Manipulation-resistant facility location mechanisms for ZV-line graphs

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Abstract

In many real-life scenarios, a group of agents needs to agree on a common action, e.g., on the location for a public facility, while there is some consistency between their preferences, e.g., all preferences are derived from a common metric space. The *facility location* problem models such scenarios and it is a well-studied problem in social choice. We study mechanisms for facility location on graphs, which are resistant to manipulations (*strategy-proof*, *abstention-proof*, and *false-name-proof*) by both individuals and coalitions and are efficient (*Pareto optimal*). We present a family of graphs, *ZV-line graphs,* which includes almost all the graphs and graph families that were studied for this problem. We show a general facility location mechanism for this family which satisfies all these desired properties. Moreover, we show that this mechanism can be computed in polynomial time, it is anonymous, and it can be equivalently defined as the first Pareto optimal location, according to some predefined order. Finally, we discuss some generalizations and limitations of the characterization.

1 Introduction

Reaching an agreement could be hard. The seminal works of Gibbard [8] and Satterthwaite [19] show that one cannot devise a general procedure for aggregating the preferences of strategic agents to a single outcome, besides trivial procedures that a-priori ignore all agents except one (that is, the outcome is based on the preference of a predefined agent) or a-priori rule out all outcomes except two (that is, regardless of the agents' preferences, the outcome is one of two predefined outcomes). The problem is that agents might act strategically aiming to get an outcome which they prefer. Note that while we refer to a *procedure* and later to a *mechanism*, this impossibility is not technical but conceptual. We identify a procedure with the conceptual mapping induced by it from the opinions of the agents to an agreement, while the procedure itself could be more complex and abstract, e.g., to have several rounds or include a deliberation process between the agents (cheap-talk). For simplicity of terms, we refer to the *direct mechanism* which implements this mapping. That is, we think of an exogenous entity, the *designer*, who receives as input the opinions of the agents and returns as output the aggregated decision.[1]

But in many natural scenarios, one does not look for a mechanism which is defined

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[1]For the properties we study in this work, this assumption does not hurt the generality, as according to the revelation principle [16] argument, any general procedure is equivalent (w.r.t. the properties we study) to such a direct mechanism.

for any profile of preferences, but instead it is assumed that the preferences satisfy some additional exogenous rationality property, giving rise to new mechanisms that are not prone to the above drawbacks. Two prominent examples are *VCG mechanisms* and *generalized-median mechanisms*. VCG mechanisms [23, 3, 18, 9] are the mechanisms which are resistant to manipulations like the ones described above for scenarios in which the agents' preferences are quasi-linear with respect to money [13, Def. 3.b.7], and monetary transfers are allowed (that is, the outcome space is closed under monetary exchanges between the agents or between the agents and the designer).

The second example, *Generalized-Median mechanisms*, do not include monetary transfers and have more of an ordinal flavor. Generalized-median mechanisms [14] are the mechanisms which are resistant to manipulations like above when it is known the preferences are *single-peaked* w.r.t. the real line [1]. That is, the outcomes are *locations* on the real line, each agent has a unique optimal location, $\ell^\star$, and her preference over the locations to the right of $\ell^\star$ is derived by the distance to $\ell^\star$, and similarly for the locations to the left of $\ell^\star$. For example, in the Euclidean single-peaked case, the preferences of all agents' are minimizing the distance to their respective optimal locations.[2]

The facility location problem

A natural generalization of the second scenario is the *facility location* problem. In this problem, we are given a metric space over the outcomes[3] and it is assumed that the preference of each of the agents is defined by the distance to her optimal outcome: An agent with an optimal outcome $\ell^\star$ prefers outcome a over outcome b if and only if a is closer to $\ell^\star$ than b. These problems arise in many real-life scenarios, such as locating a common good, e.g., a school, and in more general agreement scenarios with a common metric, e.g. partition of a common budget to different tasks. A natural way to represent this common metric space is using a weighted undirected graph: Having a vertex (location) for each outcome and weighted edges between them s.t. the distance between any two outcomes is equal to the distance between the two respective vertices. For the ease of presentation, throughout this paper we assume that there are finitely many agents and finitely many locations.

Roughly speaking, given such a graph one seeks to find a mechanism that on one hand will not a-priori ignore some of the voters or rule-out some of the locations, and on the other hand will be resistant to manipulations of the agents. Since we would also like to consider manipulations of coalitions of agents as well, we define the *preference of a coalition* as the unanimous preference of its members. That is, we say that a coalition C weakly prefers an outcome a over an outcome b if all the members of C weakly prefer a over b.[4] In this work, we seek mechanisms which satisfy the following desired properties:

Anonymity The mechanism should not a-priori ignore agents and moreover it should treat them equally in the following strong sense. The mechanism should be a function of the agents' votes (which we also refer to as *ballots*) but not their identities. Formally, the outcome of the mechanism should be invariant to voters exchanging votes, i.e., to a permutation of the ballots. In practice, most voting systems satisfy this property by first accumulating the different ballots, by that losing the voters' identities, and applying the mechanism on the identity-less ballots.

[2] Generalized-median mechanism were also proved to be the only manipulation-resistant mechanisms even when the preferences of the agents are restricted to be Euclidean [2].

[3] I.e., a distance function between outcomes.

[4] Notice that this preference is not complete. Also note that C strictly prefers a over b if ***(i)*** all the members of C weakly prefer a over b (C weakly prefers a over b), and ***(ii)*** at least one member of C strictly prefers a over b (C does not weakly prefer b over a).

Citizen sovereignty The mechanism should not a-priori rule-out a location, and each location should be an outcome of some profile (Formally, the mapping to facility locations should be onto). Moreover, we require it to respect the preferences of the agents in the following way:

Pareto optimality The mechanism should not return a location ℓ if the coalition of all agents strictly prefers a different location ℓ' over ℓ. In particular, if there exists a unique location which is unanimously most-preferred by all agents, then it must be the outcome. This property could be justified on efficiency grounds since switching (ex-post) from ℓ to ℓ' strictly increases the satisfaction of some of the agents without decreasing the satisfaction of any of the other agents. Note that in the general case it is unreasonable to require that all locations are treated equally due the inherent asymmetry induced by the graph.

Strategy-proofness An agent should not be able to change the outcome to a location she strictly prefers by reporting a location different than her true location.

Abstention-proofness[5] An agent should not be able to change the outcome to a location she strictly prefers by not casting a ballot.

False-name-proofness An agent should not be able to change the outcome to a location she strictly prefers by casting more that one ballot.

This property received less attention in the classic social choice literature, since in most voting scenarios there exists a central authority that can enforce a 'one person, one vote' principle (but cannot enforce participation or sincere voting). In contrast, many of the voting and aggregation scenarios nowadays are run in a distributed manner on some network and include virtual identities or avatars, which can be easily generated, so a manipulation of an agent pretending to represent many voters is eminent.

Resistance to group manipulations We also consider the generalizations of the above three properties and also require that a coalition should not be able to change the outcome to a location it strictly prefers by its members casting insincere ballots, abstaining, or casting more than one ballot.[6]

Related work

Several other variants of the facility location problem were also considered in the literature. For instance, Schummer and Vohra [20] considered the case of continuous graphs, Lu et al. [12, 11] studied variants in which several facilities need to be located and scenarios in which an agent is located on several locations, and Feldman et al. [6] studied the impact of constraining the input language of the agents.

False-name-proofness was first introduced by Yokoo et al. [24] in the framework of combinatorial auctions.[7] In this work, the authors showed that VCG mechanism does not satisfy false-name-proofness in the general case, and they proposed a property of the preferences

[5]In the voting literature (e.g., [4, 15, 7]) this property is also referred to as **voluntary participation** and the **no-show paradox**. This property is also equivalent to **individual-rationality** which takes the, a bit different, point of view of mechanism design.

[6]Actually, this property combined with Citizen sovereignty entails Pareto optimality. Nevertheless, we prefer to think of Pareto optimality apart from this property due to the different motivation.

[7]A similar concept was also studied later n the framework of peer-to-peer systems by Douceur [5] under the name **sybil attacks**. ([24] is based on a series of previous conference papers, and hence this non-monotone time line.)

under which this mechanism becomes false-name-proof. Later, Conitzer and Sobel [4] analyzed false-name-proof mechanisms in voting scenarios, Todo et al. [22] characterized other false-name-proof mechanisms for combinatorial auctions, and Todo et al. [21] characterized the false-name-proof mechanisms for facility location on the continuous line and on continuous trees. Todo et al. [21] also analyzed the implications of their characterization for the design of mechanisms for social-welfare maximization (both for sum-of-costs and for maximal-cost).

The characterization of manipulation-resistant mechanisms for facility location is highly related to problems in *Approximate mechanism design without money* [17]. In these problems, agents are characterized using cardinal utilities and the designer seeks to find an outcome maximizing a desired target function (e.g., sum of utilities, product of utilities, or minimal utility). These works bound the trade-of between the target function and manipulation-resistance, that is, they bound the loss to the target function due to manipulation-resistance constraints.

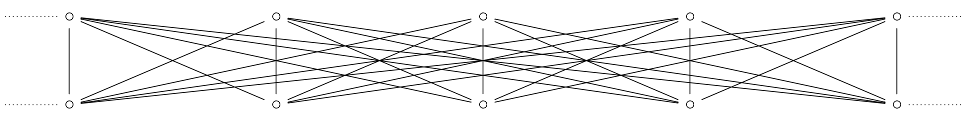
Our contribution

In this paper we present a family of unweighted graphs, ZV-line graphs, and show a general mechanism for facility location over these graphs which satisfies the desired properties. The mechanism is Pareto optimal and in particular satisfies citizen sovereignty; It is anonymous, so in particular no agent is ignored; But on the other hand, it is resistant to all the above manipulations. Roughly speaking, in ZV-line graphs we have two types of locations Z and V (and we refer to them as Z-vertices and V-vertices, respectively), and the facility is 'commonly' (except if all agents agree differently) located on a Z-vertex. For instance, the Z-vertices could represent commercial locations for locating a public mall, or a set of status-quo outcomes. We show that under some connectivity assumptions on the graph, our mechanism satisfies the desired manipulation-resistance properties. To the best of our knowledge, this is the first work to show a general false-name-proof mechanism for a general family of graphs.

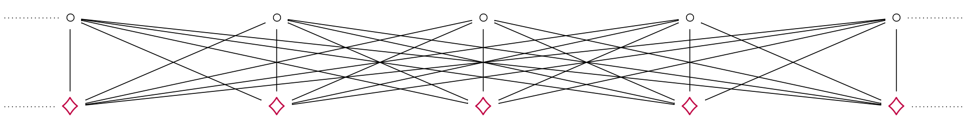
For example, consider the following family of graphs (which is a sub-family of ZV-line graphs and captures the main insight of our mechanism). Let $G = \langle \mathcal{V}, E \rangle$ be a bipartite graph with vertex set $\mathcal{V}$ and edges set E. That is, there exists a partition of the vertices $\mathcal{V} = V \dot{\cup} Z$ s.t. there are no edges between V-vertices and no edges between Z-vertices. In addition, we also require that ***(a)*** the agents agree on some predefined order of the Z-vertices (similarly to the single-peaked case [1]) and that ***(b)*** any of the V-vertices is connected to a sequence (according to the order) of Z-vertices. Our mechanism for such graphs:

- The mechanism returns the leftmost Pareto optimal location in Z, if one exists.
- If no location in Z is Pareto optimal, then necessarily all agents voted for the same location, and the mechanism returns this location.

For example, bi-cliques (full bipartite graphs)



can be represented as a ZV-line graph in which each V-vertex is connected to all the Z-vertices as follows (and we use below $\diamond$ for Z-vertices and $\circ$ for V-vertices):

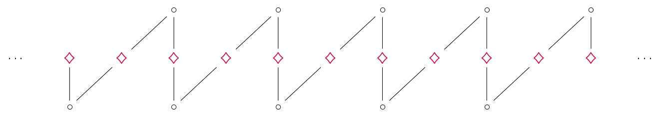


Our mechanism for this case:

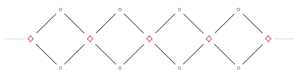
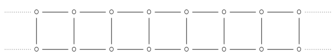
- If all agents voted unanimously for the same location, the mechanism returns this location.
- If at least two V-vertices were voted for, the mechanism returns the leftmost Z-vertex.
- Otherwise, the mechanism returns the leftmost Z-vertex that was voted for.

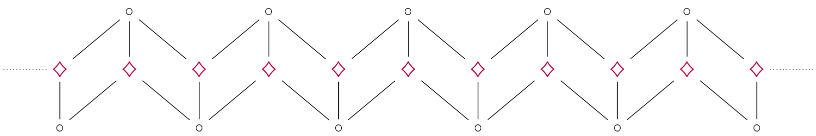
Notice that in this case the order over the Z-vertices is arbitrary (as well as the choice of one of the sides to be the Z-vertices) in the sense that it is not derived from the graph but a parameter of the mechanism (for instance, the order might represent the social norm of the society).

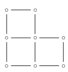
A second example is the discrete line graph, which can be represented as a ZV-line graph in which each two consecutive Z-vertices are connected by a unique V-vertex,

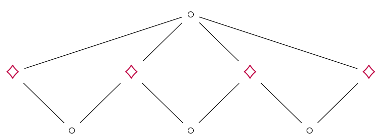
.

In particular, we show a strategy-proof, false-name-proof, Pareto optimal mechanism which is far from *generalized-median mechanisms* (for instance, in the common case the output of the mechanism belongs to a subset consisting of only half of the locations), in contrary to the characterization of these mechanisms for the continuous line [21, Thm. 2].

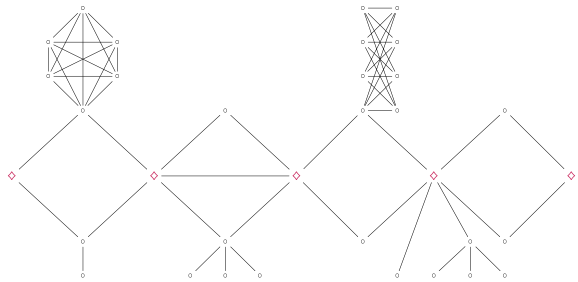
Two simple graphs that are generalizations of (the ZV-line graph representation of) the discrete line graph are , in which each two consecutive Z-vertices are connected by two V-vertices, and the $2 \times n$ grid  which can be represented as a ZV-line graph in which each three consecutive Z-vertices are connected by a unique V-vertex, i.e.,

.

A common property to all the above examples is their regularity: All the V-vertices have the same degree and similarly all the Z-vertices have the same degree. An example we encountered of a graph for which a mechanism was known (although not published) is , which can be represented as a non-regular ZV-line graph as

.

In the definition of the ZV-line graph family we extend the above family (and extend the mechanism accordingly) in two different ways: allowing edges between the Z-vertices (under a similar interval constraint), and replacing vertices by a tree, a clique, or any other graph for which a manipulation-free Pareto optimal anonymous mechanism is known. For example,



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In particular, the ZV-line graphs family includes all trees, cliques, block graphs [10], and cycles of size up to 4 (note that there is no manipulation-resistant Pareto optimal anonymous mechanism for cycles of size larger than 5).

2 Model

Consider a graph $G = \langle \mathcal{V}, E \rangle$ with a set of vertices $\mathcal{V}$ and a set of (neither weighted nor directed) edges $E \subseteq \binom{\mathcal{V}}{2}$, and we refer to the vertices $v \in \mathcal{V}$ also as *locations* and use the two terms interchangeably. We use the notations $E(S,T) = \{(s,t) \in E \mid s \in S,\ t \in T\}$ for the edges between $S \subseteq \mathcal{V}$ and $T \subseteq \mathcal{V}$, and $E(S) = E(S,S) = E \cap \binom{S}{2}$ for the edges inside S. The distance between two vertices $v, u \in \mathcal{V}$, notated $d(v,u)$, is the length of the shortest path connecting v and u,[8] and the distance between a vertex $v \in \mathcal{V}$ and set of vertices $S \subseteq \mathcal{V}$, $d(v,S)$, is defined as the minimal distance between v and a vertex in S. We define $B(v,d)$, the *ball* of radius $d \geqslant 0$ around a vertex $v \in \mathcal{V}$, to be the set of vertices of distance at most d from v

$$B(v,d) = \{u \in \mathcal{V} \mid d(v,u) \leqslant d\}.$$

We say that two vertices are *neighbors* if there is an edge connecting them. We notate by $N(v)$ the set of neighbors of a vertex v, and by $N(S)$ the set of neighbors of a set of vertices S, that is, $N(S) = \bigcup_{v \in S} N(v)$.

An instance of the *facility location problem over* G is comprised of n agents who are located on vertices of $\mathcal{V}$; Formally, we represent it by a *location profile* $\boldsymbol{x} \in \mathcal{V}^n$ where x_i is the location of Agent i. Given an instance $\boldsymbol{x}$, we would like to locate a facility on a vertex of the graph while taking into account the preferences of the agents over the locations. In this work, we assume the preference of an agent is defined by her distance to the facility: An agent located on $x \in \mathcal{V}$ strictly prefers the facility being located on $v \in \mathcal{V}$ over it being located on $u \in \mathcal{V}$ iff $d(x,v) < d(x,u)$ and she is indifferent between the two locations in case of equality.

A *general facility location mechanism* (or shortly a *mechanism*) defines for any profile of locations a location for the facility. We require the mechanism to assign a location for the facility for any profile and any number of agents. Hence, we represent the mechanism by a function $F\colon \bigcup_{t \geqslant 0} \mathcal{V}^t \to \mathcal{V}$. We also think on F as a voting procedure: Each agent votes (and we also refer to his vote as a *ballot*) for a location, and based on the ballots F returns a location for the facility. We say that a mechanism is **anonymous** if the outcome $F(\boldsymbol{x})$ does not depend on the identities of the agents, i.e., it can be defined as a function of the ballot tally, the number of votes for each of locations.

[8] For simplicity, we assume the graph is connected.

Manipulation-resistance

A strategic agent might act untruthfully if she thinks it might cause the mechanism to return a location she prefers (i.e., closer to her). In this work we consider the following manipulations: **Misreport**: An agent might report to the mechanism a location different from her real location; **False-name-report**: An agent might pretend to be several agents and submit several (not necessarily identical) ballots;[9] **Abstention**: An agent might choose not to participate in the mechanism at all. A mechanism in which no agent benefits from these manipulations, regardless to the ballots of the other agents, is said to be **strategy-proof**, **false-name-proof**, and **abstention-proof**, respectively. We also consider a generalization of these manipulations to manipulations of a coalition, and say a mechanism is **group-manipulation-resistant** (shortly manipulation-resistant) if no coalition can change the outcome, by misreporting, false-name-reporting, or abstaining, to a different location which they unanimously agree is no worse than the original outcome (i.e., if they vote sincerely) and at least one of its members strictly prefers the new location. Notice that this is a very strong definition of a resistance to manipulations - We do not require that all the members of the coalition will use the same (insincere) deviation and we do not require all of them to strictly prefer to deviate.

Definition 1 (Group-manipulation-resistant).[10] A mechanism F is not group-manipulation-resistant if there exists a vector of locations $\boldsymbol{x} \in \mathcal{V}^n$, a coalition of agents $C \subseteq \{1, \ldots, n\}$, and a set of ballots $\boldsymbol{A} \in \bigcup_{t \geqslant 0} \mathcal{V}^t$ s.t. ***(i)*** all the members of C weakly prefer $F(\boldsymbol{A}, \boldsymbol{x}_{-C})$, that is the outcome when the agents outside of C do not change their vote and the agents of C replace their ballot by $\boldsymbol{A}$, over $F(\boldsymbol{x})$ and ***(ii)*** at least one of C's members strictly prefers $F(\boldsymbol{A}, \boldsymbol{x}_{-C})$ over $F(\boldsymbol{x})$.

We note that for $C = \{i\}$ being a singleton, this general manipulation coincides with misreport for $|\boldsymbol{A}| = 1$, with false-name-report for $|\boldsymbol{A}| > 1$, and with abstention for $\boldsymbol{A} = \varnothing$.

The revelation principle

One could also consider more general mechanisms in which the agents vote using more abstract ballots, and define similar manipulation-resistance terms for the general framework. Applying a simple direct revelation principle [16] shows that any such general manipulation-resistant mechanism is equivalent to a manipulation-resistant mechanism in our framework: The two mechanisms implement the same mapping of the agents private preferences to a location of the facility, and since the above properties are defined for the mapping they are invariant to this transformation.

Efficiency

So far, we defined the desired manipulation-resistance properties for a mechanism. On the other hand, we would also like the mechanism to respect the preferences of the agents, e.g., we would like to avoid a scenario in which, after the mechanism have been used, the agents can agree that a different location is preferable. Given a location profile $\boldsymbol{x} \in \mathcal{V}^n$, the set of *Pareto optimal* locations, $PO(\boldsymbol{x})$, is the set of all locations which the agents cannot agree to rule out. Formally, given two locations $v, u \in \mathcal{V}$, we say that u *Pareto dominates* v (w.r.t. a location profile $\boldsymbol{x}$) if ***(i)*** all agents weakly prefer u over v and ***(ii)*** at least one agent strictly prefers u over v. We say that v is *Pareto optimal* ($v \in PO(\boldsymbol{x})$) if it is not Pareto dominated by any other location. We say a mechanism is **Pareto optimal** if for any

[9] A special case of false-name-voting which is considered in the literature is **double-voting**: Casting the same ballot several times to increase its impact.

[10] For simplicity of notations, we give the formal definition for anonymous mechanisms.

report profile $\boldsymbol{x}$ (and assuming truthful reporting) $F(\boldsymbol{x}) \in PO(\boldsymbol{x})$. In particular, Pareto optimality entails **unanimity**, if all the agents unanimously vote for the same location then the mechanism outputs this location, and **citizen sovereignty**, the mechanism is onto and does not a-priori rule out any location.

3 Main Result

In this work we define a family of graphs, *ZV-line graphs*, and present a general mechanism for this family.

Definition 2 (ZV-ordered partition)**.**
Given an unweighted undirected connected graph $G = (\mathcal{V}, E)$ and a sequence of non-empty sets of vertices $Z, V_1, \ldots, V_k \subseteq \mathcal{V}$, we say that the sequence $Z, V_1, \ldots, V_k$ $(k \geqslant 0)$ is a ZV-ordered partition if the following holds.

1. The sets V_i are disjoint,
$$V_i \cap V_j = \varnothing \quad \text{for } i \neq j.$$
2. The sequence is a cover of $\mathcal{V}$ and no sub-sequence of it is a cover of $\mathcal{V}$,[11]
$$\begin{cases} Z \cup V_1 \cup \cdots \cup V_k = \mathcal{V} \\ Z \nsubseteq V_1 \cup \cdots \cup V_k \\ V_i \nsubseteq Z \quad \text{for } i = 1, \ldots, k. \end{cases}$$
3. For $i = 1, \ldots, k$ there is a unique vertex in V_i which is closest to Z. We refer to it as *the root of* V_i and denote it by $\mathcal{R}(V_i)$,
$$\mathcal{R}(V_i) = \underset{v \in V_i}{\operatorname{argmin}}\, d(v, Z).$$
4. All paths between vertices of V_i and vertices outside of V_i pass through the root $\mathcal{R}(V_i)$ and through Z.[12]
5. Last, Z is equipped with an order (that is, an injective mapping from Z to $\Re$). For simplicity of description we refer to this order as an order from left to right. We call a subset A of Z an *interval* if it is a sequence of vertices according to the order, i.e., if A is the preimage of an interval in $\Re$.

We use the notions V_i-subgraphs, V-vertices, and Z-vertices for the respective sets of vertices.

Given a graph $G = (\mathcal{V}, E)$ with a *ZV-ordered partition,* $Z, V_1, \ldots, V_k \subseteq \mathcal{V}$, and a sequence of mechanisms $F_i \colon \bigcup_{t \geqslant 0} (V_i)^t \to V_i$ for $i = 1, \ldots, k$, we define the following mechanism $F^\star \colon \bigcup_{t \geqslant 0} \mathcal{V}^t \to \mathcal{V}$:

Definition 3 ($F^\star$)**.** Given a vector of reports $\boldsymbol{x} \in \bigcup_{t \geqslant 0} \mathcal{V}^t$

- ▶ If all the ballots belong to the same V_i-subgraph, return $F_i(\boldsymbol{x})$.
- ▶ Otherwise, return the leftmost Pareto optimal location in Z.

It is not hard to see the following:

$F^\star$ is well defined: If $\boldsymbol{x}$ is not included in any of the V_i-subgraphs, then there exist two locations, x_i and x_j, and a short path between them s.t. all its vertices are in $PO(\boldsymbol{x})$ and at least one of its vertices is in Z. Hence, $PO(\boldsymbol{x}) \cap Z \neq \varnothing$ and in particular the leftmost location in $PO(\boldsymbol{x}) \cap Z$ is well-defined.

[11] Since we $\{V_i\}$ are pair-wise disjoint, the third constraint is equivalent to $V_i \nsubseteq Z \cup V_1 \cup \cdots \cup V_{i-1} \cup V_{i+1} \cup \cdots \cup V_k$.

[12] Equivalently, $\forall v \in V_i \backslash \mathcal{R}(V_i) \quad N(v) \subseteq V_i$ and $N(\mathcal{R}(V_i)) \subseteq V_i \cup Z$.

$F^\star$ runs in polynomial time: Checking whether an agent weakly prefers a location a over a location b can be done in polynomial time. Hence, also checking whether all agents weakly prefer one location over another, and checking for each location whether the coalition of all agents strongly prefers some other location over it can be done in a polynomial time.

Order representation of $F^\star$: If $F_1, \ldots, F_k$ can be defined as the 'first Pareto optimal location according to some order,' then an equivalent way to define $F^\star$ is as the first Pareto optimal location in the following order:

> First, go over the vertices of Z from left to right, and then on the vertices of the V_i-subgraphs in some order s.t. for each subgraph the order over its vertices matches the order of F_i.

Next, we define ZV-line graphs by adding a connectivity constraint over the Z-vertices.

Definition 4 (ZV-line graph)**.** An unweighted undirected connected graph $G = (\mathcal{V}, E)$ is a *ZV-line graph w.r.t.* $\mathcal{V} = Z \cup (V_1 \dot\cup \cdots \dot\cup V_k)$ if $Z, V_1, \ldots, V_k$ is a *ZV-ordered partition* of G and in addition for any vertex $v \in \mathcal{V}$, $B(v, 1) \cap Z$ is an interval in Z (which might be the empty set).

For instance, for any $k \geqslant 1$ the clique over k vertices, K_k, is a ZV-line graph w.r.t. $Z = \mathcal{V}$ and any order over the vertices. A special case of ZV-line graphs is when all the V_i-subgraphs are singletons (for example, the graphs in the introduction).

Theorem 5 (Main result)**.**[13]

*Let $G = (\mathcal{V}, E)$ be a graph with a ZV-*ordered partition *$\mathcal{V} = Z \cup (V_1 \dot\cup \cdots \dot\cup V_k)$ and let $F_i \colon \bigcup_{t \geqslant 0} (V_i)^t \to V_i$ be a sequence of mechanisms s.t. for $i = 1, \ldots, k$*

- *F_i is anonymous and Pareto optimal;*
- *For an infinite number of $\tau \in \mathbb{N}$ there exists a profile $\boldsymbol{x} \in \bigcup_{t \geqslant 0} (V_i)^t$ in which all locations in V_i were voted for at least τ times and $F_i(\boldsymbol{x}) = \mathcal{R}(V_i)$; and*
- *For any vector of locations $\boldsymbol{x} \in (V_i)^n$, a coalition of agents C, and a set of ballots $\boldsymbol{A} \in \bigcup_{t \geqslant 0} (V_i)^t$,[14] $\boldsymbol{A}$ is not a beneficial deviation for C (That is, C does not strictly prefer $F_i(\boldsymbol{A}, x_{-C})$ over $F_i(\boldsymbol{x})$).* $(\bigstar)$

Then, for $F^\star \colon \bigcup_{t \geqslant 0} \mathcal{V}^t \to \mathcal{V}$ being the mechanism defined in Definition 3, $F^\star$ is an anonymous and Pareto optimal mechanism and

(*I*) *If G is a ZV-line graph w.r.t. $\mathcal{V} = Z \cup (V_1 \dot\cup \cdots \dot\cup V_k)$, then $F^\star$ satisfies $(\bigstar)$.*[15]

(*II*) *If $\mathcal{R}(V_i) \in Z$ for $i = 1, \ldots, k$, and the mechanism $F_Z \colon \bigcup_{t \geqslant 0} Z^t \to Z$ which returns the leftmost Pareto optimal location satisfies $(\bigstar)$,[15] then also $F^\star$ satisfies $(\bigstar)$.*[15]

Before proving the theorem we note the following:

- By considering singleton coalitions, we get that $F^\star$ is strategy-proof, false-name-proof, and abstention-proof.
  Moreover, no coalition can find a beneficial deviation, by assigning misreporting, false-name-reporting, or abstaining among its members.[16]
- As a corollary for the case ZV-line graphs with singleton V_i-subgraphs we get

[13] We describe the strong version of the theorem, deriving from resistance of F_i to any manipulation, the same resistance for F. The same proof shows that also weaker manipulation-resistance properties of F_i (e.g., against individual agents, against misreporting , or against abstentions) result in the same manipulation-resistance for F.

[14] Since F_i (and later $F^\star$) are anonymous mechanisms, we define $\boldsymbol{A}$ as a set of ballots ignoring identities.

[15] I.e., for any vector of locations $\boldsymbol{x} \in \mathcal{V}^n$, a coalition of agents C, and a set of ballots $\boldsymbol{A} \in \bigcup_{t \geqslant 0} \mathcal{V}^t$, $\boldsymbol{A}$ is not a beneficial deviation for C.

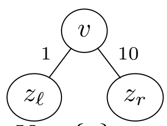
[16] Notice this is a strong notion of group manipulation-resistance. A coalition cannot even find a deviation which is beneficial for one of its members, while not hurting the other members.

Corollary 6. *Let $G = \langle \mathcal{V}, E \rangle$ be a graph and $\mathcal{V} = V \dot{\cup} Z$ a partition of the vertices to two disjoint sets with Z being equipped with an order s.t.*
- *$E(V) = \varnothing$;*
- *For any vertex $v \in V$, $N(v)$ is an interval in Z; and*
- *For any vertex $v \in Z$, $N(z) \cup \{z\}$ is an interval in Z.*

Next, let F be the following mechanism
- ▶ *If all ballots are identical, return this location as the outcome.*
- ▶ *Otherwise, return the leftmost Pareto optimal location in Z.*

Then, F is an anonymous and Pareto optimal mechanism, and the following property holds: For any vector of locations $\boldsymbol{x} \in \mathcal{V}^n$, a coalition of agents C, and a set of ballots $\boldsymbol{A} \in \bigcup_{t \geqslant 0} \mathcal{V}^t$, $\boldsymbol{A}$ is not a beneficial deviation for C.

- Last, we note that the theorem does not hold for weighted graphs. Consider the following weighted graph and a profile in which Alice is located on z_r and Bob on v. Then, the outcome is z_r, but Bob can move the facility to a preferred location z_ℓ both ***(i)*** by misreporting z_ℓ, hence $F^\star$ is not strategy-proof, and ***(ii)*** by false-name-reporting z_ℓ in addition to his sincere report, hence $F^\star$ is not false-name-proof.



$V = \{v\}$
$Z = \{z_\ell, z_r\}$

3.1 Implications: Mechanisms for recursive graph families

By applying the main result to recursive families of graphs, we can generate a recursive (and hence commonly simple) mechanisms which satisfy our desiderata. For instance, a corollary of our result is a manipulation-resistant mechanism for the following recursive family of rooted graphs (That is, $\langle \mathcal{V}, E, r \rangle$ s.t. $E \subseteq \binom{\mathcal{V}}{2}$ and $r \in \mathcal{V}$).

Definition 7 ($\mathcal{F}$)**.**
- $\langle \{v\}, \varnothing, v \rangle \in \mathcal{F}$.
- For any $k, \ell \geqslant 1$: If $\{\langle \mathcal{V}_i, E_i, r_i \rangle\}_{i=1}^{k}$ are in $\mathcal{F}$ (and the $\mathcal{V}_i$ are disjoint), then also the following graph is in $\mathcal{F}$.

$$\left(\{\widehat{r}_j\}_{j=1}^{\ell} \dot{\cup} \left(\dot{\bigcup}_{i=1}^{k} \mathcal{V}_i \right), \{\langle \widehat{r}_j, r_i \rangle\}_{i=1\ldots k, j=1\ldots \ell} \dot{\cup} \left(\dot{\bigcup}_{i=1}^{k} E_i \right), \widehat{r}_1 \right)$$

I.e., adding a new layer of pre-roots, a bi-clique between them and the roots of the graphs of the previous stage, and defining one of the pre-roots to be the new root.

Claim 8. The anonymous Pareto 0optimal mechanism $F(\boldsymbol{x}) = \operatorname{argmin}_{v \in PO(\boldsymbol{x})} d(v, r)$, which returns the Pareto optimal location closest to the root and breaks ties according to a pre-defined order, is manipulation-resistant.

Note that by setting $\ell = 1$ in the second step of the definition we get a recursive definition of *rooted trees*. Hence, we get that for any tree G the mechanism that returns the lowest common ancestor of the ballots (with regard to some root) is a manipulation-resistant mechanism (These are also the mechanisms which Todo et al. [21] characterized as the false-name-proof, anonymous, and Pareto optimal mechanisms for the continuous tree.).

Proof. We prove the claim by induction over the number of steps needed to generate G, $h(G)$.

If $h(G) = 0$, i.e., $G = \langle \{v\}, \varnothing, v \rangle$ consists of a single vertex and the trivial mechanism satisfies all the desired properties.

If $h(G) \geqslant 1$, then G is a ZV-line graph w.r.t. $Z = \{\widehat{r}_j\}_{j=1}^{\ell}$ and $V_i = \mathcal{V}_i$. Note that for all V_i-subgraphs $h(\langle \mathcal{V}_i, E_i, r_i \rangle) \leqslant h(G) - 1$. Hence, our recursive mechanism returns one of the pre-roots of the 'lowest' subgraph which includes $\boldsymbol{x}$ when ties are broken according to the (arbitrary) order over the pre-roots. □

A second example is *Connected block graphs [10].*[17] A connected graph $G = \langle \mathcal{V}, E \rangle$ is a block graph if one of the following equivalent conditions holds:

- Every biconnected component of G is a clique.
  Since for any graph the structure of its biconnected components is described by a block-cut tree,[18] connected block graphs are also called *clique trees.*
- The intersection of any two connected subgraphs of G is either empty or connected.
- For every four vertices $u, v, w, x \in \mathcal{V}$, the larger two of the distance sums

$$d(u,v) + d(w,x), \qquad d(u,w) + d(v,x), \qquad \text{and } d(u,x) + d(v,w)$$

  are equal.

Our mechanism for a connected block graph G returns the closest Pareto optimal location to an arbitrarily predefined location, breaking ties according to an arbitrarily predefined order over the locations.

Proof sketch. Let $\mathcal{T}(G)$ be the block-cut tree of G. Following the inductive structure of $\mathcal{T}(G)$, and recalling that a clique is a ZV-line graph w.r.t. all vertices of the clique being Z-vertices and any order over them, we get that our mechanism is defined by an arbitrary predefined component-vertex of $\mathcal{T}(G)$, $\mathcal{R}$, and a series of arbitrary predefined orders over the locations of each of the components. The mechanism is:

- ▶ If all ballots belong to the same component, return the first location (according to the order) that was voted for.
- ▶ Otherwise, choose the component closest to $\mathcal{R}$ s.t. one of the locations of the component is Pareto optimal, and return the first location (according to the order) in this component.

Last, we note that an equivalent definition of this mechanism is returning the closest Pareto optimal location to some location $v \in \mathcal{R}$, breaking ties according to a concatenation of the orders over the components. ☐

4 Summary & Extensions

In this work, we presented a family of graphs, ZV-line graphs, and a general anonymous Pareto optimal manipulation-resistant mechanism for the facility location problem on these graphs. To the best of our knowledge, this is the first work to show a general false-name-proof mechanism for a large family. Our construction is inductive: It derives a mechanism for a given ZV-line graph from mechanisms for its subgraphs (which might not be ZV-line graphs). Hence, it is straightforward to derive from the construction general mechanisms for recursive families of graphs.

We assumed that our graphs are connected, but it is not hard to see that the following easy extension for unconnected graphs will satisfy the same desiderata.

- ▶ At the first stage, choose the first connected component according to some predefined order s.t. at least one agent voted for a location in this component.
- ▶ At the second stage, run our mechanism taking into account only agents who voted for locations in the chosen component.

Note that, just like the mechanism for the connected case, also this mechanism can be equivalently defined as the first Pareto optimal location according to some order of the vertices - The concatenation of the respective orders for the different components.

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[17]We thank Ayumi Igarashi for suggesting us this family as an example.

[18]The *block-cut tree* of a graph G is a tree $\mathcal{T}(G)$ which is defined in the following way. In $\mathcal{T}(G)$ there is a vertex (*component-vertex*) for each maximal biconnected component of G and a vertex (*intersection-vertex*) for each vertex in G which belongs to more than one maximal biconnected component. There is an edge in $\mathcal{T}(G)$ between each component-vertex and the intersection-vertices belonging to this component.

The mechanism we presented is not the only mechanism satisfying the desired properties. Taking any other order over the Z-vertices s.t. the constraints of Def. 4 hold and defining $F^\star$ accordingly will also satisfy them. In particular, a mechanism which takes at the second stage of Def. 3 the rightmost Pareto optimal Z-vertex will also satisfy the same desiderata. We conjecture that these are the only anonymous Pareto optimal manipulation-resistant mechanisms for the facility location problem.

Conjecture 9.

*Let $G = (\mathcal{V}, E)$ be a ZV-*line *graph w.r.t. $\mathcal{V} = Z \cup (V_1 \dot\cup \cdots \dot\cup V_k)$ and let $F\colon \bigcup_{t \geqslant 0} \mathcal{V}^t \to \mathcal{V}$ be a mechanism s.t.*

- *F is anonymous and Pareto optimal; and*
- *For any vector of locations $\boldsymbol{x} \in \mathcal{V}^n$, a coalition of agents C, and a set of ballots $\boldsymbol{A} \in \bigcup_{t \geqslant 0} \mathcal{V}^t$, $\boldsymbol{A}$ is not a beneficial deviation for C.*

Then, for $i = 1, \ldots, k$: If $\boldsymbol{x} \in (V_i)^n$, i.e., all locations are in V_i, then also $F(\boldsymbol{x}) \in V_i$. Moreover,

> *F is the outcome of applying Def. 3 for F_i defined by $\boldsymbol{x} \in (V_i)^n \mapsto F(\boldsymbol{x})$ and an order over Z which satisfies the ZV-line constraints.*

Conjecture 10.

For almost all graphs $G = \langle \mathcal{V}, E \rangle$,[19] if there exists an anonymous and Pareto optimal mechanism $F\colon \bigcup_{t \geqslant 0} \mathcal{V}^t \to \mathcal{V}$ s.t.

> *For any vector of locations $\boldsymbol{x} \in \mathcal{V}^n$, a coalition of agents C, and a set of ballots $\boldsymbol{A} \in \bigcup_{t \geqslant 0} \mathcal{V}^t$, $\boldsymbol{A}$ is not a beneficial deviation for C.*

Then, there exists a sequence of non-empty sets of vertices $Z, V_1, \ldots, V_k \subseteq \mathcal{V}$ s.t. G is a ZV-line graph w.r.t. $\mathcal{V} = Z \cup (V_1 \dot\cup \cdots \dot\cup V_k)$.

Last, an important continuation of this work is analyzing the implications for Approximate mechanism design without money [17]. That is, assuming the agents are accurately represented by a cost function (e.g., the distance to the facility or a monotone function of the distance) and analyzing the implications of manipulation-resistance on the approximability of the minimization problem of natural social cost functions, e.g., the average cost (Harsanyi's social welfare), the geometric mean of the costs (Nash's social welfare), or the maximal cost (Rawls' criterion). For instance, assuming our two conjectures above, one gets that when there is a large disagreement in the population (i.e., the agents are dispersed over many V_i-subgraphs) an extreme status-quo alternative must be chosen by the mechanism, which results in a bad *price of false-name-proofness.* Nowadays, many aggregation mechanisms (e.g., over huge anonymous networks like the internet, but also in other cases in which vote frauds are easy) are highly susceptible to double voting and to more general false-name manipulations. We think that such results should open a discussion on the costs of these protocols (since the benefits are clear).

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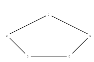
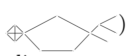
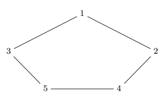
[19]The only counter example we've found to the conjecture is the cycle of size 5  (and graphs derived from it by the second result of Thm. 5, e.g., ). It is not hard to verify that a mechanism which returns the first Pareto optimal location according to one of the following orders



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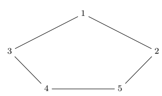
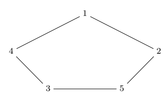


, and


(and their rotations and reflections) is a manipulation-resistant mechanism and that this graph is not a ZV-line graph. We conjecture that this is a representative extreme exception and intend to characterize the exception and replace the 'almost' with an exact statement.

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A Proof of Main Result (Thm. 5)

The anonymity of $F^\star$ is an immediate corollary of the mechanisms F_i and F_Z being anonymous mechanisms.

Notice that if all agents are in the same V_i-subgraph, then all of them strictly prefer $\mathcal{R}(V_i)$ over any location outside of V_i, so $PO(\boldsymbol{x}) \subseteq V_i$. Moreover, any location $v \in V_i \backslash PO(\boldsymbol{x})$ is Pareto dominated by a location $y \in PO(\boldsymbol{x}) \subseteq V_i$. Hence, the Pareto optimal set when considering only the locations in V_i equals to the Pareto optimal set when considering all locations. Since, the mechanisms F_i are Pareto optimal mechanisms we get that also $F^\star$ is Pareto optimal.

In order to prove the main part of the theorem, we assume towards a contradiction that there exists a vector of locations $\boldsymbol{x} \in \mathcal{V}^n$, a coalition of agents C, and a set of ballots $\boldsymbol{A} \in \bigcup_{t \geqslant 0} \mathcal{V}^t$, s.t. C can, by voting $\boldsymbol{A}$, get an outcome $F^\star(\boldsymbol{A}, \boldsymbol{x}_{-C})$ which it strictly prefers, that is, all of its members weakly prefer $F^\star(\boldsymbol{A}, \boldsymbol{x}_{-C})$ over $F^\star(\boldsymbol{x}) = F^\star(\boldsymbol{x}_C, \boldsymbol{x}_{-C})$, and at least one of C's members, Agent i for $i \in C$, strictly prefers $F^\star(\boldsymbol{A}, \boldsymbol{x}_{-C})$ over $F^\star(\boldsymbol{x})$. $F^\star(\boldsymbol{x}) \in PO(\boldsymbol{x})$ and in particular the coalition of all agents does not strictly prefer $F^\star(\boldsymbol{A}, \boldsymbol{x}_{-C})$ over $F^\star(\boldsymbol{x})$. Hence, there exists an Agent j, for $j \notin C$, who strictly prefers $F^\star(\boldsymbol{x})$ over $F^\star(\boldsymbol{A}, \boldsymbol{x}_{-C})$.

If $F^\star(\boldsymbol{x})$ is not in Z: Then necessarily, all the locations in $\boldsymbol{x}$ and $F^\star(\boldsymbol{x})$ belong to the same V_i-subgraph, w.l.o.g. V_1, so $F^\star(\boldsymbol{x}) = F_1(\boldsymbol{x})$. Since F_1 is resistant to false-name manipulations of Agent i and since Agent i can achieve $\mathcal{R}(V_1)$ by casting enough false ballots, we get that Agent i weakly prefers $F^\star(\boldsymbol{x})$ over $\mathcal{R}(V_1)$ and hence Agent i strictly prefers $F^\star(\boldsymbol{A}, \boldsymbol{x}_{-C})$ over $\mathcal{R}(V_1)$. Since for any u outside of V_1 it holds that $d(x_i, \mathcal{R}(V_1)) < d(x_i, u)$, we get that $F^\star(\boldsymbol{A}, \boldsymbol{x}_{-C}) \in V_1 \backslash \mathcal{R}(V_1) \subseteq V_1 \backslash Z$. Hence, $\boldsymbol{A} \subseteq V_1$ and $F^\star(\boldsymbol{A}, \boldsymbol{x}_{-C}) = F_1(\boldsymbol{A}, \boldsymbol{x}_{-C})$, and we get a contradiction to the false-name-proofness of F_1.

Similarly, if $F^\star(\boldsymbol{A}, \boldsymbol{x}_{-C})$ is not in Z: Then necessarily, $F^\star(\boldsymbol{A}, \boldsymbol{x}_{-C})$ and all the locations in $\boldsymbol{A}$ and $\boldsymbol{x}_{-C}$ belong to the same V_i-subgraph, w.l.o.g. V_1, so $F^\star(\boldsymbol{A}, \boldsymbol{x}_{-C}) = F_1(\boldsymbol{A}, \boldsymbol{x}_{-C})$. Since F_1 is resistant to false-name manipulations of Agent j and since Agent j can achieve $\mathcal{R}(V_1)$ by casting enough false ballots, we get that Agent j weakly prefers $F^\star(\boldsymbol{A}, \boldsymbol{x}_{-C})$ over $\mathcal{R}(V_1)$ and strictly prefers $F^\star(\boldsymbol{x})$ over $\mathcal{R}(V_1)$. Since for any u outside of V_1 it holds that $d(x_j, \mathcal{R}(V_1)) < d(x_j, u)$, we get that $F^\star(\boldsymbol{x}) \in V_1 \backslash \mathcal{R}(V_1) \subseteq V_1 \backslash Z$. Hence, hence $\boldsymbol{x} \subseteq V_1$ and $F^\star(\boldsymbol{x}) = F_1(\boldsymbol{x})$, and we get a contradiction to the false-name-proofness of F_1.

If both $F^\star(\boldsymbol{x})$ and $F^\star(\boldsymbol{A}, \boldsymbol{x}_{-C})$ are in Z: We deal with this case using two different argumentations for the two scenarios of the theorem.

(***I***) G is a ZV-line graph w.r.t. $\mathcal{V} = Z \cup (V_1 \dot\cup \cdots \dot\cup V_k)$: We first prove the following two auxiliary lemmas.

Lemma i. *For any $v \in \mathcal{V}$ and $d \geqslant 0$, $B(v, d) \cap Z$ is an interval in Z.*

Proof of Lemma i. We prove the lemma by induction over d.

For $d = 0$, $B(v, 0) \cap Z$ equals to $\{v\}$ if $v \in Z$ and to the empty set if $v \notin Z$.

For $d = 1$, $B(v, 1) \cap Z$ is either the empty set or an interval in Z.

For $d \geqslant 2$: If $d < d(v, Z)$, $B(v, d) \cap Z = \varnothing$. If $d \geqslant d(v, Z) > 1$ (in particular, $v \notin Z$ and is not a root), then there exists a location u (the root of the V_i-subgraph v belongs to) s.t. all paths from v to locations in Z pass through u, $1 \leqslant d(v, u) \leqslant d(v, Z) \leqslant d$ and

$$B(v, d) \cap Z = B(u, d - d(v, u)) \cap Z$$

which is an interval by the induction hypothesis.

Otherwise, $d(v, Z) \leqslant 1 < d$ and in particular $B(v, d) \cap Z \neq \varnothing$, and hence

$$B(v, d) \cap Z = (B(v, 1) \cap Z) \cup \left(\bigcup_{\substack{u \in N(v) \text{ s.t.} \\ d(u, Z) \leqslant 1}} B(u, d-1) \cap Z \right).$$

For any $u \in N(v)$ s.t. $d(u, Z) \leqslant 1$ we claim that $B(u, d-1) \cap Z$ and $B(v, 1) \cap Z$ intersect.

- If $u \in Z$: $u \in (B(u, d-1) \cap Z) \cap (B(v, 1) \cap Z)$.
- If $u \notin Z$: then $v \in Z$ and $v \in (B(u, d-1) \cap Z) \cap (B(v, 1) \cap Z)$.

Hence, for any $u \in N(v)$ s.t. $d(u, Z) \leqslant 1$, $B(u, d-1) \cap Z$ and $B(v, 1) \cap Z$ are intersecting intervals in Z. So $B(v, d) \cap Z$ is an interval as the union of intersecting intervals. □

Lemma ii. *Let $\boldsymbol{x}$ be a vector of locations s.t. $F^{\star}(\boldsymbol{x}) \in Z$ and let $v \in Z$ be a location s.t. Agent i strictly prefers v over $F^{\star}(\boldsymbol{x})$. Then $F^{\star}(\boldsymbol{x})$ is to the left of v.*

Proof of Lemma ii. If $x_i \in Z$ then $x_i \in PO(\boldsymbol{x}) \cap Z$ and by the definition of $F^{\star}$, $F^{\star}(\boldsymbol{x})$ is to the left of x_i. Since $F^{\star}(\boldsymbol{x}) \notin B(x_i, d(x_i, v)) \cap Z$ and since this set is an interval which includes x_i, we get that $F^{\star}(\boldsymbol{x})$ is to the left of the interval and in particular to the left of v.

Otherwise, $x_i \notin Z$ and there exists an Agent k for which x_k is not in the same V_i-subgraph as x_i. Hence, there exists a location $u \in Z$ s.t. u is on a shortest-path from x_i to x_k, $u \in Z$, and $u \in PO(\boldsymbol{x})$. Hence, $d(x_i, u) \leqslant d(x_i, v)$ and so

$$u \in B(x_i, d(x_i, u)) \cap Z \subseteq B(x_i, d(x_i, v)) \cap Z.$$

The two sets are intervals in Z, $F^{\star}(\boldsymbol{x})$ is to the left of u (or equal to it), and

$$F^{\star}(\boldsymbol{x}) \notin B(x_i, d(x_i, v)) \cap Z.$$

Hence, $F^{\star}(\boldsymbol{x})$ is to the left of v. □

By applying Lemma ii for the profile $\boldsymbol{x}$ and Agent i, we get that $F^{\star}(\boldsymbol{x})$ is to the left of $F^{\star}(\boldsymbol{A}, \boldsymbol{x}_{-C})$; and by applying Lemma ii for the profile $(\boldsymbol{A}, \boldsymbol{x}_{-C})$ and Agent j, we get that $F^{\star}(\boldsymbol{A}, \boldsymbol{x}_{-C})$ is to the left of $F^{\star}(\boldsymbol{x})$. Hence, we get a contradiction.

(II) $\mathcal{R}(V_i) \in Z$ and F_Z satisfies (★): We notice that since $\mathcal{R}(V_i) \in Z$ for all V_i-subgraphs the preference of an agent which is located in a V_i-subgraph over the locations in Z and an agent which is located on the root, $\mathcal{R}(V_i)$, are identical. Hence, for any profile $\boldsymbol{y}$ if $F^{\star}(\boldsymbol{y}) \in Z$ then $F^{\star}(\boldsymbol{y}) = F_Z(\widehat{\boldsymbol{y}})$ for $\widehat{\boldsymbol{y}}$ being the profile generated from $\boldsymbol{y}$ by replacing each ballot outside of Z with the root of its V_i-subgraph. Therefore, for the profile $\widehat{\boldsymbol{x}} \in Z^n$ the coalition C can, by voting $\widehat{\boldsymbol{A}}$, get an outcome $F_Z\left(\widehat{\boldsymbol{A}}, \widehat{\boldsymbol{x}}_{-C}\right)$ which it strictly prefers over $F_Z(\widehat{\boldsymbol{x}})$, in contradiction to F_Z satisfying (★). □