Doodle Poll Games

Svetlana Obraztsova, Maria Polukarov, Zinovi Rabinovich, Edith Elkind

Abstract

In Doodle polls, each voter may approve a subset of the available alternatives according to his preferences. While such polls can be captured by the standard models of Approval voting, Zou et al. [17] analyse real-life Doodle poll data and conclude that poll participants' behaviour seems to be affected by considerations other than their intrinsic preferences over the alternatives. In particular, they propose a model of social voting, where participants approve their top alternatives as well as additional 'safe' choices so as to appear cooperative. The appeal of their model is that it appears to be consistent with the real-life data. However, Zou et al. explicitly describe the voters' strategies rather than attempt to rationalise them in terms of voters' preferences. In this paper, we complement the work of Zou et al. by putting forward a model in which the behaviour described by Zou et al. arises as an equilibrium strategy. We do so by formally modelling the bonus that voters derive from approving additional alternatives. We consider two scenarios: one where this bonus is accumulated over all approved alternatives, and one where it is capped by a certain threshold. The former model turns out to be easier to analyse, but its predictions do not appear realistic; in contrast, trembling hand perfect Nash equilibria of the latter model behave consistently with the model of Zou et al.

1 Introduction

Scheduling group meetings is a tedious and time-consuming activity: the participants have to decide which time slots are suitable for all or most of them, and to choose one or more slots based on these constraints. A convenient online tool for this task is *Doodle*: the poll initiator creates a list of possible time slots, and all participants can then cast their vote online, indicating which time slots are acceptable for them. In the most basic version of a Doodle poll, everyone can mark each time slot as suitable or not (there is also a more flexible option, where each participant is allowed three levels of approval: 'yes', 'no', and 'if need be'), and each participant can vote 'yes' for any number of slots.

Assuming for simplicity that in the end the meeting time will be chosen among the time slots that received the largest number of votes (using a deterministic or randomised tie-breaking rule), we can view this situation as an Approval voting scenario. Now, in case of dichotomous preferences (where each voter either approves or disapproves each time slot), Approval voting is known to elicit truthful behaviour from the voters: it is a weakly dominant strategy for each voter to vote exactly for the time slots he approves. Of course, in reality voters' preferences may be more complicated: for instance, while I currently have no appointments scheduled for Monday 8:30am, I would rather not schedule a meeting for this time if at all possible. If each voter has a weak order over the time slots, a relevant notion is that of a *sincere* strategy: an Approval vote is said to be sincere if the voter weakly prefers each of the approved alternatives to each of the non-approved alternatives (see, e.g., [4]). Under reasonable assumptions on tie-breaking (and, in case of randomised tie-breaking, voters' preferences over the respective lotteries), Approval voting is known to encourage such sincere behaviour [7].

However, these theoretical results do not seem to fully explain the voters' behaviour that is observed in practice; in particular, they provide no clue as to how tvoters decide how many alternatives to approve. To address this challenge, recently Zou et al. [2015] used a large dataset from Doodle (over 14 million votes from 2 million participants in over 340,000 polls) to analyse user behaviour in Doodle polls. Their data shows that participants tend to behave differently in *closed polls* (where previously cast votes are not visible) and in *open polls* (where previously cast votes can be seen). Specifically, participants in open polls tend to approve more slots and coordinate with the previous voters. Perhaps more intriguingly, in open polls both the most popular slots and the least popular slots tend to receive more votes than in closed polls, with medium popularity slots receiving a similar number of votes.

Zou et al. explain this phenomenon by introducing the model of *social voting*. In this model, besides having intrinsic preferences for different time slots, voters would like to appear cooperative and therefore they gain extra utility from approving many slots. Thus, in addition to approving their most preferred slots, they may approve a few extra slots. It makes sense for them to choose these slots among the unpopular slots (those that have received few votes so far), to minimise the risk that these slots will actually be selected. Zou et al. show that this model is consistent with the data: they generate synthetic data according to their model and obtain voter behaviour that is qualitatively similar to what is observed in Doodle open polls.

However, the social voting model proposed by Zou et al. makes no attempt to rationalize the voters' behaviour: it simply stipulates that each voter will vote for his most preferred slots and some of his somewhat less preferred slots based on their popularity. The authors do not try to reconstruct voters' utilities that make such behaviour *rational*. This is exactly the problem that we aim to tackle in this paper.

In our model, each voter assigns some utility to each time slot (with different utilities corresponding to different *preference levels*), and the winning alternative is selected among the alternatives with the highest number of approval points using either a lexicographic tie-breaking rule (based on a fixed ordering of the alternatives) or the randomised tie-breaking rule (where the winner is chosen among the top-scoring alternatives uniformly at random). We remark that tie-breaking rules play a particularly important role in Doodle polls: since the number of alternatives is often large relative to the number of votes, ties are quite likely, and, since the decisions made by Doodle polls are typically low-stake, both tossing a fair coin and picking the lexicographically first among the top-scoring alternatives is socially acceptable. It remains to explain how voters derive utility from appearing cooperative.

A natural first attempt is to assume that, besides deriving the respective utility from the winning slot, each voter obtains a very small bonus from each time slot he approves; the second component is supposed to capture his social utility. We discuss the existence of Nash equilibria and explore the complexity of computing equilibrium voting strategies in this setting, both under lexicographic tie-breaking and under randomised tie-breaking. This problem turns out to be polynomial-time solvable under lexicographic tie-breaking (for any number of preference levels) and under randomised tie-breaking for two preference levels, but becomes computationally hard under randomised tie-breaking and three or more preference levels. However, our analysis reveals that existence of pure Nash equilibria is very unlikely in this model, both for lexicographic tie-breaking and for randomised tie-breaking: it turns out that if no alternative is in the top preference level of all voters, the existence of Nash equilibria requires a very delicate balance of voters' preferences. Thus, pure Nash equilibria in this model cannot be viewed as realistic predictions of voters' behaviour.

We then consider an alternative model, where a voter only gets the social bonus for the first κ time slots that he approves, for a value of κ that is noticeably smaller than the total number of alternatives. Indeed, it is plausible that, in practice, to appear cooperative, one only needs to approve a number of alternatives that exceeds a certain threshold, and approving further alternatives may not contribute much to the voter's perception by his peers. This is corroborated by empirical findings, which demonstrate that voters seem to approve very popular and very unpopular, but not all available alternatives [17]. This model admits a much richer set of pure strategy Nash equilibria; indeed, we obtain a plethora of 'bad' Nash equilibria where everyone approves one alternative (which may be viewed as undesirable by some or, indeed, all voters) and $\kappa - 1$ other alternatives, and approvals are distributed in such a manner that every non-winning alternative is far from becoming a winner; a similar phenomenon is well-known in the context of Plurality voting (see, e.g., [14]).

Therefore, it becomes important to come up with a suitable *equilibrium refinement* that will help us identify Nash equilibria that are more likely to occur in practice. To do so, we employ a variant of Selten's celebrated *trembling hand perfect Nash equilibrium (THPE)* [12, 15]: under this notion, each player assumes that, with small probability, other players may make a mistake when implementing their strategy and choose another (random) strategy instead. It has been argued that THPE is the most important refinement of Nash equilibrium [16]; moreover, it is particularly appealing in our setting, where the number of decisions that a voter needs to make may be large, and mistakes are rarely costly. Trembling hand perfect equilibria have been studied for a wide variety of games; in partcular, in the context of voting, different variants of THPE have been proposed to explain voter behaviour in plurality and runoff rule voting games [13], information revelation scenarios [11] and agenda-setting games [1].

In this work, we modify Selten's original definition by restricting the type of mistakes that the players may make: specifically, we assume that each player makes a mistake independently for each alternative, i.e., for each alternative he considers, with probability ε he votes 'Yes' if he intends to disapprove this alternative and 'No' if he intends to approve it, and he votes correctly with probability $1 - \varepsilon$. We then show that, in games with capped bonuses, a voter's best response to a trembling hand strategy of other players is to approve all the alternatives at his top preference level as well as some of the most unpopular alternatives, with the total number of approved alternatives never exceeding the cap. This result is consistent with the findings of [17]. The analysis of this model is quite involved, and can be seen as our main technical contribution.

From a conceptual perspective, our contribution is twofold. First, while the notion of a social bonus was put forward by [17], we are able to show that this concept can be used to *rationalize* the voters' behavior. This turns out to be a non-trivial task: the straightforward approach of incorporating the social bonus into the voters' utilities, in an uncapped fashion, results in a model with very few Nash equilibria, which are, moreover, rather unnatural.

To overcome this difficulty, we introduce the idea of a cap on the social bonus, which is both intuitively appealing and enables us to obtain results that agree with the real-life data. Another innovation is the use of trembling hand perfection as an equilibrium refinement tool: while this is an elegant and conceptually appealing notion, it has received surprisingly little attention in the algorithmic game theory literature (notable exceptions are [8, 10], which, unlike our work, focus on abstract normal-form games and obtain computational hardness results rather than efficient algorithms). Indeed, to the best of our knowledge, we are the first to approach this notion from an algorithmic perspective in the context of a specific application rather than an abstract normal-form game.

2 Model and preliminaries

We first recall the model of approval voting, and then define Doodle poll games, which extend the approval model to capture behaviour observed in Doodle polls. We conclude the section with some preliminary observations concerning equilibria of these games.

2.1 Approval voting

There is a set $V = \{v_1, v_2, \dots, v_{|V|}\}$ of *voters*, electing a single winner from a set $C = \{c_1, c_2, \dots, c_{|C|}\}$ of *alternatives*, or *candidates*. A single *vote* (or *ballot*) of voter $v \in V$ is a subset of alternatives $b^v \subseteq C$; these are the alternatives that he approves. We will also regard b^v as a $|C|$-dimensional binary vector $(b^v_1, \dots, b^v_{|C|})$ with $b^v_c = 1$ if v approves alternative $c \in C$ and $b^v_c = 0$ otherwise. A voting *profile* $\mathbf{b} = (b^v)_{v \in V}$ is a vector of ballots, one for each voter.

Given a voting profile $\mathbf{b}$, let $s_c(\mathbf{b}) = \sum_{v \in V} b^v_c$ denote the *score* of candidate c, aggregating all votes in $\mathbf{b}$; the vector $s(\mathbf{b}) = (s_c(\mathbf{b}))_{c \in C}$ is then the score vector of $\mathbf{b}$. The *(provisional) winners* of the election are given by the set of alternatives $W(\mathbf{b}) = \arg\max_{c \in C} s_c(\mathbf{b})$, and the ties among the winners are broken by a given *tie-breaking rule*. In this paper, we consider two common tie-breaking schemes: *lexicographic*, which selects the winner based on a given strict linear order over the candidates, and *randomised*, which chooses one uniformly from $W(\mathbf{b})$.

Each voter $v \in V$ has *preferences over individual candidates*, $\geq_v$, modelled as a total (but not necessarily strict) order on C. That is, $\geq_v$ is a reflexive and transitive (but not necessarily anti-symmetric) binary relation on C, which is complete. We write $x \geq_v y$ to express that voter v likes candidate x at least as much as candidate y. We write $x >_v y$ (strict preference) if $x \geq_v y$ but not $y \geq_v x$. Thus, $\geq_v$ defines a partition of C into L disjoint subsets $\{C^v_1, \dots, C^v_L\}$, so that v is indifferent among the alternatives in the same element of the partition, but strictly prefers any alternative in C^v_l over any alternative in $C^v_{l'}$, for all $l > l'$. That is, L denotes the number of *preference levels* of the voter. We allow the possibility that some, but no more than $L-2$, elements of the partition are empty. That is, no voter is indifferent between all the alternatives in C, and it is without loss of generality to assume that all voters have the same number of preference levels L. Specifically, L is the maximum number of proper (i.e., non-empty) preference levels across all the voters, and we require that for each voter v the preference levels 1 and L are not empty. A voter with exactly two proper preference levels is called *dichotomous*, and one with three proper preference levels is called *trichotomous*.

Each voter furthermore has *preferences over election outcomes*—i.e., over sets of candidates that the tie-breaking rule further selects from. Note that under deterministic tie-breaking, comparing every pair of outcomes is easy: we apply the tie-breaking rule to determine the eventual winner in each set, and compare these winners using the voter's preference order $\geq_v$. However, if ties are broken randomly, $\geq_v$ does not induce a complete order over all possible outcomes, and a common solution (see, e.g., [2, 5, 6, 9]) is to augment voters' preferences with cardinal utilities. To this end, we assume that each voter v is endowed with a function $\delta_v : C \to \mathbb{Q}$ that takes at most L distinct values and is consistent with v's preferences: $\delta_v(c) > \delta_v(c')$ if and only if $c >_v c'$. Without loss of generality, we assume that $\delta_v(c) = 1$ if $c \geq_v c'$ for all $c' \in C$ and $\delta_v(c) = 0$ if $c' \geq_v c$ for all $c' \in C$. Now, the *utility from the outcome*, $u_v(\mathbf{b})$, of voter v under ballot profile $\mathbf{b}$ is defined as follows. If the ties are broken deterministically, according to a strict linear order $\succ$, then

$$u_v(\mathbf{b}) = \delta_v(c_{\max}), \text{ where } c_{\max} \in W(\mathbf{b}) \text{ and } c_{\max} \succ c \text{ for all } c \in W(\mathbf{b}) \setminus \{c_{\max}\}. \tag{1}$$

In case of uniform tie-breaking, the utility is given by the expected value:

$$u_v(\mathbf{b}) = \frac{1}{|W(\mathbf{b})|} \sum_{c \in W(\mathbf{b})} \delta_v(c). \tag{2}$$

Assuming that each voter strives to maximise his utility from the outcome, for a fixed tie-breaking rule our setting induces a normal-form game $\Gamma = \langle V, C, (u_v)_{v\in V}\rangle$ where the set of players is given by the set of voters V, the strategy set available to each player is given by the collection of all subsets (i.e., the power set) of the set of candidates C, and the utility function of each player is given by u_v as defined by equations (1) or (2) above. We call game Γ the *approval voting game* (the tie-breaking rule will be clear from the context).

Given voter v's preference order $\geq_v$, a ballot b^v is called *sincere* if the voter prefers each approved candidate to each disapproved candidate; that is, if $x \geq_v y$ for all $x \in b^v$ and all $y \in C \setminus b^v$ [4]. Observe that according to this definition, a voter can vote sincerely in a number of ways, and abstention ballots, in particular, are also considered sincere. It is known that if a voter's preferences over individual alternatives are dichotomous, then he always has a sincere best response under approval voting, irrespective of how these individual preferences are extended to preferences over outcomes (i.e., sets)—that is, irrespective of how the utility function of game Γ is defined [3]. Under some additional assumptions on set preferences, this claim also holds for trichotomous individual preferences, but if a voter has four or more levels of preference, he may prefer to vote insincerely. Recently, [7] formulated several principles of lifting individual preferences to set preferences under which a voter will always have a sincere best response in an approval voting game, even if he has more than three proper preference levels. Importantly, the utility function as defined by equations (1) and (2) does satisfy those principles.

2.2 Doodle poll games

Doodle is an online system for scheduling events that implements the approval voting mechanism. Specifically, candidates are possible time slots, and each responder approves a subset of the slots. There are two basic types of Doodle polls: the responder either indicates his (un)availability at given time slots (dichotomous preferences), or identifies times that are (1) convenient, (2) feasible but inconvenient or (3) not feasible (trichotomous preferences). There is also a distinction between open and hidden polls: in the former a voter gets to see the responses of other participants and in the latter he does not.

The recent work by [17], who analysed over 340,000 polls (both open and hidden), has shown results that appear inconsistent with traditional models of approval voting. In particular, the authors observe that the average reported availability is higher in open polls than in hidden polls, and also open polls have higher response rates for very popular and very unpopular time slots. Motivated by these findings, here we extend the approval voting model to what we term the *Doodle poll game*, as follows.

We incorporate social rewards into voters' utility functions. Specifically, we assume that voter v gets a *social bonus*, β, for each of his approved alternatives, as long as their number is under the *cap* κ, $0 \leq \kappa \leq |C|$. This cap is the maximum number of approved alternatives that are socially rewarded. In approval games, $\kappa = 0$; if $\kappa = |C|$ then a voter gets a bonus for each alternative he approves. We assume that social bonuses never prevail over the original voter preferences:

$$0 < \beta < \frac{1}{|C|} \min_{v\in V} \min_{\substack{c,c'\in C:\\ \delta_v(c)>\delta_v(c')}} (\delta_v(c) - \delta_v(c')\}). \tag{3}$$

The *total utility* of voter v under ballot profile $\mathbf{b}$ is then composed of the utility from the outcome $u_v(\mathbf{b})$ (as defined by equation (1) or (2) depending on the tie-breaking rule), and the overall social bonus:

$$U_v(\mathbf{b}) = u_v(\mathbf{b}) + \beta \cdot \min\{|b^v|, \kappa\}. \tag{4}$$

Definition 2.1. *A* Doodle poll game (DPG) *is a normal-form game $\Gamma = \langle V, C, (U_v)_{v\in V}\rangle$ with the set of players V, where for each player v, his strategy set is the power set of the set of alternatives C and his utility function U_v is defined by equation (4).*

2.3 Equilibria in DPGs

Below we provide a few simple observations about the structure of equilibrium profiles in DPGs. In this paper, we focus on pure strategy Nash equilibria, to which we may refer simply as equilibria or Nash equilibria (NE).

Observation 2.1. *For both of our tie-breaking rules, in each DPG every voter has a best response where he approves all the alternatives at his highest level of preference.*

Indeed, if a voter fails to approve some of his most preferred alternatives, he can only increase his total utility by adding such an alternative to his ballot: by doing so he cannot lower his utility from the outcome, while his social bonus may only grow (unless capped). In fact, for $\kappa = |C|$, every best response strategy is of this form.

Observation 2.2. *For both of our tie-breaking rules, in any equilibrium of a DPG with $\kappa = |C|$ every voter approves all the alternatives at his highest level of preference.*

When $\kappa = |C|$, in most of the scenarios we consider it is beneficial for everyone to vote for the election winner.

Observation 2.3. *In any equilibrium of a DPG with $\kappa = |C|$ and lexicographic tie-breaking, all voters approve the (unique) winner of the election.*

Indeed, if there is a voter that fails to approve the current winner, he will strictly increase his utility by adding the winner to his ballot: this move will not change the winner—and hence the utility from outcome—for the voter, whereas his social bonus will grow. As for the other candidates, for similar reasons, in an equilibrium profile they all must get maximum possible support, but so that the current winner remains unchanged.

Observation 2.4. *In any equilibrium of a DPG with $\kappa = |C|$ and lexicographic tie-breaking, the candidates who are lower than the winner in the tie-breaking order are approved by all voters, while those who are higher than the winner in the tie-breaking order are approved by exactly $|V| - 1$ voters.*

A claim similar to Observations 2.3 and 2.4 holds also for games with randomised tie-breaking, in cases where equilibrium profiles result in singleton winner sets.

Observation 2.5. *In a DPG with $\kappa = |C|$ and randomised tie-breaking, if the winning set for an equilibrium profile has size one, then the unique winner is approved by all voters and every other candidate is approved by $|V| - 1$ voters.*

3 Uncapped social bonus: Lexicographic tie-breaking

We now focus on DPGs where the social bonus is uncapped, that is, a voter is socially rewarded for each alternative he approves. For lexicographic tie-breaking, we give an algorithm that for a given alternative decides in polynomial time whether there exists an equilibrium profile where this alternative wins the election. We denote the corresponding decision problem by ∃NEWIN:

> ∃NEWIN. Given a DPG with lexicographic tie-breaking and $\kappa = |C|$, and an alternative $w \in C$, is there a Nash equilibrium with winner w?

First, we argue that no voter may have zero utility from the winner in an equilibrium. This, in particular, implies that in games with dichotomous preferences, a candidate can be an equilibrium winner only if she belongs to the top preference level of each voter.

Lemma 3.1. *Consider a Doodle poll game with lexicographic tie-breaking and $\kappa = |C|$. Let $\mathbf{b}$ be an equilibrium with winner w. Then for every voter v with $b^v = C$ we have $\delta_v(w) = 1$, while for every voter v and every candidate c with $c \geq_v w$ we have $c \in b^v$.*

Lemma 3.1 (see Appendix for the proof) provides the basis for our procedure for solving ∃NEWIN (Algorithm 1). Assume for convenience that the tie-breaking order $\succ$ is given by $c_1 \succ \cdots \succ c_{|C|}$. We provide a sketch of the algorithm below; detailed descriptions of its subroutines are presented by Algorithms 2, 3 and 4.

- First, the algorithm considers the subset of candidates C^{TOP} that consists of the candidates who are at the top preference level of every voter (subroutine *Test*C^{TOP}). By Observation 2.2 and Lemma 3.1, if $C^{TOP} \neq \emptyset$ then the election winner belongs to C^{TOP}. Hence, if $C^{TOP} \neq \emptyset$, then our algorithm checks whether w is the tie-breaking winner among candidates in C^{TOP}. If yes, then it is possible to construct an equilibrium profile where w wins the election, by letting all voters vote for their top choices, and in addition approve some of their less preferred candidates, so that each candidate gets either $|V|$ or $|V| - 1$ votes, depending on her position with respect to w in the tie-breaking order. So the algorithm returns YES. Otherwise (i.e., if $w \notin C^{TOP}$, or if w is not the tie-breaking winner among the candidates in C^{TOP}), it returns NO. Finally, if C^{TOP} is empty, the algorithm proceeds to the next step.
- Now, the algorithm focuses on the set V' of voters for whom w is not among the most preferred candidates (subroutine *TestNonSupporters*). This set is non-empty, since otherwise our algorithm would have terminated at the previous step. If $|V'| > |C|$ then it is impossible to construct an equilibrium with winner w. This is because, by Lemma 3.1, in any equlibrium $\mathbf{b}$ with winner w we would have $b^v \neq C$ for each $v \in V'$,

so the total number of approvals would be at most $|C| \cdot |V| - |V'| < |C|(|V| - 1)$, but by Observation 2.4 in equilibrium the score of each alternative is at least $|V| - 1$. Hence, in this case the algorithm returns NO. Otherwise, it moves to the next step.

- Now, we can assume that the number of voters v with $\delta_v(w) < 1$ is between 1 and $|C|$. If one of these voters has w among his least preferred alternatives, then by Lemma 3.1 there will be no equilibrium with winner w, so the algorithm returns NO.
- It also returns NO if there exists a candidate c with $w \succ c$ and a voter v with $c \geq_v w$. Indeed, by Observation 2.4, in an equilibrium with winner w, both w and c have $|V|$ votes. Thus, v can make c the winner by changing her vote to $\{c\}$, as this makes c the unique candidate with $|V|$ approvals.
- Next, the algorithm decides for each voter $v \in V'$ whether it is possible to allocate him a candidate that he may disapprove (subroutime *TestDisapprovedAllocation*). By Lemma 3.1, at least one such candidate must exist in an equilibrium, and v must prefer w to this candidate. Let c be the first candidate with respect to the tie-breaking order whom v prefers to w (note that $c \succ w$, snce otherwise we would have terminated at the previous step). Then in any equilibrium v disapproves some candidate c' with $c' \succ c$. Indeed, if b^v contains all candidates c' with $c' \succ c$, then v can make c the winner by changing his vote to $\{c\}$: after this change, everyone has at most $|V| - 1$ point, and the tie-breaking rule favors c over all other candidates. Thus, the algorithm associates v with the prefix of the tie-breaking order that ends just before c; as argued above, this prefix does not contain w.
- Now, the algorithm acts greedily, in the following way. It orders the voters in V' according to the length of the associated prefix of $\succ$, from the smallest to the largest, and considers them in this order. For each $i = 1, \dots, |V'|$, if c_i belongs to the prefix of the i-th voter in this order, the algorithm assigns c_i to that voter; otherwise, it returns NO (note that this happens only if there are more than i voters in V' whose associated prefixes are contained in $c_1 \succ \cdots \succ c_i$). When all voters in V' have been processed, the algorithm proceeds to its final step.
- Let C' be the set of candidates that precede w in the tie-breaking order and have not been assigned to voters in V' in the previous step. If $C' = \emptyset$, the algorithm returns YES. Otherwise, for each candidate $c \in C'$ the algorithm seeks a voter that will disapprove c in equilibrium (we need one by Observation 2.4). To this end, the algorithm checks whether there exists a voter in V that prefers w over c. If such a voter is found for each candidate in C' (wwe can select the same voter for several candidates in C'), the algorithm returns YES. Otherwise, it returns NO.

The following theorem is then straightforward.

Theorem 3.2. *Algorithm 1 solves* $\exists$NEWIN *in time polynomial in* $|V|$ *and* $|C|$.

An easy corollary of Theorem 3.2 is that we can efficiently check whether a given Doodle poll game has a Nash equilibrium, by querying Algorithm 1 for each $w \in C$. Further, Algorithm 1 can be used to construct an example of a profile with no Nash equilibrium.

Example 3.3. *Consider an election over a candidate set C, $|C| \geq 3$, where for each candidate $c \in C$ there are two voters v, v' with $\delta_v(c) = \delta_{v'}(c) = 1$, $\delta_v(c') = \delta_{v'}(c') = 0$ for all $c' \in C \setminus \{c\}$; not that all voters are dichotomous. As $C^{TOP} = \emptyset$ and each candidate is disliked by $2|C| - 2 > |C|$ voters, our algorithm will output NO at the second step, which means that the respective Doodle poll game has no pure Nash equilibrium.*

Note that if we add voters to the election constructed in Example 3.3, the resulting election still has no Nash equilibrium. This illustrates that in elections where the number of voters is much larger than the number of candidates, the existence of Nash equilibrium is highly unlikely: essentially, there has to be an alternative that is ranked at the top preference level by (almost) all voters. On the positive side, if there is an alternative that is perfect for all voters (i.e., $C^{TOP} \neq \emptyset$), some such alternative wins in every equilibrium.

4 Uncapped social bonus: Randomised tie-breaking

In this section, we demonstrate that breaking ties randomly makes it computationally hard to determine the existence and properties of Nash equilibria in Doodle poll games. Specifically, we consider the following decision problems:

Algorithm 1 $\exists$NEWIN

Input: DPG $\Gamma = \langle V, C, (U_v)_{v \in V} \rangle$ with $\kappa = |C|$ and tie-breaking order $\succ$; candidate $w \in C$.
Output: YES if there exists an equilibrium with winner w; NO otherwise.

$answerC^{TOP} = TestC^{TOP}(\Gamma, w)$
if $answerC^{TOP} \neq$ POSSIBLE **then**
  return $answerC^{TOP}$
end if
$V' := \{v \in V \mid \delta_v(w) < 1\}$
$answerNonSupporters = TestNonSupporters(\Gamma, w, V')$
if $answerNonSupporters \neq$ POSSIBLE **then**
  return $answerNonSupporters$
end if
return $TestDisapprovedAllocation(\Gamma, w, V')$

Algorithm 2 $TestC^{TOP}$ (first stage of $\exists$NEWIN)

Input: DPG $\Gamma = \langle V, C, (U_v)_{v \in V} \rangle$ with $\kappa = |C|$ and tie-breaking order $\succ$; candidate $w \in C$.
Output: YES if there exists an equilibrium with winner w; NO if it is certain that there is no such NE; POSSIBLE otherwise.

$C^{TOP} := \{c \in C \mid \delta_v(c) = 1 \text{ for all } v \in V\}$
if $w \in C^{TOP}$ **then**
  if $w \succ w'$ for all $w' \in C^{TOP}$ **then return** YES
  else return NO
  end if
else
  if $C^{TOP} \neq \emptyset$ **then return** NO
  end if
end if
return POSSIBLE

$\exists$NE. Does a given DPG with randomised tie-breaking and $\kappa = |C|$ possess an equilibrium?

$\exists$NESINGLE. Given a DPG with randomised tie-breaking and $\kappa = |C|$, is there an equilibrium where the winning set is a singleton?

$\exists$NETIE. Given a DPG with randomised tie-breaking and $\kappa = |C|$, is there an equilibrium where the winning set is not a singleton?

We start with the simpler case of dichotomous preferences, for which all these problems turn out to be polynomial-time solvable. However, we then show that $\exists$NE and $\exists$NETIE are NP-hard if voters have three or more levels of preference.

For games with two preference levels, we separately check the existence of equilibrium with a unique winner or with multiple winners. Both turn out to be computationally easy, and hence so is $\exists$NE.

Theorem 4.1. *In games with dichotomous preferences,* $\exists$NE, $\exists$NESINGLE *and* $\exists$NETIE *are polynomial-time solvable.*

Proof. We first consider $\exists$NESINGLE. For each candidate we check if there exists a profile where this candidate is the unique election winner. Fix $w \in C$. By Observation 2.5, in an equilibrium, w is approved by all the voters, and each other candidate gets $|V|-1$ votes. Also, it must be the case that $\delta_v(w) = 1$ for all $v \in V$, as otherwise a voter with zero utility from the outcome will withdraw his approval from w and make C the winner set, thus receiving a positive utility from the outcome. Now, if for each other candidate c there exists a voter v with $\delta_v(c) = 0$, then let one such voter disapprove c, and we have an equilibrium. Otherwise, there is no equilibrium where the winning set is $\{w\}$.

Algorithm 3 $TestNonSupporters$ (second stage of ∃NEWIN)

Input: DPG $\Gamma = \langle V, C, (U_v)_{v \in V} \rangle$ with $\kappa = |C|$ and tie-breaking order $\succ$; candidate $w \in C$; the set V' of voters with $\delta_v(w) < 1$.
Output: YES if there exists an equilibrium with winner w; NO if it is certain that there is no such NE; POSSIBLE otherwise.

if $|V'| > |C|$ **then return** NO
end if
if $\exists v \in V'$ such that $\delta_v(w) = 0$ **then return** NO
end if
if $\exists v \in V'$ and $c \in C$ such that $\delta_v(c) > \delta_v(w)$ and $w \succ c$ **then return** NO
end if
return POSSIBLE

Consider now ∃NETIE. In an equilibrium, each voter v must have $\delta_v(c) = 1$ for each winning candidate c. Indeed, if v approves a candidate c with $\delta_v(c) = 0$, he can withdraw his approval from c and thereby increase his expected utility from the outcome, as each candidate from his top preference level will be selected with a higher probability. Hence, a necessary condition for the existence of an equilibrium with a non-singleton winning set is that there is a tie under strictly truthful voting, i.e., when every voter v approves exactly the candidates with $\delta_v(c) = 1$. We will now show that this condition is also sufficient. Indeed, consider the ballot vector $\mathbf{b}$ with $b^v = \{c \in C \mid \delta_v(c) = 1\}$, and let $W = W(\mathbf{b})$. Suppose that under $\mathbf{b}$ each candidate in W gets s votes. Then any profile where each voter v approves all candidates c with $\delta_v(c) = 1$ as well as some other candidates in $C \setminus W$ so that each of them gets exactly $s - 1$ votes in total, is an equilibrium. □

For games with trichotomous preferences, we obtain NP-hardness results for ∃NE and ∃NETIE. We need the following lemma.

Lemma 4.2. *In a DPG with $\kappa = |C|$, randomised tie-breaking and trichotomous preferences, if at a given profile it is beneficial for a voter to approve a candidate at the intermediate preference level, then it is beneficial for him to approve all the candidates at that level.*

Proof. Let δ denote the utility from the intermediate preference level. Assume that voter v has an incentive to approve one such candidate, and consider approving additional candidates. If these candidates do not join the winning set, then v gains the social bonus β for each such candidate. Otherwise, let k denote the size of the winning set at the original profile, and let S be the total score of the winners under this profile. The expected utility from the outcome for voter v is then given by $\frac{S}{k}$, and by our assumption, $\frac{S}{k} \leq \frac{S+\delta}{k+1}$ (or, equivalently, $\frac{S}{k} \leq \delta$). This yields $\frac{S}{k} \leq \frac{S+x\delta}{k+x}$ for any positive integer x. In case of equality, voter v still gains the social bonus. Finally, if any of the additionally approved candidates becomes a single winner, the new utility from the outcome is δ, and the claim follows. □

Now, we are ready to present our NP-hardness results.

Theorem 4.3. ∃NE *and* ∃NETIE *are NP-hard for trichotomous preferences.*

Due to space limitations, the proof of Theorem 4.3 is postponed to the Appendix. We note that the proof easily extends to more than three preference levels.

5 Capped social bonus

In this section, we consider the variant of our model where κ can be significantly smaller than $|C|$. We demonstrate that, in contrast with the case of $\kappa = |C|$ analysed in the previous sections, in this model there are many Nash equilibria, and use a variant of trembling hand perfect equilibrium to rule out 'bad' equilibria. It turns out that the voters' behaviour in trembling hand perfect equilibria of our games provides a good match to the behaviour observed in practice in Doodle polls, as described by [17]. Therefore, by capping the social bonus, we can both capture more realistic scenarios and obtain more stable outcomes.

Algorithm 4 $\mathit{TestDisapprovedAllocation}$ (third (main) stage of $\exists$NEWIN)

Input: DPG $\Gamma = \langle V, C, (U_v)_{v\in V}\rangle$ with $\kappa = |C|$ and tie-breaking order $\succ$; candidate $w \in C$; the set V' of voters with $\delta_v(w) < 1$.
Output: YES if there exists an equilibrium with winner w; NO if it is certain that there is no such NE.

```
for all v ∈ V' do
    k_v := min{k | δ_v(c_k) > δ_v(w)}
    D_v := {c_1, ..., c_{k_v - 1}}
end for
C^DIS := ∅
repeat
    pick v' from argmin_{v∈V} |D_v|
    if D_{v'} = ∅ then return NO
    else
        c' := arg min{j | c_j ∈ D_{v'}}
        C^DIS := C^DIS ∪ {c'}
        for all v ∈ V' do
            D_v := D_v \ {c'}
        end for
    end if
    V' := V' \ {v'}
until V' = ∅
C' := {c ∈ C | c ≻ w} \ C^DIS
for all c ∈ C' do
    if ∃v ∈ V s.t. δ_v(c) < δ_v(w) then
        C' := C' \ {c}
    end if
end for
if C' := ∅ then
    return YES
else
    return NO
end if
```

Consider first the Nash equilibria of a DPG with $\kappa \leq |C| - 2$. For any candidate $c \in C$, there exists a Nash equilibrium where c gets $|V|$ approvals, and every other candidate gets κ approvals. Importantly, this holds irrespective of the voters' preferences: c can be universally disliked, and some other candidate c' may be at the top preference level of all voters. This example indicates the need for an equilibrium refinement. To this end, we will now define a modified version of Selten's trembling hand perfect equilibrium in order to apply it in our setting: namely, we assume that the voters' hands tremble independently over each cell in a poll (i.e., a candidate) rather than a whole row (strategy).

Specifically, let $\varepsilon \to 0$ be the probability of voter v deviating from his intended action regarding candidate c. We call it the *trembling hand probability* or *TH probability*, in short. Then, the probability that voter v submits a ballot $\tilde{b}^v$ instead of the intended ballot b^v, termed the *ballot TH probability*, is given by:

$$P(\tilde{b}^v \mid b^v) = \varepsilon^{d(\tilde{b}^v, b^v)} (1-\varepsilon)^{|C| - d(\tilde{b}^v, b^v)}, \tag{5}$$

where $d(\tilde{b}^v, b^v)$ is the Hamming distance between binary vectors $\tilde{b}^v$ and b^v.

For a subset $S \subseteq V$ of voters, the *joint ballot TH probability* is given by the product of individual ballot trembling hand probabilities across the set S:

$$P(\tilde{\mathbf{b}}^S \mid \mathbf{b}^S) = \prod_{v\in S} P(\tilde{b}^v \mid b^v). \tag{6}$$

For a given voter v, let $-v = V \setminus \{v\}$ denote the set of his opponents in the game. Then, voter v's utility from submitting a ballot b^v against the intended joint ballot $\mathbf{b}^{-v}$ of the other voters with trembling hands is given by the expectation, under joint conditional probabilities as defined by (6):

$$\tilde{U}(b^v, \mathbf{b}^{-v}) = \sum_{\tilde{\mathbf{b}}^{-v} \in \{0,1\}^{|V|-1}} U(b^v, \tilde{\mathbf{b}}^{-v}) P(\tilde{\mathbf{b}}^{-v} \mid \mathbf{b}^{-v}). \quad (7)$$

We call this utility the *expected TH utility* of voter v.

A ballot b^v that maximises voter v's expected TH utility under his opponents' trembling hand play is a ***TH best response*** for voter v; a *trembling hand perfect equilibrium* is a voting profile where each voter's ballot is a TH best response to other voters' ballots under trembling hand play.

For brevity, we restrict our analysis of TH best responses in Doodle poll games with $\kappa << |C|$ to the case with lexicographic tie-breaking and dichotomous preferences. Given a voter v with dichotomous preferences, we say that c is *good for* v if $\delta_v(c) = 1$ and *bad for* v otherwise. We show that a voter's TH best response is to approve all of his good alternatives, and also some of the least popular bad alternatives, so that the total number of approved candidates does not exceed κ. The detailed procedure of constructing a TH best response is given in Algorithm 5.

Algorithm 5 TH best response

Input: DPG $\Gamma = \langle V, C, (U_v)_{v \in V} \rangle$ with $\kappa << |C|$ and tie-breaking order $\succ$; voter $v \in V$; joint strategy $\mathbf{b}^{-v} \in \left(\{0,1\}^{|C|}\right)^{|V|-1}$ of voters other than v.
Output: strategy $b^v \in \{0,1\}^{|C|}$ for voter v.

$C_v^{TOP} := \{c \in C \mid \delta_v(c) = 1\}$
$C_v^{APP} := C_v^{TOP}$
$C_v^{DIS} := C \setminus C_v^{TOP}$
$C_v^{SAFE} := \{c \in C \setminus C_v^{TOP} \mid s_c(\mathbf{b}^{-v}) + 1 < \max_{c' \in C} s_{c'}(\mathbf{b}^{-v})$
$\vee [s_c(\mathbf{b}^{-v}) + 1 = \max_{c' \in C} s_{c'}(\mathbf{b}^{-v}) \wedge \exists c' \in W(\mathbf{b}^{-v}), c' \succ c]\}$
repeat
  $c' \in \arg\min_{c \in C_v^{SAFE}} s_c(\mathbf{b}^{-v})$
  $C_v^{APP} := C_v^{APP} \cup \{c'\}$
  $C_v^{DIS} := C_v^{DIS} \setminus \{c'\}$
  $C_v^{SAFE} := C_v^{SAFE} \setminus \{c'\}$
until $C_v^{SAFE} = \emptyset$ or $|C_v^{APP}| = \kappa$
return b^v, where $b_c^v = 1$ for $c \in C_v^{APP}$, $b_c^v = 0$ for $c \in C_v^{DIS}$

First, the algorithm initialises the set of candidates for approval, C_v^{APP}, to be the set C_v^{TOP} of v's good alternatives; the bad candidates are placed in C_v^{DIS}. The algorithm then selects the candidates in C_v^{DIS} that are 'safe' for voter v to approve, under the joint strategy $\mathbf{b}^{-v}$ of the other players: this set, C_v^{SAFE}, consists of candidates that would not win the election should v decide to approve them. If C_v^{SAFE} is non-empty and the number of approved candidates does not exceed κ, the algorithm picks the most unpopular alternative from C_v^{SAFE} and moves it from the set of disapproved alternatives, C_v^{DIS}, to the set of approved alternatives, C_v^{APP}. This alternative also gets excluded from C_v^{SAFE}, and the algorithm terminates after C_v^{SAFE} is exhausted or the number of approvals reaches κ.

The following theorem is the main result of this section. It shows that the behaviour of a voter who plays a TH best response is similar to what is observed in practice: a voter approves all of his good candidates and a 'safe' subset of his bad candidates.

Theorem 5.1. *Algorithm 5 computes a TH best response for a voter $v \in V$ in a given DPG with $\kappa << |C|$, lexicographic tie-breaking and dichotomous preferences. Moreover, there are no other TH best responses for v.*

Proof. Take a voter $v \in V$. By Observation 2.1, v has a best response where he approves all his good alternatives. However, there may be also other best response strategies where voter v only approves a (sufficiently large) subset of his good alternatives. Now, by the same argument as before, there is also a TH best response where v approves all his good alternatives. Indeed, if a voter disapproves any of his good alternatives, he cannot lower his total utility by adding such an alternative to his ballot, independently of the choices (or mistakes made due to the trembling hand) of his opponents. This is because by doing so he can never lower his utility from the outcome, while his social bonus may only grow. Moreover, in the presence of the trembling hand, it is no longer a best response for v to approve only a subset of C_v^{TOP}, as there is a positive probability that v is a pivotal voter for one of his disapproved good candidates, who will be beaten by v's bad candidate, thus reducing v's expected utility. Hence, in a TH best response, a voter must approve all of his good alternatives.

If $|C_v^{TOP}| \geq \kappa$, the algorithm returns C_v^{TOP} as the set of approvals, and this is the only TH best response strategy for voter v. If $|C_v^{TOP}| < \kappa$, it may be beneficial for v to also approve some of his bad alternatives. However, while in the absence of the trembling hand it would be a best response strategy for v to approve any subset of C_v^{SAFE} so that the total number of approved candidates reaches or even exceeds κ, in a TH best response, it becomes unsafe to exceed the cap, as with a positive probability voter v is pivotal for some of his approved bad candidates, while the current winner may be among v's good candidates. Hence, it is clearly a dominated strategy to approve extraneous bad candidates, i.e., in a TH best response, a voter approves $\min\{|C_v^{SAFE}|, \kappa - |C_v^{TOP}|\}$ of his bad candidates.

It remains to prove that in a TH best response, a voter only approves least popular candidates from C_v^{SAFE}. Assume there are two candidates $c_i, c_j \in C_v^{SAFE}$ such that $s_{c_i}(\mathbf{b}^{-v}) > s_{c_j}(\mathbf{b}^{-v})$. It suffices to show that in the presence of the trembling hand, approving c_i is strictly less beneficial for v than approving c_j, for any combination of approvals over other candidates by voter v. For convenience, we assume that both c_i and c_j are lower in the tie-breaking order than all of v's good alternatives. The proof for other cases is analogous, but requires a lengthy case analysis.

Note that since $\delta_v(c_i) = \delta_v(c_j) = 0$, if the current winner of the election (that is, the winner under $\mathbf{b}^{-v}$ and the ballot obtained from b^v by removing approvals for c_i and c_j) is not in C_v^{TOP}, then v is indifferent between voting for c_i and voting for c_j. Therefore, we only need to consider the cases where the current winner belongs to C_v^{TOP}. In such cases, the utility from the outcome for voter v can only change (namely, decrease) if his newly approved bad candidate (c_i or c_j) beats the current winner. Thus, we need to show that the probability of c_i beating the current winner is strictly greater than the probability of c_j beating the current winner.

Let M be the number of approvals received by the election winner, and set

$$p_1 = P\left(s_{c_i}(\tilde{\mathbf{b}}^{-v}) = M \mid b^v\right), \quad p_2 = P\left(s_{c_j}(\tilde{\mathbf{b}}^{-v}) \leq M \mid b^v\right).$$

Then, $p_1 p_2$ is the probability that v makes c_i (but not c_j) the election winner. Similarly, let

$$p_3 = P\left(s_{c_j}(\tilde{\mathbf{b}}^{-v}) = M \mid b^v\right), \quad p_4 = P\left(s_{c_i}(\tilde{\mathbf{b}}^{-v}) \leq M \mid b^v\right);$$

then $p_3 p_4$ is the probability that v makes c_j (but not c_i) the election winner. It suffices to prove that $p_1 > p_3$ and $p_2 > p_4$.

To compute p_1 and p_3, for each $|V|$-dimensional binary vector with exactly M ones (that is, for each possible $\tilde{\mathbf{b}}^{-v}$ where c_i or c_j gets exactly M votes), we consider its Hamming distance to each of the vectors $\mathbf{b}_{c_i}^{-v}$ and $\mathbf{b}_{c_j}^{-v}$, representing the other voters' intended votes for c_i and c_j, respectively. For convenience, and without loss of generality, we permute the voters so that $\mathbf{b}_{c_i}^{-v}$ and $\mathbf{b}_{c_j}^{-v}$ are decreasing, that is:

$$\mathbf{b}_{c_i}^{-v} = (\underbrace{1, \ldots, 1}_{i}, 0, \ldots, 0), \quad \mathbf{b}_{c_j}^{-v} = (\underbrace{1, \ldots, 1}_{j}, 0, \ldots, 0),$$

where $i = s_{c_i}(\mathbf{b}^{-v})$, $j = s_{c_j}(\mathbf{b}^{-v})$ and $i > j$.

Note that these two vectors are identical in the first j bits. Thus, we can also divide all the vectors with M ones into disjoint subsets defined by their configuration in the first j bits, as the distance from $\mathbf{b}_{c_i}^{-v}$ and $\mathbf{b}_{c_j}^{-v}$ to any vector in a given such subset will be the same in these first bits. So, we prove separately for each subset, where we only need to consider the remaining $n = |V| - j$ bits.

Fix such a subset, and denote the number of ones in the last n bits of each of its vectors by $M' \leq M$ (different configurations of these M' ones define different vectors in the subset). Note that vector $\mathbf{b}_{c_j}^{-v}$ only has zeroes in the last n entries, and vector $\mathbf{b}_{c_i}^{-v}$ has $i' = i - j$ ones and $n - i'$ zeroes. The required probabilities would then depend on the number of those of the M' ones that fall in the first i' bits, and those that fall in the remaining $n - i'$ bits. We get

$$p_1 = \sum_{k=0}^{i'} \binom{i'}{k} \binom{n-i'}{M'-k} \varepsilon^{M'+i'-2k} (1-\varepsilon)^{n-M'-i'+2k} \tag{8}$$

and

$$p_3 = \sum_{k=0}^{i'} \binom{i'}{k} \binom{n-i'}{M'-k} \varepsilon^{M'} (1-\varepsilon)^{n-M'}. \tag{9}$$

In Equation (8), the voters make two type of mistakes: $i' - k$ of them consist of placing zeroes instead of ones, and $M' - k$ of them consist of placing ones instead of zeroes. Hence, we get $M' + i' - 2k$ deviations in total. In Equation (9), all M' ones are a result of the trembling hand.

For comparison, divide both equations by $\varepsilon^{M'}(1-\varepsilon)^{n-M'}$. We have to show that

$$\sum_{k=0}^{i'} \binom{i'}{k}\binom{n-i'}{M'-k}\varepsilon^{i'-2k}(1-\varepsilon)^{2k-i'} > \sum_{k=0}^{i'} \binom{i'}{k}\binom{n-i'}{M'-k}. \tag{10}$$

Note that $\varepsilon^{i'-2k}(1-\varepsilon)^{2k-i'} = \left(\frac{1-\varepsilon}{\varepsilon}\right)^{2k-i'} > 0$ for any k, so we can exclude the terms with $k < \frac{i'}{2}$ from the left hand side. We obtain

$$\sum_{k=\frac{i'}{2}}^{i'} \binom{i'}{k}\binom{n-i'}{M'-k}\left(\frac{1-\varepsilon}{\varepsilon}\right)^{2k-i'} > \sum_{k=0}^{i'} \binom{i'}{k}\binom{n-i'}{M'-k}. \tag{11}$$

Now, for $k = \frac{i'}{2}{}^{i'}$ we have $\left(\frac{1-\varepsilon}{\varepsilon}\right)^{2k-i'} = \left(\frac{1-\varepsilon}{\varepsilon}\right)^{0} = 1$, so we can extract the corresponding term on both sides of the inequality. Finally, using the symmetry of binomial coefficients on the right hand side, we see that

$$\sum_{k=\frac{i'}{2}+1}^{i'} \binom{i'}{k}\binom{n-i'}{M'-k}\left(\frac{1-\varepsilon}{\varepsilon}\right)^{2k-i'} > 2\sum_{k=\frac{i'}{2}+1}^{i'} \binom{i'}{k}\binom{n-i'}{M'-k}. \tag{12}$$

The inequality holds since $\frac{1-\varepsilon}{\varepsilon} > 2$ (by the assumption of $\varepsilon \to 0$) and $2k - i' > 1$.

The proof of $p_2 > p_4$ is similar: we show the inequality for each $0 \le m \le M$, and then sum up. □

Importantly, one can use Algorithm 5 to compute a trembling hand perfect equilibrium of a dichotomous Doodle poll game. To this end, we start with the profile where each voter approves his good alternatives, and then allow each voter to make a best response move in some fixed order. We obtain the following corollary.

Corollary 5.1. *Given a Doodle poll game with $\kappa << |C|$, lexicographic tie-breaking and dichotomous preferences, it is possible to compute a trembling hand perfect equilibrium in time polynomial in $|V|$ and $|C|$.*

6 Conclusions

Our aim in writing this paper was to provide a plausible explanation to the counterintuitive phenomena observed in Doodle polls. While [17] suggested that these phenomena may be caused by voters' desire to appear cooperative, they stopped short of providing a model of voters' utilities that matches the observed behaviour. Building on their work, we developed a computationally tractable model whose results agree with the real-life data. Besides the basic idea of rewarding voters for approving additional alternatives, our model has two new ingredients: placing a cap on the social bonus and identifying a suitable equilibrium refinement, namely, trembling hand perfect equilibria.

Our analysis demonstrates that the concept of trembling hand equilibria is very useful in the context of voting games, where each player (voter) has limited power and hence, we may have multiple (and often undesirable) pure Nash equilibria. It is perhaps not surprising that profiles arising at trembling hand perfect equilibria are intuitively appealing; interestingly, they also turn out to be efficiently computable for a rich class of settings (lexicographic tie-breaking and dichotomous preferences). It then may be interesting to understand the structure of trembling hand equilibria in other important application scenarios that have received significant attention in algorithmic game theory, such as, e.g., congestion games.

While we see our results for the setting with capped social bonus as our main contribution, the algorithmic results in Sections 3 and 4 help us identify important features of our model. In particular, we use the algorithm for lexicographic preferences described in Section 3 in order to build a profile with no Nash equilibrium and to argue that such profiles are quite likely in large elections. On the other hand, hardness results of Section 4 indicate that randomised tie-breaking gives rise to games where finding a Nash equilibrium is intractable, so, when studying the setting with capped social bonus, we focus on the easier case of lexicographic tie-breaking.

The main take-home message of [17] as well as of our work is that there are settings where voters engage in strategic behaviour under Approval voting. In particular, this happens when voters' utilities are different from those assumed in the classical Approval model, and in particular, take into account social effects. It would be interesting to investigate the potential impact of such effects in other voting scenarios.

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Appendix

Proof of Lemma 3.1. Let v be a voter that approves all candidates in C and assume that $\delta_v(w) < 1$. Then, there exists another candidate c with $\delta_v(c) > \delta_v(w)$. By Observation 2.3, winner w is approved by all $|V|$ voters. Let C^+ be the set of all candidates that precede w in the tie-breaking order, and let C^- be the set of all candidates that appear after w in the tie-breaking order. By Observation 2.4, all candidates in C^+ get $|V| - 1$ votes and all candidates in C^- get $|V|$ votes. Suppose that v changes her vote from C to $\{c\}$. If $c \in C^-$, she becomes the only candidate in C with $|V|$ points, so she becomes the unique winner. If $c \in C^+$, then after the change all candidates have at most $|V| - 1$ points and all candidates in $C^+ \setminus \{c\}$ have $|V| - 2$ points, so c wins by the tie-breaking rule. Thus, in both cases c becomes the new winner, so voter v will get a higher utility from the outcome. Moreover, by our assumption about the value of β, his total utility will also increase.

Suppose now that there exist a voter v and a candidate c such that $c \geq_v w$, but $c \notin b^v$. Clearly, approving c is a profitable move for v: the outcome either remains the same or changes from w to c, so his utility from the outcome does not go down, and his social bonus increases by β. ☐

Proof of Theorem 4.3. We reduce from Exact 3-Set Cover (X3C):

- X3C. Given a set $X = \{x_1, \ldots, x_{3m}\}$ and a family $Z = \{z_1, \ldots, z_n\}$ of triples $z_j = \{x_{j_1}, x_{j_2}, x_{j_3}\} \subseteq X$, $j = 1, \ldots, n$, is there a subfamily Z' of Z such that every element in X is contained in exactly one triple of Z'?

We construct an instance of ∃NE/∃NETIE as follows. Let $C = Z \cup X \cup \{w, y\}$ be the set of candidates where Z contains n elements, which correspond to triples in X3C, X is a set of $3m$ candidates corresponding to the ground set in X3C, and w and y are distinguished candidates. The voters' preferences are given by Table 1.

Table 1: ∃NE and ∃NEtie. Voters' preferences.

Block 1: n	**Block 2**: $3m$	**Block 3**: $f-m-1$	**Block 4**: 1	**Block 5**: $m-2$	**Block 6**: f
z_1 … z_n	x_1 … x_{3m}	x_1 ⋮ x_{3m} w z_1 ⋮ z_n	x_1 ⋮ x_{3m} w	x_1 ⋮ x_{3m}	y
w … w x_{1_1} … x_{n_1} x_{1_2} … x_{n_2} x_{1_3} … x_{n_3} z_2 … z_1 ⋮ … ⋮ z_n … z_{n-1}	w … w				w
y … y $X \setminus \cup_{j=1}^{n} z_j$	y … y Z	y	y Z	y Z w	Z X

There is a set V of voters, divided into 6 blocks. As can be seen from Table 1, Blocks 1 contains n voters, Block 2 has $3m$ voters, and Block 3 has $f - m - 1$ voters where f is a large constant ($f >> n >> 3m$). Furthermore, there is one voter in Block 4, $m - 2$ voters in Block 5, and f voters in Block 6.

The candidates in each block are divided into three preference levels, with the intermediate level being possibly empty. The utility from the outcome to each voter is 1 from the top level, δ from the intermediate level, and 0 from the bottom level. The intermediate utility satisfies $\beta|C| < \delta < \frac{1}{|C|}$.

By the choice of δ, if a winning set contains a candidate from the top preference level of a given voter, then it is not beneficial for that voter to add candidates from lower levels to the winning set. If, however, none of the winners belongs to the voter's top preference level, it is then beneficial to approve a candidate from the intermediate level (in fact, all such candidates, by Lemma 4.2).

Also, v can only benefit from adding a candidate c with $\delta_v(c) = 0$ to the winning set if all the current winners give v voter utility 0. However, if v benefits from adding c to the winning set, it is beneficial for him to approve all candidates in C.

Now, assume that there is an equilibrium profile under the preferences given in Table 1, where the winner is unique. Then, by Observation 2.5, all the voters approve the winner, and the other candidates receive $|V| - 1$ votes each. In our preference profile though, every candidate appears at the bottom preference level in at least one vote. Hence, such a voter has an incentive to withdraw his approval from the single winner, which has zero utility for him anyway, so that all the candidates are in a tie and he gets some positive utility from the outcome. That is, in our instance of the game an equilibrium with a single winner does not exist. Therefore, we prove the correctness of the reduction for ∃NETIE, and the result for ∃NE follows directly.

First, we show that there is no equilibrium where a winner gets more than f votes. We consider four possible cases, depending on the membership of candidates w and y in a winning set, W:

$y \in W, w \notin W$ Since w is not among the winners, each voter in Block 2 has his top choice candidates winning, so we have $X \subseteq W$. This, in turn, implies that y only gets votes from Block 6, because all voters in other blocks get positive utility from X, so they will not vote for y, which is at their bottom preference level. Since $y \in W$ and gets only f points, all members of W get f points.

$y \in W, w \in W$ In this case, if y gets more than f votes, they come from Blocks 4 and 5, and this is only possible if none of the elements of X wins. But then all voters in Block 2 vote for w, in addition to Blocks 3 and 4, for whom w is a top choice. Thus, w gets at least $f + 2m$ votes, while y obtains at most $f + m - 1$, a contradiction.

$y \notin W, w \in W$ Since y is not a winner, all the voters in Block 6 must approve w, in addition to Blocks 3 and 4 for whom w is a top choice. Hence, w gets $2f - m$ votes. The other candidates though get at most $n + f - 1$ votes from Blocks 1–5 where they are in the top, but for them to win, they need to get votes from those voters where they are in the bottom preference level. These cannot come from Blocks 1–3, 4 and 6, as those have w with positive utility and w is a winner, so only Block 5 can possibly help, but this adds only $m - 2$ votes, so that in total we get $f + n + m - 3$, which is less than the score of w, a contradiction.

$y \notin W, w \notin W$ In this case, Block 6 votes for everyone, having utility 0 from the outcome. They therefore can change the winner by removing their votes from the current winners and bringing all the candidates to a tie, a contradiction.

Hence, candidate y must be among the winners, and the winning score is exactly f.

Consider now the number of elements of Z in the winning set. If there are more than m such elements, then more than m voters from Block 1 approve elements of X, and hence they get in total more than $3m$ approvals. Hence, at least one x is approved at least twice in Block 1. This candidate x then gets $f + 1$ votes, a contradiction.

If there are fewer than m elements of Z in the winning set, then there is at least one non-winning candidate z. Hence, there must be at least one winning candidate x (as all xs get $f - 1$ votes from Blocks 2–5, and at least one such candidate will also get at least one vote from Block 1). Hence, elements of Z get no votes from Blocks 4 and 5. Thus, the score of the winning candidate z is composed of: 1 vote from Block 1 where z is on top, $l < m$ votes from Block 1 where z is at the intermediate preference level, l' votes from Block 2 where corresponding element of X is not a winner (but only in case that w does not also win), and $f - m - 1$ votes from Block 3. Consider now the score of w. She gets: l votes from Block 1, l' votes from Block 2, $f - m - 1$ votes from Block 3 and 1 vote from Block 4. Hence, the score of w is at least as high as the score of z, and so w also wins, in which case z does not get her l' votes from Block 2, a contradiction. If there are no xs among the winners, then w gets all votes from Block 2, and her score becomes $f + 2m - 1 > f$, a contradiction.

Finally, if there are exactly m elements of Z in the winning set, then exactly m votes go to the intermediate level in Block 1, in which case each element of the ground set X gets exactly one vote. This defines an exact 3-cover. □

Svetlana Obraztsova
I-Core (The Hebrew University of Jerusalem)
Jerusalem, Israel
Email: svetlana.obraztsova@gmail.com

Maria Polukarov
University of Southampton
Southampton, United Kingdom
Email: mp3@ecs.soton.ac.uk

Zinovi Rabinovich
Independent Researcher
Email: zr@zinovi.net

Edith Elkind
University of Oxford
Email: elkind@cs.ox.ac.uk