Broad Support in Two-Person Elections

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Abstract

Suppose that an award is to be given to one of two candidates by a group of voters from several different departments of an organization. The usual majoritarian model is too simplistic even for this election, because there are recognized subgroups of the electorate whose (joint) opinions may also be deemed relevant. For instance, the majority-losing candidate may receive more votes from all but one department, which votes unanimously for the other candidate.

The relevance of this hypothetical situation to the Brexit decision or the US Electoral College should be clear. However, despite voluminous literature on these complex aggregation issues, to our knowledge no one has studied systematically the *simplest* cases possible, such as with two candidates or three subgroups.

In our paper (of which this document is a greatly extended abstract), we investigate this question of *broad support* to better understand the sorts of behavior that may occur. We examine a quota-type system and a runoff-type system, looking at everything from consistency paradoxes to resoluteness. As an example of the kind of results we will present, in the simplest possible runoff system case, in two different impartiality cultures (with the usual qualifications of a quasi-polynomial and n large), 1/8 of the profiles will violate majority rule.

1 Introduction

Even in large countries, some important political decisions are made not by a representative body but by a direct vote. This notably does not include voting for the chief executive (president or prime minister, depending) of most Western democracies, but does include many referenda and votes for some heads of state. Two recent contentious examples as of the writing of this paper include the so-called 'Brexit' vote for the United Kingdom to leave the European Union and the runoff election for the presidency of France which Emanuel Macron won.

This paper considers a specific issue within such elections. Although we will frame the problem in terms of any finite number of alternatives, many of the most important decisions (such as those mentioned above) are between just two options and we will focus on this case. Given its ubiquity in legislative proceedings and voter initiatives, there is a large literature on such 'referendum' elections.

To make things concrete (and apolitical in the usual sense), consider the following example. Suppose that an academic division is tasked with giving a yearly award to one of two nominated students. Under the usual model, if one asks for Pareto efficiency along with anonymity and neutrality, May's Theorem tells us that the majoritarian system is the most 'reasonable' one for two candidates. Presumably, this is how it is usually implemented.

However, this necessarily runs into a problem of the 'tyranny of the majority'. Namely, such an academic division may be comprised of several independent departments. It is very easy to construct an example where the majority-losing candidate receives more votes from all but one department, which votes unanimously for the other candidate. In this case, which criterion should be deemed more important – that a winning candidate has 'broad appeal' among departments, or that she received a numerical majority? Suppose the 'voting' was actually aggregation by grade point average; a student who has been quite successful in most subjects would perhaps be considered a better choice for a divisional award than one who is better in just one subject, even if by a wide margin.

That this is not an idle question can be observed from many real-life examples. In the Brexit example, two of the constituent countries (Scotland and Northern Ireland) of the United Kingdom (as well as Greater London) voted strongly against the result; they are fairly small in population, and so could not affect the outcome, but many observers took this as a lack of support across the entire UK. As a result, there was substantial discussion of whether Scotland should reopen the issue of leaving the UK in order to remain with the EU. In a similar example, the nomination for Republican candidate for president of the USA must not just get the most delegates, but also win at least ten states outright.

In this paper, we consider several alternatives to strict majoritarian vote which explicitly recognize the various subgroups of the electorate whose (joint) opinions may also be deemed relevant. We will consider this both in cases where the subgroups are of equal size, as well as in cases where this is not at all so. Section 3 considers a runoff-type system which (at least with an odd number of voters) is resolute. Section 4 examines at some length a *non*-resolute family of systems involving quotas.

We wish to stress the point that one can ask all these questions with more than two candidates. For instance, in our story a very natural situation is that each department nominates a candidate, so that there are as many candidates as subgroups. Although in the French system the runoff will only ever have two alternatives, that is (as with many regional elections) only after a wide general election.

However, in focusing on the two-candidate situation, we have several aims. First, we hope to keep a clean exposition. Second, we hope to convince the reader of the richness of just this case. It will soon become evident that there are computable tradeoffs, and no clean line between 'acceptable' alternative measures; all will depend on the willingness of a particular group to adopt a procedure with different levels of violating the majority criterion. A final goal is to emphasize that it is not the number of alternatives per se, but that when a combination of criteria lead to at least three or four categories, we encounter paradox.

Finally, we would like to thank the Gordon College Provost's Office for the Summer Research Fellowship which made the REU experience possible which began this research. Special thanks go to Tommy Ratliff for giving a hint which led us to the literature viewing these questions as simple games within the economic utility literature.

2 Background, Notation, and Basics

Some of our results concerning 'how bad it can get' vis-a-vis majority rule may be interpreted in a framework of utilitarian theory for take-it-or-leave-it' decisions. (See e.g. [3].) The well-known square root rules may be approached as maximizing either utility or egalitarian considerations in the situation where one considers a 'committee of committees' or other such representative bodies. More generally, such settings may attempt to say something about 'voting power' (such as with the Banzhaf measure).

This paper may be intepreted within such a framework by setting all utilities equal, but is intended as being the two-candidate introduction to the more general question of voting in such situations. We are not trying to improve upon weighted quota rules for differently sized groups, but to investigate *other* rules for making such decisions.

Instead, this topic of 'broad support' is inspired by, and somewhat analogous to, the question of electing 'diverse committees' in several papers of Ratliff and his collaborators (see e.g. [1, 2]). There, the initial question regarded an electorate which was unable to achieve the diversity present in the pool of *candidates*, and how to design methods which might take that into account. Here, the diversity is in the pool of *voters*, and we begin investigating methods which take this into account.

We will denote the alternatives for our decision (again, such as an award) by $A, B, C, \ldots$, the set of our m voters by $M = \{1, 2, \ldots, m\}$, and the relevant subgroups of M as $G_1, G_2, \ldots, G_n$, where the $\{G_i\}_{i=1}^n$ form a partition of M with n subsets. Since the total number of voters $m = |M|$, we likewise set $m_i = |G_i|$, so that $\sum_{i=1}^n m_i = m$.

Most importantly, we set the number of (total) votes for each candidate $A, B, \ldots$ as $a, b, \ldots$, and so forth (for most of this paper, there will just be a and b). Then *within* each group G_i we will let a_i and b_i, respectively, be the number of votes received from group G_i by candidates A and B. Thus $\sum_{i=1}^n a_i = a$ and $\sum_{i=1}^n b_i = b$, while $a_i + b_i + \cdots = m_i$.

We say a *profile* for our situation is an element $p \in \{A, B\}^m$, where p_i is the choice of preferred candidate for voter i. Ordinarily of course one allows weak or strong orderings such as $A \sim B \succ C$ or $A \succ B \succ C$ in the domain (e.g. $\mathcal{L}(A)^m$) for social choice functions, but for simplicity we make this assumption.

We then say a *voting procedure* is a function $F : \{A, B\}^m \longrightarrow P(\{A, B\})$. Note that we explicitly allow *both* or *neither* candidate to win. There are several possible interpretations of this; this could include a lottery in the case of a tie, but in the case of an award the interpretation of a shared award or 'no award this year' seem very reasonable and consistent with real-life awards[1]. By contrast, a *resolute* procedure is a function with codomain $\{\{A\}, \{B\}\}$. With an even number of voters, majority is not a resolute procedure, but again one may posit a lottery (or even coin flip, as in some jurisdictions in the USA) in the event of a tie with a non-resolute procedure.

Definition 1. The *majority* procedure F_M is the procedure such that

$$F_M(p) = \begin{cases} \{A\} & a > b \\ \{B\} & b > a \\ \{A, B\} & a = b \end{cases}$$

Notice that this procedure is resolute precisely when m is odd.

Definition 2. Given a profile p and a procedure F, we say that F *violates majority* for p if $F(p) \neq F_M(p)$. We say that F *weakly violates majority* for p if it violates majority, but $F(p) \subset F_M(p)$ or $F_M(p) \subset F(p)$.

Definition 3. Given a profile p and a procedure F, define the set of 'winners' $W_F(p) = \{i \mid p_i \in F(p)\}$; the set of voters agreeing with the majority is then $W_{F_M}(p)$, or $W_M(p)$ when there is no confusion with the set of voters M. We now follow in naming the *majority deficit*

$$D_F(p) = W_M(p) - W_F(p)$$

That is 'the size of the majority camp minus the number of voters who agree with the outcome'.

The majority deficit is always nonnegative. Note that because this is not necessarily to be regarded as a simple game, we do not specify a particular distribution yet[2]. We require that the methods we investigate are neutral, but only anonymous 'by subgroup'.

[1] The Nobel Prizes in the sciences seem to practically require multiple winners, but there were some non-war years when each prize was not awarded – most recently in 1972 with the peace prize. On the other hand, "If in any year all the competitors in any category shall fail to gain a majority vote of the Pulitzer Prize Board, the prize or prizes may be withheld," as in our model; in 2012 this happened with two categories, though some older examples of this seem to have been directly connected to controversies. The Hugo awards in science fiction include 'No Award' as a voting option, which is modeled in a different way.

[2] We also would regard the proper extension to more alternatives to be a 'plurality deficit' rather than a majority deficit as such; however, given that most sensible voting procedures already have a 'plurality deficit' seen as a *feature*, such an index would have to be defined and interpreted quite narrowly.

We conclude this section with a few motivating examples to keep in mind. They should make clear why one's philosophy of how important 'broad support' should be would affect whether majority rule is considered to be the obvious choice.

Example 4. Suppose that G_1 is mathematics, G_2 is economics, and G_3 is political science, each with 10 voters (so $m_i = 10$, $m = 30$). If we have $a_1 = a_2 = 7$ and $b_3 = 10$, then by majority B would win the award, sixteen votes to fourteen. Which candidate 'represents' the division better?

Example 5. Now let $m_1 = m_2 = 10$, but $m_3 = 20$ – perhaps in the meantime there was a hiring initiative in political science. This time A does much better in G_1 and G_2, with $a_1 = a_2 = 9$, and even receives a vote $a_3 = 1$. However, B still wins, twenty-one votes to nineteen. Which candidate now 'represents' the division better?

Example 6. Suppose that instead of hiring new political scientists, instead G_4, G_5 as computer science and sociology were brought in, with all $m_i = 10$. Suppose $a_i = 6$ for $1 \leq i \leq 4$, and $b_5 = 10$. Once again by majority B would win the award, twenty-six to twenty-four. Which candidate now 'represents' the division better?

3 A Runoff System

Suppose we have everything defined as in Section 2. Then we define the following runoff system.

Definition 7. For a given profile p, let $\alpha = \{i \mid a_i > b_i\}$ and $\beta = \{i \mid b_i > a_i\}$. Then we define the *two-step majority* procedure by

$$F_{TS}(p) = \begin{cases} \{A\} & |\alpha| > |\beta| \\ \{B\} & |\beta| > |\alpha| \\ \{A\} & |\alpha| = |\beta|, a > b \\ \{B\} & |\alpha| = |\beta|, b > a \\ \{A, B\} & |\alpha| = |\beta|, a = b \end{cases}$$

Two elementary observations should be made right away. First, Examples 4, 5, and 6 are all examples where $F_{TS}(p) \neq F_M(p)$ and so F_{TS} violates majority for these profiles. Secondly, it is not at all necessary that $\alpha + \beta = n$; in fact, it is quite likely that some groups will have a tie outcome and so not influence the decision at all.

Just as with majority, F_{TS} is resolute precisely when the total number of voters is odd; consider the possibilities for the parities of the m_i when m is even. The next result is also not surprising, though it requires slightly more argument.

Proposition 8. *Suppose that $n = 2$. Then $F_{TM} = F_M$.*

Proof. In F_{TM}, the only way for a tie is a tie where $\alpha = \beta$, so we consider these cases first. If $\alpha = \beta = 1$, then this is the same as majority; the only other case is $\alpha = \beta = 0$, which means *both* groups tied, so there is no difference here either. Then suppose without loss of generality then that A is the winner. If $\alpha = 2$ then clearly the outcome is the same as majority. On the other hand it is possible that $\alpha = 1 > \beta = 0$; supposing $a_1 > b_1$ and $a_2 = b_2$, then $a_1 + a_2 > b_1 + b_2$ so again the outcome is the same. □

3.1 Majority Deficit

So far, we have not seen grave differences with F_M, but that isn't true in general.

Example 9. Suppose $n = 3$ and $m_i = 100$ for all i. Let p be a profile such that $a_1 = 80$ and $a_2 = a_3 = 48$; then $F_{TS}(p) = \{B\} \neq \{A\} = F_M(p)$. Further, $a = 176$ and $b = 124$, so that unlike in Examples 4, 5, and 6, the contradiction with majority is rather stark, as $D_F(p) = 52$, which is after all over one-sixth of the voting population. The profile p' where $a_1 = 100$ would be even worse, of course.

Indeed, it is easy to concoct an example with arbitrarily large majority deficit where $F_{TS}(p) = \{B\}$ because the smallest groups all have just one voter; for $n = 3$ as above, just let m_1 be arbitrarily large and unanimously vote for A, and $m_2 = m_3 = 1$. However, given that one's intuition is that this is a fairly unusual situation, we wish to be more precise. Our first result expresses exactly how big $D_F(p)$ can be; we omit the standard proof for this document.

Proposition 10. *Let* $h = \lceil \frac{n}{2} \rceil - 1$. *Order the group sizes* $m_1 \geq m_2 \geq \cdots \geq m_h \geq m_{\lceil \frac{n}{2} \rceil} \geq \cdots \geq m_{n-1} \geq m_n$. *Consider the* $n - h$ *smallest groups* $G_{h+1}, \ldots, G_n$, *with sizes* $m_{h+1}, \ldots m_n$, *and then define*
$e = |\{m_i \text{ even } \mid h+1 \leq i \leq n\}|$. *Then*

$$\max_p D_F(p) = \begin{cases} \sum_{i=1}^{h} m_i - (n-h) - e & n \text{ odd or } e = 0 \\ \sum_{i=1}^{h} m_i - (n-h) - (e-2) & n \text{ even and } e \neq 0 \end{cases}$$

This is a precise but less than informative statement. A common case would be where all of the groups but one have the same size.

Corollary 11. *Let* $\widetilde{m}$ *be the size of all groups but one, which has size* $t\widetilde{m}$ *(for* $1 \leq t < \infty$*). Let* h, e, *and* n *be as above. Then the maximum of* $D_F(p)$ *over all* p *subject to this constraint is given by*

$$\max D_F(p) = \begin{cases} (t+h-1)\widetilde{m} - (n-h) - e & n \text{ odd or } e = 0 \\ (t+h-1)\widetilde{m} - (n-h) - (e-2) & n \text{ even and } e \neq 0 \end{cases}$$

Corollary 12. *In the same situation, the worst possible* proportion $\frac{D_F(p)}{m}$ *of deficit to total voters is given by* $\frac{(t+\lceil \frac{n}{2} \rceil - 1)\widetilde{m} - (n - \lceil \frac{n}{2} \rceil) - e}{\widetilde{m}(t+n-1)}$.

While this can approach a ratio of 100% as noted above, more interesting is that as $\widetilde{m}$ becomes reasonably large compared to n, this is approximately $\frac{t+\lceil \frac{n}{2} \rceil - 1}{t+n-1}$. When all groups are the same size ($t = 1$) this becomes about $\frac{1}{2}$, so that no matter how many groups there are, the majority margin could be as high as 75% to 25% despite the other candidate winning. This is sort of a minimax result, because of course as t gets large compared to n the ratios can climb ever higher – even for $t = 2$, $n = 5$ it reaches 2/3.

Given these results, it is also not surprising that there is a failure aggregation in a different way.

Example 13. Suppose that we have two groups M and M', and $m_1 = m_2 = m_3 = 3 = m'_1 = m'_2 = m'_3$. If we have p such that $a_1 = a_2 = 2$ and $a_3 = 0$, and p' such that $a'_1 = a'_3 = 2$ and $a'_2 = 0$, then clearly $F_{TS}(p) = F_{TS}(p') = \{A\}$. But if we join $M \cup M'$ in such a way that G_i is the union of the respective groups for M and M', then B wins G_2 and G_3 so $F_{TS}(p \cup p') = \{B\}$.

3.2 Asymptotic Results for Different Cultures

Given this bad news, a different question one might wish to ask is what proportion of profiles do, in fact, have a nonzero majority deficit. After all, it is doubtful that even in the case of the winner of an academic prize that one would consider overruling this large of a majority, at least not if the size of the departments approximated e.g. their enrollment.

With this in mind, we wish to calculate what proportion of profiles p might exhibit a nonzero majority deficit, given some assumptions on the probability distribution of profiles. We refer to for the definitions of these distributions. Although this approach has been critiqued for the current purposes this seems to be a quite reasonable first step at determining whether such methods would even be considered acceptable.

Definition 14. Given a total of m distinct voters, in the *Impartial Culture* (IC) we consider that each voter has the same likelihood of choosing any alternative.

(This is the Bernoulli space of since in the two-alternative scenario one may view a choice of candidate as a Bernoulli trial with probability 1/2.)

Definition 15. Given a total number of voters m, in the *Impartial Anonymous Culture* (IAC) we consider each possible voting situation to be equally likely, not considering voters to be distinct.

An example of the difference between these is that with three voters, there would be $\left(\frac{1}{2}\right)^3 = 8$ total IC situations, including three of each two-to-one vote, while in IAC there would only be four total voting situations, corresponding to the possibilities $3-0$, $2-1$, $1-2$, and $0-3$ of $A-B$. So a unanimous win for A would be twice as likely under the IAC.

We will not consider the so-called Impartial Anonymous Neutral Culture for this paper, although with more candidates it would be useful. Instead, we have devised two additional cultures which are relevant in the group setting.

Definition 16. Given a total number of voters m and partition $m_1 \geq \cdots \geq m_n$ where $\sum m_i = m$, in the *Impartial Anonymous Culture for Groups* (IACG) we consider each possible voting situation *within each group* to be equally likely, but do count all possible group permutations.

Example 17. In IACG, imagine that now we have three voters in each of two groups G_1 and G_2. In IACG, we would count both $(a_1, b_1, a_2, b_2) = (3, 0, 2, 1)$ and $(a_1, b_1, a_2, b_2) = (2, 1, 3, 0)$ as situations, but the voters themselves would still be anonymous.

Definition 18. Given a total number of voters m and partition $m_1 \geq \cdots \geq m_n$ where $\sum m_i = m$, in the *Groupwise Impartial Anonymous Culture* (GIAC) we consider each possible voting situation *within each group* to be equally likely, and groups of the same size anonymously as well.

Example 19. Using the same situation as in the previous example, in GIAC we would count $(a_1, b_1, a_2, b_2) = (3, 0, 2, 1)$ and $(a_1, b_1, a_2, b_2) = (2, 1, 3, 0)$ as the same profile.

Example 20. If the three voters in our earlier examples were allocated so that two were in group G_1 and one in G_2, we then have three possible anonymous profiles in G_1, and two in G_2, for a total of six such profiles – one-sixth of which have A winning unanimously. In this case IACG and GIAC are identical cultures.

The question to investigate is how often we have a nonzero majority deficit. For IAC and GIAC it doesn't seem to matter which culture we pick, getting results of the type of Ratliff or Lepelley et al.

Proposition 21. *Suppose that there are two candidates and $n = 3$, $m_i = m'$ for all i. Under the IAC and GIAC, the number of profiles with nonzero majority deficit is a quasi-polynomial in m', and further the* proportion *of such profiles is equal to $\frac{1}{8} + r(m')$, where r is a 'quasi-rational function' (quotient of q-p's).*

For IC and IACG, on the other hand, preliminary numerical results seem to indicate a possible non-polynomial component.

4 A Quota System

A typical way to set up a coalitional voting game is with a quota of some kind. This could involve voters with different weights (such as in the European Union or the US Electoral College) needing to reach a quota, or could be a simple game defined by its winning coalitions but which is *equivalent* to a weighted game, such as the UN Security Council. Many legislatures require supermajorities for passing certain kinds of measures.

In this section, we introduce and analyze a voting procedure in our framework which has a defined quota *for each group.* We then require a winning candidate to satisfy the quota for *all* groups. This family of procedures may be quite non-resolute, but as we are not confining ourselves to merely political situations. An interesting instantiation would be where not only is an award to be given (which may be shared), but that there is a monetary value given to the award, which is *also* shared if there are multiple winners[3] (and which is not given if there are none). We do not explore the game-theoretic implications of this setup.

Definition 22. Suppose we are given a *quota ratio* $0 \leq q \leq 1$. For each group G_i of (integer) size m_i, we call the *group quota* $Q_i = qm_i$; this is not necessarily an integer. We say that a candidate X *meets quota for group* i if $x_i \geq Q_i$. Then for a given profile p, we define the *groupwise quota* procedure $F_q(p)$ by saying that $X \in F_q(p)$ if and only if X meets quota for all groups $1 \leq i \leq n$.

Perhaps surprisingly, there is a richness even in the case of $n = 2$ groups. Here is a simple case.

Example 23. Suppose $m_1 = 21$ and $m_2 = 9$, and $q = 1/3$. Then $Q_1 = 7$ and $Q_2 = 3$.

- If $a_1 = 15$ and $a_2 = 7$, then $b_1 = 6$ and $b_2 = 2$, so only A wins.
- If $a_1 = 9$ and $a_2 = 4$, then $b_1 = 12$ and $b_2 = 4$, so actually both win.
- If $a_1 = 15$ and $a_2 = 2$, then $b_1 = 6$ and $b_2 = 7$, so although A meets quota for G_1 and B meets quota for G_2, neither wins the procedure.

Proposition 24. *If $q \geq 1/2$, then $F_q(p) \subset F_M(p)$. Such an F_q may be viewed as a stronger version of majority rule.*

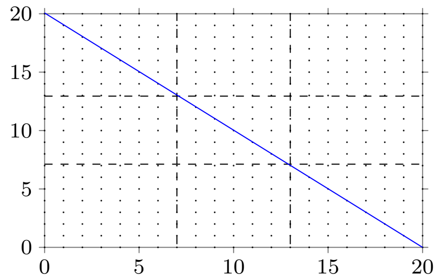
Remark 25. Note that F_q for $q = 1/2$ is not stronger than majority rule if one replaces our definition of F_M with a version that is resolute. For instance, with $n = 2$ and both m_i being even, the tie $a_i = b_i$ for all i yields a tie for $F_{1/2}$.

4.1 Same Size Groups

It is evident that these quota procedures will not necessarily agree with the majority, as there are many opportunities for tie outcomes. To understand the sorts of outcomes that are possible, we will begin with just two groups, where both have size m.

[3]This is exactly how Nobel Prize money is awarded.

Figure 1: Quota $q = 1/3$, $Q = 7$ with two subgroups each at $m = 20$



In Figure 1 we have two axes, representing the voters in each group. The key thing to note is that the point (a_1, a_2) represents exactly those voters selecting alternative A, which means that the profile at (a_1, a_2) has $b_i = m - a_i$. In this case, we have $m = 20$ and $q = 1/3$, so that $Q = 7$ is the quota for both groups. Hence the point $(15, 12)$ represents $b_1 = 5$, $b_2 = 8$, where A is an unchallenged winner although B did meet quota in G_2. The various lines represent when A or B has met quota, and the diagonal line is the majority line, so that points above and to the right of this line have A in the majority.

Note the rotational symmetry. One result that is immediate from the geometry of this situation is that an analogue to Proposition 24 is true when all groups have the same size.

Proposition 26. *If $q < 1/2$, $n = 2$, and all $m_i = m$ are the same, then $F_q(p) \subset F_M(p)$ or $F_M(p) \subset F_q(p)$.*

Remark 27. This is trivially true if one (or both) of the results is a tie, so what makes this interesting is that the concept of majority deficit would not directly apply in this case.

In Example 13 we saw that aggregating two groups with one result does not guarantee that result stays true.

Example 28. Suppose that we have two groups M and M', both with $n = n' = 2$ and $m_i = m'_i = 20$. Suppose further that, as above, $q = 1/3$ and $Q = 7$. Now if we have p with $(a_1, a_2) = (9, 14)$ and p' with $(a'_1, a'_2) = (14, 9)$, then $F_{1/3}(p) = F_{1/3}(p') = \{A\}$. However, if we join $M \cup M'$ in such a way that G_i is the union of the respective groups for M and M', then $F_{1/3}(p \cup p') = \{A, B\}$.

Although from the method it is clear this can happen, still it might be disconcerting to anyone who had done a straw poll among M and M' separately – not to mention to candidate A, who now has to share the award with B!

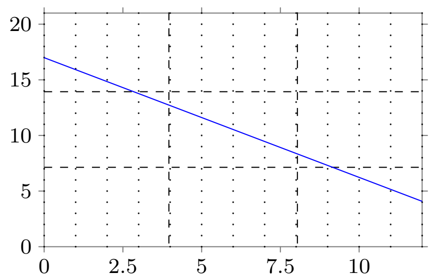
4.2 Different Size Groups

The situation becomes more interesting when $m_1 \neq m_2$. Observe that in Figure 2 there are some new regions in addition to the previous ones, which are also somewhat less symmetric than before.

Example 29. Suppose that $m_1 = 21$ and $m_2 = 12$, but $q = 1/3$ so that $Q_1 = 7$ and $Q_2 = 4$. Then for p given by $(a_1, a_2) = (14, 3)$, we have that $F_{1/3}(p) = \{B\}$ but $F_M(p) = \{A\}$, so there is a nonzero majority deficit.

One can check this is the *only* such profile with these winners.

Figure 2: Quota $q = 1/3$ with $Q_1 = 7$ and $Q_2 = 4$



A casual glance at Figure 2 would make it look as if there were always such a possibility when $m_1 \neq m_2$, since there are regions with a majority deficit; however, the question is which *integer* profiles will have this.

Example 30. Suppose that $m_1 = 21$, $m_2 = 20$, and $q = 1/3$, so that $Q_1 = 7 = Q_2$. In order for there to be a nonzero majority deficit (different winners for the procedures), a profile with $F_{1/3}(p) = \{A\}$ would have to have $b_1, b_2 < 7$, and also $b_1 + b_2 > 20$, which is clearly impossible. In fact, one can easily check that there is no q such that F_q would have this situation, given m_1 and m_2.

Proposition 31. *Suppose we have groups G_1 and G_2 such that $m_1 > m_2$. Let $\Delta m = m_1 - m_2$. The following conditions are necessary for a profile p to have a nonzero majority deficit (with $F_q(p) = \{B\}$):*

1. $a_1 + a_2 > 1/2 \cdot (m_1 + m_2)$
2. $Q_1 \leq a_1 \leq m_1 - Q_1$
3. $a_2 < Q_2$
4. $\Delta m \geq 3$
5. $q < \begin{cases} \frac{1}{2} - \frac{1}{\Delta m} & \Delta m \text{ odd} \\ \frac{1}{2} - \frac{1}{2\Delta m} & \Delta m \text{ even} \end{cases}$

Example 32. Suppose that $m_1 = 21$ and $m_2 = 12$, but $q = 1/3$ so that $Q_1 = 7$ and $Q_2 = 4$. Then for p given by $(a_1, a_2) = (14, 3)$, we have that $F_{1/3}(p) = \{B\}$ but $F_M(p) = \{A\}$, so there is a nonzero majority deficit.

The first three conditions in Proposition 31 are fairly obvious, but the last two are more useful in determining when such a situation could occur. The geometry of the quota situation makes analyzing the majority deficit easier. Both the greatest possible majority deficit and the total number of profiles yielding one have direct geometric interpretations in terms of lattice points. Most interestingly, the *ratio* of the area of the small triangle region to the overall can be analyzed by the same methods using our impartial cultures.

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