Generalized Distance Bribery

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Abstract

The bribery problem in elections asks whether an external agent can make some distinguished candidate win or prevent her from winning, by bribing some of the voters. This problem was studied with respect to the weighted swap distance between two votes by Elkind et al. [7]. We generalize this definition by introducing a bound on the distance between the original and the bribed votes. The distance measures we consider include a restriction of the weighted swap distance and variants of the footrule distance, which capture some real-world models of influence an external agent may have on the voters. We study constructive and destructive variants of distance bribery for k-approval and Borda elections, and obtain polynomial-time algorithms as well as NP-hardness results. These results answer a number of open questions from the literature.

1 Introduction

In an election in which voters influence the election's result by casting their votes they might not be the only agents interested in its outcome. A well known example of strategic influence on elections is bribery, formally introduced by Faliszewski et al. [9] (see also the book chapter by Faliszewski and Rothe [11]). By paying voters with a resource, e. g. money, to change their vote, an external agent tries to change the election outcome to her advantage. In the constructive case the aim is to make a specific candidate a winner of the election, while in the destructive case the briber tries to prevent a candidate from winning. Originally the number of voters that can be bribed is limited. However, there is no limitation to the amount of change within a vote. It is natural to assume that a voter's cost for changing her vote to a large extent is higher than the cost for making small adjustments. This intuition is modeled in the microbribery problem introduced by Faliszewski et al. [10], and in the swap-bribery problem introduced by Elkind et al. [7]. In both models the briber has to pay certain prices for swaps of adjacent candidates. In order to obtain a more realistic model, we propose to imply a bound on the distance between the truthful and the bribed votes.

The idea of limiting the difference between originally cast votes and their altered versions in strategic influences of elections is not restricted to bribery. Obraztsova and Elkind [16] have introduced distances to manipulation, where a voter changes her vote in order to subjectively improve the election outcome. They study the optimal manipulation problem with a focus on the unweighted swap distance and footrule distance, and the maximum displacement distance between the truthful and the strategic vote. We propose to extend the study of distances to both, the constructive and destructive bribery problem and use unweighted and weighted variants of the swap and footrule distance. One motivation for introducing a bound on the overall distance of the bribed profile follows from the work by Obraztsova and Elkind [16] on optimal manipulation. The distance between a truthful and a manipulative vote should be bounded since otherwise a manipulation may be detected more easily.

In a more positive sense bribery can be seen as a form of campaign management where an external actor tries to make her desired candidate win by running specific campaigns. While the swap distance counts the inversions of pairs of candidates, it is a more realistic approach to assign a weight to a specific swap. This models a campaign that focuses on the difference between these two candidates. The restriction of element-weighted distance functions (studied by Kumar and Vassilvitskii [14] for linear orders) assigns a weight to every candidate and the cost of a swap is the product of the corresponding weights. This can be seen as an indicator of how sure a voter is about the position of a candidate in the vote. The same weight functions can also be applied to the footrule distance. Instead of inversions, here the positions a candidate is shifted by are counted.

We will study the computational complexity of distance bribery for plurality, veto, k-approval and Borda elections. As distances we consider swap and footrule distances in the unweighted, element-weighted, and weighted case. Our results include polynomial-time algorithms obtained through some network-flow algorithm and NP-completeness results. A detailed overview of all results will be presented in Section 3 in Table 1, after all needed notions are formally introduced.

2 Preliminaries

In this section we first provide the basics on elections and then introduce the different forms of distances.

2.1 Elections

An election is a pair (C, V) with a set of candidates C and a list of voters $V = (v_1, \ldots, v_n)$ (also called profile), where a vote v_i is associated with a preference order $>_{v_i}$ over the candidates. If the vote v_i is clear from the context we omit the index or simply write $>_i$. The set of all linear orders over the candidates in C will be denoted by $\mathcal{L}(C)$. The position of candidate c in vote v_i will be denoted by $\mathrm{pos}(v_i, c)$, and the candidate on position k in vote v_i will be denoted by $\mathrm{cand}(v_i, k)$. For two lists $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_k)$ of votes let $A \cup B = (a_1, \ldots, a_n, b_1, \ldots, b_k)$ be the concatenation of both lists. A voting rule is a function $\mathcal{E}$ that maps an election (C, V) to an element of 2^C. In this paper tie-breaking is not an issue since we focus on the so-called *non-unique* winner model, where a bribery action is successful if the designated candidate is one of the winning candidates in the constructive case, and if she is none of the winners in the destructive case.

We will consider different voting rules, all belonging to the class of scoring rules. Every candidate gets points according to her position in each vote, and the winners are those with the maximum number of points. The **plurality** rule gives only one point for a first position, whereas the **k-approval** rule gives one point to every candidate placed in the top k positions of a vote. Accordingly the **veto** and **k-veto** rule give one point to all candidates but the candidate(s) in the last/last k positions who get zero points. In addition we will consider the **Borda** rule, where in an election with m candidates the first placed candidate gets $m - 1$ points, the second one $m - 2$, up to the last candidate who gets 0 points. By $\mathrm{score}_{(C,V)}(c)$ we denote the points that candidate c gets in the election (C, V).

2.2 Distances

Now we will introduce different distance measures. Formally, a distance measure d on a space A is a mapping $d : A \times A \to \mathbb{R}$ that fulfills the following properties for all $a, b, c \in A$: (i) $d(a, b) \geq 0$ (non-negativity), (ii) $d(a, b) = 0$ if and only if $a = b$ (identity of indiscernibles), (iii) $d(a, b) = d(b, a)$ (symmetry), and (iv) $d(a, b) + d(b, c) \geq d(a, c)$ (triangle inequality). We will focus on distances between votes. For a fixed set of candidates this is a distance of the form $d : \mathcal{L}(C) \times \mathcal{L}(C) \to \mathbb{R}$. As we will study the complexity of problems involving such distances we actually consider a family of distances $(d^m)_{m \geq 1}$ that contains one distance function for each possible candidate set size m.

The two basic distance measures we consider are the swap distance and the footrule distance. For two votes v and v' from $\mathcal{L}(C)$ the **swap distance** $\mathrm{swap}(v, v')$ counts the number of inverted pairs and is formally defined as $\mathrm{swap}(v, v') = |\{(c, b) \in C \times C \mid c >_v b \text{ and } b >_{v'} c\}|$. Beside the swap distance, we will also consider the footrule distance. Instead of counting swaps, this distance counts the positions that a candidate needs to be shifted by to obtain the target order. The **footrule distance** $\mathrm{fr}(v, v')$ is defined as: $\mathrm{fr}(v, v') = \sum_{c \in C} |\mathrm{pos}(v, c) - \mathrm{pos}(v', c)|$.

In addition to the well-known unweighted versions of these distance measures we also consider weighted forms. The most general form is the weighted swap distance which is also used for the swap bribery problem (see Elkind et al. [7]). In order to define the weighted distance we introduce the general notion of a cost function $\pi : C \times C \to \mathbb{N}_0$, that assigns a weight to each pair of candidates. We will always assume that $\pi(c, c) = 0$ for every $c \in C$.

Definition 1 (Weighted swap/footrule distance) *Let π be a cost function, then the **weighted swap distance** between two votes v, v' from $\mathcal{L}(C)$ is $\mathrm{swap}^{\mathrm{w}}_{\pi}(v, v')$ with $\mathrm{swap}^{\mathrm{w}}_{\pi}(v, v') = \sum_{(c,b)\in\{(c,b)\in C\times C \mid c>_v b \text{ and } b>_{v'} c\}} \pi(c, b)$, and the **weighted footrule distance** between two votes v, v' from $\mathcal{L}(C)$ is $\mathrm{fr}^{\mathrm{w}}_{\pi}(v, v') = \sum_{c\in C} \left| \sum_{b\in\{b\in C \mid b>_v c\}} \pi(b, c) - \sum_{b\in\{b\in C \mid b>_{v'} c\}} \pi(b, c) \right|$.*

For matters of consistency we sometimes denote the unweighted distances by fr_{π} and swap_{π}, where the cost function is $\pi(c, b) = 1$ for all $c, b \in C, c \neq b$. The following example illustrates how these distances are calculated.

Example 2 *Consider the votes $v : p > q > r > s > t$ and $v' : p > s > r > q > t$. The swap distance is $\mathrm{swap}(v, v') = |\{(q, r), (q, s), (r, s)\}| = 3$, i. e., the number of times adjacent candidates have to be swapped. For a cost function π, the weighted swap distance is the sum of the costs of these pairs: $\mathrm{swap}^{\mathrm{w}}_{\pi}(v, v') = \pi(q, r) + \pi(q, s) + \pi(r, s)$. The unweighted footrule distance is $\mathrm{fr}(v, v') = |2 - 4| + |4 - 2| = 4$, whereas the weighted footrule distance for a cost function π is $\mathrm{fr}^{\mathrm{w}}_{\pi}(v, v') = | - \pi(s, q) - \pi(r, q)| + |\pi(q, r) - \pi(s, r)| + |\pi(q, s) + \pi(r, s)|$.*

In addition to these general weighted distances we will consider a special weighted variant that corresponds to the element-weighted distances described by Kumar and Vassilvitskii [14]. In the **element-weighted swap/footrule distance** (denoted by $\mathrm{swap}^{\mathrm{ew}}_{\pi}$ and $\mathrm{fr}^{\mathrm{ew}}_{\pi}$ respectively), the function π is restricted to the form $\pi(c, b) = \varphi(c) \cdot \varphi(b)$, where $\varphi : C \to \mathbb{N}_0$ is a function that assigns a weight to every candidate. While our distances are obviously heavily inspired by the weighted distances proposed by Kumar and Vassilvitskii [14], they actually differ widely. Their distances are invariant, i. e., they are not dependent on the actual identity of each element but on the permutation in-between vote v and v'. Our distances are not invariant for a good reason. They should depend on the identity of each candidate and the measurable differences or similarities between them.

The metric properties are still fulfilled for our definition of the element-weighted swap distance since it is a special case of the weighted swap distance, for which it is not hard to show these properties. For the triangle inequality note that each inversion between a and c is part of the inversions between a and b or b and c. Non-negativity, identity of indiscernibles, and symmetry are obviously also fulfilled for the weighted footrule distance. For the triangle inequality note that (again) each inversion between a and c is part of the inversions between a and b or b and c. Splitting the sums up in-between a to b inversions and b to c inversions and calculating the absolutes of both differences individually at least increases the overall value. Note that if costs of 0 are allowed, all distance measures only fulfill the requirements for a pseudometric, i. e., the identity of indiscernibles is replaced by the property $d(a, a) = 0$.

Even though the element-weighted swap and footrule distances focus on different aspects of the differences between two votes, it can be shown that they are somehow related, and we will make use of this in our proofs. The Diaconis-Graham inequality also holds for the element-weighted distances used in this paper.

Lemma 3 *For a list of candidates C and $v, v' \in \mathcal{L}(C)$ it holds that $\mathrm{swap}^{\mathrm{ew}}_{\pi}(v, v') \leq \mathrm{fr}^{\mathrm{ew}}_{\pi}(v, v') \leq 2 \cdot \mathrm{swap}^{\mathrm{ew}}_{\pi}(v, v')$.*

While the definitions of the overall element-weighted distances differ from those by Kumar and Vassilvitskii [14], the idea of the proof of this lemma is transferable: The element-weighted votes are substituted with unweighted votes, where each candidate c is replaced by $\varphi(c)$ copies. Since the distance values for the weighted and unweighted case of the swap and footrule distance are equal, the Diaconis-Graham inequality is passed from the unweighted to the weighted case.

3 Distance Bribery

Now we will define decision problems in order to study their computational complexity. For details on computational complexity we refer to the textbooks by Papadimitriou [17] and Arora and Barak [1]. The study of the computational complexity of such manipulative actions is important, since in the context of voting in multi-agent systems it is crucial to know about the possibility of external intervention. Of course, a worst-case complexity analysis is only the first step and should be followed by a more fine-grained analysis and the design of fast algorithms for the average case.

The original bribery problem asks if for a given election a designated candidate can be turned into a winner of the election by bribing a specific number of candidates, whereas the swap bribery problem (introduced by Elkind et al. [7]) requires that the costs for the swaps of adjacent candidates may not exceed a budget. In the related shift bribery problem, also introduced by Elkind et al. [7], only swaps involving the desired candidate are allowed. For a given distance function $\mathcal{D}$ and a voting rule $\mathcal{E}$ we define a generalization of the swap bribery problem as follows:

	$(\mathcal{D}, \mathcal{E})$-DISTANCE BRIBERY
Given:	An election $E = (C, V)$, with $V = (v_1, \dots, v_n)$, a cost function π_i, $1 \leq i \leq n$, for every voter, a positive integer K, and a distinguished candidate $p \in C$.
Question:	Is there a list of votes $V' = (v'_1, \dots, v'_n)$ such that $\sum_{i \in \{1,\dots,n\}} \mathcal{D}_{\pi_i}(v_i, v'_i) \leq K$ and $p \in \mathcal{E}(C, V')$?

In the destructive variant $(\mathcal{D}, \mathcal{E})$-DESTRUCTIVE DISTANCE BRIBERY we have the same input, but we ask whether there is an alternative profile $V' = (v'_1, \dots, v'_n)$ with $\sum_{i \in \{1,\dots,n\}} \mathcal{D}_{\pi_i}(v_i, v'_i) \leq K$ and $p \notin \mathcal{E}(C, V')$. Hence we want to prevent p from being a winner of the election.

One restriction is that the cost function is identical for all voters. This follows the work on bribery where only a certain number of votes could be changed arbitrarily (see Faliszewski et al. [9]). All our polynomial-time results obviously cover this restricted version, and with the exception of Theorem 7 all NP-hardness results also hold in this special case. The most general problems are those for the weighted distances, while unweighted distances are the most restricted problems. Hence, polynomial-time algorithms for $\mathrm{swap}_\pi^{\mathrm{w}}$ and $\mathrm{fr}_\pi^{\mathrm{w}}$ carry over to the corresponding problems for $\mathrm{swap}_\pi^{\mathrm{ew}}$ and $\mathrm{fr}_\pi^{\mathrm{ew}}$ distances, and both to the corresponding problems for swap_π and fr_π distances. At the same time, NP-hardness for the unweighted distances (swap_π and fr_π) implies NP-hardness for the element-weighted distances ($\mathrm{swap}_\pi^{\mathrm{ew}}$ and $\mathrm{fr}_\pi^{\mathrm{ew}}$) which in turn imply NP-hardness for the weighted distances $\mathrm{swap}_\pi^{\mathrm{w}}$ and $\mathrm{fr}_\pi^{\mathrm{w}}$. Obviously, all decision problems are contained in NP, since the winners for all voting rules considered here can be computed in polynomial-time and as, given two votes, it is possible to compute all introduced distances in polynomial time.

Results for the constructive and destructive variants for the different distances are summarized in Table 1. Whenever results were previously known (or follow by the above described relations), the respective source is given. For all other cases, the number of the corresponding theorem, for which the proof is provided in this paper, is given in brackets. Our results solve a number of open questions from the literature. Shiryaev et al. [18] conjectured that $\mathrm{swap}_\pi^{\mathrm{w}}$ DESTRUCTIVE DISTANCE BRIBERY is hard for many scoring rules, especially Borda and k-approval for any fixed $k \geq 2$. We confirmed this conjecture for Borda (Theorem 7) but disproved it for k-approval (Theorem 5). Obraztsova and Elkind [16] posed the question whether the swap bribery problem, where only one voter can be bribed, remains tractable for scoring rules. We give a negative answer in the case of Borda elections. In our NP-hardness proof of $(\mathrm{swap}_\pi^{\mathrm{w}}, \mathrm{Borda})$-DESTRUCTIVE DISTANCE BRIBERY (Theorem 7) there is only one voter that may be bribed. Together with the remarks after the proof, this also shows that the corresponding optimal manipulation problem for weighted swap distances is NP-complete. Note that this proof can be extended to a larger class of scoring rules, where all entries of the scoring vector differ. In addition, Obraztsova and Elkind [16] suggested to extend the study of optimal manipulation to other distances. Since the optimal manipulation problem is a special case of

		swap_π	$\mathrm{swap}^{\mathrm{ew}}_\pi$	$\mathrm{swap}^{\mathrm{w}}_\pi$	fr_π	$\mathrm{fr}^{\mathrm{ew}}_\pi$	$\mathrm{fr}^{\mathrm{w}}_\pi$
Plurality	C	P$^{\blacktriangle}$	P$^{\blacktriangle}$	P$^{\blacktriangle}$	P, (10)	P, (10)	P, (10)
	D	P$^{\spadesuit}$	P$^{\spadesuit}$	P$^{\spadesuit}$	P, (6)	P, (6)	P, (6)
Veto	C	P$^{\blacktriangle}$	P$^{\blacktriangle}$	P$^{\blacktriangle}$	P, (10)	P, (10)	P, (10)
	D	P$^{\spadesuit}$	P$^{\spadesuit}$	P$^{\spadesuit}$	P, (6)	P, (6)	P, (6)
k-app.	C	P*	NP-c., (9)	NP-c.$^{\blacktriangle}$	P, (11)		
	D	P, (5)	P, (5)	P, (5)	P, (6)	P, (6)	P, (6)
Borda	C	NP-c., (12)	NP-c., (12)	NP-c.$^{\blacktriangle}$		NP-c., (13)	NP-c., (13)
	D	P$^{\spadesuit}$		NP-c., (7)	P, (8)		

Table 1: Overview of complexity results for the constructive and destructive DISTANCE BRIBERY problems for various distances, where "NP-c." stands for NP-complete. Results that were previously known (or directly follow from them) are marked by their corresponding source: $^{\blacktriangle}$ Elkind et el. [7], * Dorn and Schlotter [6], and $^{\spadesuit}$ Shiryaev et al. [18]. For all other results the reference to the corresponding theorem is given in brackets. The NP-completeness of constructive distance bribery for $\mathrm{swap}^{\mathrm{w}}_\pi$ for k-approval holds for a fixed $k \geq 2$, whereas for $\mathrm{swap}^{\mathrm{ew}}_\pi$ it holds for a fixed $k \geq 4$.

bribery where only one voter can be bribed, all our P results carry over to the corresponding optimal manipulation problems.

Related Work The use of individual prices for the change of each vote has been introduced by Faliszewski et al. [9] for the priced bribery problem. There is however one price fixed irrespective of the change in the vote. Distance restrictions on constructive and destructive bribery problems have also been studied by Yang et al. [20] for the swap distance and the Hamming distance. In our notation the swap bribery problem corresponds to the distance bribery problem with the $\mathrm{swap}^{\mathrm{w}}_\pi$ distance. The related bribery problems with distance restrictions introduced by Yang et al. [20] additionally include an upper bound on the number of votes that can be bribed. Since we follow the work on swap bribery we relate the upper bound of allowed bribery actions to the distance at hand. Furthermore, we consider the distance bound to be part of the input, whereas Yang et al. fix it in the problem name. It is obvious that their distance restricted bribery problem for an unlimited distance bound equals the ordinary bribery problem. Another difference is that Yang et al. bound the distance for each bribed voter, while we have a global bound on the distance, consistent with the swap bribery problem. Elkind et al. [7] show that the swap bribery problem is a generalization of the possible winner problem. However, there is no direct reduction for problems with other distance functions than the weighted swap distance. Even for the element-weighted swap distance their reduction no longer holds. On the one hand, our problem differs from the nonuniform bribery, which was introduced by Faliszewski [8], since they use utility-based voting, while our definition applies to profiles where votes are linear orders over the candidates. On the other hand, it should be mentioned that our model can also be applied to various other preference representations, as well as to the approval vectors used by Faliszewski [8], as long as meaningful distances can be defined on that form of representation.

Another related problem is the margin of victory in an election studied, e. g., by Magrino et al. [15] and Xia [19], and the robustness of an election studied by Shiryaev et al. [18]. They focus on the possibility to change the election winner through some changes in the votes. It is argued that the amount of change needed to obtain a different outcome can be seen as a measure for the robustness of a voting rule. Similar as for the optimal manipulation problem the motivation is again to detect fraud in elections. The latter problem is equivalent to destructive swap bribery, and the same arguments obviously apply to the bribery problem. Hence, the distance measures used in this paper also provide a way of measuring the robustness of a voting rule.

4 Results

In this section we first provide the results for the destructive case and continue by giving the results for the constructive case.

4.1 Destructive Distance Bribery

Many of our proofs for DESTRUCTIVE DISTANCE BRIBERY will make use of a non-trivial flow network designed for the distance bribery problem for k-approval elections. Before stating our results we will describe the cost flow network and some basic properties. Since this construction is fairly general it can be used to proof several different results. Due to its generality it can also be adapted to solve related problems.

Construction 4 Given a set of candidates C with $\{p, q\} \subseteq C$ and a list of votes $V = (v_1, \ldots, v_n)$, such that $\mathrm{score}_{(C,V)}(q) > \mathrm{score}_{(C,V)}(p)$. The purpose of the algorithm is to decide whether some candidate p can win against some candidate q in the current election through a bribery action. The cost function for voter v_i will be denoted by π_i. Let $\delta = \mathrm{score}_{(C,V)}(q) - \mathrm{score}_{(C,V)}(p) + 1$ be the deficit between q and p. The only significant changes a briber can perform are: move q out of the top k positions (denoted by $\overrightarrow{q}$), move p into the top k positions (denoted by $\overleftarrow{p}$), or combine both actions (denoted by $\overleftrightarrow{pq}$). We will refer to the first two as *single operations* and to the last one as *double operation*. We will denote the minimum costs of performing such an operation to vote v_i by $\mathrm{cost}_i(o)$, with $o \in \{\overrightarrow{q}, \overleftarrow{p}, \overleftrightarrow{pq}\}$. If a certain operation is not possible, we will set the cost to ∞. Since $\overrightarrow{q}$ is a single operation with the purpose of reducing the deficit by one, it is not allowed that p will also be shifted to a position giving 0 points. The same holds for the single operation $\overleftarrow{p}$.

The difficulty is that performing one of these operations excludes the other two and the double operation can cost less or more than the sum of the single operations. Each single operation decreases the deficit by 1 and each double operation by 2. To determine the optimal set of operations striking out the whole deficit we will generate an optimal set of single operations for every possible number of double operations and will pick the cost minimum with respect to the given cost. Let n_{do} and n_{so} denote the number of double/single operations. It holds that $n_{do} \in \{0, \ldots, \lfloor \frac{\delta}{2} \rfloor\}$ and $n_{so} = \delta - 2n_{do}$. We construct a minimum-cost flow network as follows. The set of vertices is $Q = \{s, t\} \cup \{v_1, \ldots v_n\} \cup \{p_1, q_1, pq_1, \ldots, p_n, q_n, pq_n\} \cup \{do, so\}$, the set of edges is $R = R_1 \cup R_2 \cup R_3$ with: $R_1 = \{(s, v_i) \mid 1 \le i \le n\} \cup \{(do, t), (so, t)\}$, $R_2 = \{(v_i, p_i), (v_i, q_i), (v_i, pq_i) \mid 1 \le i \le n\}$, and $R_3 = \{(p_i, so), (q_i, so), (pq_i, do) \mid 1 \le i \le n\}$. The question is, whether there is a flow of $n_{do} + n_{so}$ from the source s to the sink t with minimum cost. To ensure that the total number of operations is not exceeded, the capacity of the edges from do and so to the sink t are n_{do} and n_{so}, respectively, and the costs are 0 for these edges. The edges from a vote vertex v_i leading to one of the operation vertices (i. e., p_i, q_i, or pq_i) have cost $\mathrm{cost}_i(o)$ for the corresponding operation o, and capacity 1. The cost of all remaining edges is 0 and the capacity is 1. An example for such a flow network is shown in Figure 1. The first value on the edges refers to the costs, the second one to the capacity.

Obviously, the used edges leading to the operations are equivalent to the operations that have to be performed on the set of votes. A flow of $n_{do} + n_{so}$ can only be reached by using n_{do} double and n_{so} single operations, providing the optimal set of operations for a fixed number of double operations. Selecting their optimal is the overall optimal set of operations.

Solving several of these minimum-cost flow problems, provides an cost optimal bribery making p better than q. To solve the decision problem, we have to check if the computed cost optimal bribery stays inside the budget K.

Theorem 5 *($\mathcal{D}$, k-approval)*-DESTRUCTIVE DISTANCE BRIBERY *is in* P *for* $\mathcal{D} \in \{\mathrm{swap}_\pi, \mathrm{swap}_\pi^{\mathrm{w}}, \mathrm{swap}_\pi^{\mathrm{ew}}\}$.

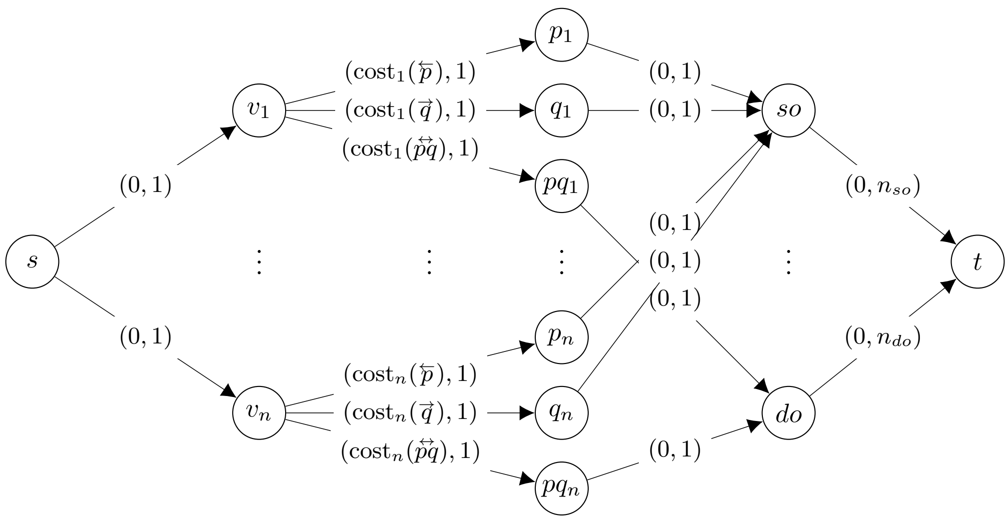

Figure 1: Minimum-cost flow network from Construction 4.

Proof. Due to the comments made above it is sufficient to show polynomial-time solvability for $(\text{swap}^{\text{w}}_{\pi}, k\text{-approval})$-DESTRUCTIVE DISTANCE BRIBERY. This proof will make use of the flow network from Construction 4. It is enough to show, that the minimum costs for the single and double operations for a given vote v_i can be computed in polynomial time. The trivial approach for operation $\overleftarrow{p}$ would be to shift p to the k-th position. While this approach works for the unweighted case, it does not always generate the cost minimum allocation for the weighted case (e. g., think of some candidate c in front of p with $\pi(c,p) \to \infty$).

Now we show how to compute the minimum costs for some fixed vote v_i. Divide the candidates into two sets A and B according to the points they get in v_i: $A = \{\text{cand}(v_i, j) \mid 1 \leq j \leq k\}$ and $B = \{\text{cand}(v_i, j) \mid k+1 \leq j \leq m\}$. In the cost minimum order v_i^* for operation o it holds that the number of candidates in $A' \subseteq A$ that are moved to a position giving 0 points equals the numbers of candidates in $B' \subseteq B$ that are moved to a position giving 1 point. Based on the operation $o \in \{\overrightarrow{q}, \overleftarrow{p}, \overleftrightarrow{pq}\}$ the sets are fulfilling the following restrictions: For $\overrightarrow{q}$: $q \in A'$ and $p \notin A'$, for $\overleftarrow{p}$: $p \in B'$ and $q \notin B'$ and for $\overleftrightarrow{pq}$: $q \in A'$ and $p \in B'$ holds.

Since v_i^* is obtained through minimal costs it satisfies $(A \setminus A') > B' > A' > (B \setminus B')$, where the candidates in the sets are in their original order given by v_i. This order is cost minimum for given A' and B' since it only contains the swaps that are unavoidable when exchanging A' and B'. If there are swaps with cost 0 the cost minimum order is not necessary unique. Hence, to determine the minimum costs for some operation we have to check each combination of possible sets A' and B' fulfilling the restrictions of the corresponding operation. Let the size of A' and B' be $s \in \{1, \ldots, k\}$. There are less than k^s different sets A' and $(m-k)^s$ different sets B'. Hence the overall number of combinations to consider is $\sum_{s=1}^{k} k^s \cdot (m-k)^s$ which is polynomial for a fixed k.

Note that for $k = 1$ this proof covers plurality, and a very similar approach can be used to show polynomial-time solvability for k-veto. ❑

Theorem 6 *$(\mathcal{D}, k\text{-approval})$-*DESTRUCTIVE DISTANCE BRIBERY *is in* P *for* $\mathcal{D} \in \{\text{fr}_{\pi}, \text{fr}^{\text{w}}_{\pi}, \text{fr}^{\text{ew}}_{\pi}\}$.

Proof. It is again sufficient to show polynomial-time solvability for $(\text{fr}^{\text{w}}_{\pi}, k\text{-approval})$-DESTRUCTIVE DISTANCE BRIBERY.

The proof relies on the same ideas as the proof of Theorem 5. Naturally, we have to argue that the given cost minimum order v_i^* regarding v_i performing some operation $o \in \{\overrightarrow{q}, \overleftarrow{p}, \overleftrightarrow{pq}\}$ with the given restrictions is also cost minimum for the weighted footrule. Hence we have to show, that (using the notation of the proof of Theorem 5) the following order v_i^* is cost-minimal for the weighted footrule distance: $(A \setminus A') > B' > A' > (B \setminus B')$. While the correction of an adjacent inversion always reduces the swap distance, the footrule distance can increase since the sum of both candidates' related term's absolutes is not necessarily lowering. In order to prove the cost minimality of the above order we will argue that correcting the leftmost or rightmost inversion in a vote v_i' always lowers the weighted footrule distance to v_i. Note that a swap of adjacent candidates only affects the footrule terms of the swapped candidates.

Assume that the pair of candidates (x, y) is the leftmost inversion in vote v_i', i. e., it holds that in vote v_i candidate x is preferred to y, while in v_i' it is the other way round. Note that, since it is the leftmost inversion, all candidates that are in front of y in v_i' are also in front of y in v_i. Hence, swapping x and y lowers the term for candidate y in $\mathrm{fr}_\pi^{\mathrm{w}}(v_i, v_i')$ by $\pi(x, y)$ since it reduces the difference of $\sum_{d \in \{d \in C \mid d >_{v_i} y\}} \pi(d, y)$ and $\sum_{d \in \{d \in C \mid d >_{v_i'} y\}} \pi(d, y)$ by $\pi(x, y)$. This holds since the former sum is, due to the previous observation increased by at least $\pi(x, y)$ compared to the latter sum. So increasing the latter sum by $\pi(x, y)$ through swapping x and y lowers the difference by exactly $\pi(x, y)$. The x-term could be increased by at most $\pi(x, y)$ so the overall distance will be lower or equal. We can use a similar approach for the rightmost inversion of candidates (x, y), where in vote v_i candidate x is preferred to y and in vote v_i candidate y is preferred to x. This time each candidate that is in front of x in v_i is in front of x in v_i'. Thus, $\sum_{d \in \{d \in C \mid d >_{v_i'} x\}} \pi(d, x)$ is at least increased by $\pi(x, y)$ compared to $\sum_{d \in \{d \in C \mid d >_{v_i} x\}} \pi(d, x)$. Swapping x and y is therefore reducing the former sum by $\pi(x, y)$, lowering the difference by $\pi(x, y)$, and increasing the y-term by at most $\pi(x, y)$. Again the overall distance will be lower or equal.

Consequently, each vote v_i' where A' and B' are switched that is deviating from the above order v_i^* is having a higher or the same distance to v_i compared to the distance between v_i and v_i^*. ❑

This result directly implies polynomial time solvability for the weighted footrule variants of DESTRUCTIVE DISTANCE BRIBERY for k-approval. We continue, by considering the destructive variant of distance bribery for Borda elections. It is known that destructive shift bribery[1] and destructive unweighted swap bribery are in P for Borda (see, e. g. Kaczmarczyk and Faliszewski [13] and Shiryaev et al. [18]). In contrast we will show that DESTRUCTIVE DISTANCE BRIBERY is NP-complete for Borda when the weighted swap distance is used.

Theorem 7 *($\mathrm{swap}_\pi^{\mathrm{w}}$, Borda)-*DESTRUCTIVE DISTANCE BRIBERY *is* NP*-complete.*

Proof. This proof is inspired by a proof of Elkind et al. [7] for the constructive swap bribery problem. The reduction is from the BALANCED BICLIQUE problem. The input is a bipartite graph $G = (U, W, E)$ with the sets of vertices $U = \{u_1, u_s, \ldots, u_N\}$ and $W = \{w_1, w_2, \ldots, w_N\}$, the set of edges $E \subseteq U \times W$, and a positive integer $k \leq N$. It is a yes-instance if the graph has a bi-clique of size k (see Garey and Johnson [12]). We construct an instance of the destructive bribery problem as follows. The set of candidates is $A = \{u_1, \ldots, u_N, w_1, \ldots, w_N, p, q, d\}$, the cost limit is $K = N - k$, and the briber's goal is to prevent q from being a winner of the election. There are only three voters v_1, v_2, and v_3 with the following preferences: $v_1 : q > d > w_1 > \cdots > w_N > p > u_1 > \cdots > u_N$, $v_2 : p > d > q > u_N > \cdots > u_1 > w_N > \cdots > w_1$, and $v_3 : p > q > w_1 > \cdots w_N > u_1 > \cdots u_N > d$. The cost functions π_1 and π_2 are set such that a swap between two candidates is $K + 1$ and hence exceeds the budget. The cost function π_3 is as follows: $\pi_3(u_i, u_j) = \pi_3(w_i, w_j) = 0$, for $1 \leq i, j \leq N$, $i \neq j$, $\pi_3(w_i, q) = 1$, and $\pi_3(u_i, q) = 0$, for $1 \leq i \leq N$, $\pi_3(u_i, w_j) = \begin{cases} 0 & \text{if } (u_i, w_j) \in E \\ K+1 & \text{otherwise} \end{cases}$, all remaining costs will be set to $K + 1$, making a swap impossible. The current winner of the election is q with $6N + 3$ points followed by p

[1] In destructive shift bribery the only possible operation is to move the distinguished candidate to a lower position.

with $5N+4$ points. Note that due to the forbidden swaps no other candidate can become an election winner. In order to turn q into a non-winner one has to move q back to position $N+2$ in v_3. It is straightforward to see that this is only possible while staying in the distance limit if and only if G contains a bi-clique of size k. ❑

Please note that this proof also covers the constructive case, since the question of whether q loses or p wins are equivalent for this construction. (Non-)uniquess can be adjusted by shifting p or q by one position in v_1 or v_2.

Theorem 8 $(\mathrm{fr}_\pi, \textit{Borda})$-DESTRUCTIVE DISTANCE BRIBERY *is in* P.

Proof. Again, it is the briber's aim to prevent q from winning the election. Let p be the candidate that is supposed to beat q in the election and consider all $p \in C \setminus \{q\}$ successively. Given two profiles $V = (v_1, v_2, \ldots, v_n)$ and $V' = (v'_1, v'_2, \ldots, v'_n)$ it holds due to the definition of the footrule distance that

$$\begin{aligned}\sum_{i=1}^{n} \mathrm{fr}_\pi(v_i, v'_i) &= \sum_{i=1}^{n} \sum_{c \in C} |\mathrm{pos}(v_i, c) - \mathrm{pos}(v'_i, c)| \\ &= \sum_{c \in C} \left|\mathrm{score}_{(C,V)}(c) - \mathrm{score}_{(C,V')}(c)\right|,\end{aligned}$$

where $\mathrm{score}_{(C,V)}(c)$ is the Borda score of c in the election (C, V). We denote the deficit of p and q by $\delta = \mathrm{score}_{(C,V)}(q) - \mathrm{score}_{(C,V)}(p) + 1$ and define the function $cost_i(k)$ as the minimum footrule distance between a vote v_i and its modified version v'_i with the deficit reduced by at least k. Note that we do not consider the exact reduction of δ by k as this might not minimize the costs. If in the vote v_i candidate p is ranked on a better position than q it holds $cost_i(k) = 2k$. If however p is ranked on a lower position than q it holds that $cost_i(k) = 2k$ if $k < y - x$, $cost_i(k) = 2(y - x)$ if $y - x \leq k \leq 2(y - x)$, and $cost_i(k) = 2(y - x) + 2(k - 2(y - x))$ if $k > 2(y - x)$, where x is q's and y is p's position in v_i.

The optimal destructive bribery action making p score higher than q needs a budget of at least δ. We try to reduce the deficit as much as possible only by switching p and q in the individual votes which is not always sufficient. The idea is to use the pseudo-polynomial subset-of-sums algorithm (see the book by Garey and Johnson [12]) to find those votes which reduce the deficit to an integer boundary b as much as possible by only switching p and q. The subset-of-sums algorithm takes a set of integers S and an integer value b as input. It searches for a subset of S with a sum closest to b. For each vote v'_j in which q is preferred to p, add $2 \cdot (y - x)$ to S, where x is q's and y p's position in the current vote v'_j. The bound b equals δ. Note that in the constructed instance the sum of the values in S is bounded by the number of candidates and the number of voters, and hence we have a polynomial-time algorithm. Switching p and q in the votes corresponding to the subset with the sum closest to δ minimizes the costs. On the one hand, if the algorithm's result is less than δ, reducing the difference between this sum and δ by one increases the costs by 2. On the other hand, if the algorithm's result is greater than δ, reducing the difference does not minimize the costs, and we already payed a cost of two times the difference additionally.

Note that if there were another profile with smaller costs, this would be a contradiction to the optimality of the subset-of-sums result which concludes the proof. ❑

4.2 Constructive Distance Bribery

We now turn to the constructive case. Some of the following results refer to proofs seen above.

Theorem 9 $(\mathrm{swap}^{\mathrm{ew}}_\pi, k\textit{-approval})$-DISTANCE BRIBERY *is* NP*-complete for any fixed k, $k \geq 4$, even if the cost functions are identical for all voters.*

Proof. NP-hardness will be shown via a reduction from the NP-complete problem X3C (see Garey and Johnson [12]). An X3C instance consists of a set $U = \{u_1, \ldots, u_{3q}\}$ of items and a family $S = \{S_1, \ldots, S_r\}$ of 3-element subsets of U. It is a yes-instance if there is a subfamily $S' \subseteq S$ with $|S'| = q$ and $\bigcup_{S_i \in S'} S_i = U$.
Given an X3C instance (U, S) with $U = \{u_1, \ldots, u_{3q}\}$ and $S = \{S_1, \ldots, S_r\}$ we construct an instance of ($\text{swap}_\pi^{\text{ew}}$, k-approval)-DISTANCE BRIBERY as follows. The set of candidates is $C = \{u_1, \ldots, u_{3q}, p\} \cup D \cup X$, where $D = \{d_1, \ldots, d_4\}$ is a set of dummy candidates and X is a set of polynomially many blocking candidates. The list of votes is $V = V_s \cup V_d$. V_s contains one vote v_i for each set $S_i \in S$ with $S_i = \{u_{i_1}, u_{i_2}, u_{i_3}\}$ of the following form: $v_i : d_1 > d_2 > d_3 > d_4 > u_{i_1} > u_{i_2} > u_{i_3} > p > X$. The candidates in X can be in an arbitrary order. The set V_d contains polynomially many dummy votes to achieve the following scores: $\text{score}_{(C,V)}(u_i) = r - 1$, for $1 \leq i \leq 3q$, $\text{score}_{(C,V)}(p) = r - q$, $\text{score}_{(C,V)}(d_i) = r$, for $1 \leq i \leq 4$, all blocking candidates in X get at most r, and there is at least one blocking candidate that gets exactly r points. In the dummy votes there is always a blocking candidate of X at the fifth position. This is obviously possible with polynomially many blocking candidates. The weight function $\pi(p, d) = \varphi(p) \cdot \varphi(d)$ is identical for all voters and it holds that $\varphi(d_i) = 1$, for $1 \leq i \leq 4$, $\varphi(u_i) = \varphi(p) = 64$, for all $1 \leq i \leq 3q$, and all other weights are set to $1024q + 1$. The distance bound is $K = 1024q$, implying that only swaps involving candidates of C and D are possible. In particular, the candidates of X in the fifth position of the dummy votes prevents that a swap in these votes leads to changes in the scores. Candidate p has to gain at least q points to win the election.

If there exists a successful bribery, p must have at least r points. The cheapest way to gain one point in a vote from V_s is to move all candidates of U that are in front of p and p herself to the first four positions with a total swap cost of 1024. Any swap involving two candidates of U or a candidate of U and p has a cost of 4096, which does not leave enough budget to gain $q - 1$ additional points from other votes. Hence, the bribery affects exactly q votes, and each candidate of U may be shifted only once to one of the first 4 positions, since otherwise she beats p. However, if there is an exact cover S' for the X3C instance it suffices to bribe the v_i such that $S_i \in S'$ in a way that the corresponding candidates of U and p gain an additional point. The cost for this bribery is $1024q$, and all candidates in U, candidate p, and a subset of the blocking candidates win the election with r points. This proof can be adapted to k-approval for every fixed k, $k > 4$, by adding blocking candidates to the first positions of every vote in V_s with a weight that prevents swaps. ❑

Theorem 10 *($\mathcal{D}$, plurality)*-DISTANCE BRIBERY *and ($\mathcal{D}$, veto)*-DISTANCE BRIBERY *are in* P *for* $\mathcal{D} \in \{\text{fr}_\pi, \text{fr}_\pi^{\text{w}}, \text{fr}_\pi^{\text{ew}}\}$.

Proof. We show polynomial-time solvability for the weighted variant, which directly implies the results for the other two variants. Elkind et al. [7] argue that for the bribery problem with the weighted swap distance for plurality and veto elections it suffices to calculate the minimum costs of replacing the top or rearmost candidate with another specific candidate and use this value as input for the polynomial-time nonuniform-bribery algorithm of Faliszewski [8], which needs the costs for transferring a point from one alternative to another as input. Hence it is sufficient to show that these costs can be computed in polynomial time for the weighted footrule distance.

The much more general proof of Theorem 6 implies that for plurality the most cost efficient way of transferring the points from candidate $\text{cand}(v_i, 1)$ to candidate $\text{cand}(v_i, j)$ with $2 \leq j \leq m$ is to shift $\text{cand}(v_i, j)$ to the top position while moving each candidate in between one position downwards. For veto the most cost efficient way of transferring one point from $\text{cand}(v_i, j)$ with $1 \leq j \leq m - 1$ to $\text{cand}(v_i, m)$ is to shift $\text{cand}(v_i, j)$ downwards to position m while moving each candidate in between one position upwards. ❑

Note that this approach does not extend to k-approval since the costs of transferring one point to another candidate is not independent from the other changes in the vote.

Theorem 11 $(\mathrm{fr}_\pi, k\text{-}approval)$-DISTANCE BRIBERY *is in* P.

Proof. This proof makes use of an algorithm proposed by Dorn and Schlotter [6, Theorem 1] to compute a swap_π cost minimum bribery for k-approval. Let $V^* = (v_1^*, ..., v_n^*)$ be the swap_π cost minimum bribery computed by the above mentioned algorithm. Since V^* is swap_π cost minimum each vote v_i^* fulfills the order proposed in the proof of Theorem 5. Since each v_i^* is 3-inversion free,[2] it holds due to Diaconis and Graham [5] that $\mathrm{fr}_\pi(v_i, v_i^*) = 2 \cdot \mathrm{swap}_\pi(v_i, v_i^*)$, which extends to the overall costs: $\sum_{i=1}^n \mathrm{fr}_\pi(v_i, v_i^*) = 2 \cdot \sum_{i=1}^n \mathrm{swap}_\pi(v_i, v_i^*)$. Assume there is a different fr_π cost minimum bribery $V' = (v_1', ..., v_n')$ with $\sum_{i=1}^n \mathrm{fr}_\pi(v_i, v_i') < \sum_{i=1}^n \mathrm{fr}_\pi(v_i, v_i^*)$. Again, since V' is cost minimum, by the proof of Theorem 6 we know that each v_i' fulfills the 3-inversion free order proposed in the proof, so $\sum_{i=1}^n \mathrm{fr}_\pi(v_i, v_i') = 2 \cdot \sum_{i=1}^n \mathrm{swap}_\pi(v_i, v_i')$ holds. Substituting fr_π with swap_π in the above inequality leads to $\sum_{i=1}^n \mathrm{swap}_\pi(v_i, v_i') < \sum_{i=1}^n \mathrm{swap}_\pi(v_i, v_i^*)$ which is a contradiction to the swap_π cost minimality of V^*. ❑

Theorem 12 $(\mathrm{swap}_\pi, Borda)$-DISTANCE BRIBERY *and* $(\mathrm{swap}_\pi^{\mathrm{ew}}, Borda)$-DISTANCE BRIBERY *is* NP-*complete, even if the cost functions are identical for all voters.*

Proof. Hardness will be shown by a reduction from the Borda shift bribery problem with unit prices. This problem was shown to be NP-hard by Bredereck et al. [3]. It is very similar to the $(\mathrm{swap}_\pi, \mathrm{Borda})$-DISTANCE BRIBERY problem since each swap increases the distance by 1, but only swaps moving p forward are allowed. The reduction is based on the idea, that the only reasonable action a briber can perform on a voters vote is a swap that involves moving p a position forward.

Assume it is possible to make p win the election through K swaps shifting p forward. Then p can obviously also win by using the same K swaps in the variant where all swaps are allowed.

If it is possible to make p win the election with K unrestricted swaps, we will show that K swaps involving p are sufficient to make p the winner. First of all note, that swapping p forward involving $c \in C \setminus \{p\}$, gives p a relative advantage of 2 points compared to c and a relative advantage of 1 point to all candidates in $C \setminus \{c, p\}$. A swap involving two candidates $x, y \in C \setminus \{p\}$ with $x >_j y$ for some $j \in \{1, ..., n\}$ gives y a relative advantage of 1 point compared to p and p a relative advantage of 1 point compared to x. Assume the briber performed s_i general swaps to vote v_i. Shifting p exactly $s_i' = \min\{s_i, \mathrm{pos}(v_i, p)\}$ positions forward and distributing the remaining $s_i - s_i'$ shifts to other votes is at least as effective as the original swaps since each candidate that has been swapped with p will also be swapped with p by shifting p forward, which gives p the original 2 points relative advantages. By the above observation each other swap can arbitrarily be replaced with a swap moving p forward in any vote. NP-hardness for $(\mathrm{swap}_\pi^{\mathrm{ew}}, \mathrm{Borda})$-DISTANCE BRIBERY then follows immediately. ❑

Note that if it is not required, that the distance functions of the voters are identical, NP-hardness for $(\mathrm{swap}_\pi^{\mathrm{ew}}, \mathrm{Borda})$-DISTANCE BRIBERY and $(\mathrm{fr}_\pi^{\mathrm{ew}}, \mathrm{Borda})$-DISTANCE BRIBERY can be shown by a direct reduction from the corresponding COALITIONAL MANIPULATION problem.

Theorem 13 $(\mathrm{fr}_\pi^{\mathrm{ew}}, Borda)$-DISTANCE BRIBERY *and* $(\mathrm{fr}_\pi^{\mathrm{w}}, Borda)$-DISTANCE BRIBERY *are* NP-*complete, even if the cost functions are identical for all voters.*

Proof. Hardness will be shown by a reduction from the unweighted coalitional manipulation problem for the Borda rule. This problem was shown to be NP-hard independently by Betzler at al. [2] and Davies et al. [4]. It is defined as follows. Given an election $E = (C, V)$, a distinguished candidate $p \in C$, and a number k of manipulators, the question is whether there exists a list W with $|W| = k$ of votes, such that p is the unique Borda winner of the election $(C, V \cup W)$. Given an instance of the manipulation problem with the set of candidates $C = \{c_1, \ldots, c_{m-1}, p\}$, the list of

[2] There is a 3-inversion between v and v^* if $a >_v b >_v c$ and $c >_{v^*} > b >_{v^*} a$ for some candidates a, b, and c.

votes $V = (v_1, \dots, v_n)$, and the bound k, we construct an instance for our DISTANCE BRIBERY problem as follows. The set of candidates is $C' = C \cup B$ with $B = \{b_1, \dots b_{2m}\}$. The distance limit is $K = km(m-1)$. The cost function π is the same for all voters with $\pi(c,d) = \varphi(c) \cdot \varphi(d)$, where $\varphi(c_i) = \varphi(p) = 1$ for all $c_i \in C$ and $\varphi(b_i) = K + 1$ for all $b_i \in B$. The list of votes is $V' = V'_c \cup V'_b$. V'_c contains the following two votes v'_i and v''_i for each $v_i \in V$: $v'_i : \mathrm{cand}(v_i, 1) > b_1 > b_2 > \mathrm{cand}(v_i, 2) > b_3 > b_4 > \cdots > \mathrm{cand}(v_i, m) > b_{2m-1} > b_{2m}$ and $v''_i : \mathrm{cand}(v_i, m) > b_{2m} > \mathrm{cand}(v_i, m-1) > b_{2m-1} > \cdots > \mathrm{cand}(v_i, 1) > b_{m+1} > b_m > \cdots > b_1$. The other list of votes $V'_b = (v^*_1, \dots, v^*_k)$ contains one vote v^*_j, $1 \le j \le k$, for each manipulator of the form $v^*_j = c_1 > c_2 > \cdots > c_{m-1} > p > b_1 > b_2 > \cdots > b_{2m}$. The scores from the votes in V'_c are $\mathrm{score}_{(C',V'_c)}(c_j) = n(3m+1) + \mathrm{score}_{(C,V)}(c_j)$, $\mathrm{score}_{(C',V'_c)}(p) = n(3m+1) + \mathrm{score}_{(C,V)}(p)$, and $\mathrm{score}_{(C',V'_c)}(b_j) \le n(3m-1)$. Since the weights of the candidates in B exceed the budget K, no swap with these candidates is possible due to the lower bound of the Diaconis-Graham inequality from Lemma 3. Due to the construction only swaps in the votes from V'_b are possible. Furthermore, the budget K allows every possible order of the candidates from C at the first m position of these votes. This also implies that no candidate of B can win the election.

Now we will show the soundness of this reduction. Assume that there is a list of votes W such that p wins in $E = (C, V)$. Then it holds that $\mathrm{score}_{(C,V \cup W)}(c) > \mathrm{score}_{(C,V \cup W)}(c_i)$ for all $c_i \in C \setminus p$. Since the relative scores for the candidates from C are the same in the elections (C, V) and (C', V'_c), the votes in V'_b can be bribed such that the order of the candidates from C equals the votes in W. If there exists a successful bribery, setting the votes in W such that they correspond to the order of the candidates from C in the votes from V'_b results in a successful manipulation due to the same arguments as above. ❑

As the number of votes and candidates in our reduction is linearly dependent on the number of original votes, manipulators, and candidates we also receive NP-hardness for a constant number of votes, since the same holds for unweighted coalitional Borda manipulation problem.

5 Conclusions

We introduced various forms of distances, namely weighted and unweighted versions of the swap and footrule distance, to the bribery problem in elections and studied the computational complexity of the destructive and constructive variants of these problems. In this context we drew a distinction to other distance measures and noted relations among the problems at hand and inferences from known results. For plurality and veto elections we obtain a complete picture, but for k-approval and Borda elections some cases are still open. We conjecture that all open cases for the footrule distance where the corresponding problem for swap distances is NP-complete, since the ideas may be transferable. This leaves open the cases for destructive distance bribery in Borda elections for element-weighted footrule and swap distances. This paper shows that in general the problems with an unweighted distance are easier to solve than those with weighted distances. It would be interesting to explore exactly which distance properties are causing this difficulty. Our study was restricted to k-approval and Borda elections, so the next step would be to enlarge the study to the whole class of scoring rules and also to other voting rules. Another direction for future work is to extend our study to other distances based on position weights that were proposed by Kumar and Vassilvitskii [14]. In addition the study of weighted distances can be extended to other election problems, such as manipulation or control. Combining these ideas, it would be interesting to compare the results of weighted distance measures to unweighted versions and getting a general picture of the influence distances have on elections.

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