Trading Transforms of Non-weighted Simple Games and Integer Weights of Weighted Simple Games

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Abstract

This paper is concerned with simple games. One of the fundamental questions regarding simple games is that of what makes a simple game a weighted majority game. Taylor and Zwicker (1992) showed that a simple game is non-weighted if and only if there exists a trading transform of finite size. They also provided an upper bound on the size of such a trading transform, if it exists. Gvozdeva and Slinko (2009) improved on that upper bound. Their proof employs a property of linear inequalities demonstrated by Muroga (1971). We provide a new proof of the existence of a trading transform when a given simple game is non-weighted. Our proof employs Farkas' lemma (1894), and yields an improved upper bound on the length of a trading transform.

We also discuss an integer weights representation of a weighted simple game, and improve on the bounds obtained by Muroga (1971). We show that our bounds are tight when the number of players is less than or equal to five, based on the computational results obtained by Kurz (2012).

We deal with the problem of finding a minimum integer weights representation under the assumption that we have minimal winning coalitions and maximal losing coalitions. We propose a polynomial time probabilistic algorithm for finding an approximate solution.

1 Introduction

A simple game consists of a pair $G = (N, \mathcal{W})$ where N is a finite set of players and $\mathcal{W} \subseteq 2^N$ is an arbitrary collection of subsets of N. Throughout this paper, we denote $|N|$ by n. Usually, the property

$$\text{(monotonicity): if } S' \supseteq S \in \mathcal{W}, \text{ then } S' \in \mathcal{W}, \tag{1}$$

is assumed. Subsets in $\mathcal{W}$ are called *winning coalitions*. We denote $2^N \setminus \mathcal{W}$ by $\mathcal{L}$, and subsets in $\mathcal{L}$ are called *losing coalitions*. A simple game $(N, \mathcal{W})$ is said to be *weighted* if there exists a weight vector $\boldsymbol{w} \in \mathbb{R}^N$ and $q \in \mathbb{R}$ satisfying

$$\text{(weightedness): for any } S \subseteq N, S \in \mathcal{W} \text{ if and only if } \sum_{i \in S} w_i \geq q. \tag{2}$$

The necessary and sufficient conditions that guarantee the weightedness of a simple game are known. Elgot [2] and Chow [1] investigated the theory of threshold logic, and showed a condition in terms of *asummability*. Muroga [15] described a proof of the sufficiency of asummability based on the theory of linear inequality systems and discussed some variations of their results in cases of a few variables. Taylor and Zwicker [18, 19] obtained necessary and sufficient conditions independently in terms of a *trading transform*. A *trading transform* of size j is a coalition sequence $(X_1, X_2, \ldots, X_j; Y_1, Y_2, \ldots, Y_j)$, which may contain repetitions of coalitions, satisfying that $\forall p \in N$, $|\{i \mid p \in X_i\}| = |\{i \mid p \in Y_i\}|$. A simple game will be called *k-trade robust* if there is no trading transform of size j satisfying (1) $1 \leq j \leq k$, (2) $X_1, X_2, \ldots, X_j \in \mathcal{W}$, and (3) $Y_1, Y_2, \ldots, Y_j \in \mathcal{L}$. A simple game will be called *trade robust* if it is k-trade robust for all positive integers k.

Taylor and Zwicker showed that a given simple game G with n players is weighted if and only if G is 2^{2^n}-trade robust. In 2009, Gvozdeva and Slinko [7] showed that a given simple game G is weighted if and only if G is $(n+1)n^{n/2}$-trade robust. The relations between results in the field of threshold logic and the field of simple games are clarified in Freixas, Freixas and Kurz [6].

In Section 2, we show that a given simple game G is weighted if and only if G is α_{n+1}-trade robust where α_{n+1} denotes the maximal value of determinants of $(n+1) \times (n+1)$ 0-1 matrices. It is wellknown that $\alpha_{n+1} \leq (n+2)^{\frac{n+2}{2}}(1/2)^{(n+1)}$.

Our definition of a weighted simple game allows for an arbitrary real number of weights. However, it is easy to see that any weighted simple game is represented by integer weights. An *integer weights representation* of a weighted simple game consists of an integer vector $\boldsymbol{w} \in \mathbb{Z}^N$ and some $q \in \mathbb{Z}$ satisfying property (2). Isbell [10] found an example of weighted simple game of 12 players without a unique minimum sum integer weights representation. Examples for 9, 10, or 11 players are given in [4, 5]. In the field of threshold logic, examples of threshold functions requiring large weights are discussed in [16, 15, 9]. There exist some previous studies that enumerate (minimal) integer weights representations of simple games with small numbers of players (e.g., [13, 20, 14, 11]). In the case of $n = 9$ players, we refer the reader to Kurz's work [12]. In general, Muroga [15] showed that every weighted simple game has an integer weight representation satisfying (1) $w_i \leq (n+1)^{\frac{n+1}{2}}(1/2)^n$ $(\forall i \in N)$ and (2) $q \leq n(n+1)^{\frac{n+1}{2}}(1/2)^n$ simultaneously.

In Section 3, we slightly improve Muriga's result and show that every weighted simple game has an integer weight representation $(\boldsymbol{w}, q) \in \mathbb{Z}^N \times \mathbb{Z}$ satisfying (1) $w_i \leq \alpha_n$ $(\forall i \in N)$ and (2) $q \leq \alpha_{n+1}$ simultaneously. Based on the computational results of Kurz [12], we also demonstrate the tightness of our bounds (1) and (2) when $n \leq 5$. Here we note that α_n denotes the maximal value of determinants of $n \times n$ 0-1 matrices, and satisfies $\alpha_n \leq (n+1)^{\frac{n+1}{2}}(1/2)^n$.

When we have a family of minimal winning coalitions, Peled and Simone [17] proposed a polynomial time algorithm for checking the weightedness of a given simple game. They also showed that for weighted simple games represented by minimal winning coalitions, all maximal losing coalitions can be computed in polynomial time. When we have minimal winning coalitions and maximal losing coalitions, linear programming finds a weight vector $\boldsymbol{w} \in \mathbb{R}^N$ and $q \in \mathbb{R}$ satisfying property (2) in polynomial time. However, it is less straightforward to find an integer weights representation where all the weights are integers as the problem transforms from linear programming to integer programming.

In Section 4, we deal with the problem of finding an integer weights representation under the assumption that we have minimal winning coalitions and maximal losing coalitions. More precisely, we discuss three problems; P_q of minimizing the quota, P_w of minimizing the maximum weight, and P_Σ of minimizing the sum of weights. For each problem, we show that there exists an integer weights representation whose objective value is less than or equal to $((2-\sqrt{2})n + (\sqrt{2}-1))z(\overline{\mathrm{P}})$ where $z(\overline{\mathrm{P}})$ denotes the optimal value of the corresponding linear relaxation problem. We also propose a probabilistic algorithm for finding an integer weights representation.

2 Trading Transforms of Non-weighted Simple Games

In this section, we discuss the size of a trading transform that guarantees the non-weightedness of a given simple game. Throughout this section, we will not need to assume the monotonicity property (1). First, we introduce a linear inequality system for determining the weightedness of a given simple game. For any nonempty family of player subsets $\emptyset \neq \mathcal{N} \subseteq 2^N$, we introduce a 0-1 matrix $A(\mathcal{N}) = (a(\mathcal{N})_{Si})$ whose rows are indexed by subsets in $\mathcal{N}$ and columns are indexed by players in N defined by

$$a(\mathcal{N})_{Si} = \begin{cases} 1 & (\text{if } i \in S \in \mathcal{N}), \\ 0 & (\text{otherwise}). \end{cases}$$

It is obvious that a given simple game $G = (N, \mathcal{W})$ is weighted if and only if the linear inequality system

$$\begin{array}{rcl} \text{P1:} \begin{pmatrix} A(\mathcal{W}) & \mathbf{1} & \mathbf{0} \\ -A(\mathcal{L}) & -\mathbf{1} & -\mathbf{1} \end{pmatrix} \begin{pmatrix} \boldsymbol{w} \\ -q \\ \varepsilon \end{pmatrix} & \geq & \mathbf{0}, \\ \varepsilon & > & 0, \end{array}$$

is feasible, where $\mathbf{0}$ ($\mathbf{1}$) denotes a zero vector (all-one vector) of the appropriate dimension.

Farkas' Lemma [3] says that P1 is infeasible if and only if the system

$$\text{D1:} \begin{pmatrix} A(\mathcal{W})^\top & -A(\mathcal{L})^\top \\ \mathbf{1}^\top & -\mathbf{1}^\top \\ \mathbf{0}^\top & -\mathbf{1}^\top \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 0 \\ -1 \end{pmatrix}, \quad \boldsymbol{x} \geq \mathbf{0}, \ \boldsymbol{y} \geq \mathbf{0},$$

is feasible. For simplicity, we denote the linear inequality system D1 by $A_1 \boldsymbol{z} = \boldsymbol{c}, \boldsymbol{z} \geq \mathbf{0}$ where

$$A_1 = \begin{pmatrix} A(\mathcal{W})^\top & -A(\mathcal{L})^\top \\ \mathbf{1}^\top & -\mathbf{1}^\top \\ \mathbf{0}^\top & -\mathbf{1}^\top \end{pmatrix}, \ \boldsymbol{z} = \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix}, \text{ and } \boldsymbol{c} = \begin{pmatrix} \mathbf{0} \\ 0 \\ -1 \end{pmatrix}.$$

In the following, we consider the case that the linear inequality system $A_1 \boldsymbol{z} = \boldsymbol{c}, \boldsymbol{z} \geq \mathbf{0}$ is feasible. Let $\widetilde{A_1} \boldsymbol{z} = \widetilde{\boldsymbol{c}}$ be a linear equality system obtained from $A_1 \boldsymbol{z} = \boldsymbol{c}$ by repeatedly removing redundant equalities. It is wellknown that if the linear inequality system $\widetilde{A_1} \boldsymbol{z} = \widetilde{\boldsymbol{c}}, \boldsymbol{z} \geq \mathbf{0}$ is feasible, then it has a basic feasible solution, i.e., there exists an index subset J of the index set of the vector $\boldsymbol{z}$ satisfying that (1) a column submatrix B of $\widetilde{A_1}$ consisting of column vectors of $\widetilde{A_1}$ indexed by J is a square invertible matrix and (2) $B^{-1}\widetilde{\boldsymbol{c}} \geq \mathbf{0}$. We introduce a subvector $\boldsymbol{z}_J$ ($\boldsymbol{z}_{\overline{J}}$) consisting of components of $\boldsymbol{z}$ indexed by indices in J (not in J). Then, we have a basic feasible solution

$$\boldsymbol{z}^* = \begin{pmatrix} \boldsymbol{z}^*_J \\ \boldsymbol{z}^*_{\overline{J}} \end{pmatrix} = \begin{pmatrix} B^{-1}\widetilde{\boldsymbol{c}} \\ \mathbf{0} \end{pmatrix}$$

when $A_1 \boldsymbol{z} = \boldsymbol{c}, \boldsymbol{z} \geq \mathbf{0}$ is feasible. By Cramer's rule, $z^*_j = \det(B_j)/\det(B)$ for each $j \in J$ where B_j is obtained from B with the column that is indexed by j replaced by $\widetilde{\boldsymbol{c}}$. Because B_j is an integer matrix, $\det(B) z^*_j = \det(B_j)$ is an integer for any $j \in J$. Thus, $|\det(B)|\boldsymbol{z}^*$ is an integer feasible solution of $A_1 \boldsymbol{z} = |\det(B)|\boldsymbol{c}, \boldsymbol{z} \geq \mathbf{0}$. Let $\begin{pmatrix} \boldsymbol{x}^* \\ \boldsymbol{y}^* \end{pmatrix}$ be an integer vector corresponding to $|\det(B)|\boldsymbol{z}^*$. Let us recall that $\boldsymbol{x}^*$ is indexed by $\mathcal{W}$ and $\boldsymbol{y}^*$ is indexed by $\mathcal{L}$. Then, the pair of integer vectors $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ satisfies that

$$A(\mathcal{W})^\top \boldsymbol{x}^* = A(\mathcal{L})^\top \boldsymbol{y}^*, \quad \sum_{S \in \mathcal{W}} x^*_S = \sum_{S \in \mathcal{L}} y^*_S = |\det(B)|, \quad \boldsymbol{x}^* \geq \mathbf{0}, \quad \boldsymbol{y}^* \geq \mathbf{0}.$$

In the following, we construct a trading transform corresponding to the pair $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$. Let $\mathcal{X} = (X_1, X_2, \ldots, X_{|\det(B)|})$ be a sequence of winning coalitions satisfying that each winning coalition $S \in \mathcal{W}$ appears in $\mathcal{X}$ x^*_S-times. Similarly, we introduce a sequence $\mathcal{Y} = (Y_1, Y_2, \ldots, Y_{|\det(B)|})$ satisfying that each losing coalition $S \in \mathcal{L}$ appears in $\mathcal{Y}$ y^*_S-times. Then, the equality $A(\mathcal{W})^\top \boldsymbol{x}^* = A(\mathcal{L})^\top \boldsymbol{y}^*$ implies that $(\mathcal{X}; \mathcal{Y})$ is a trading transform of size $|\det(B)|$. From the above discussion, we have shown that if D1 is feasible, then a given simple game $G = (N, \mathcal{W})$ is not $|\det(B)|$-trade robust.

Finally, we provide an upper bound on $|\det(B)|$. Let α_n be the maximum of the determinant of an $n \times n$ 0-1 matrix. If a column of B is indexed by a component of $\boldsymbol{x}$ (i.e., indexed by a winning coalition), then each component of the column is either 0 or 1. Otherwise, a column (of B) is indexed by a component of $\boldsymbol{y}$ (i.e., indexed by a losing coalition) whose components are either 0 or -1. Now, we apply elementary matrix operations to B. For each column of B indexed by a component $\boldsymbol{y}$, we multiply the column by (-1). Then, the resulting matrix, denoted by B', is a 0-1 matrix satisfying $|\det(B)| = |\det(B')|$.

Let us recall that B is a submatrix of A_1, and the number of rows of A_1 is equal to $n+2$. Thus, the number of rows (columns) of the basis matrix B is less than or equal to $n+2$. In case that the number of rows (columns) of B is less than $n+2$, we obtain the desired result that $|\det(B)| = |\det(B')| \leq \alpha_{n+1}$. Consider the case that the basis matrix B has $n+2$ rows. Then, B has a row vector corresponding the equality $\mathbf{1}^\top \boldsymbol{x} - \mathbf{1}^\top \boldsymbol{y} = 0$, which satisfies that each component is either 1 or -1, and thus B' has an all-one row vector. (Because B' is invertible, B' includes exactly one all-one row vector.) Now, we apply the following elementary row operations to B'. For each row vector of B' except a unique all-one row vector, if the first component is equal to one, then we multiply the row by (-1) and add the all-one row vector. Then, the obtained matrix, denoted by B'', is an $(n+2) \times (n+2)$ 0-1 matrix satisfying that $|\det(B)| = |\det(B')| = |\det(B'')|$ and the first column is a unit vector. Thus, it is obvious that $|\det(B'')| \leq \alpha_{n+1}$.

From the above discussion, we obtain the following theorem (without the assumption of monotonicity property (1)).

Theorem 1 *A given simple game $G = (N, \mathcal{W})$ with n players is weighted if and only if G is α_{n+1}-trade robust, where α_n is the maximum of determinants of $n \times n$ 0-1 matrices.*

Proof. If a given simple game is not α_{n+1}-trade robust, then it is not trade robust and thus not weighted, as shown by Taylor and Zwicker [18, 19]. We have discussed the inverse implication that a given simple game $G = (N, \mathcal{W})$ is not weighted. Then, the linear inequality system P1 is infeasible. Farkas' lemma [3] implies that D1 is feasible. From the above discussion, we have a trading transform $(X_1, \ldots, X_j; Y_1, \ldots Y_j)$ satisfying that (1) $j \leq \alpha_{n+1}$, (2) $X_1, \ldots, X_j \in \mathcal{W}$, and (3) $Y = 1, \ldots, Y_j \in \mathcal{L}$. Q.E.D.

The application of Hadamard's evaluation [8] of the determinant leads to the following.

Theorem 2 *For any positive integer n, $\alpha_n \leq (n+1)^{\frac{n+1}{2}}(1/2)^n$.*

The exact values of α_n for small positive integers n appear in "The On-Line Encyclopedia of Integer Sequences A003432 [21]" and Table 1.

In the above discussion, we have shown the following property.

Corollary 1 *If B is an $n \times n$ 0-1 matrix satisfying that $n \geq 2$ and exactly one row vector of B is the all-one vector, then $|\det(B)| \leq \alpha_{n-1}$.*

3 Integer Weights of Weighted Simple Games

In this section, we discuss the integer weight representations of weighted simple games. Throughout this section, we will not need to assume the monotonicity property (1), except in Table 1.

Theorem 3 *For any weighted simple game $G = (N, \mathcal{W})$, there exists an integer vector $\boldsymbol{w} \in \mathbb{Z}^N$ and some $q \in \mathbb{Z}$ satisfying*

(W0) $\forall S \subseteq N,\ \ S \in \mathcal{W}$ *if and only if* $\sum_{x \in S} w(x) \geq q$,

(W1) $|w_i| \leq \alpha_n$ $(\forall i \in N)$, *and*

(W2) $|q| \leq \alpha_{n+1}$,

simultaneously.

Proof. It is easy to show that a given simple game $G = (N, \mathcal{W})$ is weighted if and only if the linear inequality system

$$\text{P2:} \quad \begin{array}{rcl} A(\mathcal{W})\boldsymbol{w} & \geq & q\mathbf{1}, \\ A(\mathcal{L})\boldsymbol{w} & \leq & q\mathbf{1} - \mathbf{1}, \end{array}$$

is feasible. We define

$$A_2 = \begin{pmatrix} A(\mathcal{W}) & \mathbf{1} \\ -A(\mathcal{L}) & -\mathbf{1} \end{pmatrix}, \quad \boldsymbol{v} = \begin{pmatrix} \boldsymbol{w} \\ -q \end{pmatrix}, \quad \boldsymbol{d} = \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix},$$

and simply denote the inequality system P2 by $A_2\boldsymbol{v} \geq \boldsymbol{d}$. In the following, we discuss the case that $A_2\boldsymbol{v} \geq \boldsymbol{d}$ is feasible.

Given a pair (I, J) of subsets of row indices and column indices of the matrix A_2, $A_2(I, J)$ denotes a submatrix of A_2 indexed by $I \times J$, $A_2(*, J)$ denotes a column submatirx of A_2 consisiting of column vectors of A_2 indexed by J, and $\boldsymbol{d}_I$ denotes a subvector of $\boldsymbol{d}$ indexed by I. When the inequality system $A_2\boldsymbol{v} \geq \boldsymbol{d}$ is feasible, it is wellknown that there exists a basic feasible solution, i.e., there exists a pair (I, J) of subsets of row and column indices of matrix A_2 satisfying that (1) $A_2(I, J)$ is a square invertible matrix, and (2) a vector $A_2(I, J)^{-1}\boldsymbol{d}_I$ satisfies $A_2(*, J)A_2(I, J)^{-1}\boldsymbol{d}_I \geq \boldsymbol{d}$. We introduce a subvector $\boldsymbol{v}_J$ ($\boldsymbol{v}_{\overline{J}}$) of $\boldsymbol{v}$ consisting of components indexed by indices in J (not in J). In the remainder of this proof, we denote $A_2(I, J)$ by B for simplicity. Then, we have a basic feasible solution

$$\boldsymbol{v}^* = \begin{pmatrix} \boldsymbol{v}^*_J \\ \boldsymbol{v}^*_{\overline{J}} \end{pmatrix} = \begin{pmatrix} B^{-1}\boldsymbol{d}_I \\ \mathbf{0} \end{pmatrix}$$

of the inequality system $A_2\boldsymbol{v} \geq \boldsymbol{d}$. By Cramer's rule, $v^*_j = \det(B_j)/\det(B)$ for each $j \in J$ where B_j is obtained from B with the column indexed by j replaced by $\boldsymbol{d}_I$. Because B_j is an integer matrix, $\det(B)v^*_j = \det(B_j)$ is an integer for any $j \in J$. Thus, $|\det(B)|\boldsymbol{v}^*$ is an integer vector satisfying $A_2|\det(B)|\boldsymbol{v}^* = |\det(B)|A_2\boldsymbol{v}^* \geq |\det(B)|\boldsymbol{d} \geq \boldsymbol{d}$.

For each $j \in J$, we apply the following elementary matrix operations to B_j. First, we multiply the j-th column of B_j (which is equal to $\boldsymbol{d}_I$) by (-1). Next, we multiply every row vector of the obtained matrix by (-1), if it is indexed by a losing coalition. Then, the obtained matrix, denoted by B'_j, is 0-1 valued and satisfies $|\det(B_j)| = |\det(B'_j)|$. If we denote the number of rows (columns) of B by n', it is obvious that $n' \leq n + 1$ and thus

$$||\det(B)|v^*_j| = |\det(B)v^*_j| = |\det(B_j)| = |\det(B'_j)| \leq \alpha_{n'} \leq \alpha_{n+1} \ \ (\forall j \in J).$$

Because $\boldsymbol{v}^*_{\overline{J}} = \mathbf{0}$, we obtain that

$$-\alpha_{n+1}\mathbf{1} \leq -\alpha_{n'}\mathbf{1} \leq |\det(B)|\boldsymbol{v}^* \leq \alpha_{n'}\mathbf{1} \leq \alpha_{n+1}\mathbf{1}. \tag{3}$$

Let $\begin{pmatrix} \boldsymbol{w}^* \\ -q^* \end{pmatrix}$ be an integer vector corresponding to $|\det(B)|\boldsymbol{v}^*$. Because $|\det(B)|\boldsymbol{v}^*$ is a feasible solution of $A_2\boldsymbol{v} \geq \boldsymbol{d}$, the pair $\boldsymbol{w}^*$ and q^* is feasible for P2, and satisfies property (W0). The inequalities in (3) directly imply that $|q^*| \leq \alpha_{n+1}$, and thus the property (W2) holds. When n', the number of rows (columns) of B, is less than or equal to n, the inequalities in (3) imply that $|w_i^*| \leq \alpha_n$ $(\forall i \in N)$. Finally, we demonstrate the property (W1) in the case that $n' = n+1$. In this case, B is a row submatrix of A_2, and thus contains a column corresponding to original variable $-q$, whose entries consist of 1 or -1. We only need to consider the case that the column index $j \in J$ corresponds to a player $i \in N$, which implies that the index j does not correspond to the original variable $-q$, and thus the matrix B_j includes a column whose entries consisting of 1 or -1. Then, the matrix B'_j contains a column vector that is equal to the all-one vector. Corollary 1 implies that

$$|w_i^*| = ||\det(B)|v_j^*| = |\det(B_j)| = |\det(B'_j)| = |\det((B'_j)^\top)| \leq \alpha_{n'-1} = \alpha_n$$

and thus we obtain the desired result. QED

Kurz [12] exhaustively generated all weighted voting games satisfying the monotonicity property (1) for up to nine voters. Table 1 shows maxima of the exact values of minimal integer weights representations obtained in Kurz [12] and our upper bounds. The table shows that our bounds are tight when $n \leq 5$.

Table 1: Exact values of integer weights representations.

n	1	2	3	4	5	6	7	8	9	10	11
α_n [21]	1	1	2	3	5	9	32	56	144	320	1458
$\max_{(N,\mathcal{W})} \min_{[q;w]} \max_i w_i$ [12]	1	1	2	3	5	9	18	42	110		
Our bound (α_n)	1	1	2	3	5	9	32	56	144	320	1458
$\max_{(N,\mathcal{W})} \min_{[q;w]} q$ [12]	1	2	3	5	9	18	40	105	295		
Our bound (α_{n+1})	1	2	3	5	9	32	56	144	320	1458	

4 Approximation Algorithms

This section deals with the problem of finding integer weights representations. Throughout this section, we assume the monotonicity property (1). In this section, a weighted simple game is given by a triplet $(N, \mathcal{W}^{\mathrm{m}}, \mathcal{L}^{\mathrm{M}})$ where $\mathcal{W}^{\mathrm{m}}$ and $\mathcal{L}^{\mathrm{M}}$ denote the set of minimal winning coalitions and the set of maximal losing coalitions, respectively. We also assume that the empty set is a losing coalition, the set of all players N is a winning coalition, and every player in N is not a null player, and thus there exists an integer weights representation satisfying that $q \geq 1$ and $w_i \geq 1$ $(\forall i \in N)$.

In this section, we discuss the following three problems;

$$\begin{aligned}
\mathrm{P}_q\text{:} \quad & \text{minimize} && q \\
& \text{subject to} && \sum_{i \in S} w_i \geq q \quad (\forall S \in \mathcal{W}^{\mathrm{m}}), && (4) \\
& && \sum_{i \in S} w_i \leq q - 1 \quad (\forall S \in \mathcal{L}^{\mathrm{M}}), && (5) \\
& && q \geq 1, \quad w_i \geq 1 \quad (\forall i \in N), && (6) \\
& && q \in \mathbb{Z}, \quad w_i \in \mathbb{Z} \quad (\forall i \in N), && (7)
\end{aligned}$$

$$\mathrm{P}_w:\quad \text{minimize } w \text{ subject to } (4),(5),(6),(7),\ \ w \geq w_i\ \ (\forall i \in N),$$
$$\mathrm{P}_\Sigma:\quad \text{minimize } \sum_{i\in N} w_i \text{ subject to } (4),(5),(6),(7).$$

Let P be a problem in $\{\mathrm{P}_q, \mathrm{P}_w, \mathrm{P}_\Sigma\}$. A linear relaxation problem $\overline{\mathrm{P}}$ is obtained from P by dropping the integer constraints (7). The optimal values of P and $\overline{\mathrm{P}}$ are denoted by $z(\mathrm{P})$ and $z(\overline{\mathrm{P}})$, respectively.

Our proof in the previous section provides a method for generating an integer weights representation of P whose objective value is equal to $|\det(B)|z(\overline{\mathrm{P}})$ where B is an optimal base matrix of linear relaxation problem $\overline{\mathrm{P}}$. When $|\det(B)| > n$, there exists a simple method for generating a smaller integer weights representation. Let the pair $(q^*; \boldsymbol{w}^*)$ be an optimal solution of $\overline{\mathrm{P}}$. For any weight vector $\boldsymbol{w} = (w_1, w_2, \ldots, w_n)^\top$, we denote the integer vector $(\lfloor w_1 \rfloor, \lfloor w_2 \rfloor, \ldots, \lfloor w_n \rfloor)^\top$ by $\lfloor \boldsymbol{w} \rfloor$. We introduce an integer vector $\boldsymbol{w}' = \lfloor n\boldsymbol{w}^* \rfloor$ and an integer $q' = \lfloor n(q^* - 1) \rfloor + 1$. For any minimal winning coalition $S \in \mathcal{W}^{\mathrm{m}}$, we have that

$$\begin{aligned}\sum_{i\in S} w'_i &> \sum_{i\in S}(nw^*_i - 1) = n\sum_{i\in S} w^*_i - n \geq nq^* - n = n(q^*-1) \geq \lfloor n(q^*-1) \rfloor \\ \sum_{i\in S} w'_i &\geq \lfloor n(q^*-1) \rfloor + 1 = q'.\end{aligned}$$

Each maximal losing coalition $S \in \mathcal{L}^{\mathrm{M}}$ satisfies that

$$\begin{aligned}\sum_{i\in S} w'_i &\leq \sum_{i\in S} nw^*_i \leq n(q^*-1) \\ \sum_{i\in S} w'_i &\leq \lfloor n(q^*-1) \rfloor = q' - 1.\end{aligned}$$

Thus, the pair $\boldsymbol{w}'$ and q' gives an integer weights representation and the corresponding objective value is less than or equal to $nz(\overline{\mathrm{P}})$.

In the remainder of this section, we show that there exists an integer weights representation whose objective value is less than or equal to $((2-\sqrt{2})n + (\sqrt{2}-1))z(\overline{\mathrm{P}})$. A feasible solution of P is called a *θ-approximate solution* when the corresponding objective function value is less than or equal to $\theta z(\mathrm{P})$. Since $nz(\overline{\mathrm{P}}) \leq nz(\mathrm{P})$, the above procedure gives an n-approximate solution for each problem P in $\{\mathrm{P}_q, \mathrm{P}_w, \mathrm{P}_\Sigma\}$. In the following, we improve the approximation ratio from n to $(2-\sqrt{2})n + (\sqrt{2}-1) < 0.5858n + 0.4143$.

Theorem 4 *For each problem* P *in* $\{\mathrm{P}_q, \mathrm{P}_w, \mathrm{P}_\Sigma\}$*, we have the following. Let* $(q^*; \boldsymbol{w}^*)$ *be an optimal solution of* $\overline{\mathrm{P}}$*. We define* $\ell_1 = (2-\sqrt{2})n - (\sqrt{2}-1)$ *and* $u_1 = (2-\sqrt{2})n + (\sqrt{2}-1)$*. Then, there exists a real number* $\lambda^\bullet \in [\ell_1, u_1]$ *satisfying that the pair* $Q = \lfloor \lambda^\bullet(q^*-1) \rfloor + 1$ *and* $\boldsymbol{W} = \lfloor \lambda^\bullet \boldsymbol{w}^* \rfloor$ *gives an integer weights representation satisfying* $(Q, \boldsymbol{W}^\top) \leq u_1(q^*, \boldsymbol{w}^{*\top})$.

Proof. For any positive real λ, it is easy to see that each maximal losing coalition $S \in \mathcal{L}^{\mathrm{M}}$ satisfies that

$$\begin{aligned}\sum_{i\in S} \lfloor \lambda w^*_i \rfloor &\leq \sum_{i\in S} \lambda w^*_i \leq \lambda(q^*-1) \\ \sum_{i\in S} \lfloor \lambda w^*_i \rfloor &\leq \lfloor \lambda(q^*-1) \rfloor \end{aligned} \qquad (8)$$

We introduce a function $g(\lambda) = \lambda - \sum_{i\in N}(\lambda w^*_i - \lfloor \lambda w^*_i \rfloor)$. In the latter part of this proof, we will show that if we choose $\Lambda \in [\ell_1, u_1]$ uniformly at random, then $\mathrm{E}[g(\Lambda)] \geq 0$. This implies that $\exists\lambda^\bullet \in [\ell_1, u_1]$ satisfying $g(\lambda^\bullet) > 0$, because $g(\lambda)$ is not a constant function. Then, for any minimal winning coalition $S \in \mathcal{W}^{\mathrm{m}}$, we have that

$$
\begin{aligned}
\lambda^\bullet &> \sum_{i\in N}(\lambda^\bullet w_i^* - \lfloor \lambda^\bullet w_i^* \rfloor) \geq \sum_{i\in S}(\lambda^\bullet w_i^* - \lfloor \lambda^\bullet w_i^* \rfloor) \\
\sum_{i\in S} \lfloor \lambda^\bullet w_i^* \rfloor &> \sum_{i\in S} \lambda^\bullet w_i^* - \lambda^\bullet = \lambda^\bullet \left(\sum_{i\in S} w_i^* - 1 \right) \geq \lambda^\bullet (q^* - 1) \geq \lfloor \lambda^\bullet (q^* - 1) \rfloor \\
\sum_{i\in S} \lfloor \lambda^\bullet w_i^* \rfloor &\geq \lfloor \lambda^\bullet (q^* - 1) \rfloor + 1.
\end{aligned}
$$

Finally, we show that $\mathrm{E}[g(\Lambda)] \geq 0$. It is obvious that

$$
\begin{aligned}
\mathrm{E}[g(\Lambda)] &= \mathrm{E}[\Lambda] - \sum_{i\in N} \mathrm{E}[(\Lambda w_i^* - \lfloor \Lambda w_i^* \rfloor)] = \frac{\ell_1 + u_1}{2} - \sum_{i\in N} \frac{\displaystyle\int_{\ell_1}^{u_1} (\lambda w_i^* - \lfloor \lambda w_i^* \rfloor) \mathrm{d}\lambda}{u_1 - \ell_1} \\
&= (2 - \sqrt{2})n - \sum_{i\in N} \frac{\displaystyle\int_{\ell_1}^{u_1} (\lambda w_i^* - \lfloor \lambda w_i^* \rfloor) \mathrm{d}\lambda}{u_1 - \ell_1}.
\end{aligned}
$$

Now we discuss the last term appearing above. By substituting μ for λw_i^*, we obtain that

$$
\frac{\displaystyle\int_{\ell_1}^{u_1} (\lambda w_i^* - \lfloor \lambda w_i^* \rfloor) \mathrm{d}\lambda}{u_1 - \ell_1} = \frac{\displaystyle\int_{\ell_1 w_i^*}^{u_1 w_i^*} (\mu - \lfloor \mu \rfloor) \mathrm{d}\mu}{w_i^*(u_1 - \ell_1)} \leq \frac{\displaystyle\int_{-w_i^*(u_1-\ell_1)}^{0} (\mu - \lfloor \mu \rfloor) \mathrm{d}\mu}{w_i^*(u_1 - \ell_1)} = \frac{\displaystyle\int_{-x}^{0} (\mu - \lfloor \mu \rfloor) \mathrm{d}\mu}{x}
$$

where the last equality in obtained by setting $x = w_i^*(u_1 - \ell_1)$. Since $u_1 - \ell_1 = 2(\sqrt{2} - 1)$ and $w_i^* \geq 1$, it is clear that $x = w_i^*(u_1 - \ell_1) \geq 2(\sqrt{2} - 1)$. Here we introduce a function $f(x) = \frac{\int_{-x}^{0} (\mu - \lfloor \mu \rfloor) \mathrm{d}\mu}{x}$. By numerical calculations (see Figure 1), the inequality $x \geq 2(\sqrt{2} - 1)$ implies that $f(x) \leq 2 - \sqrt{2}$.

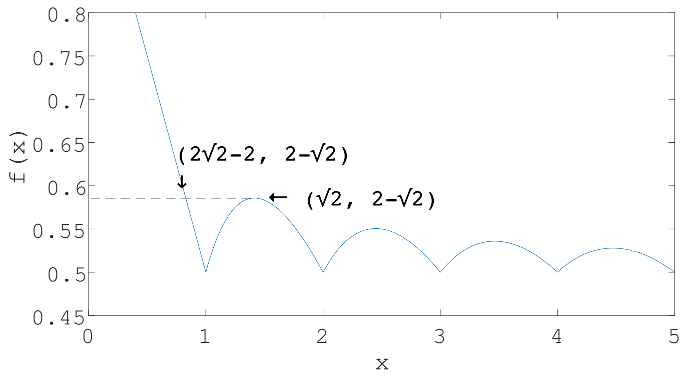

Figure 1: picture

From the above, we obtained the desired result

$$\mathrm{E}[g(\Lambda)] \geq (2-\sqrt{2})n - \sum_{i\in N}(2-\sqrt{2}) = (2-\sqrt{2})n - (2-\sqrt{2})n = 0.$$

QED

Next, we discuss algorithmic aspects of our result. For any positive real $\varepsilon > 0$ and positive integer M, we introduce a randomized rounding procedure:

<u>RR(ε, M)</u>

Step 0: Set $m := 1$.

Step 1: Choose $\Lambda_m \in [\ell_1+\varepsilon, u_1+\varepsilon]$ uniformly at random.
If $g(\Lambda_m) > 0$ then output $Q = \lfloor \Lambda_m(q^*-1)\rfloor + 1$, $\boldsymbol{W} = \lfloor \Lambda_m \boldsymbol{w^*} \rfloor$ and stop.

Step 2: If $m = M$ then output "Fail" and stop. Else update $m := m+1$ and go to Step 1.

Theorem 5 *For any $\varepsilon > 0$ and $0 < \delta < 1$, we have the following. If we set $M \geq \dfrac{(2(\sqrt{2}-1)+n)^2 \ln(1/\delta)}{2\varepsilon^2}$ then* RR(ε, M) *outputs an integer weights representation $(Q, \boldsymbol{W}^\top) \leq (u_1+\varepsilon)(q^*, \boldsymbol{w}^{*\top})$ with a probability greater than or equal to $1-\delta$.*

Proof. It is obvious that if RR(ε, M) outputs a pair $(Q, \boldsymbol{W}^\top)$, then the pair gives an integer weights representation satisfying $(Q, \boldsymbol{W}^\top) \leq (u_1+\varepsilon)(q^*, \boldsymbol{w}^{*\top})$. Similarly to the proof of Theorem 4, the properties $w_i \geq 1$ $(\forall i \in N)$ and $(u_1+\varepsilon)-(\ell_1+\varepsilon) = 2(1-\sqrt{2})$ imply that if we choose $\Lambda \in [\ell_1+\varepsilon, u_1+\varepsilon]$ uniformly at random, then

$$\mathrm{E}[g(\Lambda)] = \mathrm{E}[\Lambda] - \sum_{i\in N} \frac{\int_{\ell_1+\varepsilon}^{u_1+\varepsilon} (\lambda w - \lfloor \lambda w \rfloor)\mathrm{d}\lambda}{u_1-\ell_1} \geq (2-\sqrt{2})n + \varepsilon - (2-\sqrt{2})n = \varepsilon.$$

First, we define a random variable $X_m = -g(\Lambda_m)$ at Step 1. Since $\Lambda_m \in [\ell_1+\varepsilon, u_1+\varepsilon]$ and $g(\lambda) = \lambda - \sum_{i\in N}(\lambda w_i^* - \lfloor \lambda w_i^* \rfloor)$, it is clear that $-(u_1+\varepsilon) \leq X_m \leq -(l_1+\varepsilon-n)$ $(\forall m)$. If algorithm RR(ε, M) fails then $g(\Lambda_m) \leq 0$ $(\forall m)$, and thus $S_M = X_1 + X_2 + \cdots + X_M \geq 0$. By employing Hoeffding's inequality, we obtain an upper bound of the failure probability of algorithm RR(ε, M) as follows;

$$\begin{aligned}
\Pr[\forall m, g(\Lambda_m) \leq 0] &\leq \Pr[S_M \geq 0] = \Pr[S_M - \mathrm{E}[S_M] \geq -\mathrm{E}[S_M]] \\
&= \Pr\left[S_M - \mathrm{E}[S_M] \geq -\sum_{m=1}^{M} \mathrm{E}[-g(\Lambda_m)]\right] \leq \Pr\left[S_M - \mathrm{E}[S_M] \geq M\varepsilon\right] \\
&\leq \exp\left(-\frac{2(M\varepsilon)^2}{M((u_1+\varepsilon)-(\ell_1+\varepsilon-n))^2}\right) = \exp\left(-\frac{2M\varepsilon^2}{(2(\sqrt{2}-1)+n)^2}\right).
\end{aligned}$$

By setting $M \geq \dfrac{(2(\sqrt{2}-1)+n)^2 \ln(1/\delta)}{2\varepsilon^2}$, we obtain the desired result. QED

From theoretical point of view, a computer does not actually sample uniformly from an interval of real numbers. Thus, we need a polynomial time deterministic version of our algorithm. However, the existence of derandomization technique for our algorithm remains open.

5 Conclusion

In this paper, we discussed the smallest value k^*, such that every k^*-trade robust simple game to be weighted. We provided a new proof of the existence of a trading transform when a given simple game is non-weighted. Our proof yields an improved upper bound on the required length of a trading transform. More precisely, we showed that $k^* \leq (n+2)^{\frac{n+2}{2}}(1/2)^{(n+1)}$, which improves the existing bound $k^* \leq (n+1)n^{n/2}$ obtained by Gvozdeva and Slinko [7].

Next, we discussed upper bounds for the maximum possible integer weights and quota needed to represent any weighted simple game on n players. We obtained upper bounds based on the maximal value of determinants of $n \times n$ 0-1 matrices and demonstrate the tightness of our bounds when $n \leq 5$.

Lastly, we described a simple n-approximation algorithm for generating a minimal integer weights representation. We also proposed a probabilistic algorithm which improves the approximation ratio from n to $(2-\sqrt{2})n + (\sqrt{2}-1) < 0.5858n + 0.4143$.

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