1}{n})$, which in turn implies that $\rho \geq \frac{10}{27} + \Omega(\frac{1}{n})$. ■

E Proofs concerning polynomial time algorithms

In this section, we demonstrate how to overcome the need for computing the MMS or APS value, which is an NP-hard task, in the bidding strategies presented in Theorem 3 and Theorem 4. Namely, we show polynomial time allocation algorithms that output allocations that approximate the APS and MMS within ratios almost as good as those shown in Theorems 3 and 4. Our proof is based on a technique that was introduced in [GHS+18].

Consider an allocation instance I with n agents, m items, integer valued valuation functions, with values of bundles in the range $[0, K]$ (that is, $K = \max_{i \in \mathcal{N}} v_i(\mathcal{M})$). (Alternatively, for a valuation function v_i that is not integer valued, K denotes an upper bound on $\frac{v_i(\mathcal{M})}{v_i(S)}$, over all sets $S \subset \mathcal{M}$ for which $v_i(S) > 0$. Namely, K is an upper bound on the ratio between the values of the most valuable bundle and least valuable bundle of positive value.) We assume that all allocation algorithms have value query access to the valuation functions of the agents. We say that an allocation algorithm runs in polynomial time if both its running time and the number of value queries that it makes are polynomial in n, m and $\log K$. In the following presentation, we consider the APS as our share notion, and allow for agents of arbitrary entitlement. We remark that the same principles apply (with straightforward modifications) to settings with equal entitlement, and the MMS as a fairness notion.

Remark 32. *Definition 33 and Theorem 34 (that will be presented shortly) involve an approximation ratio ρ. All results extend without any change in the proofs to settings in* *which ρ is not a fixed constant, but instead a function of the entitlement (such approximation ratios appear in Theorem 3, for example). That is, for agent i with entitlement b_i, the approximation ratio is $\rho(b_i)$.*

Definition 33. *For $\rho > 0$, we say that an allocation algorithm is a ρ-APS algorithm for a class C of valuations, if given any allocation instance with valuations from the class C, the algorithm outputs an allocation $(A_1, \ldots, A_n)$ in which every agent i gets a bundle A_i of value $v_i(A_i) \geq \rho \cdot APS_i$. We say that an allocation algorithm is a* conditional *ρ-APS algorithm for a class C of valuations, if given any allocation instance with valuations from the class C, and given any vector $(t_1, \ldots, t_n)$, the algorithm outputs an allocation $(A_1, \ldots, A_n)$ that satisfies the following property: for every agent i, if it happens that $t_i \leq APS_i$, then $v_i(A_i) \geq \rho \cdot t_i$.*

The proof of the following theorem is similar to a proof of a related theorem proved in [GHS+18]. We present its proof for completeness.

Theorem 34. *Fix arbitrary $\rho > 0$ and arbitrary $\varepsilon > 0$. Then every polynomial time conditional ρ-APS algorithm for a class C of valuations can be transformed into a polynomial time (unconditional) $(1-\varepsilon)\rho$-APS algorithm for the class C of valuations. Here, the dependence on ε of the running time of the unconditional algorithm is a multiplicative factor of $O(\frac{1}{\varepsilon})$.*

Proof. In Algorithm 1 we give an unconditional $(1-\varepsilon)\rho$-APS algorithm, using a conditional ρ-APS algorithm as a blackbox.

Algorithm 1: An *unconditional* $(\rho - \varepsilon)$-APS algorithm using a *conditional rho*-APS algorithm as a blackbox

Data: $\mathcal{M}, \mathcal{N}, \langle v_1, \ldots, v_n \rangle, \varepsilon, K$
1 For every $i \in \mathcal{N}$, $t_i \leftarrow v_i(\mathcal{M})$;
2 **while** *true* **do**
3   Run *conditional*-ρ-APS algorithm with guesses $\langle t_1, \ldots, t_n \rangle$, resulting $\mathcal{A} = \langle A_1, \ldots, A_n \rangle$;
4   **if** $\exists i$, *such that* $v_i(A_i) < \rho_i \cdot t_i$ **and** $t_i \geq v_i(\mathcal{M}) \cdot \frac{1}{K}$ **then**
5         $i = i$ which satisfies the condition;
6         $t_i \leftarrow (1-\varepsilon)t_i$;
7   **else**
8   Return $\mathcal{A}$ and exit;
9   **end**
10 **end**

Remark 35. *In Algorithm 1, we require having K (the maximum ratio between largest and smallest value bundles) as an input. However, in the case of agents with submodular valuations, K is not needed as input. Instead, K can be computed efficiently as it equals $\max_{i \in \mathcal{N}}\{\max_{\{e \in \mathcal{M} | v_i(e) > 0\}}\{\frac{v_i(\mathcal{M})}{v_i(e)}\}\}$.*

Claim 36. *During the whole run of Algorithm 1, for every agent i, $t_i \geq (1-\varepsilon)APS_i$.*

Proof. Fix an agent i. At the beginning of the algorithm, $t_i = v_i(\mathcal{M}) \geq APS_i$. during the algorithm we only reduce the value of t_i each time by a factor of $(1-\varepsilon)$. Consider the first time when $t_i < APS_i$. Then, $t_i \geq (1-\varepsilon)APS_i$ (since in its previous value, the variable t_i was greater than APS_i). Since $t_i < APS_i$, by Definition 33, every time we run command 3, the bundle of agent i in the resulted allocation $\mathcal{A}$ is of value $\geq \rho t_i$, so we will not reduce the value of t_i. ■

We prove the correctness of Algorithm 1. If the algorithm terminates, then it returns an allocation $\mathcal{A} = \langle A_1, \ldots, A_n \rangle$ with the property that for every agent i with $t_i \geq v_i(\mathcal{M}) \cdot \frac{1}{K}$, $v_i(A_i) \geq \rho_i \cdot t_i$. By Claim 36, we have that each variable t_i holds during the algorithm a value greater than $(1-\varepsilon)APS_i$. Thus, given that the algorithm terminates, each such agent i gets a bundle A_i of value at least $(1-\varepsilon)\rho APS_i$. It remains to consider those agents i with $t_i < v_i(\mathcal{M}) \cdot \frac{1}{K}$, and for which furthermore, $APS_i > 0$. The assumption that $APS_i > 0$ implies that $APS_i \geq v_i(S)$, where S is the bundle of minimum value. Also, $t_i < v_i(\mathcal{M}) \cdot \frac{1}{K} \leq v_i(S)$. Thus, by Claim 36, $v_i(A_i) \geq \rho t_i \geq v_i(S) \geq t_i$ (where the middle inequality holds because every bundle of positive value has value at least $v_i(S)$). As $APS_i < \frac{t_i}{1-\varepsilon}$ (otherwise, the condition of step 4 of the algorithm would not allow the value t_i to be reached), we have that $v_i(A_i) \geq (1-\varepsilon)APS_i$.

We turn to analyze algorithm's time complexity. Let $T(n, m, \log K)$ denote the running time of the *conditional*-ρ-APS algorithm (called in command 3 of Algorithm 1). The number of times that command 3 is run is at most $n \cdot \log_{(1-\varepsilon)} K = n \cdot \frac{1}{\varepsilon} \cdot \log K$ (because in each iteration, at least one t_i drops by a factor of $1-\epsilon$, and the total drop in value of t_i is at most a factor of K). Thus the overall runtime of Algorithm 1 is $O\left(n \cdot \log(K) \cdot \frac{1}{\varepsilon} \cdot T(n, m, \log K)\right)$, yielding a polynomial time algorithm (under the assumption that $T(n, m, \log K)$ is polynomial).

■

The following theorem, Theorem 37, is a relatively straightforward consequence of Theorems 3 and 4.

Theorem 37. *There is a polynomial time conditional ρ-APS algorithm for submodular valuations, with $\rho = \frac{1}{3-2b_i}$. For the equal entitlement case, there is a polynomial time conditional ρ-MMS algorithm for submodular valuations, with $\rho = \frac{10}{27} + \Omega(\frac{1}{n})$.*

Proof. As described in the *proportional bidding strategy*, both in the original and *altruistic* version of the bidding game, an agent i executes the *proportional bidding strategy* is required to know/compute their APS (MMS). Based on her APS_i (MMS_i), the agent knows how to bid. Consider a modification of the bidding strategy, in which agent i receives as an auxiliary input a value t_i (instead of computing her true APS (MMS) value), and truncates her valuation function at t_i, namely $v_i \leftarrow v_i^{t_i}$ (Definition 11). Then, the agent infers her bidding using the value of t_i instead of APS_i (MMS_i).

The conditional ρ-APS algorithm (or ρ-MMS algorithm) is simply to simulate the bidding game (or altruistic bidding game) with the t_i values as auxiliary inputs to the agents, and having each agent follow her respective modified proportional bidding strategy. Now, giving value queries, the bidding game is simulated in polynomial time, where we break ties arbitrarily. It remains to show the correctness of the algorithm. For this, notice that given a value $t_i \leq APS_i$ (MMS_i), then by Claim 12 the APS (MMS) value of agent i is reduced to t_i, and the truncation preserves submodularity. Now the conditions of Theorem 3 (Theorem 4) are met, and agent i gets a bundle of value at least ρ-APS (-MMS), as desired. ■

Combining theorems 34 and 37 we prove Corollary 5, which is restated here for convenience.

Corollary 5. *For agents with submodular valuations, there are polynomial time algorithms offering the following guarantees. In the case of arbitrary entitlements, each agent gets at least $\frac{1}{3}$-APS. For the case of equal entitlements, each agents gets at least $\frac{10}{27}$-MMS.*

Proof. Theorem 37 states that there is a conditional ρ-APS algorithm for submodular valuations with $\rho = \frac{1}{3-2b_i}$. We can assume $b_i \geq \frac{1}{m}$ (as otherwise the APS of agent i

is 0, implying that $\rho = \frac{1}{3} + \Omega(\frac{1}{m})$. Thus, setting ε to be $O(\frac{1}{m})$ in Algorithm 1 yields the existence of an unconditional $\frac{1}{3}$-APS polynomial time algorithm, as desired. In a similar way for the MMS, by setting ε to be $O(\frac{1}{n})$ we obtain an unconditional $\frac{10}{27}$-MMS polynomial time algorithm. ■

Finally, we restate and prove Corollary 6.

Corollary 6. *There is a polynomial time allocation algorithm, which simultaneously guarantees for submodular agents $\frac{1}{3}$-APS, for additive agents $\frac{3}{5}$-APS, and for Unit-demand agents 1-APS.*

Proof. For the sake of this proof, we consider a modified version of the original bidding game. When an agent wins a round (i.e., she is the highest bidder), instead of picking one item, she can pick $k \geq 1$ of the remaining items and pay k times her bid. We consider this version of bidding game because this is the version for which it was previously shown (in [BEF21]) that additive agents have a bidding strategy that guarantees themselves $\frac{3}{5}$ of their APS. That strategy is implementable in polynomial time.

We show that in this version of the bidding game, any submodular, additive, or unit-demand agent has a strategy that guarantees herself the relevant guarantee of her APS.

- Submodular agents have a strategy that guarantees $\frac{1}{3}+\Omega(\frac{1}{n})$ fraction of their APS. The key property that enables the proportional bidding strategy presented in Theorem 3 to maintain its guarantee also in the current version of he bidding game is the fact the bidding sequence of a submodular agent executing the proportional bidding strategy is non-increasing. (Observation 22). Therefore, if another agent, o, wins a round and decides to pick $k > 1$ items, for the submodular agent perspective, this is equivalent to k rounds in which o raises the same bid, and all other agents raise a bid of zero. Since the bids of the submodular agent are non-increasing, the submodular agent will not raise a bid strictly greater than o's bid. Thus, with adversarial tie-breaking, we can guarantee that in both cases of the bidding game, the bundle of the submodular agent will remain the same, and the $\frac{1}{3} + \Omega(\frac{1}{n})$ guarantee from Theorem 3 holds.

- Additive agents have a strategy that guarantees a $\frac{3}{5}$ fraction of their APS. The proof is presented in [BEF21]. In addition, [BEF21] presented a polynomial time implementation of this strategy, that does not require knowing the APS value of an agent.

- Unit-demand agents have a strategy that guarantees a 1-APS. Consider a unit-demand agent p with entitlement b_p. It is easy to verify that the APS of p is the $\lceil\frac{1}{b_p}\rceil$th most valuable item, and 0 if there are fewer than $\lceil\frac{1}{b_p}\rceil$ items. We claim that the simple strategy of p bidding her entire budget in each round and, upon winning, taking the most valuable remaining item guarantees p her APS. Since p bids her entire budget until she wins an item, in every round she does not win, at least b_p of the total budget of agents is spent. Hence, p must win one of the first $\lfloor\frac{1}{b_p}\rfloor$ rounds. By that, she guarantees herself one of the $\lfloor\frac{1}{b_p}\rfloor$ most valuable items, which is at least her APS. Note that this strategy does not require agent p to know her APS value, and given access to value queries, her strategy is polynomial-time.

It remains to show a transformation from the existence of approximate fair allocation (induced by the above strategies) to a polynomial time algorithm. The proof of this is similar to the proof of Corollary 5, and is omitted. ■

F Negative examples

We restate and prove Proposition 17.

Proposition 17. *For every constant $\rho > \lim_{k\to\infty} \rho_k \simeq 0.3716$ (where for each $k \in \mathbb{N}$ we will define ρ_k in the proof), there is an allocation instance with equal entitlements and an adversarial run of the* altruistic *version of bidding game, in which an agent p that has a submodular valuation function and uses the proportional bidding strategy gets a bundle of value smaller than ρMMS_p.*

Proof. We present a series of instances in which agent p with a submodular valuation function executes the proportional bidding strategy.

The instances are parameterized by $k \in \mathbb{N}$. The kth instance will be as follows: Define

$$q_1 = 2$$
$$q_k = 1 + \prod_{i=1}^{k-1} q_i$$

(This sequence is known as the Sylvester sequence)

The number of agents will be: $n_k = q_{k+1} - 1$ (for example, for $k = 2$, $n_k = 2 \cdot 3 \cdot 7 = 43$).

The set of item is $\mathcal{M} = \{e_{i,j}\}$ for $1 \le i \le k+1$, $1 \le j \le n$ ($n \cdot (k+1)$ items).

If we think of $e_{i,j}$ as arranged in a matrix, then all the items in a row are copies of the same item and are substitutes. The value of items from different rows is additive.

For any $1 \le i \le k$ and for any j, $v_p(e_{i,j}) = \frac{1}{q_i - 1}$. For $i = k+1$ and any j, $v_p(e_{k+1,j}) = 1$. For example, if $k = 3$, there are 43 agents and columns, and in each column j, $v_p(e_{1,j}) = 1$, $v_p(e_{2,j}) = \frac{1}{2}$, $v_p(e_{3,j}) = \frac{1}{6}$, $v_p(e_{4,j}) = 1$.

- v_p is submodular. The marginal value of each item is weakly decreasing (the marginal value of item $e_{i,j}$ to a set S is either $v_p(e_{i,j})$ or 0, depending on whether the set S already contains an item from the i'th row).
- The columns C_j of the matrix $\{e_{i,j}\}$ form an MMS partition. The value of every bundle is at most $v_p(\mathcal{M})$, and in this partition, the value of each bundle (column) is exactly $v_p(\mathcal{M})$.
- $APS_p = MMS_p = v_p(\mathcal{M}) = v_p(C_j) = v_p(e_{k+1,j}) + \sum_{i=1}^{k} v_p(e_{i,j}) = 1 + \sum_{i=1}^{k} \frac{1}{q_i - 1}$
- q_i divides n, for every $i \le k$.

For convenience, we assume w.l.o.g that the budget of each agent equal her MMS.

For every k, we first show a run of the bidding game with adversarial bidding of the other agents, in which agent p executes the proportional bidding strategy with $\rho_k = \frac{1}{MMS_p} = \frac{1}{1+\sum_{i=1}^{k} \frac{1}{q_i - 1}}$, and she receives a value of precisely 1 (she gets the bundle that consists only of items from row $k+1$), which is a $\frac{1}{MMS_p}$ of her MMS. The series of ρ_k is monotonically decreasing and bounded by 0, so $\lim_{k\to\infty} \rho_k$ exists. (Sylvester's sequence grows at a doubly exponential rate. Hence, the sequence of ρ_k converges very fast.)

Then, by *proportional*(ρ_k), in each round, agent p bids the highest marginal value of the remaining items. Moreover, each agent who spends more than 1 value from her budget becomes inactive. We now present the adversarial run.

In round 1, p bids 1, and is allowed to win. She selects an item of value 1 from row $k+1$.

In each of the next n rounds, at least one of the first $\frac{n}{2}$ other agents bids 1, and upon winning (note that p bids 1 in each of these rounds, and the algorithm is assumed to brake the ties adversarially), takes an item from the first row (i.e., $e_{1,j}$). All items of the first row are taken by $\frac{n}{2}$ of the other agents. These $\frac{n}{2}$ agents surpass a payment of 1 and become inactive.

In each of the next n rounds, at least one of the next $\frac{n}{3}$ other agents bids $\frac{1}{2}$, and upon winning (note that p bids $\frac{1}{2}$ in each of these rounds), takes an item from the second row (i.e., $e_{1,j}$). Each such agent surpasses a payment of 1 exactly when winning her 3rd item, and becomes inactive.

The run proceeds in the same way, where for every i, $\frac{n}{q_i}$ of the other agents bid $\frac{1}{q_i-1}$, win all the items in the i'th row, and become inactive. Note that each such agent surpasses a payment of 1 exactly when winning her q_ith item, and becomes inactive. Note that we use the property of $q_i \mid n$ for every $i \leq k$.

Thus, the number of other agents that take all items from rows 1 to k is:

$$\sum_{i=1}^{k} \frac{n}{q_i} = n \cdot \sum_{i=1}^{k} \frac{1}{q_i} = n \cdot (1 - \frac{1}{q_{k+1}-1}) = n \cdot (1 - \frac{1}{n}) = n - 1$$

Thus, there are sufficiently many other agents to take all items from rows 1 to k, and agent p gets items only from row $k+1$. As they are substitutes, the total value received by p is 1.

Consider for $\rho' > \rho_k$ the altruistic version of the bidding game in which an agent becomes inactive after spending a ρ' fraction of her budget, and suppose that p executes the *proportional*(ρ') bidding strategy. On the instance I_k described above, the same run of the bidding game holds, and p does not get a bundle of value $\rho' APS_p$, but rather only $\rho_k APS_p$. Hence, I_k serves as an example showing that our proof of Theorem 3 does not extend to values of ρ larger than ρ_k. The same holds for every $\rho > \lim_{k\to\infty} \rho_k$ (by enlarging k, we can make ρ_k as close as we wish to $\lim_{k\to\infty} \rho_k$). ■

Remark 38. *The negative example in Proposition 17 can be modified by replacing (for every j) the single item $e_{k+1,j}$ of value 1 by $q_k - 1$ items, each of value $\frac{1}{q_k-1}$ (the same value as that of item $e_{k,j}$). In this modified version, the adversarial run is changed so that other agents win all items e_{ij} for $i \leq k$ (their budgets exactly suffice for this), and agent p can take the remaining items from one of the bundles of the MMS partition (the remaining items in different MMS bundles are substitutes to each other and do not provide additional marginal value), getting a value of 1. This modified example is useful in illustrating that for certain variations of the bidding game (considered by the authors but omitted here), bidding strategies similar to the ones considered in the proof of Theorem 4 do not lead to approximation ratios that are significantly better than those proved in Theorem 4).*

We restate Proposition 18.

Proposition 18. *For every constant $\rho > \frac{1}{3}$, there is an allocation instance with equal entitlements and an adversarial run of the bidding game, in which an agent p that has a submodular valuation function and uses the proportional(ρ) bidding strategy gets a bundle of value smaller than ρMMS_p.*

The proof of this proposition is similar to the proof of Proposition 17, with some relatively straightforward modifications. For completeness, we present here the proof in a self-contained way.

Proof. We present a series of instances in which agent p with a submodular valuation function executes the proportional bidding strategy.

The instances are parameterized by $k \in \mathbb{N}$. The kth instance will be as follows: Define

$$q_1 = 2$$
$$q_k = 1 + \prod_{i=1}^{k-1} q_i$$

(This sequence is known as the Sylvester sequence)

The number of agents will be: $n_k = q_{k+1} - 1$ (for example, for $k = 2$, $n_k = 2 \cdot 3 \cdot 7 = 43$)

The set of item is $\mathcal{M} = \{e_{i,j}\}$ for $1 \le i \le k+1$, $1 \le j \le n$ ($n \cdot (k+1)$ items)

If we think of $e_{i,j}$ as arranged in a matrix, then all the items in a row are copies of the same item and are substitutes. The value of items from different rows is additive.

For any $1 \le i \le k$ and for any j, $v_p(e_{i,j}) = \frac{2}{q_i}$. For $i = k+1$ and any j, $v_p(e_{k+1,j}) = 1$. For example, if $k = 3$, there are 43 agents and columns, and in each column j, $v_p(e_{1,j}) = 1$, $v_p(e_{2,j}) = \frac{2}{3}$, $v_p(e_{3,j}) = \frac{2}{7}$, $v_p(e_{4,j}) = 1$.

- v_p is submodular. The marginal value of each item is weakly decreasing (the marginal value of item $e_{i,j}$ to a set S is either $v_p(e_{i,j})$ or 0, depending on whether the set S already contains an item from the i'th row).
- The columns C_j of the matrix $\{e_{i,j}\}$ form an MMS partition. The value of every bundle is at most $v_p(\mathcal{M})$, and in this partition, the value of each bundle (column) is exactly $v_p(\mathcal{M})$.
- $APS_p = MMS_p = v_p(\mathcal{M}) = v_p(C_j) = v_p(e_{k+1,j}) + \sum_{i=1}^{k} v_p(e_{i,j}) = 1 + \sum_{i=1}^{k} \frac{2}{q_i} = 1 + 2\sum_{i=1}^{k} \frac{1}{q_i} = 1 + 2 \cdot (1 - \frac{1}{q_{k+1}-1}) \underset{*}{=} 3 - \frac{2}{q_{k+1}-1}$, where equality * is a known property of the partial sums of Sylvester's inverse series (this can be proved by induction, Wikipedia value of Sylvester sequence).
- q_i divides n, for every $i \le k$.

For convenience, we assume w.l.o.g that the budget of each agent is 2.

For every k, we first show a run of the bidding game with adversarial bidding of the other agents, in which agent p executes the proportional bidding strategy with $\rho_k = \frac{1}{APS_p}$, and she receives a value of precisely 1 (she gets the bundle that consists only of items from row $k+1$) which is a $\frac{1}{APS_p}$ of her APS. For that instance, $\rho_k = \frac{1}{3 - \frac{2}{q_{k+1}-1}}$. The series of ρ_k is monotonically decreasing to a limit of $\frac{1}{3}$. (Sylvester's sequence grows at a doubly exponential rate. Hence, the sequence of ρ_k converges very fast.)

Consider the I_k instance parameterized by k. For convenience, assume the budget of each agent is 2 (which is $2\rho_k APS_p$). Then, by $proportional(\rho_k)$, in each round, agent p bids the highest marginal value of the remained items. We now present the adversarial run.

In round 1, p bids 1, and is allowed to win. She selects an item of value 1 from row $k+1$.

In each of the next n rounds, at least one of the first $\frac{n}{2}$ other agents bids 1, and upon winning (note that p bids 1 in each of these rounds, and the algorithm is assumed to brake the ties adversarially), takes an item from the first row (i.e., $e_{1,j}$). All items of the first row are taken by $\frac{n}{2}$ of the other agents. These $\frac{n}{2}$ agents exhaust their budget and become inactive.

In each of the next n rounds, at least one of the next $\frac{n}{3}$ other agents bids $\frac{2}{3}$, and upon winning (note that p bids $\frac{2}{3}$ in each of these rounds), takes an item from the second row (i.e., $e_{1,j}$). Each such agent becomes inactive after taking three items.

The run proceeds in the same way, where for every i, $\frac{n}{q_i}$ of the other agents bid $\frac{2}{q_i}$, win all the items in the i'th row, and become inactive. Note that we use the property of $q_i \mid n$ for every $i \leq k$.

Thus, the number of other agents that take all items from rows 1 to k is:

$$\sum_{i=1}^{k} \frac{n}{q_i} = n \cdot \sum_{i=1}^{k} \frac{1}{q_i} = n \cdot (1 - \frac{1}{q_{k+1} - 1}) = n \cdot (1 - \frac{1}{n}) = n - 1$$

Thus, there are sufficiently many other agents to take all items from rows 1 to k, and agent p gets items only from row $k+1$. As they are substitutes, the total value received by p is 1.

Notice that if p executes $proportional(\rho')$ with $\rho' > \rho_k$, then the bids of p in each round are strictly smaller than those described above. Hence same run of the algorithm holds, and p does not get a bundle of value $\rho' APS_p$, but rather only $\rho_k APS_p$. Hence, I_k serves as an example showing for every $\rho' > \rho_k$ that $proportional(\rho')$ does not guarantee p a value of $\rho' APS_p$.

Since ρ_k is close to $\frac{1}{3}$ as we wish, for any $\rho > \frac{1}{3}$, there exist a witness I_k on which $proportional(\rho)$ does not guarantee p a (ρ)-fraction of her APS. ■