Closing the Gap of Control Complexity in Borda Elections: Solving Twelve Open Cases[1]

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Abstract

We consider the problem of control in elections where an election chair seeks to either make a designated candidate win, or prevent her from winning, via actions such as adding, deleting, or partitioning either candidates or voters. These scenarios have been studied for many voting systems and the related control problems have been classified in terms of their complexity. However, for one of the most prominent natural voting systems, the Borda Count, complexity results are known for only half of these cases. We settle the complexity for twelve missing cases in the unique-winner model, leaving just one case open. We also show that Borda is vulnerable to control for this one open case in the *non*unique-winner model. An interesting consequence is that Borda is vulnerable to another type of control in the nonunique-winner model, yet it is resistant to it in the unique-winner model. This is one of the few known cases where the complexity of control problems differs depending on the winner model chosen.

1 Introduction

Much work has been done in computational social choice to show that complexity can help to protect election outcomes from being tampered with by manipulation, control, and bribery attacks. For a comprehensive overview of related results, we refer to the book chapters by Conitzer and Walsh [5], Faliszewski and Rothe [11], and Baumeister and Rothe [3]. Here, we focus on the standard control scenarios in elections—including adding, deleting, or partitioning either candidates or voters—introduced by Bartholdi et al. [2] and Hemaspaandra et al. [16].[2] In particular, Bartholdi et al. [2] defined *constructive* control scenarios where an election chair seeks to make a given candidate win an election, while Hemaspaandra et al. [16] introduced the corresponding *destructive* control scenarios where the chair seeks to ensure that a given candidate does not win.

Each of these scenarios has been thoroughly discussed in the literature (for example, in the book chapters mentioned above), and motivating real-world applications have been presented for each scenario. While they have been studied intensively for many voting systems, such as for plurality, Condorcet, approval [2, 16], Copeland [10], Bucklin [7], Schulze [27], certain variants of approval and range voting [9, 24], and veto [23], one of the most prominent natural voting systems, the Borda Count, is still heavily underexplored. The purpose of this paper is to fill this gap.

The eleven results previously known for control in Borda are due to Russel [29], Elkind et al. [6], Loreggia et al. [22], Chen et al. [4], and Hemaspaandra and Schnoor [20]. Table 1 presents an overview of their results (marked in grey) and of our twelve new results (marked in boldface). In this table, an "R" stands for *resistance* (which in Section 2 is defined as NP-hardness of the corresponding control problem) and "V" stands for *vulnerability* (which in Section 2 is defined as polynomial-time solvability of the corresponding control problem). Further, we use the standard names of the control problems that correspond to the standard control scenarios (see, e.g., [3, 11]). For example, CCDC stands for "constructive control by deleting candidates" and DCDC denotes the destructive variant of this problem. Each control problem for which we provide a new result in Borda elections will be formally defined in Sections 3 and 4, and the unique-winner versus the nonunique-winner model will be discussed in Section 2.

[1]This paper combines two earlier papers that appeared in the proceedings of AAAI-2017 [26] and ICTCS-2017 [25].

[2]To take certain restrictions (e.g., geographical constraints) into account, other models of control have been proposed and studied by Puppe and Tasnádi [28], Erdélyi et al. [8], Lewenberg and Lev [21], and Bachrach et al. [1].

control type	CAUC C	CAUC D	CAC C	CAC D	CDC C	CDC D	CPC-TE C	CPC-TE D	CPC-TP C	CPC-TP D	CRPC-TE C	CRPC-TE D	CRPC-TP C	CRPC-TP D	CAV C	CAV D	CDV C	CDV D	CPV-TE C	CPV-TE D	CPV-TP C	CPV-TP D
Borda	**R**†	**V**‡	R$	V£	R❄	V£	**R**♣	**V**♮	**R**∘	?	**R**♢	**V**¶	**R**△	**R**□	R§	V§	R♠	V§	**R**♡	V§	**R**⊕	**R**⊗

Table 1: Control complexity in Borda elections (unique-winner model), with standard notation of control types [3, 11]. "R" means resistance, "V" vulnerability, and "?" indicates an open question. **New results are in boldface: Thm. 1 (†), Thm. 2 (‡), Thm. 3 (♢), Thm. 4 (♣), Thm. 5 (♮), Cor. 1 (¶), Thm. 6 (□), Thm. 8 (∘), Cor. 3 (△), Thm. 9 (♡), Thm. 10 (⊕), and Thm. 11 (⊗).** Previously known results are grey and due to Russel [29] (marked by §), Elkind et al. [6] ($), Loreggia et al. [22] (£), Chen et al. [4] (❄), and Hemaspaandra and Schnoor [20] (♠).

As Table 1 shows, Borda is now known to be resistant to every standard type of constructive control, whereas it is vulnerable to most of the destructive control types (resistance is known only for destructive control by run-off partition of candidates and by partition of voters, both in the so-called "ties-promote" (TP) model formally defined in Section 3). One case of destructive candidate control remains open (namely, DCPC-TP, marked by "?" in Table 1) in the unique-winner model. Interestingly, we can show Borda to be vulnerable to this control type in the *non*unique-winner model (Theorem 7). Even more interesting is a consequence of this result (Corollary 2): In the *non*unique-winner model, Borda-DCRPC-TP (which is known to coincide with Borda-DCPC-TP in the nonunique-winner model [15] but not to coincide with it in the unique-winner model) is in P as well, yet it is NP-hard in the unique-winner model (Theorem 6).

2 Preliminaries

An *election* is a pair (C,V) that contains a set C of candidates and a list V of votes describing the voters' preferences—as strict linear orders—over the candidates. We will represent a vote over C as a string that ranks the candidates from left (most preferred) to right (least preferred); for example, if $C = \{a,b,c,d\}$, a vote $c\ d\ b\ a$ means that this voter prefers c to d, d to b, and b to a. A *voting rule* determines a set of winners from each given election. Positional scoring rules are an important class of such rules, and among those we will only consider the perhaps most prominent one, the *Borda Count*, which works as follows: Given m candidates, every candidate in position i of the voters' rankings scores $m-i$ points, and all candidates scoring the most points win.

Let $score_{(C,V)}(x)$ denote the number of points candidate x obtains in a Borda election (C,V), and let $dist_{(C,V)}(x,y) = score_{(C,V)}(x) - score_{(C,V)}(y)$. For a subset $X \subseteq C$ of candidates, $\overrightarrow{X}$ in a vote denotes a ranking of these candidates in an arbitrary but fixed order, $\overleftarrow{X}$ denotes their ranking in reverse order, and we write X when the order of the candidates in X does not matter in this vote.

The control types considered here will be formally defined in Sections 3 and 4, and we refer to the book chapters by Faliszewski and Rothe [11] and Baumeister and Rothe [3] (and to the references therein) for all other standard control types and for real-world scenarios that motivate them.

A voting rule is said to be *susceptible to a type of control* (e.g., constructive control by adding candidates) if there is some election for which the chair can reach her goal (e.g., turning a nonwinning candidate into a winner) by exerting this type of control. If a voting rule is not susceptible to a control type, it is said to be *immune to it*. Borda is susceptible to each standard control type, in particular to those considered here. A voting system that is susceptible to some type of control is said to be *vulnerable to it* if the associated control problem can be solved in polynomial time, and it is said to be *resistant to it* if the associated control problem is NP-hard.

Our control problems will be defined in Sections 3 and 4 in the *unique-winner model* (see also Table 1). That is, a constructive (destructive) control action is viewed as being successful only if the designated candidate can be made a *unique* winner (*not a unique* winner) by this action. We note, however, that using essentially the same constructions and slightly modifying the arguments in our proofs, most of our results also work in the *nonunique-winner model*, which means that for a constructive (destructive) control action to be successful, it is enough to make the designated candidate only *a* winner (she can be made *not even a* winner) by this action. The only exception is destructive control by run-off partition of candidates in the ties-promote model (to be defined in Section 3) to which Borda will be shown resistant in the unique-winner model (Theorem 6), yet vulnerable in the nonunique-winner model (Corollary 2).[3]

In our proofs, we will sometimes use the following result due to Hemaspaandra et al. [15] (see their technical report [14] for the proof of Fact 1), which shows that some of the destructive control cases (to be defined in the next section) can collapse depending on the chosen winner model.

Fact 1 (Hemaspaandra et al. [15]) DCRPC-TE $=$ DCPC-TE *in the unique-winner model and* DCRPC-TE $=$ DCPC-TE *and* DCRPC-TP $=$ DCPC-TP *in the nonunique-winner model.*

3 Candidate Control in Borda Elections

We solve all open problems for candidate control in Borda elections except one, starting with constructive control by adding an unlimited number of candidates.

Borda-CCAUC and Borda-DCAUC. Elkind et al. [6] showed that Borda is resistant to constructive control by adding a *limited* number of candidates (i.e., a bound k on the number of candidates that may be added is part of the problem instance), and Loreggia et al. [22] showed that Borda is vulnerable to the destructive variant of this control type (see Table 1). Originally, however, Bartholdi et al. [2] defined control by adding candidates in an *unlimited* variant where no such bound is given. The definition of the limited variant is due to Faliszewski et al. [10], who also proved that the two variants of the problem can have different complexity: Two special cases of Copeland$^\alpha$ elections (namely, Copeland^0 and Copeland^1, the latter a.k.a. Llull elections) are resistant to the constructive, limited variant (the corresponding problem denoted by CCAC), whereas they are vulnerable to the constructive, unlimited variant, which we define below.

In the problem Borda-CONSTRUCTIVE-CONTROL-BY-ADDING-AN-UNLIMITED-NUMBER-OF-CANDIDATES (Borda-CCAUC) we ask, given a set C of candidates, an additional set A of candidates, $C \cap A = \emptyset$, a set V of voters with preferences over $C \cup A$, and a distinguished candidate $p \in C$, whether there is a subset $A' \subseteq A$ such that p is the unique Borda winner of $(C \cup A', V)$.[4] The proof of Theorem 1 makes use of a reduction from X3C to Borda-CCAUC. First, Lemma 1 below, which was proven by Elkind et al. [6, Lemma B.3],[5] allows us to construct votes conveniently.

Lemma 1 (Elkind et al. [6]) *Let* $C = \{c_1, \ldots, c_{2t-1}, d\}$*,* $t \geq 2$*, be a set of candidates and let* $A = \{a_1, \ldots, a_s\}$ *be a set of spoiler candidates. Let* $L = 2t-1$*. Then there is a polynomial-time computable preference profile* $\mathscr{R} = (R_1, \ldots, R_{2L})$ *over* $C \cup A$ *such that for each* $A' \subseteq A$ *the Borda scores in the election* $(C \cup A', \mathscr{R})$ *are as follows: (a) For each* $c_i \in C$*,* $score(c_i) = L(2|A'| + |C| - 1) + 1$*; (b)* $score(d) = L(2|A'| + |C| - 1) - L$*; and (c) for each* $a_i \in A'$*,* $score(a_i) \leq L(2|A'| + |C| - 1) - 2L$*.*

Theorem 1 *Borda is resistant to constructive control by adding an unlimited number of candidates.*

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[3]Our only other result explicitly shown in the nonunique-winner model is Theorem 7: Borda is vulnerable to destructive control by partition of candidates in the ties-promote model. The corresponding case in the unique-winner model is still open (again, see Table 1).

[4]For convenience, whenever we have a list V of votes over a set $C \cup A$ and then consider an election with fewer candidates, $C \cup A'$ with $A' \subset A$, we use $(C \cup A', V)$ to denote the election with the votes in V tacitly assumed to be restricted to $C \cup A'$.

[5]The original lemma by Elkind et al. [6] is slightly more general in that they consider nonnegative integers $\ell_1, \ldots, \ell_{2t-1}$ with $L = \sum_{i=1}^{2t-1} \ell_i$. For our purpose, it is enough to set $\ell_1 = \cdots = \ell_{2t-1} = 1$, so $L = 2t-1$.

Proof. To show NP-hardness, we provide a reduction from X3C to Borda-CCAUC. Let $(X, \mathscr{S})$ be a given X3C instance with $X = \{x_1, \dots, x_m\}$, $m = 3k$, $k > 1$, and $\mathscr{S} = \{S_1, \dots, S_n\}$ with $S_i \subseteq X$ and $|S_i| = 3$ for each i, $1 \leq i \leq n$. Without loss of generality, we assume that k is even and $k > 2$ (this can be achieved by duplicating the instance if necessary). Construct from $(X, \mathscr{S})$ a Borda-CCAUC instance $((C,V), p, k)$ as follows. Let $C = X \cup \{u, p\}$ with p being the distinguished candidate and $A = \{a_1, \dots, a_n\}$ a set of spoiler candidates. Define V to consist of the following votes:

1. For each i, $1 \leq i \leq n$, there are two votes: $\overrightarrow{S_i}\ u\ p\ \overrightarrow{X \setminus S_i}\ A$ and $\overleftarrow{X \setminus S_i}\ p\ u\ a_i\ \overleftarrow{S_i}\ A \setminus \{a_i\}$.
2. Three times, there are two votes of the form $u\ \overrightarrow{A}\ p\ \overrightarrow{X}$ and $\overleftarrow{X}\ p\ u\ \overleftarrow{A}$.
3. All votes we obtain by applying Lemma 1 to the candidate set C with each x_i taking the role of a c_i, p that of c_{3k+1}, and u that of d. (Here, we need k to be even.)

We claim that $(X, \mathscr{S})$ is a yes-instance of X3C if and only if (C, A, V, p) is a yes-instance of Borda-CCAUC.

From left to right, suppose there is an exact cover $\mathscr{S}' \subseteq \mathscr{S}$. Let $A' = \{a_j \in A \mid S_j \in \mathscr{S}'\}$.[6] Then we have $dist(p, x_i) = |\{a_j \in A' \mid x_i \in S_j\}| = 1$ for every $x_i \in X$, since every $x_i \in X$ is contained in one element of the exact cover $\mathscr{S}'$ of X exactly once. Furthermore, we have $k = |\mathscr{S}'| = |A'|$. Thus $dist(p, u) = 3k + 2 - 3k = 2$, so p defeats every candidate and is the only Borda winner of $(C \cup A', V)$.

From right to left, suppose that p can be made the only Borda winner by adding the candidates of a subset $A' \in A$. Therefore, p defeats every candidate in $(C \cup A', V)$, so we have $dist(p, u) > 0$ and $dist(p, x_i) > 0$ for every $x_i \in X$ (recall that p always defeats every $a_j \in A'$). Since $dist(p, x_i) = |\{a_j \in A' \mid x_i \in S_j\}| > 0$ for every $x_i \in X$, the subfamily $\mathscr{S}' = \{S_j \in S \mid a_j \in A'\}$ covers X. Thus we have $|\mathscr{S}'| \geq k$, as there are $3k$ elements in X and every subset of $\mathscr{S}$ contains three elements. Furthermore, we have $dist(p, u) = 3k + 2 - 3|A'| > 0$, so $|\mathscr{S}'| = |A'| \leq k$. Overall, we have that $\mathscr{S}'$ covers X and $|\mathscr{S}'| = k$, which means that $\mathscr{S}'$ is an exact cover of X. ❑

For the destructive variant, we provide a polynomial-time algorithm to show that Borda-DCAUC is in P.

Theorem 2 *Borda is vulnerable to destructive control by adding an unlimited number of candidates.*

Proof. The following polynomial-time algorithm solves the problem Borda-DCAUC. Given a Borda-DCAUC instance (C, A, V, p), the algorithm works as follows. If p is not a unique Borda winner of $(C \cup A, V)$, accept. Otherwise, do the following steps:

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for all candidates x ∈ (C ∪ A) \ {p} do
  distdiff(x) ← 0
  D = ∅
  for all candidates y ∈ A \ {x} do
    Let w_{x,y} be the number of votes with ··· p ··· y ··· x ···
    Let ℓ_{x,y} be the number of votes with ··· x ··· y ··· p ···
    deldiff(x,y) ← w_{x,y} − ℓ_{x,y}
  end for
  for i ← 1 to |A \ {x}| do
    y ← argmax_{a ∈ A \ (D ∪ {x})} deldiff(x,a)
    if deldiff(x,y) > 0 then
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[6]Note that p ranks ahead of every $a_j \in A'$ in all but three votes in the second group of voters. The point deficit from those three votes is always offset by the other votes in this group, so we can disregard the points of every $a_j \in A'$ from now on, since p always defeats them. For the point differences of p to the other candidates in the election $(C \cup A', V)$ for any $A' \subseteq A$, we have $dist(p, u) = 3k + 2 - 3|A'|$ and $dist(p, x_i) = |\{a_j \in A' \mid x_i \in S_j\}|$. If $A' = \emptyset$, we have $dist(p, x_i) = 0$, so p is not winning (C, V) alone.

$D = D \cup \{y\}$
$distdiff(x) \leftarrow distdiff(x) + deldiff(x,y)$
end if
end for
if $dist_{(C\cup A,V)}(p,x) \leq distdiff(x)$ **then**
print control possible
end if
end for
print control impossible

If the output is "control possible," accept the input, else reject it. The algorithm runs in polynomial-time with runtime $O((|C|+|A|)\cdot|A|\cdot|V|)$ and is correct. Correctness of step 1 is obvious. For step 2, the algorithm tries to find a candidate $x \in C \cup A$ that can defeat or tie p, if some candidates of A are deleted from the election $(C \cup A, V)$. To do that, for every candidate $y \in A \setminus \{x\}$, it computes how a deletion of y would affect the point balance of p and x. If there is a vote of the form $\cdots\ p\ \cdots\ y\ \cdots\ x\ \cdots$, p would lose a point with respect to x by deleting y, and for every vote of the form $\cdots\ x\ \cdots\ y\ \cdots\ p\ \cdots$, p would gain a point with respect to x by deleting y. For all other votes, deleting y would cause both p and x to either lose a point or gain a point. Note that this is independent of who else from A is participating in the election. Then the algorithm deletes all candidates of A that after deletion change the point balance of p and x in favor of x (let D be the set of those candidates) and checks whether x ties or defeats p in $(C \cup (A \setminus D))$. If this is true, p can be tied or defeated by adding the candidates of $A \setminus D$. If it is false for all candidates $x \in C \cup A$, p always wins alone. ❑

Borda-CCRPC-TE and Borda-CCPC-TE. In the problem Borda-CONSTRUCTIVE-CONTROL-BY-RUN-OFF-PARTITION-OF-CANDIDATES-TE (Borda-CCRPC-TE) we ask, given an election (C,V) and a distinguished candidate $p \in C$, whether the candidate set C can be partitioned into two subsets C_1 and C_2 such that p is the unique Borda winner of the final run-off among the Borda winners of subelections (C_1,V) and (C_2,V), where only unique subelection winners move forward in the ties-eliminate (TE) model. The proof of Theorem 3 below makes use of a reduction from the standard NP-complete satisfiability problem (3SAT) [12]: Given a boolean formula φ in 3-CNF (i.e., with exactly three literals per clause), does there exist a satisfying truth assignment to φ? For a boolean formula φ, we denote by $\#_i$ the number of literals occurring in the ith clause that are negated variables.

Theorem 3 *Borda is resistant to constructive control by run-off partition of candidates in the ties-eliminate model.*

Proof. To show NP-hardness, we now provide a reduction from 3SAT to Borda-CCRPC-TE. Given a 3SAT instance $\varphi(x_1,\dots,x_n)$, construct a Borda-CCRPC-TE instance $((C,V),p)$ as follows. Let $X = \{x_1,x_2,\dots,x_n\}$ be the set of variables and let $K = \{K_1,\dots,K_m\}$ be the set of clauses of φ, where $K_i = (\ell_i^{(1)} \vee \ell_i^{(2)} \vee \ell_i^{(3)})$, $1 \leq i \leq m$. Furthermore, let $D = \{d_1,\dots,d_6\}$ and $D_i = \{d_j \mid 1 \leq j \leq i\} \subseteq D$. Define the candidate set by $C = X \cup K \cup \{p,r,r^*\} \cup D$ with p being the distinguished candidate the chair wants to make a unique winner. Define V to consist of the following votes:

1. For each i, $1 \leq i \leq m$, there are two votes: $\overrightarrow{C \setminus (\{p,K_i\} \cup D)}\ p\ D_{2\#_i}\ K_i\ D \setminus D_{2\#_i}$ and $K_i\ d_6\ p\ \overleftarrow{C \setminus (\{p,K_i\} \cup D)}\ D_5$.

2. For each i, $1 \leq i \leq m$, and for each literal $\ell_i^{(1)}$, $\ell_i^{(2)}$, and $\ell_i^{(3)}$, there are four votes: either twice $K_i\ x_j\ p\ \overrightarrow{C \setminus (\{K_i,x_j,p\} \cup D)}\ D$ and twice $\overleftarrow{C \setminus (\{K_i,x_j,p\} \cup D)}\ p\ K_i\ x_j\ D$ if $\ell_i^k = \overline{x_j}$ is a negated variable, or twice $\overrightarrow{C \setminus (\{K_i,x_j,p\} \cup D)}\ p\ x_j\ K_i\ D$ and twice $K_i\ p\ \overleftarrow{C \setminus (\{K_i,x_j,p\} \cup D)}\ x_j\ D$ if $\ell_i^k = x_j$ is a positive variable.

3. There are m votes of the form $r^*\ r\ \overrightarrow{K}\ \overrightarrow{D}\ p\ X$ and m votes of the form $r\ p\ \overleftarrow{D}\ \overleftarrow{K}\ r^*\ X$.

Since $dist_{(C,V)}(p,r) = m(-6-m-2) = -m(m+8) < 0$, p does not win in (C,V). Note that p and r score the same number of points in the first two groups of votes. Later on, we will also need the following argument. Consider a clause candidate K_i. In the first group of votes, p scores $2\#_i - 1$ points more than candidate K_i, with $\#_i$ being the number of negated variables in clause K_i. In the second group of votes, p gains two more points with respect to candidate K_i for each positive variable in clause K_i, and p loses two points with respect to candidate K_i for each negated variable in clause K_i. Since p and K_i score the same number of points in the third group of votes, we have $dist_{(C,V)}(p,K_i) = -2\#_i + 2(3-\#_i) + (2\#_i - 1) = 5 - 2\#_i$. Assuming that one variable candidate x_j is assigned to the other subelection than p and K_i, if x_j is a negated variable in clause K_i then p gains two points with respect to candidate K_i, and if x_j is a positive variable in clause K_i then p loses two points with respect to K_i. Further, if C' is the set of candidates obtained by removing from C all variable candidates corresponding to positive variables in clause K_i, then $dist_{(C',V)}(p,K_i) = 5 - 2\#_i - 2(3-\#_i) = -1$ because p is losing as many points with respect to K_i as there are positive variables in clause K_i. That is, p is defeated by K_i in their subelection if all variable candidates corresponding to positive variables in clause K_i are removed from the subelection containing p and K_i (and are assigned to the other subelection) and all variable candidates corresponding to negated variables in clause K_i remain in the subelection with p and K_i. For p to defeat K_i, either the subelection containing them also contains at least one variable candidate corresponding to a positive variable in clause K_i, or the other subelection contains at least one variable candidate corresponding to a negated variable in clause K_i, or both.

We show that φ is a yes-instance of 3SAT if and only if $((C,V),p)$ is a yes-instance of Borda-CCRPC-TE.

From left to right, suppose there is a satisfying truth assignment to the variables of $\varphi(x_1,\ldots,x_n)$, say α. Let $X_+ \subseteq X$ denote the set of variables set to *true* under α, and let $X_- \subseteq X$ denote the set of variables set to *false* under α. Partition C into $C_1 = \{p\} \cup D \cup K \cup X_+$ and $C_2 = \{r, r^*\} \cup X_-$. The Borda winners of subelection (C_2,V) are r and r^*, since they score more points than the candidates in X_- due to the third voter group and the same number of points in the other two voter groups. Due to TE, no candidate proceeds to the final run-off from this subelection. In subelection (C_1,V), p defeats all candidates from D, since p scores more points than these candidates in the first voter group and the same number of points in the other two voter groups. p also defeats all candidates from X_+, since p scores at least $m(m+5)$ points more than any candidate in X_+ in the third voter group, at most m points fewer than any candidate from X_+ in the second voter group (which is the case if some positive variable occurs in all clauses), and the same number of points in the first voter group. What about the clause candidates? The truth assignment (giving rise to X_+ and X_-) satisfies φ, so each clause K_i of φ is satisfied. Thus, for every i, $1 \le i \le m$, at least one positive variable in K_i is assigned to *true* or at least one negated variable in K_i is assigned to *false*. In the former case, the corresponding variable candidate is in X_+ and thus in the same subelection as p; in the latter case, the corresponding variable candidate is in X_- and thus not in the same subelection as p. By the above argument, p scores more points than K_i. Summing up, since p defeats all other candidates in her subelection and no one moves forward to the final run-off from the other subelection, p alone is the overall Borda winner.

From right to left, suppose that p is the unique overall Borda winner for some partition of the candidates. This implies that p also is the unique Borda winner of one subelection. Since r scores more points than p due to the third voter group, p and r must be in different subelections (regardless of who else participates in them). Without loss of generality, assume that p is in C_1 and r is in C_2. Consider C_2 first. r cannot be the unique Borda winner in subelection (C_2,V), since otherwise p would not win the run-off. Therefore, there must be candidates that either tie or defeat r in (C_2,V). Clause candidates, variable candidates, and candidates from D lose too many points in the third voter group (that cannot be made up for in the first and second voter groups) to tie-or-defeat r. Only candidate r^* remains. However, r^* cannot be the unique Borda winner of subelection (C_2,V), since

p and r^* would score the same number of points in the run-off, contradicting that p is the *unique* run-off winner. Thus there must be a tie between r and r^* in (C_2,V), which prevents them both from proceeding to the run-off due to the TE model. Therefore, neither candidates from D nor from K can be in C_2, for otherwise the balance of points between r and r^* would be perturbed due to the third voter group. Variable candidates, however, may be in C_2, since they get fewer points than either r and r^* and would not interfere with their point balance. Thus C_1 contains p and all candidates from D and K and some variable candidates. Let X_+ denote the set of variable candidates in C_1. Note that p defeats the candidates in D by the first voter group and the candidates in X_+ by the third voter group. Since p also defeats each clause candidate K_i, the variable candidates must be distributed among C_1 and C_2 according to the argument given earlier. Now, if we assign the value *true* to all variables corresponding to variable candidates in X_+ and the value *false* to all variables corresponding to variable candidates not in X_+, we obtain a satisfying truth assignment to $\varphi(x_1,\ldots,x_n)$. ❑

Borda-Constructive-Control-by-Partition-of-Candidates-TE (Borda-CCPC-TE) is defined as follows. Given an election (C,V) and a distinguished candidate $p \in C$, we ask whether the candidate set C can be partitioned into two subsets C_1 and C_2 such that p is the unique Borda winner of the final election in which the Borda winner of subelection (C_1,V)—if there exists one (again, in model TE, only unique subelection winners move forward)—faces all candidates from C_2.

The proof of Theorem 4 is deferred to the appendix and so are the upcoming proofs of Theorems 6 and 11.

Theorem 4 *Borda is resistant to constructive control by partition of candidates in the ties-eliminate model.*

Borda-DCPC-TE and Borda-DCRPC-TE. We now turn to the destructive variants of the previous two problems. Unlike in the constructive case, we can give a polynomial-time algorithm for Borda-DCPC-TE. By Fact 1, Borda-DCPC-TE is the same as Borda-DCRPC-TE in the unique-winner model, which gives Corollary 1.

Theorem 5 *Borda is vulnerable to destructive control by partition of candidates in the ties-eliminate model.*

Proof. Our algorithm uses the result of Loreggia et al. [22] that Borda-DCDC is in P (see Table 1). Given an election (C,V) and a distinguished candidate $p \in C$ the algorithm works as follows: If p is not a unique Borda winner, accept because control is possible via the partition $(C,\emptyset)$. If $|C| = 1$, control is impossible since p is the only candidate and always wins, so reject. Let $k = |V| - 2$. If $((C,V),p,k))$ is a yes-instance of Borda-DCDC, which can be checked in polynomial time [22], accept; otherwise, reject.

This algorithm runs in polynomial time and is correct. In step 1, if p is not a unique Borda winner, p is at most tied with some candidates, so she can be either beaten or eliminated by the tie-handling rule in a subelection. Correctness of step 2 is obvious. For step 3, if the constructed instance is a yes-instance, there is a subset $C' \subseteq C$, $|C'| \leq k = |V| - 2$, $p \notin C'$, so that p is at most tied in the election $(C \setminus C',V)$. Therefore, we can eliminate p in the subelection by partitioning C into $C \setminus C'$ and C', so control is possible. If the instance is a no-instance, p cannot be beaten or tied even if all but one candidate other than p are deleted from the election. That means for every $C' \subseteq C$, $|C'| \leq k = |V| - 2$, $p \notin C'$, that p is the sole winner of $(C \setminus C',V)$, so p cannot be eliminated in a subelection. In this case, p wins any subelection and reaches the run-off. There may be a set of other candidates that reached the final, say $C^\star \subseteq C$, $p \notin C^\star$. If some of those candidates beat or at least tie p in this run-off, destructive control would have still been achieved. But this will never happen because then p could have been eliminated in a subelection by partitioning C into $\{p\} \cup C^\star$ and $C \setminus (\{p\} \cup C^\star)$. As stated above, this is impossible since the constructed Borda-DCDC instance is a no-instance, so p alone wins the run-off and control is impossible. ❑

Corollary 1 *Borda is vulnerable to destructive control by run-off partition of candidates in the ties-eliminate model.*

Borda-DCPC-TP and Borda-DCRPC-TP. Next, we consider the same two problems as above but with the ties-promote (TP) instead of the ties-eliminate rule, which means that *all* subelection winners move forward to the final round. The proof of Theorem 6 is deferred to the appendix.

Theorem 6 *Borda is resistant to destructive control by run-off partition of candidates in the ties-promote model.*

Borda-DCPC-TP is the only problem considered here (see Table 1) whose complexity in the unique-winner model remains open. However, we can show the following in the nonunique-winner model.

Theorem 7 *In the nonunique-winner model, Borda is vulnerable to destructive control by partition of candidates in the ties-promote model.*

Proof. To prove P membership of the problem, the algorithm from the proof of Theorem 5 can be used with some slight modifications. Let (C,V) be a given election and $p \in C$ the distinguished candidate. Apart from the trivial cases, we only need to check whether there is a candidate who beats p in an election with a subset of candidates $C' \subseteq C \setminus \{p\}$, which can be done in polynomial time by slightly modifying an algorithm of Loreggia et al. [22]. If this is the case, we can prevent p from winning by eliminating her in the subelection $(C' \cup \{p\}, V)$. Otherwise, p is a winner in every election $(C' \cup \{p\}, V)$ with $C' \subseteq C \setminus \{p\}$, so control is impossible. ❑

By Fact 1, Borda-DCPC-TP and Borda-DCRPC-TP are identical in the nonunique-winner model. Therefore, Theorem 7 implies Corollary 2. In light of Theorem 6, this is somewhat surprising, as it shows that the complexity of Borda-DCRPC-TP starkly differs depending on the winner model.

Corollary 2 *In the nonunique-winner model, Borda is vulnerable to destructive control by run-off partition of candidates in the ties-promote model.*

Borda-CCPC-TP and Borda-CCRPC-TP. Finally, we turn to the constructive variants of the above two problems. Note that a slight modification of the proof of Theorem 8 yields Corollary 3.

Theorem 8 *Borda is resistant to constructive control by partition of candidates in the ties-promote model.*

Proof. To show NP-hardness, we provide a reduction from X3C to Borda-CCPC-TP. Let $(X, \mathscr{S})$ be a given X3C instance with $X = \{x_1, \ldots, x_m\}$, $m = 3k$, $k > 1$, and $\mathscr{S} = \{S_1, \ldots, S_n\}$ with $S_i \subseteq X$ and $|S_i| = 3$ for each i, $1 \leq i \leq n$. Again we assume that every $x_i \in X$ appears in exactly three subsets $S_j \in \mathscr{S}$, so $n = 3k$. Construct from $(X, \mathscr{S})$ a Borda-CCPC-TP instance $((C,V), p)$ as follows. Let $C = \{p, r, r^*\} \cup X \cup \mathscr{S}$ with p being the distinguished candidate. Define V to consist of the following votes:

1. There are $2k$ votes of the form $r\ \overrightarrow{X}\ p\ r^*\ \overrightarrow{\mathscr{S}}$ and $2k$ votes of the form $r\ \overleftarrow{X}\ p\ r^*\ \overleftarrow{\mathscr{S}}$.
2. There are 2 votes $r^*\ \overrightarrow{\mathscr{S}}\ r\ p\ \overrightarrow{X}$ and $r^*\ \overleftarrow{\mathscr{S}}\ r\ p\ \overleftarrow{X}$.
3. For every $S_i = \{x', x'', x'''\} \in \mathscr{S}$, there are $(2k-1)(3k+3)+1$ votes of the form $\overrightarrow{X \setminus S_i}\ r\ r^*\ p\ S_i\ x'\ x''\ x'''\ \overrightarrow{\mathscr{S} \setminus \{S_i\}}$ and $(2k-1)(3k+3)+1$ votes of the form $x'''\ x''\ x'\ p\ r^*\ r\ S_i\ \overleftarrow{X \setminus S_i}\ \overleftarrow{\mathscr{S} \setminus \{S_i\}}$.

It is easy to see that p is beaten by r in every possible subelection and therefore is not winning in (C,V).

We claim that $(X,\mathscr{S})$ is a yes-instance of X3C if and only if $(C,V),p)$ is a yes-instance of Borda-CCPC-TP.

From left to right, suppose there is an exact cover $\mathscr{S}' \subseteq \mathscr{S}$. Then partition C into $C_1 = \{r,r^*\} \cup \mathscr{S} \setminus \mathscr{S}'$ and $C_2 = \{p\} \cup \mathscr{S}' \cup X$. All candidates in C_2 are directly qualified for the final election. Furthermore, r^* is the unique Borda winner of the subelection (C_1,V), since $dist_{(C_1,V)}(r^*,r) = -4k+4k+2 = 2$ and $dist_{(C_1,V)}(r^*,S_i) > 0$ for every $S_i \in \mathscr{S} \setminus \mathscr{S}'$. In the final election (C',V) with $C' = \{p,r^*\} \cup \mathscr{S}' \cup X$, we have that p beats every candidate in $\mathscr{S}'$ and the candidate r, since $dist_{(C',V)}(p,r^*) = 4k-2k-2 > 0$ (recall that we required $k > 1$). Because $\mathscr{S}'$ is an exact cover, every x_i is contained in exactly one element of $\mathscr{S}'$. Therefore,

$$dist_{(C',V)}(p,x_i) = -2k(3k+1)+(3k+1)+(2k-1)(3k+1)+1 = 1$$

and p is the unique Borda winner of the final election.

From right to left, suppose that p can be made the only Borda winner by partitioning the candidates. Since p is beaten by r in every possible subelection, r and p need to be in different parts of the partition (say, r is in (C_1,V)), and r needs to be eliminated in the subelection (C_1,V). It is easy to see that r beats all candidates from X and $\mathscr{S}$ in all possible subelections as well. Therefore, $r^* \in C_1$. For subsets $X' \subseteq X$ and $\mathscr{S}' \subseteq \mathscr{S}$, the point balance of r^* and r is $dist_{(\{r,r^*\}\cup X' \cup \mathscr{S}',V)}(r^*,r) = -4k|X'| - 4k + 2|\mathscr{S}'| + 2$. In order for r^* to beat r, no candidate from X may be in the subelection and at least $2k$ candidates from $\mathscr{S}$ need to participate in it. This leaves candidates $C_1 = \{r^*,p\} \cup X \cup \mathscr{S}'$ with $\mathscr{S}' \subseteq \mathscr{S}$ and $|\mathscr{S}'| \leq k$ in the final election. Note that p beats all candidates in $\mathscr{S}'$ and the candidate r^*, since $|\mathscr{S}'| \leq k$. Without any $S_i \in \mathscr{S}'$, p has a point deficit of $-2k(3k+1)+(3k+1)$ on every $x_j \in X$. For every $S_i \in \mathscr{S}'$, p gains $(2k-1)(3k+1)+1$ points on every $x_j \in S_i$. Therefore, for p to beat all candidates in X, every $x_i \in X$ needs to be in at least one element of $\mathscr{S}'$. Since $|\mathscr{S}'| \leq k$, we have that $\mathscr{S}'$ is an exact cover. ❑

Corollary 3 *Borda is resistant to constructive control by run-off partition of candidates in the ties-promote model.*

4 Control by Partition of Voters in Borda Elections

In this section we solve the only three problems that still were open for voter control in Borda elections (recall Table 1): constructive control by partition of voters when ties promote or ties eliminate and destructive control by partition of voters when ties promote.

In Borda-CONSTRUCTIVE-CONTROL-BY-PARTITION-OF-VOTERS-TE ($\mathscr{E}$-CCPV-TE) we ask, given an election (C,V) and a candidate p in C, whether V can be partitioned into V_1 and V_2 such that p is the unique Borda winner of the two-stage election where only unique Borda winners of subelections (C,V_1) and (C,V_2) proceed to the final run-off. We will make use of a reduction from the well-known NP-complete problem PARTITION [12]: Given a set $A = \{1,\ldots,n\}$ and a list $s = (s_1,\ldots,s_n)$ of nonnegative integers, can A be partitioned into two subsets A_1 and A_2 such that $\sum_{i\in A_1} s_i = \sum_{i\in A_2} s_i$?

Theorem 9 *Borda is resistant to constructive control by partition of voters in the ties-eliminate model.*

Proof. To prove NP-hardness, we now provide a reduction from PARTITION to Borda-CCPV-TE. Given a PARTITION instance (A,s) with $A = \{1,\ldots,n\}$, a list $s = (s_1,\ldots,s_n)$ of nonnegative integers, and $K = \sum_{i\in A} s_i$, construct a Borda-CCPV-TE instance $((C,V),p)$ as follows. Let $C = B^{(1)} \cup \cdots \cup B^{(n)} \cup D \cup T \cup \{p,r,r^*\}$ (with p being the distinguished candidate the chair wants to make a unique winner) contain n sets $B^{(i)} = \{b_1^{(i)}, \ldots, b_{2s_i-1}^{(i)}\}$, $1 \leq i \leq n$, a set $D = \{d_1, \ldots, d_{2K}\}$, and a set $T = \{t_1, \ldots, t_{2K+2}\}$.

As a notation, we write $D_{i,j} = \{d_i, d_{i+1}, \ldots, d_{j-1}, d_j\}$, where $1 \leq i \leq j \leq 2K$. Construct V to consist of $n+2$ votes:

$$\begin{aligned} v_i &= r\ B^{(i)}\ r^*\ p\ T\ D\ B^{(1)} \cdots B^{(i-1)}\ B^{(i+1)} \cdots B^{(n)},\ i \in A, \\ v_{n+1} &= p\ D_{1,K}\ r^*\ D_{K+1,2K}\ r\ T\ B^{(1)} \cdots B^{(n)}, \\ v_{n+2} &= r^*\ D_{1,K-1}\ r\ T\ p\ D_{K,2K}\ B^{(1)} \cdots B^{(n)}. \end{aligned}$$

Since $dist_{(C,V)}(p,r) = 2K+2-(2K+3)-(2K+2n) < 0$, p is not a Borda winner of (C,V). We claim that (A,s) is in PARTITION if and only if $((C,V),p)$ is in Borda-CCPV-TE.

From left to right, suppose there is a partition of A into two sets, A_1 and A_2, such that $\sum_{i \in A_j} s_i = K/2$ for $j \in \{1,2\}$. Assign v_{n+1} to V_1 and v_{n+2} to V_2. Add v_i to V_1 for each $i \in A_1$, and add the remaining votes v_j with $j \in A_2$ to V_2.

In subelection (C,V_1), p scores $K+1$ points more than r^* and $2K+2$ points more than r due to vote v_{n+1} alone. Candidate r^* scores at most $n-1$ points more than p by the other votes in V_1, so $dist_{(C,V_1)}(p,r^*) \geq K+1-(n-1) > 0$, since $K \geq n$ (if $s_1 = \cdots = s_n = 1$, we have $K = n$, otherwise we have $K > n$.) And r scores $2s_i$ points more than p for each vote $v_i \in V_1$ with $1 \leq i \leq n$. Since $\sum_{i \in A_1} s_i = K/2$, $dist_{(C,V_1)}(p,r) = 2K+2-2K/2 = K+2 > 0$, so p scores more points than r and r^* in (C,V_1). Note also that p is preferred to all candidates of D and T in the votes of V_1. A candidate $b_j^{(i)}$ is preferred to p in at most one vote in V_1 and thus can score at most $2\max_{i \in A}\{s_i\}$ points more than p. However, p scores at least $|T| = 2K+2$ points more than $b_j^{(i)}$ in v_{n+1} and thus has a higher score in total than $b_j^{(i)}$ in (C,V_1) because $K \geq \max_{i \in A}\{s_i\}$. It follows that p is the unique Borda winner of subelection (C,V_1) and proceeds to the final run-off.

In subelection (C,V_2), r^* scores K points more than r due to vote v_{n+2} alone. By the other votes in V_2, however, r scores $2K/2 = K$ points more than r^*, since $\sum_{i \in A_2} s_i = K/2$. Thus $dist_{(C,V_2)}(r,r^*) = K-K = 0$, so r and r^* are tied in (C,V_2). In the votes from V_2, (a) both r and r^* are preferred to p and to all candidates from T, (b) r is preferred to each $b_j^{(i)} \in B^{(i)}$, and (c) r^* is preferred to each $d_j \in D$. Overall, both r and r^* win subelection (C,V_2) and thus are both eliminated by the tie-handling rule. It follows that no candidate moves forward to the final run-off from subelection (C,V_2).

Being the only participant, p alone wins the run-off.

From right to left, suppose now that p can be made the only Borda winner by some partition of V into V_1 and V_2. Thus p is the only Borda winner of at least one of the subelections (C,V_1) and (C,V_2). Without the vote v_{n+1}, however, p cannot win a subelection, since both r and r^* are preferred to p in all other votes. Let (C,V_1) be the subelection (with $v_{n+1} \in V_1$) that p is the only Borda winner of. Note that $v_{n+2} \notin V_1$, since otherwise p would lose too many points compared to r and r^* that cannot be regained via votes v_i, $1 \leq i \leq n$. Thus $v_{n+2} \in V_2$. Due to the tie-handling rule, at most two candidates can take part in the final run-off. In direct comparison, p is defeated by r and r^*, since $dist_{(\{p,r\},V)}(p,r) = -n$ and $dist_{(\{p,r^*\},V)}(p,r^*) = -n$. Therefore, also some votes v_i, $1 \leq i \leq n$, must belong to V_2, for otherwise r^* would win (C,V_2) and would then defeat p in the run-off. Since neither candidates from D nor T nor some $b_j^{(i)} \in B^{(i)}$ can win (C,V_2) by adding votes v_i, $1 \leq i \leq n$, to V_2, r and r^* must tie so as to make sure that no candidate can proceed from (C,V_2) to the final run-off. We have $dist_{(C,\{v_{n+2}\})}(r,r^*) = -K$ and $dist_{(C,\{v_i\})}(r,r^*) = 2s_i$ for each i, $1 \leq i \leq n$. Thus we need to have $dist_{(C,V_2)}(r,r^*) = -K + \sum_{v_i \in V_2} 2s_i$. Hence, $dist_{(C,V_2)}(r,r^*) = 0$ requires $\sum_{v_i \in V_2} 2s_i = 2\sum_{v_i \in V_2} s_i = 2K/2 = K$ to hold. Let $A_2 = \{i \mid v_i \in V_2\}$, so $\sum_{i \in A_2} s_i = \sum_{v_i \in V_2} s_i = K/2$, and with $A_1 = A \setminus A_2$ we obtain a partition of A such that $\sum_{i \in A_1} s_i = \sum_{i \in A_2} s_i = K/2$. ❑

Borda-CCPV-TP and Borda-DCPV-TP. Lastly, we consider the same problem as above, but with the ties-promote (TP) instead of the ties-eliminate rule and its destructive variant also with the ties-promote (TP) rule.

Theorem 10 *Borda is resistant to constructive control by partition of voters in the ties-promote model.*

Proof. To show NP-hardness, we provide a reduction from X3C to Borda-CCPV-TP. Let $(X,\mathscr{S})$ be a given X3C instance with $X=\{x_1,\dots,x_m\}$, $m=3k$, $k>1$, and $\mathscr{S}=\{S_1,\dots,S_n\}$ with $S_i\subseteq X$ and $|S_i|=3$ for each i, $1\le i\le n$. Note that we again assume that every $x_i\in X$ appears in exactly three subsets $S_j\in\mathscr{S}$. From this restriction it follows that $n=3k$. Construct from $(X,\mathscr{S})$ a Borda-CCPV-TP instance $((C,V),p)$ as follows. First we construct a large but polynomial number of buffer candidates $B=B_1\cup B_2\cup\dots\cup B_{6k+3}$ with B_{2i}, $1\le i\le 3k$, containing $6k(3k+2)-1$ candidates; B_{2i-1}, $1\le i\le 3k$, containing $9k(3k+2)+4$ candidates; B_{6k+1} containing $6k(3k+2)(2k-1)$ candidates; B_{6k+2} containing $3k(9k(3k+2)+4+6k(3k+2))$ candidates; and B_{6k+3} containing $6k(3k+2)(k+1)-1$ candidates. Note that all B_i, $1\le i\le 6k+3$ are pairwise disjunct. Let $C=\{p,r,r^*\}\cup X\cup B$ with p being the distinguished candidate. Define V to consist of the following groups votes V_1, V_2, V_3, and V_4:

1. V_1 contains a single vote of the form $r\ B_{6k+1}\ r^*\ B_{6k+2}\ p\ \overrightarrow{X}\ B\setminus(B_{6k+1}\cup B_{6k+2})$.
2. V_2 contains a single vote of the form $r\ B_{6k+3}\ r^*\ \overleftarrow{X}\ p\ B\setminus B_{6k+3}$.
3. V_3 contains a vote v_j of the form $X\setminus S_i\ p\ B_{2j-1}\ r^*\ B_{2j}\ r\ x'\ x''\ x'''\ B\setminus(B_{2j-1}\cup B_{2j})$ for every $S_j=\{x',x'',x'''\}\in\mathscr{S}$.
4. V_4 contains $3k$ votes of the form $r\ \overleftarrow{X}\ p\ r^*\ B$.

Note that in the way these votes are set up, every buffer candidate $b_j\in B$ is behind some candidate from $C\setminus B$ in every vote (as a matter of fact, b_j is behind every candidate from $C\setminus B$ in all votes but one). This lets us conveniently disregard all buffer candidates, since they are eliminated in all possible subelection and can never reach the final.

Note that p is not winning in (C,V), since $dist_{(C,V)}(p,r)\le-(3k(9k(3k+2)+4+6k(3k+2))+1)+3k(6k(3k+2)+9k(3k+2)+4)<0$. We claim that $(X,\mathscr{S})$ is a yes-instance of X3C if and only if (C,A,V,p) is a yes-instance of Borda-CCPV-TP.

Suppose there exists an exact cover $\mathscr{S}'\subseteq\mathscr{S}$. Let $\widehat{V}=\{v_j\mid S_j\in\mathscr{S}'\}$. Partition V into $V'=V_1\cup(V_3\setminus\widehat{V})\cup V_4$ and $V''=V_2\cup\widehat{V}$. In the subelection (C,V'), r^* beats every other candidate, since $dist_{(C,V')}(r^*,r)=3k(3k+2)-1-(3k+2)>0$, $dist_{(C,V')}(r^*,p)=3k(9k(3k+2)+6k(3k+2)+5)-2k(9k(3k+2)+5)-3k>0$, and $dist_{(C,V')}(r^*,x_i)\ge dist_{(C,V')}(r^*,p)+1-(2k-2)(3k-3)-9k^2>0$ for every $x_i\in X$. In the other subelection (C,V''), p is the only Borda winner, since $dist_{(C,V'')}(p,r^*)=k(9k(3k+2)+5)-(3k+1)>0$, $dist_{(C,V'')}(p,r)=-6k(k+1)(3k+2)-(3k+1)+5k(3k(3k+2)+1)>0$ and $dist_{(C,V'')}(p,x_i)\ge-3k-(k-1)(3k-3)+15k(3k+2)+6>0$. In the final election $(\{p,r^*\},V)$, p is the only Borda winner, since $dist_{(\{p,r^*\},V)}(p,r^*)=6k-2>0$.

For the converse, suppose there is no exact cover. We now show that p cannot be made the only Borda winner by partitioning the votes. Since no buffer candidate reaches the final, for a subset $X'\subseteq X$ only the following final elections with p participating are possible: $(\{p,r,r^*\}\cup X',V),(\{p,r\}\cup X',V),(\{p,r^*\}\cup X',V)$ and $(\{p\}\cup X',V)$. It is easy to see that p wins alone only if r^* participates and $X'=\emptyset$. Without loss of generality, assume that $V_1\subseteq V'$. Then p cannot win (C,V'), since the deficit of $2k(6k(3k+2))+3k(15k(3k+2)+5)$ to r cannot be made up for, not even with all the votes from V_3. Therefore, p can only win (C,V''). For p to beat every $x_i\in X$ in (C,V''), there need to be votes $\widehat{V}\subseteq V_3$ in V'' so that for every $x_i\in X$ there is a $v_j\in\widehat{V}$ with $x_i\in S_j$. Otherwise, p would be behind x_i in every vote of V''. Since there is no exact cover, we need at least $k+1$ to ensure that p is not beaten by a candidate $x_i\in X$ in (C,V''). Now, for r^* to reach the final, she needs to either tie with p in (C,V'') or win (C,V'). Since $\widehat{V}\subseteq V_3$, r^* cannot make up the deficit of at least $k(9k(3k+2)+4)$ points to p, as she is ahead of r^* in all votes of V_4 and the vote of V_2 would give r^* only $3k+1$ points more than p. So r^* needs to win (C,V'). With $V_1\subseteq V'$, it follows that $V_2\subseteq V''$, or else the point deficit of r^* to r in (C,V') from votes of V_1 and V_2 cannot be made up for by at most $2k-1$ votes from

V_3, since $-(6k(3k+2)(2k-1)+1)-(6k(3k+2)(k+1))+6k(3k+2)(2k-1)<0$. But still, with only $2k-1$ votes of V_3 in V' and any number of votes from V_4 in V', we have $dist_{(C,V')}(r^*,r)<0$, so r^* is not winning in (C,V') and cannot reach the final. Therefore, without an exact cover, either p or r^* cannot reach the final. ❑

The proof of Theorem 11, again, is deferred to the appendix.

Theorem 11 *Borda is resistant to destructive control by partition of voters in the ties-promote model.*

5 Conclusions and Future Work

We have solved twelve open problems about the complexity of standard control scenarios in Borda elections (recall Table 1), leaving just one case open: destructive control by partition of candidates in the ties-promote model. In particular, complementing previous results, we have now shown that Borda is resistant to every standard type of constructive control, whereas it is vulnerable to most of the destructive control types. We have also identified one of the rare cases where the complexity of a control problem in the unique-winner model parts company from that in the nonunique-winner model.

As future work for control in Borda elections, we propose (a) to solve the one open question mentioned above, (b) to provide a parameterized complexity analysis of the cases where resistance is known, and (c) to study online control for sequential Borda elections (see Hemaspaandra et al. [18, 19] for the model of online control in sequential elections). Another challenging task is to settle the complexity of control for all scoring rules, ideally by establishing dichotomy results in the style of Hemaspaandra et al. [17, 20].

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A Deferred Proofs

Proof of Theorem 4. To show NP-hardness, since instances of Borda-CCRPC-TE and Borda-CCPC-TE are defined identically, we can use the same construction as in the proof of Theorem 3 that yields a reduction from 3SAT to Borda-CCRPC-TE. That is, given a 3SAT instance $\varphi(x_1,\ldots,x_n)$, construct the same candidates and votes as in the proof of Theorem 3. Note that the argument on the point balance of p and a clause candidate K_i (after the construction in that proof) still holds.

To show correctness of the construction, we only outline the most important arguments to highlight the slight differences to the argumentation in that proof. To prove the equivalence from left to right, suppose there is a satisfying truth assignment α to the variables of $\varphi(x_1,\ldots,x_n)$. Partition C into C_1 and C_2 so that C_1 contains r, r^*, and all variable candidates that are set to *false* in α, and C_2 contains all the other candidates. r and r^* tie in subelection (C_1,V) and are eliminated by the tie-handling rule. Candidates in C_2 get a bye to the final run-off in which p then beats all other candidates (in particular, the clause candidates) from C_2 because α is a satisfying truth assignment.

For the right-to-left direction, suppose that p is the unique overall Borda winner for some partition of the candidates. r had to be eliminated in the subelection; otherwise, r would have beaten p in the run-off. This can only be achieved by r^*, who can tie (but not beat) r in the subelection if the candidates in D, K, and p are not participating.

Thus C_1 contains r, r^*, and some variable candidates, and C_2 contains p and all candidates from D, K, p, and the remaining variable candidates. All candidates from C_2 advance directly to the run-off, and in subelection (C_1,V) all winners are tieing and, therefore, are eliminated by the tie-handling rule. Since p beats all clause candidates in the run-off, the variable candidates must have been distributed among C_1 and C_2 according to the above argument. This leads to a satisfying truth assignment if every variable candidate in C_2 is assigned to *true*, and all the others to *false*. ❑ Theorem 4

Proof of Theorem 6. To show NP-hardness we provide a reduction from X3C to Borda-DCRPC-TP. Let $(X,\mathscr{S})$ be a given X3C instance with $X=\{x_1,\ldots,x_m\}$, $m=3k$, $k>1$, and $\mathscr{S}=\{S_1,\ldots,S_n\}$ with $S_i\subseteq X$ and $|S_i|=3$ for each i, $1\leq i\leq n$. Note that we assume that every $x_i\in X$ appears in exactly three subsets $S_j\in\mathscr{S}$ (thus $n=3k$). This restricted version of X3C was proven to be NP-complete by Gonzalez [13]. Construct from $(X,\mathscr{S})$ a Borda-DCRPC-TP instance $((C,V),p)$ as follows. Let $C=\{p,r\}\cup X\cup S$ with p being the distinguished candidate. For every $x_i\in X$, let $S_{x_i}=\{S_j \mid x_i\in S_j\}$. Note that $|S_{x_i}|=3$ for every $x_i\in X$. Define V to consist of the following votes:

1. There are $3k+1$ votes of the form $p\ \overrightarrow{\mathscr{S}}\ r\ \overrightarrow{X}$ and $3k+1$ votes of the form $p\ \overleftarrow{\mathscr{S}}\ r\ \overleftarrow{X}$.
2. There is a vote $r\ \overrightarrow{X}\ p\ \overrightarrow{\mathscr{S}}$ and a vote $r\ \overleftarrow{X}\ p\ \overleftarrow{\mathscr{S}}$.
3. For every x_i, there are $(3k+1)(3k+2)$ votes $x_i\ S_{x_i}\ \overrightarrow{X\setminus\{x_i\}}\ r\ p\ \overrightarrow{\mathscr{S}\setminus S_{x_i}}$ and there are $(3k+1)(3k+2)$ votes $p\ S_{x_i}\ \overleftarrow{\mathscr{S}\setminus S_{x_i}}\ r\ \overleftarrow{X\setminus\{x_i\}}\ x_i$.
4. There are $3k(3k+1)(3k+2)$ votes $p\ \overrightarrow{X}\ r\ \overrightarrow{\mathscr{S}}$ and $3k(3k+1)(3k+2)$ votes $r\ \overleftarrow{X}\ p\ \overleftarrow{\mathscr{S}}$.

Before we proceed to prove that the reduction is correct, we need the following.

Lemma 2 *For subsets $X'\subseteq X$ and $\mathscr{S}'\subseteq\mathscr{S}$, p is the unique Borda winner of the election (C',V) with $C'=\{p,r\}\cup X'\cup\mathscr{S}'$ if $|X'|<3k$ or $|\mathscr{S}'|>0$, and a Borda winner only tied with r otherwise.*

Proof. For subsets $X'\subseteq X$ and $\mathscr{S}'\subseteq\mathscr{S}$ we have the following point balances in the election (C',V) with $C'=\{p,r\}\cup X'\cup\mathscr{S}'$:

- $dist_{(C',V)}(p,r)=(2|\mathscr{S}'|+2)(3k+1)-2(|X'|+1)$.

- For each $x_i \in X'$, $dist_{(C',V)}(p,x_i) \geq (3k+1)(2|\mathscr{S}'|+2)+3k(|X'|+1) > 0$.
- For each $S_j \in \mathscr{S}'$, $dist_{(C',V)}(p,S_j) \geq (3k+1)(3k+2)((3k-3)|X'|+6k+3k|\mathscr{S}'|-9) > 0$.

We can see that p always beats all $x_i \in X'$ and $S_j \in \mathscr{S}'$ and ties r only if $X' = X$ and $\mathscr{S}' = \emptyset$. Note that even when r is removed from the election, p is the only Borda winner for any $X' \subseteq X$ and $\mathscr{S}' \subseteq \mathscr{S}$. ❑ Lemma 2

From Lemma 2 we see that p is the only Borda winner of election (C,V).

Lemma 3 *For subsets $X' \subseteq X$ and $\mathscr{S}' \subseteq \mathscr{S}$, p is the unique Borda winner of the election (C',V) with $C' = \{p\} \cup X' \cup \mathscr{S}'$.*

Proof. For subsets $X' \subseteq X$ and $\mathscr{S}' \subseteq \mathscr{S}$ we have the following point balances in the election (C',V) with $C' = \{p,r\} \cup X' \cup \mathscr{S}'$:

- For each $x_i \in X'$, $dist_{(C',V)}(p,x_i) \geq (3k+1)(2|\mathscr{S}'|)+3k(|X'|+1) > 0$.
- For each $S_j \in \mathscr{S}'$, $dist_{(C',V)}(p,S_j) \geq (3k+1)(3k+2)((3k-3)|X'|+3k+3k|\mathscr{S}'|-6) > 0$.

We can see that p always beats all $x_i \in X'$ and $S_j \in \mathscr{S}'$. ❑ Lemma 3

We claim that $(X,\mathscr{S})$ is a yes-instance of X3C if and only if (C,A,V,p) is a yes-instance of Borda-DCRPC-TP.

From left to right, suppose there is an exact cover $\mathscr{S}' \subseteq \mathscr{S}$. We partition C into $C_1 = \{p\} \cup (\mathscr{S} \setminus \mathscr{S}')$ and $C_2 = \{r\} \cup X \cup \mathscr{S}'$. From Lemma 3, p is the only Borda winner of (C_1,V) and reaches the final. In the second subelection, r ties every $x_i \in X$, since $dist_{(C_2,V)}(r,x_i) = (3k+1)^2+(3k+1)-(3k+1)(3k+2) = 0$. Furthermore, r beats every $S_j \in \mathscr{S}'$, since $dist_{(C_2,V)}(r,S_j) > -(3k+1)(3k+2)(4k)+3k(3k+1)(3k+2)(5k) > 0$. Due to the ties-promote model, r and every $x_i \in X$ reach the final. Since p, r, and all candidates in X participate in the final, but no candidates in $\mathscr{S}$ do, from Lemma 2 we can conclude that p is tied with r and thus prevented from being a unique Borda winner.

From right to left, suppose that p can be prevented from being the unique Borda winner by partitioning the set of candidates. From Lemma 2 and Lemma 3 we can conclude that r, all candidates in X and no candidate in $\mathscr{S}$ reach the final. Furthermore, p cannot participate in a subelection with r or some candidates X as it would prevent at least one of them from reaching the final. Without loss of generality, assume that $p \in C_1$ and $\{r\} \cup X \subseteq C_2$. It is easy to see that p is the only Borda winner of (C_1,V) and so reaches the final. In $((\{r\} \cup X),V)$, r beats every $x_i \in X$ by $(3k+1)(3k+2)$ points. For every $S_j \in \mathscr{S}$ that is added to C_2, every $x_i \in S_j$ gains $(3k+1)(3k+2)$ onto r. For r and all candidates from X to proceed to the final, candidates $\mathscr{S}' \in \mathscr{S}$ need to be added to C_2 so that every $x_i \in X$ is contained in exactly one element of $\mathscr{S}'$. Therefore, $\mathscr{S}'$ is an exact cover of X. Note also that r beats all those candidates $\mathscr{S}'$ in (C_2,V). ❑ Theorem 6

Proof of Theorem 11. To show NP-hardness, we provide a reduction from X3C to Borda-DCPV-TP. Let $(X,\mathscr{S})$ be a given X3C instance with $X = \{x_1,\dots,x_m\}$, $m = 3k$, $k > 1$, and $\mathscr{S} = \{S_1,\dots,S_n\}$ with $S_i \subseteq X$ and $|S_i| = 3$ for each i, $1 \leq i \leq n$. Again, we assume that every $x_i \in X$ appears in exactly three subsets $S_j \in \mathscr{S}$ (recall that this implies $n = 3k$). Construct from $(X,\mathscr{S})$ a Borda-DCPV-TP instance $((C,V),p)$ as follows.

We start by constructing a large but polynomial number of buffer candidates $B = B_1 \cup B_2 \cup \cdots \cup B_{3k+2} \cup \{\widehat{b}_1,\widehat{b}_2,\widehat{b}_3\}$ with B_i, where $1 \leq i \leq 3k$ and $S_i = \{x_p,x_q,x_r\}$ with $p < q < r$, containing $rk-3$ candidates; B_{3k+1} containing $6k+3$ candidates; and B_{3k+2} containing $9k^2$ candidates. Note that all B_i, $1 \leq i \leq 3k+2$, and $\{\widehat{b}_1,\widehat{b}_2,\widehat{b}_3\}$ are pairwise disjoint.

Let $C = \{p,r,r^*\} \cup X \cup B$ with p being the distinguished candidate. For a more convenient construction of votes, we introduce additional notation. If we write $\overrightarrow{X_{|\{x_i,x_j,x_\ell\}}}$ for some $\{x_i,x_j,x_\ell\} \subseteq$

X, the candidates of X appear in the vote in the usual order but x_i, x_j, and x_ℓ are replaced with $\widehat{b}_1$, $\widehat{b}_2$, and $\widehat{b}_3$. It is important to note that when $\overrightarrow{X}$ appears in a vote, the candidates of X are ordered from lowest to highest index, whereas they are ordered from highest to lowest index in case of $\overleftarrow{X}$.

Now, define V to consist of the following groups of votes V_1, V_2, V_3, and V_4:

1. V_1 contains $3k+2$ votes of the form $p\ B_{3k+2}\ r\ \overleftarrow{X}\ B \setminus B_{3k+2}$.
2. V_2 contains two votes of the form $r\ \widehat{b}_1\ \overrightarrow{X}\ p\ B \setminus \{\widehat{b}_1\}$.
3. For every $S_j = \{x_p, x_q, x_r\} \in \mathscr{S}$ with $p < q < r$, V_3 contains a vote v_j that is constructed in the following way: Set x_r on position one, then $(r-q)k-1$ buffer candidates from B_j, then x_q (at position $(r-q)k+1$), then $(q-p)k-1$ buffer candidates from B_j, then x_p (at position $(r-p)k+1$), then the remaining $pk-1$ buffer candidates from B_j, and from position $rk+1$ onwards the vote has the form $p\ r\ \overrightarrow{X_{|\{x_p,x_q,x_r\}}}\ B \setminus (B_j \cup \{\widehat{b}_1, \widehat{b}_2, \widehat{b}_3\})$.
4. V_4 contains $3k+1$ votes of the form $r\ p\ B_{3k+2}\ \overrightarrow{X}\ B \setminus B_{3k+2}$ and $3k+1$ votes of the form $p\ r\ B_{3k+2}\ \overleftarrow{X}\ B \setminus B_{3k+2}$.

The votes in V_3 are set up in a way so that for a vote $v_j \in V_3$, if $x_i \in S_j$ then $dist_{(C,\{v_j\})}(r, x_i) = -(ik+1)$, and if $x_i \notin S_j$ then $dist_{(C,\{v_j\})}(r, x_i) = i$.

Note that every buffer candidate $b_i \in B$ is behind one candidate from $C \setminus B$ in all votes. Therefore, no buffer candidate b_i ever survives a subelection and we can disregard their scores.

Lemma 4 *For any partition of votes (V', V''), p always is the unique Borda winner of one subelection.*

Proof. Let (V', V'') be a partition of V. Since V_1 contains $3k+2$ votes, there is at least one part of the partition with at least k votes of V_1, let us say V'. From these votes, p is ahead of r by at least $k(6k+4)$ points and ahead of every $x_i \in X$ by at least $k(6k+5)$ points in (C, V'). Even if the other votes in V' all rank r ahead of p, r can only gain at most $3k+1+6k+4$ points on p, which is not enough to at least tie p. For each $x_i \in X$, if the other votes in V' all rank x_i ahead of p then x_i can only gain at most $2+3k^2+1$ points on p, which is not enough to at least tie p. Therefore, p is the unique Borda winner of (C, V'). ❑ Lemma 4

In election (C,V), p is the unique Borda winner because $dist_{(C,V)}(p,r) = (3k+2)(6k+4) - 2(3k+2) + 3k > 0$ and $dist_{(C,V)}(p,x_i) \geq (3k+2)(6k+5) - 2 - 3(3k^2) + (3k+1)(18k^2+3k+2) > 0$ for every $x_i \in X$.

We claim that $(X, \mathscr{S})$ is a yes-instance of X3C if and only if (C, A, V, p) is a yes-instance of Borda-DCPV-TP.

From left to right, suppose there is an exact cover $\widehat{\mathscr{S}} \subseteq \mathscr{S}$. Let $\widehat{V} = \{v_j \in V_3 \mid S_j \in \widehat{\mathscr{S}}\}$. Partition V into $V' = V_1 \cup \widehat{V} \cup V_4$ and $V'' = V_2 \cup (V_3 \setminus \widehat{V})$

In (C, V''), $dist_{(C,V'')}(r,p) = 6k+4-2k = 4k+4 > 0$ and $dist_{(C,V'')}(r,x_i) = 2i+2-2(ik+1)+(2k-2)i = 0$ for every $x_i \in X$, since $\widehat{\mathscr{S}}$ is an exact cover, so every $x_i \in X$ is covered exactly twice in $S \setminus \widehat{\mathscr{S}}$. Therefore, r and all $x_i \in X$ tie and proceed to the final according to the TP rule. From Lemma 4 it follows that p wins (C, V') alone. r ties p in the final election, since $dist_{(\{p,r\} \cup X, V)}(p,r) = 2 \cdot 3k - (2|X|) = 0$.

From right to left, suppose that p can be made the only Borda winner by partitioning the votes. From Lemma 4 it follows that p always reaches the final election. Since no buffer candidate survives the subelections, for a subset $X' \subseteq X$ the only possible final elections are $(\{p,r\} \cup X', V)$ and $(\{p\} \cup X', V)$. In $(\{p,r\} \cup X', V)$, $dist_{(\{p,r\} \cup X', V)}(p,r) = 2 \cdot 3k - (2|X'|)$ and $dist_{(\{p,x_i\} \cup X', V)}(p,x_i) \geq 2(3k+2) - 11 > 0$ for every $x_i \in X'$. Therefore, p is not the only winner if r and all candidates in X

reach the final. In $(\{p\} \cup X', V)$, $dist_{(\{p,x_i\} \cup X', V)}(p, x_i) \geq (3k+2) - 11 + (3k+1) > 0$ for every $x_i \in X'$. Therefore, p wins the final election alone if r or any x_i fail to reach the final. From Lemma 4 it follows that r and all $x_i \in X$ need to tie in one subelection. Without loss of generality, assume that this subelection is (C, V'). Then no vote from V_1 can be in V' (or else p would beat r), and no vote from V_4 can be in V' (or else p would beat at least one $x_i \in X$). Thus V' consists of votes from V_2 and V_3. If there is no vote from V_2 in V' then r is beaten by p; otherwise, r always beats p. If V' contains one vote from V_2 then r scores $i+1$ points more than every $x_i \in X$. To tie r and all x_i, there need to be votes $\widehat{V} \subseteq V_3$ in V'. Let $\widehat{S} = \{S_j \mid v_j \in \widehat{V}\}$ and $S_{x_i} = \{S_j \in \widehat{S} \mid x_i \in S_j\}$. If $|S_{x_i}| = 0$ for a $x_i \in X$ then $dist_{(C,V')}(r, x_i) > 0$. Therefore, $|\widehat{V}| = |\widehat{S}| \geq k$. If $|S_{x_i}| = 3$ then $dist_{(C,V')}(r, x_i) = -3(ik+1) + i + 1 + (|\widehat{S}| - 3)i \leq -(3k-1)i - 2 + (3k-3)i = -2i - 2 < 0$. If $|S_{x_i}| = 2$ then $dist_{(C,V')}(r, x_i) = -2(ik+1) + i + 1 + (|\widehat{S}| - 2)i = -(2k-1)i - 1 + (\widehat{S} - 2)i \neq 0$ for $i \neq 1$. If $|S_{x_i}| = 1$ then $dist_{(C,V')}(r, x_i) = -(ik+1) + i + 1 + (|\widehat{S}| - 1)i = 0$ for $|\widehat{S}| = k$ and all i. Therefore, if V' consists of one vote of V_2, r and all $x_i \in X$ can only tie if there exists an exact cover.

If V' contains both votes from V_2 then r scores $2i+2$ points more than every $x_i \in X$. If $|S_{x_i}| = 0$ for a $x_i \in X$ then $dist_{(C,V')}(r, x_i) > 0$. If $|S_{x_i}| = 3$ then $dist_{(C,V')}(r, x_i) = -3(ik+1) + 2i + 2 + (|\widehat{S}| - 3)i \leq -(3k-2)i - 1 + (3k-3)i = -i - 1 < 0$. If $|S_{x_i}| = 2$ then $dist_{(C,V')}(r, x_i) = -2(ik+1) + 2i + 2 + (|\widehat{S}| - 2)i = -(2k-2)i + (|\widehat{S}| - 2)i = 0$ for $|\widehat{S}| = 2k$ and all i. This would mean that $\mathscr{S} \setminus \widehat{S}$ is an exact cover. If $|S_{x_i}| = 1$ then $dist_{(C,V')}(r, x_i) = -(ik+1) + 2i + 2 + (|\widehat{S}| - 1)i = -(k-2)i + 1 + (|\widehat{S}| - 1)i \geq -(k-2)i + 1 + (k-1)i = 1 + i > 0$. Therefore, if V' consists of two votes of V_2, r and all $x_i \in X$ can only tie if there exists an exact cover. ❑ Theorem 11