Condorcet Domains, Median Graphs and the Single Crossing Property

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Abstract

Condorcet domains are sets of linear orders with the property that, whenever the preferences of all voters belong to this set, the majority relation has no cycles. We observe that, without loss of generality, such a domain can be assumed to be closed in the sense that it contains the majority relation of every profile with an odd number of individuals whose preferences belong to this domain. We show that every closed Condorcet domain is naturally endowed with the structure of a median graph and that, conversely, every median graph is associated with a closed Condorcet domain (which may not be a unique one). Maximality of a Condorcet domain imposes additional restrictions on the underlying median graph. We prove that among all trees only the chains can be induced by maximal Condorcet domains, and we characterise the chains that in fact do correspond to maximal Condorcet domains.

1 Introduction

The problem of finding and characterizing preference domains on which pairwise majority voting never admits cycles—the so-called Condorcet domains—has a long history in social choice theory. In their seminal contributions, Black [5] and Arrow [3] noticed that the domain of all strict orders that are single peaked with respect to some underlying linear spectrum form a Condorcet domain. Later, Sen [28] provided a characterization of Condorcet domains in terms of the well-known condition of value restriction. Since this early work some progress has been made in understanding the structure of Condorcet domains; see [1, 2, 6, 9, 10, 14, 15, 16] for important contributions, and [21] for an excellent survey. However, with the exception of the paper of Danilov and Koshevoy [10], the bulk of the results established in the literature pertain only to the special case of the so-called connected domains.

The present paper provides a unifying general approach by establishing a close connection between Condorcet domains and median graphs (see a comprehensive survey about these in [20]) on the one hand, and the well-studied class of *single crossing* domains on the other hand (see [25, 17, 27], among others).

First, we observe that if one adds to a Condorcet domain of strict orders the (transitive) majority relations of all profiles with an odd number of voters, one obtains again a Condorcet domain. We may thus assume without loss of generality that Condorcet domains are *closed* in the sense that pairwise majority voting among an odd number of individuals always yields an order within the given domain. In particular, all *maximal* Condorcet domains are necessarily closed. Our main result shows that

(i) every closed Condorcet domain on a finite set of alternatives equipped with its natural neighborhood relation[1] is a median graph, and

(ii) for every finite median graph there exists a set of alternatives and a closed Condorcet domain on this set that is isomorphic to the given median graph.

[1]Two linear orders in a domain are neighbors if there is no other linear order in this domain that is between them. An order is *between* two orders if it agrees with all binary comparisons on which the two linear orders agree, cf. [18].

Our analysis is related to prior work by Nehring and Puppe [24], Demange [11] and Clearwater et al. [7]. Nehring and Puppe [24] introduced a general notion of a median space and demonstrate its usefulness in aggregation theory. Demange [11] showed that, if it is possible to assign linear orders of a certain domain to the vertices of a median graph in such a way that orders that lie on a shortest path between two other orders in the graph are also between them in the Kemeny-Snell sense of betweenness ([19]), then the majority relation of any profile with preferences from this domain admits no cycles, and, if the number of voters in the profile is odd, coincides with the preferences of one voter from the domain (but not necessarily participating in the selected profile). However, [11] takes both the median graph and the preference profile as given and does not address the issue under which conditions the required construction can indeed be carried out. Part (i) of our main theorem shows that in fact *every* closed Condorcet domain (with its respective neighborhood relation) is a median graph.[2] Clearwater et al. [7] generalise the concept of classical single crossing domain to single crossing domains on trees and consider several related algorithmic problems.

While all median graphs give rise to closed Condorcet domains, it is not true that every median graph corresponds to a *maximal* Condorcet domain. It turns out that, in fact, certain *types* of median graphs never enable maximality of the respective Condorcet domains. In particular, we prove that among all trees only (some) forkless trees (chains) can be associated with maximal Condorcet domains.

Condorcet domains, whose median graphs are chains, have been studied quite extensively in economics under the name of single-crossing domains. They are characterized by the *single-crossing property which* stipulates that the orders of the domain can be arranged in a chain so that, for any ordered pair of alternatives, the set of all orders that rank one alternative strictly above the other form an interval in this chain. It is well-known that single-crossing domains have the *representative voter property* (cf. [26]), i.e., in any profile with an odd number of voters whose preferences belong to the given domain there is one voter whose preference order coincides with the majority relation.

A maximal single-crossing domain must obviously contain two completely reversed orders. Interestingly, not all maximal single-crossing domains are maximal Condorcet domains, i.e., typically it is possible to add further preference orders to a maximal single-crossing domain without generating cycles in the majority relation of any profile. Here, we provide a simple necessary and sufficient condition of when a maximal single-crossing domain is also a maximal Condorcet domain. The condition requires that the 'switched' pairs of alternatives associated with any two consecutive orders of the domain have one element in common.

The remainder of the paper is organized as follows. In the following Section 2, we introduce the concept of Condorcet domain and observe some of its fundamental properties. In particular, we show that closed Condorcet domains are exactly the median stable subsets of the space of all strict orders on a given set of alternatives. Section 3 introduces median graphs and states our main result establishing the correspondence between closed Condorcet domains and median graphs. Section 4 provides the characterization of single crossing domains, and discusses a weaker version of the single crossing property, namely, single crossingness on trees. Section 5 addresses maximality of Condorcet domains, an issue that has already received attention in the literature [21]. In particular, we prove that trees different from paths are never associated with maximal Condorcet domains, and we characterize the single crossing domains that are maximal Condorcet domains.

[2]Importantly, this median graph is in general *not* a subgraph of the graph corresponding to the natural neighborhood relation on the universal domain of all strict orders, see Section 3 below for detailed explanation.

2 Preliminaries

In this section, we introduce the main notions of this paper: Condorcet domains, median domains and median graphs. We show that the class of domains that are closed under the operation of taking the majority relation for any of their profiles are precisely the median stable subsets of the space of linear orderings endowed with the natural betweenness relation. To the best of our knowledge, the central notion of this paper of a closed Condorcet domain has not yet been formally introduced in the literature. On the other hand, a majority of results in this section are not original. In Theorem 1 we gathered a number of classical characterisations of Condorcet domains and Theorem 2 can be derived from the analysis in Nehring and Puppe [24] who investigated a more general class of median spaces. Nevertheless, we believe that our exposition, and, in particular, the new short proof of the main characterization result, which is Theorem 2 below, will help to clarify and unify several different approaches in the literature.

2.1 Condorcet Domains

Consider a finite set of alternatives X and the set $\mathcal{R}(X)$ of all *linear (strict) orders* (i.e., complete, transitive and antisymmetric binary relations) on X. A subset $\mathcal{D} \subseteq \mathcal{R}(X)$ will be called a *domain of preferences* or simply a *domain*. A *profile* $\rho = (R_1, \dots, R_n)$ *on* $\mathcal{D}$ is an element of the Cartesian product $\mathcal{D}^n$ for some number $n \in \mathbb{N}$ of 'voters', where the linear order R_i represents the preferences of the ith voter over the alternatives from X. A profile with an odd number of voters will simply be referred to as an *odd profile.* Frequently, we will denote linear orders simply by listing the alternatives in the order of decreasing preference, e.g., a linear order that ranks a first, b second, c third, etc., is denoted by $abc\dots$.

The *majority relation* associated with a profile ρ is the binary relation P^{maj}_ρ on X such that $xP^{\mathrm{maj}}_\rho y$ if and only if more than half of the voters rank x above y. Note that, according to this definition, the majority relation is asymmetric and for any odd profile ρ and any two distinct alternatives $x, y \in X$, we have either $xP^{\mathrm{maj}}_\rho y$ or $xP^{\mathrm{maj}}_\rho y$. An asymmetric binary relation P is *acyclic* if there does not exist a subset $\{x_1, \dots, x_m\} \subseteq X$ such that x_1Px_2, x_2Px_3, ... , $x_{m-1}Px_m$ and x_mPx_1. The class of all domains $\mathcal{D} \subseteq \mathcal{R}(X)$ such that, for all n, the majority relation associated with any profile $\rho \in \mathcal{D}^n$ is acyclic has received significant attention in the literature, see the survey of [21] and the references therein. In the following, we will refer to any such domain as a *Condorcet domain.*[3]

The following result prepares the ground for our analysis, providing some well-known characterizations of Condorcet domains (cf. [21, p. 142]). In particular, condition d) below is Sen's 'value restriction'; condition e) has been introduced by [29] as the 'absence of a Latin square' (in other terminology, it requires the absence of a 'Condorcet cycle').

Theorem 1. *Let X be finite, and let $\mathcal{D} \subseteq \mathcal{R}(X)$ be a subset of the space of all linear orders on X. The following statements are equivalent.*

a) $\mathcal{D}$ is a Condorcet domain, i.e., the majority relation corresponding to every profile over $\mathcal{D}$ is acyclic.

b) For every profile over $\mathcal{D}$, the corresponding majority relation is a strict partial order (i.e., irreflexive, transitive and asymmetric binary relation).

c) For every odd profile over $\mathcal{D}$, the corresponding majority relation is a linear order, i.e., an element of $\mathcal{R}(X)$.

[3] Fishburn [14] calls them *acyclic sets of linear orders.*

d) In any triple $x, y, z \in X$ of pairwise distinct alternatives and any restriction of an order from $\mathcal{D}$ to the set $\{x, y, z\}$, there exists one element in $\{x, y, z\}$ that has either never rank 1, or never has rank 2, or never has rank 3.

e) For no triple $R_1, R_2, R_3 \in \mathcal{D}$, and no triple $x, y, z \in X$ of distinct alternatives xR_1yR_1z, yR_2zR_2x and zR_3xR_3y are true simultaneously.

We will say that a Condorcet domain $\mathcal{D}$ is *closed* if the majority relation corresponding to any odd profile over $\mathcal{D}$ is again an element of $\mathcal{D}$, and we will say that a Condorcet domain $\mathcal{D}$ is *maximal* if no Condorcet domain (over the same set of alternatives) is a proper superset of $\mathcal{D}$. The following simple observation will be very useful.

Lemma 2.1. *Let $\mathcal{D}$ be a Condorcet domain and $R \in \mathcal{R}(X)$ be the majority relation corresponding to an odd profile over $\mathcal{D}$. Then $\mathcal{D} \cup \{R\}$ is again a Condorcet domain. In particular, every Condorcet domain is contained in a closed Condorcet domain.*

Proof. By Theorem 1 e), it suffices to show that $\mathcal{D} \cup \{R\}$ does not admit three orders R_1, R_2, R_3 and three elements $x, y, z \in X$ such that xR_1yR_1z, yR_2zR_2x and zR_3xR_3y. Assume on the contrary that it does; then, evidently, not all three orders R_1, R_2, R_3 belong to $\mathcal{D}$. Thus, one of them, say R_3, is the majority relation of an odd profile $\rho \in \mathcal{D}^n$. Consider the profile $\rho' = (nR_1, nR_2, \rho) \in \mathcal{D}^{3n}$ that consists of n voters having the order R_1, n voters having the order R_2 and the n voters of the profile ρ. Then voters of the subprofile (nR_1, nR_2) will unanimously prefer y to z, which forces the majority relation $P^{\mathrm{maj}}_{\rho'}$ corresponding to ρ' to have the same ranking of y and z. At the same time, the voters of this subprofile are evenly split in the ranking of any other pair of alternatives from $\{x, y, z\}$. Hence, the majority relation $P^{\mathrm{maj}}_{\rho'}$ yields the cycle $zP^{\mathrm{maj}}_{\rho'}xP^{\mathrm{maj}}_{\rho'}yP^{\mathrm{maj}}_{\rho'}z$, in contradiction to the assumption that $\mathcal{D}$ is a Condorcet domain. □

This observation allows us to concentrate our attention on closed Condorcet domains without loss of generality, and we do so for the rest of the paper. Note, in particular, that by Lemma 2.1 all maximal Condorcet domains are closed.

2.2 Betweenness and Median Domains

The universal domain $\mathcal{R}(X)$ is naturally endowed with the following betweenness structure. An order Q is *between* orders R and R' if $Q \supseteq R \cap R'$, i.e., Q agrees with all binary comparisons in which R and R' agree ([19]).[4] The set of all orders that are between R and R' is called the *interval* spanned by R and R' and is denoted by $[R, R']$. The domain $\mathcal{R}(X)$ endowed with this betweenness relation is referred to as the *permutahedron*.

A subset $\mathcal{D} \in \mathcal{R}(X)$ of the permutahedron is called *median stable* if, for any triple of elements $R_1, R_2, R_3 \in \mathcal{D}$, there exists an element $R^{\mathrm{med}} = R^{\mathrm{med}}(R_1, R_2, R_3) \in \mathcal{D}$, the *median order* corresponding to R_1, R_2, R_3, such that

$$R^{\mathrm{med}} \in [R_1, R_2] \cap [R_1, R_3] \cap [R_2, R_3].$$

Proposition 2.1. *The median order of a triple $R_1, R_2, R_3 \in \mathcal{R}(X)$, if it exists, is unique.*

Proof. If a triple R_1, R_2, R_3 admits two different median orders, say R and R', these must differ on the ranking of at least one pair of alternatives. Suppose they disagree on the ranking of x and y. In this case, not all three orders of the triple agree on the ranking of x versus y. Hence, exactly two of them, say R_1 and R_2, must agree on the ranking of x versus y; but then, either R or R' is not between R_1 and R_2, a contradiction. □

[4] Some authors such as, e.g., Grandmont [18] and Demange [11] refer to orders that are between two others in this sense as 'intermediate' orders.

In the following, we will refer to median stable subsets of $\mathcal{R}(X)$ as *median domains.* Evidently, not every subset of $\mathcal{R}(X)$ is a median domain; for instance, the universal domain $\mathcal{R}(X)$ itself is not a median domain whenever $|X| \geq 3$. This can be verified by considering any three orders of the form $R_1 = \ldots a \ldots b \ldots c \ldots$, $R_2 = \ldots b \ldots c \ldots a \ldots$, and $R_3 = \ldots c \ldots a \ldots b \ldots$. Since any linear order R in $[R_1, R_3]$ has aRb, any linear order R in $[R_1, R_2]$ has bRc, and any linear order R in $[R_2, R_3]$ has cRa, we obtain $[R_1, R_2] \cap [R_1, R_3] \cap [R_2, R_3] = \emptyset$ due to the transitivity requirement.

Prominent examples of median domains include the well-studied single crossing domains.

Example 1 (Classical single crossing domains). There are several equivalent descriptions of single crossing domains (see, e.g., [17, 27]). The following will be useful for our purpose. A domain $\mathcal{D} \subseteq \mathcal{R}(X)$ is said to have the *single crossing property* if $\mathcal{D}$ can be linearly ordered, say according to $R_1 > R_2 > \ldots > R_m$, so that, for all pairs x, y of distinct elements of X, the sets $\{R_j \in \mathcal{D} \mid xR_jy\}$ and $\{R_j \in \mathcal{D} \mid yR_jx\}$ are connected in the ordering $>$. Thus, for each pair x, y of distinct elements, there is exactly one 'cut-off' order R_k such that either (i) xR_jy for all $j \leq k$ and yR_jx for all $j > k$, or (ii) yR_jx for all $j \leq k$ and xR_jy for all $j > k$. It is easily verified that, for any triple with $R_i > R_j > R_k$ the median order exists and coincides with the middle ordering, i.e., $R^{\text{med}}(R_i, R_j, R_k) = R_j$.

The close connection between Condorcet domains and median domains to be established in Theorem 2 below stems from the following simple but fundamental observation.

Observation 1. *A triple $R_1, R_2, R_3 \in \mathcal{R}(X)$ admits a median order if and only if the majority relation of the profile $\rho = (R_1, R_2, R_3)$ is acyclic, in which case the median order $R^{med}(R_1, R_2, R_3)$ and the majority relation of ρ coincide.*

Proof. If the majority relation P_ρ^{maj} is acyclic, and hence is an element of $\mathcal{R}(X)$, it belongs to each interval $[R_i, R_j]$ for all distinct $i, j \in \{1, 2, 3\}$. Indeed, if both R_i and R_j rank x higher than y, then so does the majority relation. Conversely, if R is the median of the triple R_1, R_2, R_3, then for any pair $x, y \in X$ at least two orders from this triple agree on ranking of x and y. Then R must agree with them, hence it is the majority relation for this triple. □

Corollary 1. *Any closed Condorcet domain is a median domain.*

Proof. Suppose $\mathcal{D}$ is a closed Condorcet domain and let R_1, R_2, R_3 be any triple of orderings from $\mathcal{D}$. The majority relation R corresponding to the profile $(R_1, R_2, R_3) \in \mathcal{D}^3$ by Theorem 1(c) is an element of $\mathcal{R}(X)$, and by the assumed closedness it is in fact an element of $\mathcal{D}$. By Observation 1, R is the median order of the triple R_1, R_2, R_3. □

A subset $\mathcal{C} \subseteq \mathcal{D}$ of a domain $\mathcal{D} \subseteq \mathcal{R}(X)$ will be called *convex* if $\mathcal{C}$ contains with any pair $R, R' \in \mathcal{C}$ the entire interval spanned by R and R', that is, $\mathcal{C}$ is convex if

$$\{R, R'\} \subseteq \mathcal{C} \;\Rightarrow\; [R, R'] \subseteq \mathcal{C}.$$

A family $\mathbb{F}$ of subsets of a set is said to have the *Helly property* if the sets in any subfamily $\mathbb{F}' \subseteq \mathbb{F}$ have a non-empty intersection whenever their pairwise intersections are non-empty, i.e., if $\mathcal{C} \cap \mathcal{C}' \neq \emptyset$ for each pair $\mathcal{C}, \mathcal{C}' \in \mathbb{F}'$ implies $\cap \mathbb{F}' \neq \emptyset$. For us this property will be important when $\mathbb{F}$ is the set of all convex subsets.

Proposition 2.2 (Helly property and median domains). *A domain $\mathcal{D}$ is a median domain if and only if $\mathcal{D}$ has the Helly property for convex subsets of $\mathcal{D}$.*

Proof. Let $\mathcal{D}$ be median domain and $\mathbb{F}$ be a family of convex subsets with pairwise non-empty intersection. We proceed by induction over $m = |\mathbb{F}|$. If $m = 2$, there is nothing to prove, thus let $m = 3$, i.e., $\mathbb{F} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\}$. Choose any ordering $R_1 \in \mathcal{C}_1 \cap \mathcal{C}_2$, $R_2 \in \mathcal{C}_2 \cap \mathcal{C}_3$ and $R_3 \in \mathcal{C}_3 \cap \mathcal{C}_1$, and consider the median order $R = R^{\text{med}}(R_1, R_2, R_3)$. By convexity of the sets $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ we have $R \in \mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{C}_3$ which, in particular, shows that $\cap\mathbb{F}$ is non-empty.

Now consider $\mathbb{F} = \{\mathcal{C}_1, \ldots, \mathcal{C}_m\}$ with $m > 3$ elements, and assume that the assertion holds for all families with less than m elements. Then, the family $\{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \cap \ldots \cap \mathcal{C}_m\}$ constitutes a family of three convex subsets with pairwise non-empty intersections. By the preceding argument, we thus have $\cap\mathbb{F} \neq \emptyset$.

Conversely, consider a domain $\mathcal{D}$ such that any family of convex subsets of $\mathcal{D}$ has the Helly property. Consider any three orders $R_1, R_2, R_3 \in \mathcal{D}$. Since, evidently, all intervals are convex, the Helly property applied to the intervals $[R_1, R_2]$, $[R_1, R_3]$, $[R_2, R_3]$ implies the existence of a median. □

For any domain $\mathcal{D}$ and any pair $x, y \in X$ of alternatives, denote by $\mathcal{V}_{xy}^{\mathcal{D}}$ the set of orders in $\mathcal{D}$ that rank x above y, i.e.,

$$\mathcal{V}_{xy}^{\mathcal{D}} := \{R \in \mathcal{D} \mid xRy\}.$$

Note that, for all distinct $x, y \in X$, the sets $\mathcal{V}_{xy}^{\mathcal{D}}$ and $\mathcal{V}_{yx}^{\mathcal{D}}$ form a partition of $\mathcal{D}$. Also observe that the sets of the form $\mathcal{V}_{xy}^{\mathcal{D}}$ are convex for all pairs $x, y \in X$. We will now use the Helly property applied to this family of convex sets to show that every median domain is a closed Condorcet domain. The following is the main result of this section.

Theorem 2. *The classes of median domains and closed Condorcet domains coincide, i.e., a domain is a median domain if and only if it is a closed Condorcet domain.*

Proof. In the light of Corollary 1, it suffices to show that every median domain is a closed Condorcet domain. Thus, let $\mathcal{D}$ be a median domain and consider an odd profile $\rho = (R_1, \ldots, R_n) \in \mathcal{D}^n$. For any two alternatives $x, y \in X$, let $\mathcal{U}_{xy} = \{R_i \mid xR_iy\}$, and observe that obviously, $\mathcal{U}_{xy} \subseteq \mathcal{V}_{xy}^{\mathcal{D}}$. Let z, w also be alternatives in X, not necessarily distinct from x and y. If $xP_\rho^{\text{maj}}y$ and $zP_\rho^{\text{maj}}w$, then $\mathcal{U}_{xy} \cap \mathcal{U}_{zw} \neq \emptyset$ and hence $\mathcal{V}_{xy}^{\mathcal{D}} \cap \mathcal{V}_{zw}^{\mathcal{D}} \neq \emptyset$. By Proposition 2.2 we have

$$\bigcap_{xP_\rho^{\text{maj}}y} \mathcal{V}_{xy}^{\mathcal{D}} \neq \emptyset,$$

hence there is a linear order in $\mathcal{D}$ that coincides with the majority relation of ρ.

2.3 Median Graphs

The term median graph was coined by Nebesky [23]. For a comprehensive survey on median graphs see [20].

Let $\Gamma = (V, E)$ be a connected graph. The *distance* $d(u, v)$ between two vertices $u, v \in V$ is the smallest number of edges that a path connecting u and v may contain. While the distance is uniquely defined, there may be several shortest paths from u to v. We say that a vertex w is *geodesically between* the vertices u and v if w lies on a shortest path that connects u and v or, which is the same, $d(u, v) = d(u, w) + d(w, v)$. A *(geodesically) convex* set in a graph $\Gamma = (V, E)$ is a subset $C \subseteq V$ such that for any two vertices $u, v \in C$ all the vertices of any shortest path between u and v in Γ lie entirely in C. A connected graph $\Gamma = (V, E)$ is called a *median graph* if, for any three vertices $u, v, w \in V$, there is a *unique* vertex $\text{med}(u, v, w) \in V$ which lies simultaneously on shortest paths from u to v, from u to w and from v to w.

Figure 1 shows some examples of median graphs.

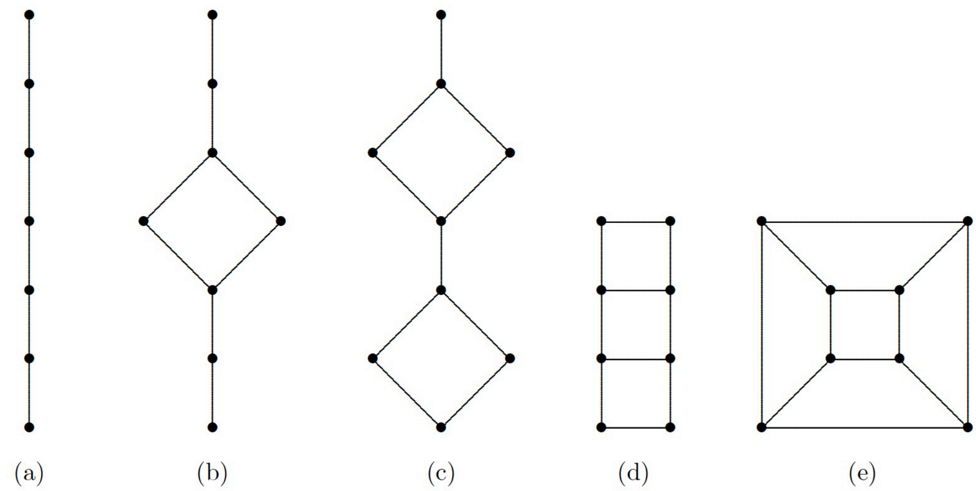

Figure 1: Examples of median graphs

As we will see all of them are induced by maximal Condorcet domains on four alternatives.

To characterize the structure of an arbitrary median graph we recall the concept of *convex expansion*. For any two subsets $S, T \subseteq V$ of the set of vertices of the graph Γ, let $E(S, T) \subseteq E$ denote the set of edges that connect vertices in S and vertices in T.

Definition 1. *Let $\Gamma = (V, E)$ be a graph. Let $W_1, W_2 \subset V$ be two subsets with a non-empty intersection $W_1 \cap W_2 \neq \emptyset$ such that $W_1 \cup W_2 = V$ and $E(W_1 \setminus W_2, W_2 \setminus W_1) = \emptyset$. The* expansion *of Γ with respect to W_1 and W_2 is the graph Γ' constructed as follows:*

- *each vertex $v \in W_1 \cap W_2$ is replaced by two vertices v^1, v^2 joined by an edge;*
- *v^1 is joined to all the neighbours of v in $W_1 \setminus W_2$ and v^2 is joined to all the neighbours of v in $W_2 \setminus W_1$;*
- *if $v, w \in W_1 \cap W_2$ and $vw \in E$, then v^1 is joined to w^1 and v^2 is joined to w^2.*
- *if $v, w \in W_1 \setminus W_2$ or if $v, w \in W_2 \setminus W_1$, they will be joined by an edge in Γ' if and only if they were joined in Γ; if $v \in W_1 \setminus W_2$ and $w \in W_2 \setminus W_1$, they remain not joined in Γ'.*

If W_1 and W_2 are convex, then Γ' will be called a convex expansion *of Γ.*

Example 2 (Convex expansion). In the graph Γ shown on the left of Figure 1 we set $W_1 = \{a, b, c, d\}$ and $W_2 = \{c, d, e, f\}$. These are convex and their intersection $W_1 \cap W_2 = \{c, d\}$ is not empty. On the right we see the graph Γ' obtained by the convex expansion of Γ with respect to W_1 and W_2.

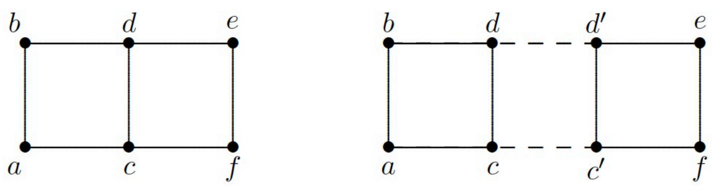

Figure 2: Example of a convex expansion of median graphs

The following important theorem about median graphs is due to Mulder [22].

Theorem 3 (**Mulder's convex expansion theorem**). *A graph is median if and only if it can be obtained from a trivial one-vertex graph by repeated convex expansions.*

3 Condorcet Domains and Median Graphs

This section contains the main results of this paper, which explicate the connection between closed Condorcet domains and median graphs. In Subsection 3.1 we prove that every closed Condorcet domain naturally induces a median graph. In Subsection 3.2 we show that, conversely, for every median graph one can construct a (non-unique) closed Condorcet domain that induces the given graph.

3.1 Closed Condorcet Domains Induce Median Graphs

To each domain $\mathcal{D} \subseteq \mathcal{R}(X)$ one can associate a graph $\Gamma_{\mathcal{D}}$ on $\mathcal{D}$ as follows. Say that two distinct orders $R, R' \in \mathcal{D}$ are *neighbors in* $\mathcal{D}$, or simply $\mathcal{D}$*-neighbors*, if $[R, R'] \cap \mathcal{D} = \{R, R'\}$, and define $\Gamma_{\mathcal{D}}$ to be the (undirected) graph on $\mathcal{D}$ that connects each pair of $\mathcal{D}$-neighbors by an edge. Note that $\Gamma_{\mathcal{D}}$ is always connected, i.e., any two orders in $\mathcal{D}$ are connected by a path in $\Gamma_{\mathcal{D}}$. Evidently, two $\mathcal{D}$-neighbors need not be $\mathcal{R}(X)$-neighbors, but two $\mathcal{R}(X)$-neighbors R, R' are always $\mathcal{D}$-neighbors whenever $R, R' \in \mathcal{D}$. For any $\mathcal{D} \subseteq \mathcal{R}(X)$, the order Q is $\Gamma_{\mathcal{D}}$*-geodesically between* the orders R and R' if Q lies on a shortest $\Gamma_{\mathcal{D}}$-path that connects R and R'. Finally, say that a domain $\mathcal{D}$ itself is *connected* if the graph $\Gamma_{\mathcal{D}}$ is a subgraph of $\Gamma_{\mathcal{R}(X)}$, i.e., if all $\mathcal{D}$-neighbors differ in the ranking of exactly *one* pair of alternatives.

The following theorem is central to our approach. It states that the natural betweenness on a domain coincides with the geodesic betweenness of its induced graph for three natural classes of domains: (i) all median domains (ii) all connected domains, and (iii) all domains for which the associated graph is acyclic.

Theorem 4. *The betweenness relation on a domain $\mathcal{D} \subseteq \mathcal{R}(X)$ coincides with the geodesic betweenness of the induced graph $\Gamma_{\mathcal{D}}$, i.e., for all $R, R', Q \in \mathcal{D}$,*

$$Q \in [R, R'] \;\Leftrightarrow Q \textit{ is } \Gamma_{\mathcal{D}}\textit{-geodesically between } R \textit{ and } R', \tag{1}$$

if one of the following conditions is satisfied:

(i) $\mathcal{D}$ *is a Condorcet domain,*

(ii) $\mathcal{D}$ *is connected,*

(iii) $\Gamma_{\mathcal{D}}$ *is acyclic (i.e., a tree).*

Theorem 4 will be proven using a result by Bandelt and Chepoi [4]. The statement of this result requires some additional definitions, in particular, the notion of a geometric interval operator. An *interval operator* on a (finite) set V is a mapping that assigns to each pair $(v, w) \in V \times V$ a non-empty subset $[v, w] \subseteq V$, the *interval spanned by v and w*, such that, for all $v, w \in V$, $v \in [v, w]$ and $[v, w] = [w, v]$. An interval operator is called *geometric* if it satisfies, in addition, the following properties: for all $t, u, v, w \in V$,

$$[v, v] = \{v\}, \tag{2}$$

$$u \in [v, w] \Rightarrow [v, u] \subseteq [v, w], \tag{3}$$

$$t, u \in [v, w] \;\&\; t \in [v, u] \Rightarrow u \in [t, w]. \tag{4}$$

A pair v, w is called an *edge* if $v \neq w$ and $[v, w] = \{v, w\}$. These edges form a graph Γ on the vertex set V.

We first verify the geometricity of the natural interval operator induced by every domain of orders. This will also give us an example of geometric operator.

Lemma 3.1. *For any domain $\mathcal{D} \subseteq \mathcal{R}(X)$, the interval operator that assigns to every pair $R, R' \in \mathcal{D}$ the interval $[R, R'] \cap \mathcal{D}$ is geometric.*

Proof. Properties (2) and (3) are easily verified. To verify (4), consider $T, U, V, W \in \mathcal{D}$ as required in the antecedent. Let $x, y \in X$ be such that xTy and xWy. We have to show that then xUy. Since x, y were arbitrarily chosen, this would imply $U \in [T, W]$, as desired. Assume, by contradiction, that yUx; then, we must have xVy since by assumption $T \in [V, U]$. But this contradicts the assumption that $U \in [V, W]$. □

We will also need the following lemma.

Lemma 3.2. *Consider a geometric interval operator on V, and let $u \in [v, w]$. Then, there exist pairwise distinct $t_1, \ldots, t_m \in [v, w]$ such that $t_1 = v$, $t_m = w$, $t_k = u$ for some $k \in \{1, \ldots, m\}$, and such that*

$$[t_1, t_2] \subset [t_1, t_3] \subset \ldots \subset [t_1, t_m]$$

is a maximal chain. The graph induced by a geometric interval operator is connected.

Proof. The existence of a maximal chain of the required form follows at once from condition (3). The pairs $t_k t_{k+1}$ must form an edge for $k = 1, \ldots, m-1$ by maximality of the chain. Thus, any two vertices are connected by a path. □

An interval operator is called *graphic* if, for all $u, v, w \in V$, $u \in [v, w]$ if and only if u is geodesically between v and w in the induced graph Γ; note that this is exactly condition (1) in Theorem 4 above. An interval operator is said to satisfy the *triangle condition* if, for all triples $u, v, w \in V$ such that

$$[u, v] \cap [v, w] = \{v\} \text{ and } [v, w] \cap [w, u] = \{w\} \text{ and } [w, u] \cap [u, v] = \{u\}, \tag{5}$$

all three intervals are edges whenever one of them is. Observe that (5) can be satisfied only if either all three elements u, v, w coincide, or are pairwise distinct. The following result is due to [4, Th.1].

Proposition 3.1. *Any geometric interval operator satisfying the triangle condition is graphic.*

We will apply Proposition 3.1 in the proof of Theorem 4. This proof shows, among other things, that the triangle condition (5) is a powerful sufficient condition for an interval operator to be graphic. This condition is satisfied in all three cases considered in Theorem 4 (sometimes vacuously).

Proof of Theorem 4(i). In case of a Condorcet domain, which by Theorem 2 is also a median domain, the triangle condition is vacuously satisfied, since by the median property there can obviously be no triples of pairwise distinct elements satisfying (5). By Proposition 3.1 the equivalence (1) is satisfied for any median domain.

(ii). Next consider any connected domain $\mathcal{D} \subseteq \mathcal{R}(X)$. There can exist triples satisfying (5), but we shall show that in this case none of the three intervals can form an edge, hence the triangle condition is again satisfied. Thus, suppose that the three pairwise distinct orders $U, V, W \in \mathcal{D}$ satisfy (5) and, by contradiction, that one of the three intervals is an edge, say $[U, V] = \{U, V\}$. Since $\mathcal{D}$ is connected, there exists $x, y \in X$ such that U and V differ only in the ranking of x versus y, say xUy and yVx, while U and V agree in the ranking of all other pairs of alternatives. There are two possibilities: either xWy or yWx. In the first case, we have $U \in [V, W]$ and hence $[U, V] \cap [V, W] \supseteq \{U, V\}$; in the second case, $V \in [W, U]$ and hence $[W, U] \cap [U, V] \supseteq \{U, V\}$. In both cases, we obtain a contradiction to assumption (5). By Proposition 3.1 the equivalence (1) is satisfied for $\mathcal{D}$.

(iii). Finally, assume that $\mathcal{D}$ is such that $\Gamma_{\mathcal{D}}$ is acyclic, i.e., a tree. As in part (i), we show that there cannot exist triples satisfying (5) hence again the triangle condition is satisfied vacuously.[5] Assume, by way of contradiction, that the pairwise distinct orders $U, V, W \in \mathcal{D}$ satisfy (5). By Lemma 3.2, there exists a path π_{UV} in $\Gamma_{\mathcal{D}}$ connecting U and V that stays entirely in $[U, V]$; in particular, π_{UV} does not contain W. Similarly, there exists a path π_{VW} in $\Gamma_{\mathcal{D}}$ connecting V and W and staying entirely in $[V, W]$, and a path π_{WU} connecting W and U and staying entirely in $[W, U]$. But then the union $\pi_{UV} \cup \pi_{VW} \cup \pi_{WU}$ forms a cycle, which contradicts the assumed acyclicity of $\Gamma_{\mathcal{D}}$. Thus, again, by Proposition 3.1 the equivalence (1) is satisfied for the domain $\mathcal{D}$. □

From Theorem 2 and Theorem 4(i), we immediately obtain

Theorem 5. *The induced graph $\Gamma_{\mathcal{D}}$ of any closed Condorcet domain $\mathcal{D}$ is a median graph. Moreover the betweenness relation on a domain $\mathcal{D}$ coincides with the geodesic betweenness of $\Gamma_{\mathcal{D}}$.*

Note that not all median domains are connected, as exemplified by the domain to the left of Fig. 3. It is also worth noting that Theorem 4 does *not* imply that $\mathcal{D}$ is a median domain whenever the associated graph $\Gamma_{\mathcal{D}}$ is median graph. A counterexample is the domain $\mathcal{D}$ in the middle of Fig. 3 which is not a median domain despite the fact that its induced graph $\Gamma_{\mathcal{D}}$ is a median graph. However, we have the following corollary.

Corollary 2. *Let $\mathcal{D} \subseteq \mathcal{R}(X)$ be a connected domain. Then, $\mathcal{D}$ is a median domain if and only if the induced graph $\Gamma_{\mathcal{D}}$ is a median graph.*

Proof. The induced graph of any median domain is a median graph by Theorem 4(i). Conversely, if $\mathcal{D}$ is connected the geodesic median of any triple of vertices with respect to $\Gamma_{\mathcal{D}}$ is also the median with respect to the betweenness in $\mathcal{D}$ by Theorem 4(ii). Thus, if $\Gamma_{\mathcal{D}}$ is a median graph, $\mathcal{D}$ is a median domain. □

Example 3. (Condorcet domains and their graphs.) The five Condorcet domains on four alternative given below are maximal and their corresponding graphs are listed in Figure 1:

(a) abcd; bacd; bcad; bcda; bdca; dbca; dcba;

(b) abcd; bacd; bcad; cbad; bcda; cbda; cdba; dcba;

(c) abcd; abdc; bacd; badc; bdac; bdca; dbac; dbca; dcba;

(d) abcd; bacd; bcad; cbad; dabc; dbac; dbca; dcba;

(e) abcd; abdc; bacd; badc; cdab; dcab; cdba; dcba.

Here is what we can prove about graphs of maximal Condorcet domains.

Theorem 6. *Let $\mathcal{D}$ be a maximal Condorcet domain. If $\Gamma_{\mathcal{D}}$ is a tree, it is, in fact, a chain.*

[5] Note, however, that we cannot use part (i) directly since we do not know yet whether $\mathcal{D}$ is a median domain.

Proof. Let $\mathcal{D}$ be a maximal Condorcet domain, and assume that $\Gamma_\mathcal{D}$ is a tree but not a chain. Then there exists a vertex R in $\Gamma_\mathcal{D}$ of degree at least 3. Consider now any three neighbors of R in $\Gamma_\mathcal{D}$, say R_1, R_2 and R_3. It is possible to show that $\mathcal{D}$ is connected, hence there are three distinct ordered pairs (x_i, y_i), $i = 1, 2, 3$, of alternatives such that $R_i = R \setminus \{(x_i, y_i)\} \cup \{(y_i, x_i)\}$. We will say that R_i is obtained from R by *switching* the pair of adjacent alternatives (x_i, y_i). Moreover, since in every pair (x_i, y_i), $i = 1, 2, 3$, the alternatives are adjacent in R, there must exist at least two pairs that have no alternative in common, say $\{x_1, y_1\} \cap \{x_2, y_2\} = \emptyset$.

Now let R' be the order that coincides with R except that both pairs (x_1, y_1) and (x_2, y_2) in R' are switched, i.e., $y_1 R' x_1$ and $y_2 R' x_2$, and consider the domain $\mathcal{D} \cup \{R'\}$. Since x_1, y_1 and x_2, y_2 are neighbors in each of the orders R, R_1, R_2, R', for every three alternatives $\{a, b, c\}$ no new order among them appears in R' which has not yet occurred in R, R_1, or R_2. Hence, by Theorem 1d), $\mathcal{D} \cup \{R'\}$ is a Condorcet domain. By the maximality of $\mathcal{D}$, this implies $R' \in \mathcal{D}$. But in this case, the graph $\Gamma_\mathcal{D}$ evidently contains the 4-cycle $\{R, R_1, R', R_2\}$, contradicting the assumed acyclicity $\Gamma_\mathcal{D}$. Hence, there cannot exist a vertex of degree 3 or larger, i.e., $\Gamma_\mathcal{D}$ is a chain. □

3.2 Every Median Graph is Induced by Some Closed Condorcet Domain

Theorem 7. *For every (finite) median graph $\Gamma = (V, E)$ there exists a closed Condorcet domain $\mathcal{D} \subseteq \mathcal{R}(Y)$ on a finite set of alternatives Y with $|Y| \leq |V|$ such that $\Gamma_\mathcal{D} = \Gamma$.*

Proof. We apply Mulder's theorem. Since the statement is true for the trivial graph consisting of a single vertex, arguing by induction, we assume that the statement is true for all median graphs with k vertices or less. Let $\Gamma' = (V', E')$ be a median graph with $|V'| = k+1$. By Mulder's theorem Γ' is a convex expansion of some median graph $\Gamma = (V, E)$ relative to convex subsets W_1 and W_2, where $|V| = \ell \leq k$. By induction there exists a domain $\mathcal{D} \subseteq \mathcal{R}(X)$ with $|X| \leq k$ such that $\Gamma_\mathcal{D}$ is isomorphic to Γ with the mapping $v \mapsto R_v$ associating a linear order R_v to a vertex $v \in V$.

To obtain a new domain $\mathcal{D}'$ such that $\Gamma_{\mathcal{D}'}$ is isomorphic to Γ' we clone an arbitrary alternative $x \in X$ and introduce a clone $y \notin X$ of x.[6] The mapping $v \mapsto R'_v$ that associates vertices of Γ' to linear orders will be constructed as follows. If v is a vertex of $W_1 \setminus W_2$, to obtain R'_v we replace x with xy in R_v, placing x higher than y, and to obtain R'_u for $u \in W_2 \setminus W_1$ we replace x by yx in R_u, placing y higher than x. Let v now be in $W_1 \cap W_2$. In the convex expansion this vertex is split into v^1 and v^2. To obtain R'_{v^1} we clone the linear order R_v replacing x by xy and to obtain R'_{v^2} we clone the same linear order R_v replacing x by yx. The number of alternatives has increased by one only, so it is not greater than $|V'| = \ell + 1 \leq k + 1$.

To complete the induction step, we have to show that the betweenness relation in $\mathcal{D}'$ is exactly as in the expansion Γ' of Γ. First, we need to show that there is no edge between R'_u and R'_v if $v \in W_1 \setminus W_2$ and $u \in W_2 \setminus W_1$. This follows from the fact that R_u and R_v were not neighbors in $\Gamma_\mathcal{D}$. Hence there was a linear order $R_w \in [R_u, R_v]$ between them. In $\mathcal{D}'$ this linear order will be cloned to R'_w and, no matter how we place x and y there, we obtain $R'_w \in [R'_u, R'_v]$, hence R'_u and R'_v are not neighbors in $\Gamma_{\mathcal{D}'}$ as well. Secondly, we have to check that R'_{v^1} and R'_{v^2} are linked by an edge. This holds because these orders differ in the ranking of just one pair of alternatives, namely x and y, hence they are neighbors in $\Gamma_{\mathcal{D}'}$. To prove that the betweenness relation in $\mathcal{D}'$ is exactly as in Γ' we need to consider several cases. For example, let $u \in W_1 \setminus W_2$ and $v \in W_2 \setminus W_1$, and let us prove that all linear orders

[6] We say that x and y are clones if they are neighbors in any linear order of the domain, cf. [12, 13].

on a shortest path between u and v are between R'_u and R'_v. As we have shown, $W_1 \setminus W_2$ and $W_2 \setminus W_1$ are disconnected, hence any shortest path between u and v will contain an edge $e = w^1w^2$ with $w \in W_1 \cap W_2$. There is a corresponding path in Γ, where the edge e is contracted to w, and all linear orders on that path are between R_u and R_v. The way we placed y in these linear orders will not disturb the betweenness since all linear orders between R'_u and R_{w^1} (inclusive) will place x above y, and all others will place y above x. The other cases follow similarly. □

It is worth noting that [8] showed that for a median graph with k vertices we might need exactly k alternatives to construct a closed Condorcet domain. The star-graph represents the worst case scenario.

The next result exactly specifies when a maximal single crossing domain is a maximal Condorcet domain. To formulate the result, observe that in a maximal single crossing domain every edge corresponds to *exactly* one pair $x, y \in X$ of distinct alternatives such that all orders on one side of the given edge rank x above y, and all orders on the other side rank y above x (if there were more pairs of this kind we could extend the ordering by putting an additional vertex on this edge as we have done in the proof of Theorem 6 above. Let us call such pair the *switching pair* of the given edge.

Theorem 8. *A single crossing domain $\mathcal{D} \subseteq \mathcal{R}(X)$ constitutes a maximal Condorcet domain if and only if it satisfies the following three conditions: (i) $\mathcal{D}$ is connected, (ii) $\mathcal{D}$ contains two completely reversed orders, and (iii) the switching pairs of any two adjacent edges of $\Gamma_{\mathcal{D}}$ have one element in common.*

Proof. Necessity of conditions (i) - (iii) has already been observed in the main text. Their sufficiency follows using [16, Th. 2] and [30, Lemma 2.2]. The argument requires some elementary facts from group theory. Let us start with noting that the symmetric group S_n is generated by the permutations $s_1 = (1\,2), \ldots, s_{n-1} = (n{-}1\,n)$, i.e., any other permutation is a product of these.

Consider a word $w = a_1a_2 \cdots a_n$ with n symbols (presently, it does not matter which symbols we take but later they will be $1, 2, 3, \ldots, n$). A permutation σ from S_n acts on w as $w\sigma = a_{\sigma(1)}a_{\sigma(2)} \cdots a_{\sigma(n)}$. In particular, s_i swaps the ith and $(i + 1)$th symbols of the word whatever they are. Let ω be the permutation such that $\omega(i) = n - i$, i.e., the permutation that reverses all the symbols, i.e., $w\omega = a_na_{n-1} \cdots a_1$. We are interested in presentations of ω of the form:

$$\omega = s_{i_1}s_{i_2} \cdots s_{i_k}, \tag{6}$$

where

1. $k = n(n - 1)/2$, and
2. when $s_{i_1}s_{i_2} \cdots s_{i_k}$ acts on $w = 12 \cdots n$ it swaps each pair (i, j) exactly once.

[16] call (6) a maximal reduced decomposition. If we identify the word $a_1a_2 \cdots a_n$ with the linear order $a_1 < a_2 < \ldots < a_n$, [16] show that linear orders $w_0 = 12 \cdots n$, $w_1 = ws_{i_1}, \ldots,$ $w_k = ws_{i_1}s_{i_2} \cdots s_{i_k}$ encode a maximal chain in the weak Bruhat order.

Two maximal chains (maximal reduced decompositions) are equivalent if they can be obtained from each other by using relations $s_is_j = s_js_i$ with $|i - j| > 1$. [16] showed that the permutations visited by an equivalence class of maximal reduced decomposition form a maximal Condorcet domain.

In our case the maximal chain is the only one in the equivalent class since (6) does not have neighboring s_i and s_j with $|i - j| > 1$. □

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