

Analyzing Games with Ambiguous Player Types using the MINthenMAX Decision Model

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Abstract

In many common interactive scenarios, participants lack information about other participants, and specifically about the preferences of other participants. In this work, we model an extreme case of incomplete information, which we term *games with type ambiguity*, where a participant lacks even information enabling him to form a belief on the preferences of others. Under type ambiguity, one cannot analyze the scenario using the commonly used Bayesian framework, and therefore he needs to model the participants using a different decision model.

In this work, we present the MINthenMAX decision model under ambiguity. This model is a refinement of Wald's MiniMax principle, which we show to be too coarse for games with type ambiguity. We characterize MINthenMAX as the finest refinement of the MiniMax principle that satisfies three properties we claim are necessary for games with type ambiguity. This prior-less approach we present here also follows the common practice in computer science of worst-case analysis.

Finally, we define and analyze the corresponding equilibrium concept assuming all players follow MINthenMAX. We demonstrate this equilibrium by applying it to two common economic scenarios: coordination games and bilateral trade. We show that in both scenarios, an equilibrium in pure strategies always exists and we analyze the equilibria.

1 Introduction

In many common interactive scenarios participants lack information about other participants, and specifically about the preferences of other participants. The extreme case of such partial information scenario is termed *ambiguity*, and in our case ambiguity as to the preferences of other participants. In these scenarios, not only does a participant not know the preferences of other participants, but he cannot even form a belief on them (that is, he lacks the knowledge to form a probability distribution over preferences). Hence, one cannot analyze the scenario using the Bayesian framework, which is the common practice for analyzing partial-information scenarios, and new tools are needed.¹ Similarly, in the computer science literature, many times algorithms, agents, and mechanisms are analyzed without assuming a distribution on the input space or on the environment. In this paper, we define and analyze equilibria under ambiguity regarding the information of other players, namely their *type*, and we concentrate on equilibria of *games with type ambiguity*, i.e., games with ambiguity regarding other players' preferences. Our equilibria definition is based on a refinement of Wald's MiniMax principle, which corresponds to the common practice in computer science of worst-case analysis.

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Due to space considerations, some of the proofs and the full description of one of the examples are omitted. They can be found in the full version (<http://arxiv.org/abs/1603.01524>) and in the appendix.

¹Clearly, if a player has information that he can use to construct a belief regarding the others, we expect the player to use it. In this work, we study the extreme case in which one has no reason to assume the players hold such a belief.

In Section 2, we define a general model of games with ambiguity, similar to Harsanyi’s model of games with incomplete information [17],² and derive from it the special case of *games with type ambiguity*. In this model, the knowledge of Player i on Player j is represented by a set of types \mathcal{T} . Player i knows that the type of Player j belongs to \mathcal{T} , but has no prior distribution on this set, and no information that can be used to construct one.

Next, we present a novel model for decisions under ambiguity: MINthenMAX preferences. We characterize MINthenMAX as a decision model in the general framework of decision under partial information, as the unique finest preference that satisfies a few natural properties. Specifically, we claim these properties are satisfied by players when facing games with type ambiguity, thereby justifying choosing MINthenMAX as the analysis tool.

Finally, we derive the respective equilibrium concept, dubbed MINthenMAX-NE, and present some of its properties, both in the context of the general model of games with type ambiguity and in two common economic scenarios’ models.

Wald’s MiniMax principle A common model for decision under ambiguity is the MiniMax principle presented by Wald [27], which we refer to as MIN preferences (in contrast to the MINthenMAX preferences that we present).³ In this model, similarly to the worst-case analysis in computer science, the preference of the decision maker over actions is based solely on the set of possible outcomes (in games with type ambiguity, the possible outcomes are the consequence of playing the game with the possible types of the other players). An action a is preferred to another action b if the worst possible outcome (for the decision maker) from taking action a is better than the worst outcome from taking action b . This generalizes the classic preference maximization model: if there is no ambiguity, there is a unique outcome for each action, and the MIN decision model coincides with preference maximization. The MIN model has been used for analyzing the expected behavior in scenarios of decision under ambiguity regarding some parameters of the environment, e.g., ambiguity regarding the distribution of prizes (the multi-prior model) [16], and ambiguity of the decision maker regarding his own utility [10]. MIN has also been used for analyzing interactive scenarios with ambiguity, e.g., first-price auctions under ambiguity of both the bidders and the seller regarding the ex-ante distribution of bidders’ values [9], and for designing mechanisms assuming ambiguity of the players regarding the ex-ante distribution of other players’ private information [28].

As we show shortly, as part of the coordination games example, the MIN decision model is too coarse and offers too little predictive value in some scenarios involving ambiguity regarding other players. We show a natural scenario (a small perturbation of the battle of the sexes game [19]), in which almost all action profiles are Nash equilibria according to MIN. Hence, we are looking for a refinement of the MIN model that breaks indifferences in some reasonable way in cases in which two actions result in equivalent worst outcomes. In Section 3, we show that naïvely breaking indifferences by applying MIN recursively⁴ does not suit scenarios with ambiguity regarding other players either. We present two game scenarios, and we claim that they are equivalent in a very strong sense: a player cannot distinguish between these two scenarios, even if he has enough information to know the outcomes of all of his actions. Hence, we claim that a rational player should act the same way in these two scenarios. Yet, we show that a player that follows the recursive MIN decision model does not play the same way in the two scenarios. We claim that a decision model for scenarios with type ambiguity should not be susceptible to this problem.

The MINthenMAX decision model In this work we suggest a refinement of Wald’s MiniMax principle that is not susceptible to the above problems, and we term this preference

²As described, for example, in [21, Def. 10.37 p. 407].

³Wald measures actions by their losses while we analyze actions by their possible gains. This is the reason that this rule, which aims to maximize the worst (minimal) gain, was named by Wald the MiniMax principle.

⁴I.e., when the decision maker faces two actions that have equivalent worst outcomes, he decides according to the second-worst outcome, and so on.

MINthenMAX. According to MINthenMAX the decision maker (DM) picks an action having an optimal worst outcome (just like under MIN), and breaks indifferences according to the best outcome. We characterize MINthenMAX as the unique finest refinement of MIN that satisfies three desired properties (Section 3): monotonicity in the outcomes, state symmetry, and independence of irrelevant information. *Monotonicity in the outcomes* is a natural rationality assumption stating that the DM (weakly) prefers an action a to an action b if in every state of the world (in our framework, a state is a vector of types of the other players), a results in an outcome that is at least as good as the outcome of b . *State symmetry* asserts that the decision should not depend on the names of the states and should not change if the names are permuted. *Independence of irrelevant information* asserts that the DM should not suffer from susceptibility to irrelevant information bias, which we describe above. That is, the DM’s decision should depend only on state information that is relevant to his utility. Specifically, it requires that if two states of the world have the same outcomes for each of the actions, the distinction between the two should be irrelevant for the DM, and his preference over actions should not change in case he considers these two states as a single state. We show that these properties characterize the family of preferences that are determined by only the worst and the best outcomes of the actions. Moreover, we show that MINthenMAX is the finest refinement of MIN in this family: for any preference P that satisfies the three properties, if P is a refinement of MIN,⁵ then MINthenMAX is a refinement of P .

Equilibrium under MINthenMAX preferences In Section 2, we define MIN-NE to be the equilibria under MIN preferences, that is, the set of action profiles in which each player best-responds to the actions of other players, and similarly we define MINthenMAX-NE to be the equilibria under MINthenMAX preferences.

We show that for every game with ambiguity, a MIN-NE in mixed strategies always exists (Thm. 4). On the other hand, we show there are generic games with ambiguity in which the set of MIN-NE is unrealistic and too large to be useful. This holds even for cases in which the ambiguity is symmetric (all players have the same knowledge) and is only regarding other players’ preferences. Here, once again is our motivation for studying the equilibria under MINthenMAX. On the other hand, we present a simple generic two-player game with type ambiguity for which no MINthenMAX-NE exists. We note that since MINthenMAX is the unique finest refinement of MIN (which satisfies some properties), the equilibria of a game with ambiguity under any other refinement of MIN is a super-set of the MINthenMAX-NE set. Hence, one can think of MINthenMAX-NE as the set of equilibria that do not depend on assumptions regarding the tie-breaking rule over MIN applied by the players.

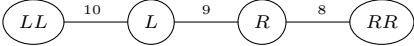
We show that finding a MIN-NE is a *PPAD-complete* problem [25, 24], same as the computational complexity of finding a Nash equilibrium when there is no ambiguity.

Applications of MINthenMAX-NE to economic scenarios To understand the benefits of analysis using the MINthenMAX model, we apply MINthenMAX-NE to two well-studied economic scenarios while introducing ambiguity: coordination games and bilateral trade games. We show that in both scenarios, a MINthenMAX-NE in pure strategies always exists, and we analyze these equilibria.

Coordination games In coordination games, the players simultaneously choose a location for a common meeting. All players prefer to choose a location that maximizes the number of players they meet, but they differ in their tie-breaking rule, i.e., their preference over the possible locations. This is a generalization of the game battle of the sexes [19, Ch. 5, Sec. 3], and it models economic scenarios where the players need to coordinate a common action, like agreeing on a meeting place, choosing a technology (e.g., cellular company), and locating a public good when the cost is shared, as well as Schelling’s focal point

⁵That is, for any two actions a and b , if a is strongly preferred to b according to MIN, then a is strongly preferred to b according to P too.

experiments [26, p. 54–57]: the parachuters’ problem and meeting in NYC problem.

For example, consider a linear street with four possible meeting locations: LL , L , R , and RR (see ) and two players who choose locations simultaneously, trying to meet each other. Both players prefer the meeting taking place to not taking place, but have different preferences over the meeting places (and both are indifferent regarding their action if the meeting does not take place). We also assume that the preference of each player is by the distance to a location he is located in.⁶

Consider a scenario in which each player can be located in each of the four locations, but each player does not know the location of the other; i.e., there is ambiguity regarding the types of the players (and hence regarding their preference).

First, we notice that profiles in which both players choose, regardless of their types, the same location are MINthenMAX-NE of the game. From the perspective of Player 2, if all the types of Player 1 choose the same location, then choosing this location and meeting Player 1 for sure is strictly preferred (regardless of the position of Player 2) to any other choice, which would surely result in no meeting. The analysis for Player 1 is identical, proving that these profiles are indeed MINthenMAX-NE.

Next, consider a case in which Player 1 goes either to LL or to RR (i.e., the locations chosen by the types of Player 1 are these two locations). From Player 2’s perspective, all of his actions are equivalent in terms of their worst outcome, as there is always a possibility of not meeting Player 1 (when facing a type of Player 1 who chose a different location). Thus, Player 2 chooses according to the best outcome for him. Only the actions LL and RR result in a possibility to meet Player 1, i.e., meet one of the types of Player 1, and hence these actions are strictly preferred to L and R , and so Player 2 will choose between LL and RR according to his preference over them. Following this reasoning, we show that the profile in which each type of each of the players goes to the location closest to him out of LL and RR is MINthenMAX-NE, and that any other profile in which one of the players plays both LL and RR (i.e., the locations played by his types are these two locations) is not MINthenMAX-NE.

This simple scenario also demonstrates the drawback of using the MIN model to analyze games with type ambiguity. When Player 1 goes to either LL or RR , Player 2 is indifferent between the worst outcomes of all of his actions, and so according to the MIN model Player 2 is indifferent between all of his actions. Hence, the profile in which each type goes to the location **farthest** from him out of LL and RR is MIN-NE. This seems highly unrealistic: we would expect a rational Player 2 of type LL to prefer playing LL to RR .⁷

We show that, in general, for every coordination game with type ambiguity the pure Nash equilibria of the no-ambiguity case, in which all players choose the same unique location are also MINthenMAX-NE.⁸ Note that the set of equilibria when there is no ambiguity does not depend on the players’ types (which are only a tie-breaking rule between two locations having the same number of other players). Yet, we get that ambiguity regarding the types gives rise to new equilibria, which we characterize. We show that an equilibrium is uniquely defined by a set of meeting locations (LL and RR in the example above) to be the action profile in which each type of each player chooses his optimal location in this set. Finally, we also characterize the equilibria for several cases in which we assume a natural homogeneity constraint on the type sets. The constraint we choose, taking the type sets to be single-peaked consistent w.r.t. a line, restricts the ambiguity regarding other players in a natural way, and hence its impact on the set of equilibria is interesting.

⁶For example, the preference of a player located in L is $L \succ R \succ LL \succ RR$.

⁷In general, for a coordination game over m locations and n players s.t. each of them has at least t types, more than $(1 - 1/m^{t-1})^n$ -fraction of the pure action profiles are MIN-NE of the game.

⁸In general, if an action profile a is a Nash equilibrium for all the possible combinations of types, then a is also a MINthenMAX-NE, since we expect the players to follow the profile when the information regarding the types is irrelevant to their decision.

Bilateral trade The second scenario we analyze (in the full version) is bilateral trade. These are two-player games between a seller owning an item and a buyer who would like to purchase the item. Both players are characterized by the value they attribute to the item (their respective willingness to accept and willingness to pay). In the mechanism that we analyze, both players simultaneously announce a price and if the price announced by the buyer is higher than the one announced by the seller, then a transaction takes place and the price is the average of the two.⁹ For simplicity, we assume that a player has the option not to participate in the trade.¹⁰

When there is no ambiguity, an equilibrium that includes a transaction consists of a single price, which is announced by both players. When there is ambiguity regarding the values, we show that in addition a new kind of equilibrium emerges. For instance, consider the case in which the value of the buyer can be any value between 20 and 40 and the value of the seller can be any value between 10 and 30. First, we notice that there are equilibria that are based on one price as above,¹¹ but in any such equilibrium there will be types (either of the buyer or of the seller) that will prefer not to participate. If, for example, the price is 25 or higher, then there are types of the buyer that value the item at less than this price and will prefer not to participate; similarly, for prices below 25 there are possible sellers that value the item at more than 25 and hence will prefer not to participate. We show also MINthenMAX-NE with two prices, 15 and 35, and in which both players participate regardless of their type: the seller announces 35 if his value is higher than 15 and 15 otherwise, and the buyer announces 35 if his value is higher than 35 and 15 otherwise.¹² In this profile, a buyer that values the item at more than 35 prefers buying the item at either price to not buying it, and hence he best-responds by announcing 35 and buying the item for sure. A buyer that values the item at less than 35 prefers buying the item at 15 to not buying it, and not buying the item to buying it at 35. The best worst-case outcome he can guarantee is not buying the item (e.g., by announcing any value between 15 and his value). Based on the worst outcome the buyer is indifferent between these announcement, hence choosing between these announcements according to the best-outcome (i.e., meeting a seller who announces 15), he best-responds by announcing 15. Similar analysis shows that also the seller best-responds in this profile.

We characterize the set of MINthenMAX-NE for bilateral trade games, and in particular we show that for every bilateral trade game, an equilibrium consists of at most two prices. As a corollary we characterize the cases for which there exists a full-participation MINthenMAX-NE, i.e., equilibria in which both players choose to participate regardless of their value (but their bid in these equilibria might depend on the value).

2 Model

We define a more general model - game with ambiguity¹³ and then derive from it as a special case games with type ambiguity. A *game with ambiguity* is a vector $\langle \mathcal{N}, (\mathcal{A}^i)_{i \in \mathcal{N}}, \Omega, (u^i)_{i \in \mathcal{N}}, (\mathcal{T}^i)_{i \in \mathcal{N}} \rangle$, where:

- $\langle \mathcal{N}, (\mathcal{A}^i)_{i \in \mathcal{N}} \rangle$ is an n -player game form. That is, \mathcal{N} is a finite set of *players* $N = \{1, \dots, n\}$; and \mathcal{A}^i is a finite set of *actions* of

⁹Our result also holds for a more general case than setting the price to be the average.

¹⁰This option is equivalent to the option of the seller declaring an extremely high price that will not be matched (and similarly for the buyer).

¹¹An equilibrium in which both players choose (as a function of their value) either to announce a price common to both or not to participate.

¹²I.e., the seller announces the lower of the two if both are acceptable to him, and the higher otherwise; And the buyer announces the higher of the two if both are acceptable to him, and the lower otherwise.

¹³For simplicity we define a finite game but the definitions extend to infinite cases as well. Our definitions of preferences and results also extend under minor technical assumptions.

Player i , and we denote by \mathcal{A} the set of *action profiles* $\times_{i \in \mathcal{N}} \mathcal{A}^i$.

- Ω is a finite set of *states of nature*.
- $u^i: \Omega \times \mathcal{A} \rightarrow \mathfrak{R}$ is a *utility function* for a Player i specifying his utility from every state of nature and profile of actions. We identify u^i with its linear extension to mixed actions - $u^i: \Omega \times \Delta(\mathcal{A}^i) \rightarrow \mathfrak{R}$, where $\Delta(\mathcal{A}^i)$ is the set of mixed actions over \mathcal{A}^i .¹⁴
- $\langle \mathcal{N}, \Omega, (\mathcal{T}^i)_{i \in \mathcal{N}} \rangle$ is an Aumann model of incomplete information. That is, \mathcal{T}^i is a partition of Ω to a finite number of partition elements ($\Omega = \cup_{t^i \in \mathcal{T}^i} t^i$). We refer to $t^i \in \mathcal{T}^i$ as a *type* of Player i .¹⁵

The above is commonly known by the players. A game proceeds as follows:

- Nature chooses (arbitrarily) a state of the nature $\omega \in \Omega$.
- Each player is informed about his own partition element $t^i \in \mathcal{T}^i$ satisfying $\omega \in t^i$.
- The players play their actions simultaneously: Player i , knowing his type t^i , selects a (mixed) action $a^i \in \Delta(\mathcal{A}^i)$.
- Every player gets a payoff according to u : Player i gets $u^i(t, a)$, where $a = (a^1, a^2, \dots, a^n)$ is the action profile, and $t = (t^1, t^2, \dots, t^n)$ is the type profile.

Notice that the difference between this model and the common model of games with incomplete information [17] (e.g., as described in [21, Def. 10.37 p. 407]), is that in the latter it is assumed that the players also have posterior distributions on t^i (or equivalently, they have subjective prior distributions on Ω).

In this work we are interested in *games with type ambiguity*. In these games the states of nature are the type vectors $\Omega = \times_{i=1}^n \mathcal{T}^i$, so a player cannot deduce from his type information regarding the type of others. For this restricted model, we justify our choice of MINthenMAX preferences. Note that we prove the existence of a mixed MIN-NE (Thm 4) for every game with ambiguity.

A strategy of a player states his action for each of his types $\sigma^i: \mathcal{T}^i \rightarrow \Delta(\mathcal{A}^i)$. Given a type profile $t = (t^1, \dots, t^n)$ and a strategy profile $\sigma = (\sigma^1, \dots, \sigma^n)$, we notate by t^{-i} the types of the players besides Player i and by $\sigma^{-i}(t^{-i})$ their actions under t and σ . I.e., $\sigma^{-i}(t^{-i}) = (\sigma^1(t^1), \dots, \sigma^{i-1}(t^{i-1}), \sigma^{i+1}(t^{i+1}), \dots, \sigma^n(t^n))$. We note that given $\sigma^{-i}(t^{-i})$ the utility of Player i of a given type is not affected by the actions of other types of Player i . Therefore, it is possible to model Player i choice of action as a vector of choice, one for each of his types (best-responding to the others). We refer to these problems as the decision process done by a type.

PREFERENCES under Ambiguity

Decision theory ([19, Ch. 13], [15]) deals with scenarios in which a single decision maker (DM) needs to choose an action from a given set \mathcal{A} when his utility from an action $a \in \mathcal{A}$ depends also on an unknown state of nature $\omega \in \Omega$, when his preference is represented by a utility function $u: \mathcal{A} \times \Omega \rightarrow \mathfrak{R}$. Player i (of type t^i) looks for a response (an action) to a profile σ^{-i} . This response problem is of the same format as the DM problem: he needs to choose an action while not knowing the state of nature ω (the types of his opponents t^{-i} and their actions $\sigma^{-i}(t^{-i})$ are derived from ω).

We define the two preference orders over actions, MIN and MINthenMAX, in the framework of Decision Theory. We define them by defining the pair-wise comparison relation, and it is easy to see this relation is indeed an order. The first preference we define corresponds to Wald's MiniMax decision rule [27].

¹⁴An implicit assumption here is that the players have a vNM preferences; They evaluate a mixed action profile by its expectation. This does not restrict the modeling of preferences under ambiguity. Using the terminology of Anscombe and Aumann [2], we distinguish between roulettes and horse races.

¹⁵For a full definition of Aumann's model and its descriptive power, see (for example) [4, 5] and [21, Def. 9.4 p. 323]. As described in [5] this model is equivalent to defining \mathcal{T}^i using signal functions or by knowledge operators (the systematic approach).

Definition 1 (MIN preference). A DM strongly prefers an action a to an action a' according to MIN, if the worst outcome possible when playing a is preferred to the worst outcome of playing a' .¹⁶ I.e., $\min_{\omega \in \Omega} u(a, \omega) > \min_{\omega \in \Omega} u(a', \omega)$.

The second preference we introduce is a refinement of the MIN preference, it breaks ties in cases MIN states indifference between actions.

Definition 2 (MINthenMAX preference). A DM strongly prefers an action a to an action a' according to MINthenMAX, if either $\min_{\omega \in \Omega} u(a, \omega) > \min_{\omega \in \Omega} u(a', \omega)$, or when he is indifferent between the two respective worst outcomes and he prefers the best outcome of playing a to the best outcome if playing a' .¹⁷ I.e., $\min_{\omega \in \Omega} u(a, \omega) = \min_{\omega \in \Omega} u(a', \omega)$ and $\max_{\omega \in \Omega} u(a, \omega) > \max_{\omega \in \Omega} u(a', \omega)$.

Returning to our framework of games with ambiguity, we define the corresponding *best-response* correspondences: MIN-BR and MINthenMAX-BR. The best-response of a (type of a) player is a function that maps any action profile of the others to the actions that are optimal according to the preference. It is easy to see that a best-response according to MINthenMAX is also a best-response according to MIN, that is, MINthenMAX-BR is a refinement of MIN-BR. We show that the two best-response notions are well-defined and exist for any (finite) game.¹⁸

Lemma 3. *The correspondences pure MIN-BR, mixed MIN-BR, pure MINthenMAX-BR, and mixed MINthenMAX-BR are non-empty.*

Equilibria Under Ambiguity

Next we define the corresponding (interim) Nash equilibrium (NE) concepts as the profiles of strategies in which each type best-responds to the strategies of the others. From the definition of MINthenMAX it is clear that any equilibrium according to MINthenMAX is also an equilibrium according to MIN. Hence we regard MINthenMAX-NE as an equilibrium selection notion or a refinement of MIN-NE, in cases in which we find MIN-NE to unreasonable. Our main theorem for this section is showing that any game with ambiguity has an equilibrium according to MIN (MIN-NE) in mixed strategies.

Theorem 4. *Every game with ambiguity has a MIN-NE in mixed actions.*

Proof idea: We take \mathbb{S} to be the set of all profiles of mixed strategies of the types and define the following set-valued function $F : \mathbb{S} \rightarrow \mathbb{S}$. Given a strategy profile s , $F(s)$ is the product of the best-responses to s (according to MIN) of the different types. We prove the existence of a mixed MIN-NE by applying Kakutani's fixed point theorem [18] to F . A fixed point of F is a profile s satisfying $s \in F(s)$, i.e., each type best-responds to the others in the profile s , so s is a MIN-NE. \square

Since the existence of MIN-NE is the result of applying Kakutani's fixed point theorem on the best-response function,¹⁹ we get a characterization of the complexity of computing MIN-NE.

¹⁶The MIN preference is representable by a utility function $U(a) = \min_{\omega \in \Omega} u(a, \omega)$.

¹⁷The MINthenMAX preference is not representable by a utility function, from the same reason the lexicographic preference over \mathfrak{R}^2 is not representable by a function [20, Ch. 3.C, p. 46].

¹⁸This lemma can be easily extended to infinite number of states case by assuming some structure on the actions set and the utility function.

¹⁹The best-response can be computed in polynomial time. $\text{BR}(s) = \text{argmax}_{\sigma_i} \min_{\omega \in \Omega} \mathbb{E}_{a \sim \sigma_i} C_{a, \omega}$ for $C_{a, \omega} = u(\omega, a, s^{-i}(t^{-i}(\omega)))$. The maximal value a player can guarantee himself, $v^* = \max_{\sigma_i} \min_{\omega \in \Omega} \mathbb{E}_{a \sim \sigma_i} C_{a, \omega}$, is the solution to the linear program: $\max v$ s.t. $\forall \omega \mathbb{E}_{a \sim \sigma_i} C_{a, \omega} \geq v$, that can be solved in polynomial time. Given v^* , $\text{BR}(s)$ is the intersection of $|\Omega|$ hyperplanes of the form $\mathbb{E}_{a \sim \sigma_i} C_{a, \omega} \geq v^*$.

Corollary 5. *Finding a MIN-NE is in PPAD [25, 24]. Moreover, it is a PPAD-complete problem since a private case of it, computing a Nash equilibrium, is a PPAD-hard problem [12].*

Next we show that there are games that have no equilibrium according to MINthenMAX. We show this is true even for a simple generic game – a two-player game with type ambiguity on one side only.

Lemma 6. *There are games for which there is no MINthenMAX-NE.*

Proof. Let G be the following two-player game with two actions for each of the players.

The Row player utility is $\frac{T}{B} \begin{array}{c|c|c} L & R \\ \hline 0 & -1 \\ \hline -1 & 1 \end{array}$. The Column player is one the two types: either

$\frac{T}{B} \begin{array}{c|c|c} L & R \\ \hline 0 & 1 \\ \hline 0 & -2 \end{array}$ or $\frac{T}{B} \begin{array}{c|c|c} L & R \\ \hline 0 & 2 \\ \hline 0 & -1 \end{array}$ (and the Row player does not know which).

Then, in the unique MIN-NE the first type of the Column player is mixing $\frac{1}{2}L + \frac{1}{2}R$, the second type of the Column player is playing R , and the Row player is mixing $\frac{2}{3}T + \frac{1}{3}B$ (all his mixed actions give him a worst-case payoff of 0). But this is not a MINthenMAX-NE since the Row player prefers to deviate to playing B for the possibility of getting 1, and hence the game does not have MINthenMAX-NE in mixed strategies.²⁰ \square

3 Axiomatization of MINthenMAX

We justify using equilibria under MINthenMAX preferences for the analysis of games with type ambiguity by presenting three properties for decision making under ambiguity and characterizing MINthenMAX as the finest refinement of MIN which satisfies them (Thm. 10). We claim these properties are necessary for modeling decisions under ambiguity regarding other players' types, by that justifying our application of MINthenMAX-NE.

The framework

Let Ω be a finite set of states of nature. We characterize a preference, i.e., a total order, of a decision maker (DM) over the set actions \mathcal{A} where an action is a function $a: \Omega \rightarrow \mathfrak{R}$ that returns a utility for each state of nature.

Our first two properties are natural and we claim that any reasonable preference under ambiguity should satisfy them. The first property we present is the basic rationality assumption - monotonicity. It requires if that an action a results in a higher or equal utility than an action b in all states of nature, then the DM should weakly prefer a to b .

Axiom 7 (Monotonicity). *For any two actions a and b , if $a(\omega) \geq b(\omega)$ for all $\omega \in \Omega$, then either the DM is indifferent between the two or he prefers a to b .*

The second property, state symmetry, states that the DM should decide based on properties of the actions and not of the states. I.e., if we permute the states' names, his preference

²⁰A technical comment: The reason Kakutani's theorem cannot be applied here (besides its result being wrong) is twofold:

- The best-response set is not convex: Consider a player that needs to choose between two actions, having a utility $\frac{T}{B} \begin{array}{c|c|c|c} L & M & R \\ \hline 0 & 1 & 2 \\ \hline 0 & 2 & 1 \end{array}$, and faces one of three opponent's types. Then, when facing a profile of the opponent playing the three actions, respectively, he is indifferent between his two options (Both give him 0 on the worst case and 2 on the best case), but strictly prefers the two pure actions on any mixture of the two (giving him less than 2 in the best case).
- The best-response function is not upper semi continuous: In the example in the lemma, when the Row player is facing one type playing the pure strategy R and other mixing $(\frac{1}{2} + \epsilon)L + (\frac{1}{2} - \epsilon)R$, his unique best-response is playing T for any $\epsilon > 0$, but playing B for $\epsilon = 0$.

should not change. Since we can assume that the states themselves have no intrinsic utility beyond the definition of the actions, this property formalizes the property that the DM, having ambiguity regarding the state, should satisfy the *Principle of Insufficient Reason* and treat the states symmetrically.²¹

Axiom 8 (State symmetry). *For any two actions a and b and a bijection $\psi: \Omega \rightarrow \Omega$: if a is preferred to b , then $a \circ \psi$ is preferred to $b \circ \psi$ ($a \circ \psi(\omega)$ is defined to be $a(\psi(\omega))$).*

The last property we present is independence of irrelevant information. This property requires that if the DM considers one of the states of nature as two, by considering some a new parameter, his preference should not change. We show the desirability of this property to games with type ambiguity by the following example. Consider the following variant of Battle of the Sexes game between Alice and Bob that need to decide on a joint activity - either a Bach concert (B) or a Stravinsky concert (S). Taking the perspective of Alice, assume she faces one of two types of Bob: Bob^B that she expects to choose B or Bob^S that she expects to choose S . Assume that Alice prefers B , so her valuation of actions is:

$$\begin{array}{cc} & \begin{array}{cc} Bob^B & Bob^S \end{array} \\ \begin{array}{c} B: \\ S: \end{array} & \begin{array}{cc} 2 & 0 \\ 0 & 1 \end{array} \end{array} \quad (0 \text{ if they do not meet and } 2 \text{ or } 1 \text{ if they jointly go to a concert}).$$

But there might be other information Alice does not know regarding Bob. In case he prefers (and chooses) S , she also does not know his favorite soccer team.²² So she might actually think

$$\text{on the situation as } \begin{array}{ccc} & \begin{array}{ccc} Bob^B & Bob^S & Bob^S \\ B: & 2 & 0 & 0 \\ S: & 0 & 1 & 1 \end{array} \end{array}.$$

Since this new soccer information is irrelevant to the game, it should not change the action of a rational player. Notice that in case Alice chooses according to the recursive MIN rule we described in the introduction, she will choose according to the second-worst outcome and hence choose B in the first scenario and S in the second scenario. We find a decision model of a rational player which is susceptible to this problem ill-defined.

Axiom 9 (Independence of irrelevant information). *Let a and b be two actions on Ω s.t. $a \preceq b$, and let $\hat{\omega} \in \Omega$ be a state of nature. Define a new state space $\Omega' = \Omega \dot{\cup} \{\hat{\omega}\}$ and let a' and b' be two actions on Ω' satisfying $a'(\omega) = a(\omega)$ and $b'(\omega) = b(\omega)$ for all states $\omega \in \Omega \setminus \{\hat{\omega}\}$, $a'(\hat{\omega}) = a'(\hat{\omega}) = a(\hat{\omega})$, and $b'(\hat{\omega}) = b'(\hat{\omega}) = b(\hat{\omega})$. Then $a' \preceq b'$.*

We show that MINthenMAX is the finest refinement of MIN that satisfies the above three axioms.

Theorem 10. *MINthenMAX is the unique preference that satisfies*

- *Monotonicity*
- *State symmetry*
- *Independence of irrelevant information*
- *It is a refinement of MIN.*²³
- *It is the finest preference that satisfies the above three properties. That is, it is a refinement of any preference that satisfies the above properties.*

4 Coordination games

In this section we study a second application of MINthenMAX-NE to an economic scenario: Analyzing coordination games with type ambiguity. These games model scenarios in which the participants prefer to coordinate their actions with others, e.g., due to positive

²¹This property rules out any subjective expectation maximization preference, except for expectation according to the uniform distribution.

²²Of course, favorite soccer team is clear in case he prefers Bach.

²³That is, for any two actions a and b , if a is strongly preferred to b by a DM holding a MIN preference, then a is strongly preferred to b by a DM holding a MINthenMAX preference.

externalities. For example, choosing a meeting place (a generalization the game battle of the sexes [19, Ch. 5, Sec. 3]), choosing a cellular company, and placing a public good or bad when the cost is shared. We analyze coordination games in which all players prefer to maximize the number of other players they coordinate with (and indifferent regarding their identity), but they might differ in their tie-breaking rule between two maximizing actions.

Definition 11 (Coordination games with type ambiguity²⁴). A (finite) coordination game of n players over m locations is a game in which all players have the set set of actions of size m (and we refer to the actions as *locations*), and the preference of each player (his type) over the action profiles is defined by a strict (ordinal) preference α over the locations in the following way: Player i , holding a preference α over the locations, strongly prefers an action profile $a = (a^1, \dots, a^n) \in \mathcal{A}$ to an action profile $b = (b^1, \dots, b^n) \in \mathcal{A}$ if either he meets more players under a than under b ($|\{j \neq i \mid a^j = a^i\}| > |\{j \neq i \mid b^j = b^i\}|$) or if he meets the same (non-zero) number of players under both profiles and he prefers the meeting location in a to the one in b (a^i is preferred to b^i according to α). I.e., the set of types of Player i , \mathcal{T}^i , is a set of strict preferences over the m locations. In particular, a player is indifferent between the outcomes in which he does not meet any of the other players.

We show that every MINthenMAX-NE profile a is uniquely defined by the set of locations chosen in a , $L(a) = \{l \mid \exists i \in \mathcal{N}, t^i \in \mathcal{T}^i \text{ s.t. Player } i \text{ of type } t^i \text{ plays } l \text{ in the profile } a\}$:

Lemma 12. *Let G be a coordination game and let L be a non-empty set of locations. There exists a MINthenMAX-NE profile a s.t. $L(a) = L$ if and only if for all i the mapping $f^i : \mathcal{T}^i \rightarrow L$ that maps a type to his best location in L is onto.*

Moreover, the action profile a in which every type of every player chooses his best location in L is the unique pure MINthenMAX-NE that satisfies $L(a) = L$.

Abusing notation, we say that a location set L is a MINthenMAX-NE ($L \in \text{MINthenMAX-NE}(G)$), if there exists a MINthenMAX-NE profile a s.t. $L(a) = L$. An immediate corollary from the lemma is that any location set $L \in \text{MINthenMAX-NE}$ satisfies $|L| \leq \min_i |\mathcal{T}^i|$, and in particular, if there is no ambiguity for the type of at least one player ($\exists i \text{ s.t. } |\mathcal{T}^i| = 1$), then the only MINthenMAX-NE are the ones in which all types of all players choose the same location. Notice that for any vector of type sets $\{\mathcal{T}^i\}$ an action profile in which all types of all players choose the same location is always a MINthenMAX-NE. In these profiles, any deviation of a (type of a) player results in the deviator not meeting any of the other players and hence hurting him.

From now on we assume that $|\mathcal{T}^i| > 1$ for all i (there is real ambiguity regarding the preference each of the players), and study the equilibria (location sets) according to MINthenMAX. We analyze the non-trivial equilibria that emerge due to the type ambiguity. We call an equilibrium profile a *non-trivial* if $|L(a)| > 1$,²⁵ and call a location set *non-trivial* if it includes at least two locations.

Coordination Games with single-peaked consistent preferences

Next, we study the equilibria of coordination games under type ambiguity for several special cases in which the type sets satisfy some natural constraints. In this work, we present cases in which the set of preferences $\cup_i \mathcal{T}^i$ is single-peaked consistent with regard to a line.

Definition 13 (single-peaked consistent preferences with regard to a line [7]). A preference α over a set of locations $S \subseteq \mathfrak{R}$ is said to be *single-peaked with regard to \mathfrak{R}* , if there exists a

²⁴Most of the current literature (e.g., [26, p. 54–74], [19, p. 90–91], [23, p. 15–16]) deal with two-player coordination games (games in which the best-response of a player is to copy the other player’s action). We use the same name here for multi-player (generalized) coordination games, which capture the same kind of scenarios.

²⁵I.e., there exist at least two different locations that each of them is chosen by some type of some player.

utility function $f: \mathfrak{R} \rightarrow \mathfrak{R}$ s.t. for any two locations y and z , y is preferred to z if $f(y) > f(z)$ (i.e., f represents α); the top ranked location x^* in α is the unique maximizer of f ; and for any two locations y and z , if $x^* < y < z$ or $x^* > y > z$, then y is preferred to z .

A set of preferences over a set of locations S is *single-peaked consistent w.r.t. a line*, if there exists an embedding function $e: S \rightarrow \mathfrak{R}$ (which we refer to as the *order* of the locations) s.t. for any preference α in the set, $e(\alpha)$ is single-peaked with regard to \mathfrak{R} .

For ease of presentation, we state the results for $S \subseteq \mathfrak{R}$ (so the embedding is the identity function). For example, consider the scenario of decision on locating a common good on a linear street. It is known that each player holds an ideal location (his location), and that his preference is monotone in the path between this ideal location to the common good. Yet, there might be ambiguity regarding the ideal location, or regarding the preference between locations that are not on the same side of the ideal location (e.g, the preference might be a function of properties of the path).

We interpret the single-peakedness assumption as constraining the game in two different ways. First, it constrains the type ambiguity in a natural way: Player i knows the order over the locations (common to all types of Player j), and hence he has some information he can use to anticipate the action of other players. Second, this assumption limits the disagreement between the players (as captured by the characterization of Arrow [3, p. 77]), since they agree on the underlining order of locations. Hence, we expect to get more strict characterizations utilizing the single-peaked assumption.²⁶ Note that a set of single-peaked consistent preferences might be single-peaked with regard to more than one order of the locations (i.e., an embedding).²⁷ In cases $\cup_i \mathcal{T}^i$ is single-peaked consistent w.r.t. several orders of the locations (which can be thought of stronger homogeneity constraint), our results hold w.r.t. any of the orders, giving rise to stricter characterizations.

In the examples we describe below, we limit the ambiguity further (adding more information on players), but not adding to the inter-player agreement.

The first case we analyze is when assuming no ambiguity regarding the players' ideal location. That is, for every player there exists a location (his ideal location) x^* shared by all his type in \mathcal{T}^i . We show that in this case there can be at most two locations in the equilibrium set, one more extreme to the right than any of the ideal locations and one more extreme to the left than any of them.

Lemma 15. *Let G be a coordination game with single-peaked consistent preferences s.t. there is no ambiguity regarding the players ideal locations, and let x^i be Player i 's ideal location. Then any (non-trivial) MINthenMAX-NE L satisfies that there exist two different locations α and β s.t. $L = \{\alpha, \beta\}$ and $\alpha < \min_i x^i \leq \max_i x^i < \beta$.*

*This condition is tight. Any set L satisfying the above condition is a MINthenMAX-NE of the game in case the type sets \mathcal{T}^i are rich enough.*²⁸

Proof. Notice that the result is equivalent to requiring that for every player i there is at most one location α in L s.t. $\alpha < x^i$ and at most one location β in L s.t. $x^i < \beta$. Assume for contradiction that there are two locations in L either in $(-\infty, x^i]$ or in $[x^i, \infty)$. Then one

²⁶Constraining the disagreement by single-peaked consistency is a common assumption in the social choice literature, and it is known to have an impact on characterizations, e.g., of preference aggregation rules [8], and of facility location mechanisms [22].

²⁷**Example 14.** The preferences $1 \succ 2 \succ 3 \succ 4 \succ 5$, $2 \succ 1 \succ 3 \succ 4 \succ 5$, $2 \succ 3 \succ 1 \succ 4 \succ 5$, and $3 \succ 2 \succ 1 \succ 4 \succ 5$, are single-peaked w.r.t. the following four orders (and their inverse): $1-2-3-4-5$, $5-1-2-3-4$, $5-1-2-3-4$, and $5-4-1-2-3$.

Escoffier et al. [13] proved that for any number of locations n and $r \leq 2^{n-1}$, there exist $\frac{1}{r}2^{n-1}$ different preferences that are single-peaked consistent w.r.t r different orders (for the tight bounds see [13]).

²⁸In particular, if for all players \mathcal{T}^i is the set of all single-peaked preferences with top x^i , these are all the (non-trivial) MINthenMAX-NE.

of them must be on the path between x and the other, and hence preferred to it regardless of the type of Player i , contradicting L being an equilibrium.

In order to prove tightness, given a set $L = \{\alpha, \beta\}$ as above, for any player i there are single-peaked preferences with x^i at the top in which α is preferred to β , and there are single-peaked preferences with x^i at the top in which β is preferred to α . If the set \mathcal{T}^i includes both a type that prefers α to β , and a type that prefers β to α , then the mapping $f^i : \mathcal{T}^i \rightarrow L$ that maps a type to his preferred location in L is onto. Hence, if this condition is satisfied by all type sets, then L is a MINthenMAX-NE. \square

We see that under this homogeneity constraint, the (non-trivial) equilibria are constrained to be two extreme locations. This result can be interpreted as the power of extreme players in these scenarios. Note that when the types are single-peaked w.r.t. several orders, the equilibria should satisfy the above characterization w.r.t. all of them. For instance, consider the scenario of two players and type sets $\mathcal{T}^1 = \mathcal{T}^2 = \{2 \succ 1 \succ 3 \succ 4 \succ 5, 2 \succ 3 \succ 1 \succ 4 \succ 5\}$ (a subset of the set of preferences of Example 14). These preferences are single-peaked w.r.t. the order 5–1–2–3–4, but $L = \{4, 5\}$ is not a MINthenMAX-NE although satisfying the property of the lemma because both types prefer 4 to 5 and will not choose 5, and indeed $\{4, 5\}$ does not satisfy the property w.r.t. the order 1–2–3–4–5.

The second case we analyze is having ambiguity regarding the ideal location only, and assuming a structure on the preferences. A simple example of such structure is Euclidean preferences. A Euclidean preference is uniquely derived from its ideal location by ordering the locations according to their distance from the ideal location. Notice that also this restriction can be stated as homogeneity constraint on the preferences given the common embedding: it can be stated as a common metric on the locations set shared by all preferences.

Lemma 16. *Let G be a coordination game on the real line s.t. all players have Euclidean preferences. Then a (non-trivial) location set $L = \{l_1 < l_2 < \dots < l_k\}$ is a MINthenMAX-NE if and only if for every player i there are types derived by possible ideal locations $x_1 < x_2 < \dots < x_k$ s.t. for $t = 1, 2, \dots, k - 1$: any location m satisfying $m \in \operatorname{argmax}_p |d(p, l_t) - d(p, l_{t+1})|$,²⁹ is (strictly) between x_t and x_{t+1} .*

A special case of the lemma is when there are exactly two types of each player, derived by two possible ideal locations $x^i < y^i$. In this scenario we get that a (non-trivial) location set L is a MINthenMAX-NE if and only if there exist two locations α and β s.t. $L = \{\alpha, \beta\}$, and any location m satisfying $m \in \operatorname{argmax}_p |d(p, \alpha) - d(p, \beta)|$ is (strictly) between x^i and y^i for all players. Hence, a (non-trivial) MINthenMAX-NE exists if and only if the intersection of the segments (x^i, y^i) is non-empty.

Proof. \Rightarrow : Let $L = \{l_1 < l_2 < \dots < l_k\}$ be an equilibrium and let i be a player. Since L is an equilibrium, there are types of Player i ,³⁰ $x_1 < x_2 < \dots < x_k$, s.t. Player i chooses l_t when his type is x_t . In particular, for $t < k$ his preference between l_t and l_{t+1} when his type is x_t is different from his preference when his type is x_{t+1} . Hence, x_t and x_{t+1} lie on different sides of the median between l_t and l_{t+1} .

\Leftarrow : Following the same reasoning, if L satisfies the property of the lemma, then Player i of type x_t prefers l_t to any other location in L , hence L is an equilibrium. \square

²⁹I.e., m is the median between l_t and l_{t+1} .

³⁰We identify between the types and their ideal locations.

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A Bilateral trade

In order to demonstrate this new notion of equilibrium, MINthenMAX-NE, we apply it to two economic scenarios that have type ambiguity. The first scenario we analyze is bilateral trade games. Bilateral trade is one of the most basic economic models, which models many common scenarios. It describes an interaction between two players, a *seller* and a *buyer*. The seller has in his possession a single indivisible item that he values at v_s (e.g., the cost of producing the item), and the buyer values the item at v_b . We assume that both values are private information, i.e., each player knows only his own value, and we would like to study the cases in which the item changes hands in return for money, i.e., a transaction occurs.³¹ Chatterjee and Samuelson [11] presented bilateral trade as a model for negotiations between two strategic agents, such as settlement of a claim out of court, union-management negotiations, and of course a negotiation on transaction between two individuals and trade in financial products. The important feature they note is that an agent, while certain of the potential value it places on the transaction, has only partial information concerning the value of the other player.³² Bilateral trade, and moreover a multi-player generalization of it - *double auction*,³³ is also of a theoretical importance. These models have been used as a tool to get insights on how to organize trade between buyers and sellers, as well as study how prices are determined in markets.

In this section we assume that there is ambiguity regarding the values of the players (their type), and study trading mechanisms, i.e., procedures for deciding whether the item changes hands, and how much the buyer pays for it. We assume that the players are strategic, and hence a mechanism should be analyzed by its expected outcomes in equilibrium.

We concentrate on a family of simple mechanisms (a generalization of the bargaining rules of Chatterjee and Samuelson [11]): The seller and the buyer post simultaneously their respective bids, a_s and a_b , and if $a_s \leq a_b$ the item is sold for $x(a_s, a_b)$, for x being a known monotone function satisfying $x(a_s, a_b) \in [a_s, a_b]$. For ease of presentation, we add to the action sets of both players a “no participation” action \perp , that models the option of a player not to participate in the mechanism, i.e., there is no transaction whenever one of the players plays \perp . This simplifies the presentation by grouping together profiles in which a player chooses extreme bids that would not be matched by the other player. Hence, the utilities of a seller of type v_s and a buyer of type v_b from an action profile (a_s, a_b) are (w.l.o.g., we normalize the utilities of both players to zero in case there is no transaction):

$$u_s(v_s; a_s, a_b) = \begin{cases} a_s \leq a_b & x(a_s, a_b) - v_s \\ a_s > a_b & 0 \\ a_s = \perp \vee a_b = \perp & 0 \end{cases}$$

$$u_b(v_b; a_s, a_b) = \begin{cases} a_s \leq a_b & v_b - x(a_s, a_b) \\ a_s > a_b & 0 \\ a_s = \perp \vee a_b = \perp & 0 \end{cases}$$

Under full information (the values v_s and v_b are commonly known), there is essentially only one kind of equilibrium: *one-price equilibrium*. If $v_s \leq v_b$, the equilibria in which there is a transaction are all the profiles (a_s, a_b) s.t. $a_s = a_b \in [v_s, v_b]$ (i.e., the players agree on

³¹There is another branch of the literature on bilateral trade, which studies the process of bargaining (getting to a successful transaction). Since we would like to study the impact of ambiguity, we restrict our attention to the outcome.

³²For instance, in haggling over the price of a used car, neither buyer nor seller knows the other’s walk-away price.

³³In double auction [14], there are several sellers and buyers, and we study mechanisms and interactions matching them to trading pairs.

a price), and the equilibria in which there is no transaction are all profiles in which both players choose not to participate regardless of their type.

Introducing type ambiguity, we define the seller type set V_s and the buyer type set V_b , where each set holds the possible valuations of the player for the item. We show that under type ambiguity, there are at most three kinds of equilibria, and we characterize fully the equilibria set. We show that in addition to the above no-transaction equilibria and one-price equilibria, we get a new kind of equilibrium - *two-prices equilibria*. In such an equilibrium, both the seller and the buyer always participate (i.e., both players participate regardless of their valuations), and bid one of two possible prices p_L and p_H . For some type sets, V_s and V_b , these two-prices are the only full-participation equilibria, i.e., equilibria in which both players choose to announce a price and participate regardless of their value.

Lemma 17. ³⁴

Let G be a bilateral trade game defined by a price function $x(a_s, a_b)$ and two type sets V_s and V_b , both having a minimum and maximum.³⁵ Then all the MINthenMAX-NE of G are of one the following classes:

1. **No transaction equilibria** (These equilibria exist for any two sets V_s and V_b)
In these equilibria, both players do not participate (play \perp , or play a bid too extreme for all types of the other player) regardless of their valuations.
2. **One-price equilibria** (These equilibria are defined only when $\min V_s \leq \max V_b$. I.e., when an ex-post transaction is possible.)
In a one-price equilibrium, both the seller and the buyer choose to participate for some of their types. It is defined by a price $p \in [\min V_s, \max V_b]$ s.t. the equilibrium strategies are:

The seller bids p for valuations $v_s \leq p$, and \perp otherwise (the second clause might be vacuously true).

The buyer bids p for valuations $v_b \geq p$, and \perp otherwise (the second clause might be vacuously true).

Hence, the outcome is

	Seller	Low: $v_s \leq p$	High: $v_s > p$
Buyer	Low: $v_b < p$	no transaction	no transaction
	High: $v_b \geq p$	p	no transaction

3. **Two-prices equilibria** (These equilibria are defined only when $\min V_s \leq \min V_b$ and $\max V_s \leq \max V_b$. I.e., when there is a value for the seller s.t. an ex-post transaction is possible for any value of the buyer, and vice versa)

In a two-prices equilibrium, all types of both the seller and the buyer choose to participate, and their bids depend on their valuation. It is defined by two prices $p_L < p_H$ s.t

$$\begin{cases} \min V_s \leq p_L < \max V_s \leq p_H \\ p_L \leq \min V_b < p_H \leq \max V_b \end{cases} \text{ and the equilibrium strategies are:}$$

The seller bids p_L for valuations $v_s \leq p_L$, and p_H otherwise.

The buyer bids p_H for valuations $v_b \geq p_H$, and p_L otherwise.

Hence, the outcome is

	Seller	Low: $v_s \leq p_L$	High: $v_s > p_L$
Buyer	Low: $v_b < p_H$	p_L	no transaction
	High: $v_b \geq p_H$	$x(p_L, p_H) \in (p_L, p_H)$	p_H

Proof idea: It is easy to verify that these profiles are indeed equilibria. We will prove that these are the only equilibria.

³⁴This result is also valid, and even more natural, for infinite type sets.

³⁵We state the result here for the case when both sets have a minimal valuation and a maximal valuation. Dropping this assumption does not change the result in an essential way (some of the inequalities are changed to strict inequalities).

Assume for contradiction there is another MINthenMAX-NE profile of actions (i.e., bids or \perp), (a_s, a_b) . We define P_s to be the set of bids that are bid by the seller, i.e., $P_s = \{a_s(v_s) \mid a_s(v_s) \neq \perp\}$, and similarly we define $P_b = \{a_b(v_b) \mid a_b(v_b) \neq \perp\}$. Both these sets are not empty since (a_s, a_b) is not a no-transaction equilibrium. First, we notice that if P_s is of size one, i.e., whenever the seller participates he announces p , then if the buyer chooses to participate (based on his valuation) he chooses to match p in order to minimize the price (and vice versa). Analyzing the valuations for which they choose to participate, proves this profile is a one-price equilibrium. Now, assume that both these sets are of size at least two. If both players participate regardless of their value (never choose \perp), then the worst and best cases for a player are facing the highest and lowest bidder of the other player. Hence his best-response will be to match one of the two, and we get a two-prices equilibrium. If the seller chooses whether to participate based on his value, i.e., there is a value v_s for which he chooses \perp , then the buyer cannot guarantee himself more than zero (for instance, if he meets v_s), so he will choose one of the actions that guarantee him zero on the worst case (e.g., \perp), and will break ties among these actions by the best case (meeting the lowest bidding type of the seller). Hence, given his value, the buyer either chooses \perp , or matches the lowest bidding type of the seller. This proves this equilibrium is either a no-transaction equilibrium or a one-price equilibrium. The case in which the buyer chooses whether to participate based on his value is symmetrical. \square

We find it interesting that the set of equilibria depends on the possible types of the players, and not on the price mechanism $x(a_s, a_b)$. In addition to the two classic equilibria, no-transaction equilibria and one-price equilibria, we get a new kind of equilibrium. We see that in this equilibrium each of the players announces one of two bids, essentially announcing whether his value is above some threshold or not. This decision captures the (non-probabilistic) trade-off a player faces, whether to trade for sure, i.e., with all types of the other player, or to get a better price. E.g., the buyer decides whether to bid the high price and buy the item for sure, but maybe paying more than his value; or to bid the low price, buying at a lower price, but taking the risk of not buying at all. Since MINthenMAX is a function of the worst-case and best-case outcomes only, it does not seem surprising that we get this dichotomous trade-off and at most two bids (messages) for a player in equilibrium. This might capture scenarios in which a participant needs to choose which of two markets to attend, e.g., florists that choose whether to sell in a highly competitive auction, or in an outside market, while not knowing the demand for this day. In a continuation work, we follow this story, and analyze double auctions with several buyers and sellers, in order to check this intuition.

B Proof of Lemma 3 (Best-response is well-defined)

This lemma can be easily extended to infinite number of states case by assuming some structure on the actions set and the utility function.

Lemma. *The correspondences pure MIN-BR, mixed MIN-BR, pure MINthenMAX-BR, and mixed MINthenMAX-BR are non-empty.*

Proof. We prove the lemma in the more general decision theory framework. We show that if there are finitely many states of the nature and finitely many pure actions, there is always at least one optimal mixed action. Since the number of players and the number of types of each player are finite, a player faces one of a finite number of profiles (states of the nature) and we get the desired result.

The existence of an optimal pure action is trivial since there are finitely many pure actions. Next, we prove the existence of optimal mixed actions. The set of optimal actions according to $\text{MIN OptAct}_{\text{MIN}}$ is

$$\text{OptAct}_{\text{MIN}} = \operatorname{argmax}_{\sigma \in \Delta(\mathcal{A})} \min_{\omega \in \Omega} u(\sigma, \omega).$$

For every state of the nature $\omega \in \Omega$, $u(\sigma, \omega)$ is a continuous function in σ (it is a linear transformation). $\min_{\omega \in \Omega} u(\sigma, \omega)$ is also continuous in σ as the minimum for finitely many continuous functions. Hence, $\operatorname{argmax}_{\sigma \in \Delta(\mathcal{A})} \min_{\omega \in \Omega} u(\sigma, \omega)$ is non-empty as the maxima of a continuous function over a simplex. Moreover, $\text{OptAct}_{\text{MIN}}$ is a compact set.

Similarly, the set of optimal actions according to MINthenMAX, $\text{OptAct}_{\text{MINthenMAX}} = \operatorname{argmax}_{a \in \text{OptAct}_{\text{MIN}}} \max_{\omega \in \Omega} u(\sigma, \omega)$, is non-empty since $\text{OptAct}_{\text{MIN}}$ is compact and $\max_{\omega \in \Omega} u(\sigma, \omega)$ is continuous in σ . \square

C Proof of Theorem 4 (MIN-NE existence)

Theorem. *Every game with ambiguity has a MIN-NE in mixed actions.*

Proof. We prove the existence of a mixed MIN-NE by applying Kakutani's fixed point theorem [18]³⁶ to the following set \mathbb{S} and set-valued function F . We set $\mathbb{S} \subseteq \mathfrak{R}^K$ to be the set of all profiles of mixed strategies of the types,³⁷ i.e., a cross product of the corresponding simplexes. Given a strategy profile s , we define $F(s)$ to be the product of the best-response correspondences of the different types to the profile s .

Clearly, \mathbb{S} is a convex compact non-empty set. We proved (Lemma 3) that the best-response always exists and hence $F(s)$ is non-empty for all $s \in \mathbb{S}$. We show that the best-response correspondence of a type is an upper semi-continuous function (and hence also F) by applying Berge's Maximum theorem [6, Thm. 2, p. 116]³⁸ to³⁹

$$f(\sigma^i, s) = \min_{\omega \in \Omega} u(\omega, \sigma^i, s^{-i}(t^{-i}(\omega)))$$

We notice that the best-response of a type of Player i to a strategy profile $s \in \mathbb{S}$ is the maxima over his mixed strategies σ^i of $f(\sigma^i, s)$; that u is a linear function in σ^i and in s ; and that $f(\sigma^i, s)$ is a continuous function in s and σ^i . Hence, we get that the best-response is an upper semi-continuous correspondence.

Next, we show that for any profile s and a type of Player i , the best-response set is convex, and hence also $F(s)$ is a convex set. In order to prove the convexity of the best-response set, let σ and τ be two best-responses and let

$$m = \min_{\omega \in \Omega} u(\omega, \sigma^i, s^{-i}(t^{-i}(\omega)))$$

be their worst-case value. For any convex combination ζ of σ and τ , on one hand we get that

$$\min_{\omega \in \Omega} u(\omega, \zeta^i, s^{-i}(t^{-i}(\omega))) \leq m$$

from the optimality of σ , and on the other hand

$$u(\omega, \zeta^i, s^{-i}(t^{-i}(\omega))) \geq m \text{ for all } \omega \in \Omega.$$

³⁶Let $\mathbb{S} \subseteq \mathfrak{R}^n$ be a non-empty compact convex set. Let $F: \mathbb{S} \rightarrow \mathbb{S}$ be a set-valued upper semi-continuous function on \mathbb{S} such that $F(s)$ is non-empty and convex for all $s \in \mathbb{S}$. Then F has a fixed point, i.e. a point $s \in \mathbb{S}$ such that $s \in F(s)$.

³⁷for K being the sum of the number of types over the players.

³⁸If $f: X \times Y \rightarrow \mathfrak{R}$ is a continuous function, then the mapping $\mu: X \rightarrow Y$ defined by $\mu(x) = \operatorname{argmax}_{y \in Y} f(y)$ is an upper semi-continuous mapping.

(The statement of the theorem is taken from [1, Lemma 17.31, p. 570])

³⁹Where $t^{-i}(\omega)$ are the types Player i is facing according to ω , and $s^{-i}(t^{-i}(\omega))$ are their actions according to s .

since we assumed this lower-bound both on σ and on τ . Hence,

$$\min_{\omega \in \Omega} u(\omega, \zeta^i, s^{-i}(t^{-i}(\omega))) = m$$

and ζ is a best-response.

By applying Kakutani's fixed point theorem we get that there is a profile s s.t. $s \in F(s)$. That is, in the profile s every type best-responds to the others, so s is a MIN-NE. \square

D Proof of Theorem 10 (Axiomatization of MINthenMAX)

Theorem. MINthenMAX is the unique preference that satisfies

- Monotonicity
- State symmetry
- Independence of irrelevant information
- It is a refinement of MIN.⁴⁰
- It is the finest preference that satisfies the above three properties. That is, it is a refinement of any preference that satisfies the above properties.

Proof. In order to prove this theorem we use the following lemma (proved next) that any preference that satisfies the first three properties can be defined using the worst (minimal) and best (maximal) outcomes of the actions.

Lemma 18. Any preference that satisfies monotonicity, state symmetry, and independence of irrelevant information, can be defined as a function of the worst and the best outcomes of the actions.

Using this lemma, we turn to prove the characterization of MINthenMAX.

It is easy to verify that MINthenMAX indeed satisfies the first four properties. We note that the uniqueness is an immediate consequence of the uniqueness of a finest refinement and prove that indeed MINthenMAX is a finest refinement. Let P be an arbitrary refinement of MIN that satisfies Monotonicity, state symmetry, and Independence of Irrelevant Information. Let a and b be two actions s.t. a DM holding a P preference strongly prefers a to b , and we'll show that a is preferred to b also by a DM holding a MINthenMAX preference. Applying Lemma 18, we get that P can be defined using the minimal and maximal outcomes. We denote the respective minimal and maximal outcomes of a by m_a and M_a , and of b by m_b and M_b .

Assume for contradiction that a DM holding a MINthenMAX preference weakly prefers b to a . Since P is a refinement of MIN, it cannot be that $m_b > m_a$, and hence we get that $m_b = m_a$ and $M_b \geq M_a$. Both P and MINthenMAX can be defined as a function of the minimal and maximal outcome, so with no loss of generality we can assume that a and b result in the same minimal outcome m_a in some state ω_m , result in their respective maximal outcomes in the same state ω_M , and result in the same intermediate outcomes in all other states. Since P is monotone, we get that a DM holding a P preference weakly prefers b to a and by that get a contradiction. \square

D.1 Proof of lemma

Lemma. Any preference that satisfies monotonicity, state symmetry, and independence of irrelevant information, can be defined as a function of the worst and the best outcomes of the actions.

Proof. For this proof it is easier to use the following property that is equivalent (when Ω is finite) to state symmetry.

⁴⁰That is, for any two actions a and b , if a is strongly preferred to b by a DM holding a MIN preference, then a is strongly preferred to b by a DM holding a MINthenMAX preference.

Axiom. For any action a and bijection $\psi: \Omega \rightarrow \Omega$, the DM is indifferent between a and $a \circ \psi$.

Let a and a' be two actions over the states set Ω s.t. $\min_{\omega \in \Omega} a(\omega) = \min_{\omega \in \Omega} b(\omega)$ and $\max_{\omega \in \Omega} a(\omega) = \max_{\omega \in \Omega} b(\omega)$. We show that the DM is indifferent between the two, and notice that this will prove the lemma. Assuming this claim, given any two pairs of actions: a and a' that have the same minimal and maximal outcome, and b and b' that have the same minimal and maximal outcome, a is preferred to b if and only if a' is preferred to b' , because the DM is indifferent between a and a' , and between b and b' .

Assume for contradiction that (w.l.o.g) the DM strictly prefers a' to a . Since the preference satisfies state symmetry, we can assume that a and a' are co-monotone, that is, for any two states ω and ω' , the actions satisfy that $(a(\omega) - a(\omega'))(a'(\omega) - a'(\omega')) \geq 0$. We denote by m and M the minimal and maximal outcomes of a and by ω_m and ω_M the two respective states of nature.

	ω_m	other states	ω_M
a	m	$\in [m, M]$	M
a'	m	$\in [m, M]$	M

We define two new actions over Ω . An action b that results in m in all states besides ω_M and M otherwise, and an action b' that results in M in all states besides ω_m and m otherwise.

	ω_m	other states	ω_M
a	m	$\in [m, M]$	M
b	m	m	M
a'	m	$\in [m, M]$	M
b'	m	M	M

Due to monotonicity the DM (weakly) prefers a to b , and b' to a' , and hence the DM strictly prefers b' to b . Next, we define an auxiliary space $\hat{\Omega} = \{\omega_m, \omega_o, \omega_M\}$ and two actions on it c and c' by unifying the middle states into one as follows:

	ω_m	ω_o	ω_M
c	m	m	M
c'	m	M	M

Since the preference satisfies Independence of Irrelevant Information we get that the DM strictly prefers c' to c . Now we define a new action c'' (see below), and by the state symmetry property we get that also c'' is strictly preferred to c .

	ω_m	ω_o	ω_M
c	m	m	M
c'	m	M	M
c''	M	M	m

Using a collapsing argument similar to the one above, we get that for the following collapsed space and two actions

	ω_m	ω_M
d	m	M
d''	M	m

d'' is strictly preferred to d , but this contradicts the state symmetry property. \square

E Proof of Lemma 17 (Bilateral trade games)

Lemma. Let G be a bilateral trade game defined by a price function $x(a_s, a_b)$ and two type sets V_s and V_t , both having a minimum and maximum.⁴¹ Then all the MINthenMAX-NE of G are of one the following classes:

1. **No transaction equilibria** (These equilibria exist for any two sets V_s and V_b)
In these equilibria, both players do not participate (play \perp , or play a bid too extreme for all types of the other player) regardless of their valuations.
2. **One-price equilibria** (These equilibria are defined only when $\min V_s \leq \max V_b$. I.e., when an ex-post transaction is possible.)

In a one-price equilibrium, both the seller and the buyer choose to participate for some of their types. It is defined by a price $p \in [\min V_s, \max V_b]$ s.t. the equilibrium strategies are:

The seller bids p for valuations $v_s \leq p$, and \perp otherwise (the second clause might be vacuously true).

The buyer bids p for valuations $v_b \geq p$, and \perp otherwise (the second clause might be vacuously true).

Hence, the outcome is

	Seller	
Buyer	Low: $v_s \leq p$	High: $v_s > p$
Low: $v_b < p$	no transaction	no transaction
High: $v_b \geq p$	p	no transaction

3. **Two-prices equilibria** (These equilibria are defined only when $\min V_s \leq \min V_b$ and $\max V_s \leq \max V_b$. I.e., when there is a value for the seller s.t. an ex-post transaction is possible for any value of the buyer, and vice versa)

In a two-prices equilibrium, all types of both the seller and the buyer choose to participate, and their bids depend on their valuation. It is defined by two prices $p_L < p_H$ s.t.

$$\begin{cases} \min V_s \leq p_L < \max V_s \leq p_H \\ p_L \leq \min V_b < p_H \leq \max V_b \end{cases} \text{ and the equilibrium strategies are:}$$

The seller bids p_L for valuations $v_s \leq p_L$, and p_H otherwise.

The buyer bids p_H for valuations $v_b \geq p_H$, and p_L otherwise.

Hence, the outcome is

	Seller	
Buyer	Low: $v_s \leq p_L$	High: $v_s > p_L$
Low: $v_b < p_H$	p_L	no transaction
High: $v_b \geq p_H$	$x(p_L, p_H) \in (p_L, p_H)$	p_H

Proof. We will prove that these profiles are equilibria (MINthenMAX-NE) and that they are the only equilibria.

- No-transaction profiles

In these profiles for at least one of the players does not participate (\perp , or bids an extreme bid) regardless of his type.

A player, facing a profile in which all the types of the other player do not participate, is indifferent between all his actions (all of them result in no-transaction for sure), and hence he best-responds. Facing any other profile, at least for some of his types (e.g., the maximal value for the buyer, and minimal value of the seller), choose to participate. Hence, the only no-transaction profile that is an equilibrium, is when both players do not participate.

- One-price profiles

In these profiles, both the seller and the buyer choose to participate for some of their

⁴¹We state the result here for the case when both sets have a minimal valuation and a maximal valuation. Dropping this assumption does not change the result in an essential way (some of the inequalities are changed to strict inequalities).

types, and when they choose to participate, both bid the same price p .

First, we prove that given a one-price profile as described in the lemma, it is indeed an equilibrium.

Consider a seller of type v_s facing a profile in which there are types of the buyer that choose p and (maybe) others that choose not to participate.

- If $v_s \leq p$: He prefers bidding p to any bid $a_s < p$, because both bids will result in a transaction with the same types of the buyer, but a_s will result in a lower price (Monotonicity of MINthenMAX).

He prefers bidding p to any bid $a_s > p$, because a_s will result in no-transaction with all types of the buyer, while p results in a profitable transaction with some of them (Monotonicity of MINthenMAX).

We notice that since $p \geq \min V_s$ there are types of the seller that bid p .

- If $v_s > p$: Any bid that results in a transaction with some types of the buyer, will be at a price lower than v_s , and hence the transaction will be a losing one.

Hence, he prefers not to participate at all (Monotonicity of MINthenMAX).

Similarly, consider a buyer of type v_b facing a profile in which there are types of the seller that choose p and (maybe) others that choose not to participate.

- If $v_b < p$: Any bid that results in a transaction with some types of the seller, will be at a price higher than v_b , and hence the transaction will be a losing one.

Hence, he prefers not to participate at all (Monotonicity of MINthenMAX).

- If $v_b \geq p$: He prefers bidding p to any bid $a_b > p$, because both bids will result in a transaction with the same types of the seller, but a_b will result in a higher price (Monotonicity of MINthenMAX).

He prefers bidding p to any bid $a_b < p$, because a_b will result in no-transaction with all types of the seller, while p results in a profitable transaction with some of them (Monotonicity of MINthenMAX).

We notice that since $p \leq \max V_b$ there are types of the buyer that bid p .

Next, we prove that any single-price profile other than those described in the lemma, cannot be an equilibrium.

First we claim, that in equilibrium it cannot be that the prices of participating types of one of the sides vary, while all types of the other player bid the same. If there is a unique price p bid by participating types of the seller, based on the analysis above, all participating types of the buyer bid p . Similarly, if there is a unique price p bid by participating types of the buyer, all participating types of the seller bid p .

Hence, in a single-price profile, there exists a price p s.t. both some types of the seller and some types of the buyer choose p , and all other types (might be an empty set) choose not to participate.

If $p \in [\min V_s, \max V_b]$, by the analysis above, we see there is only one single-price equilibrium supporting p .

If $p \notin [\min V_s, \max V_b]$, by the analysis above, either all types of the buyer or all types of the seller choose not to participate. In either case this is a no-transaction profile.

- All-participating profiles

In these profiles, all types of both the seller and the buyer choose to participate (did not choose \perp).

First we prove that given an all-participating profile as described in the lemma as two-prices equilibria, it is indeed an equilibrium.

Consider a seller of type v_s facing a profile in which all the types of the buyer participate, and let p_L be the infimum of the bids chosen by types of the buyer and p_H the

supremum of the bids.

- If $v_s \leq p_L$: He prefers bidding p_L to any bid $a_s < p_L$, because both bids will result in a transaction with all types of the buyer, but a_s will result in a lower price with each of them (Monotonicity of MINthenMAX).

He prefers bidding p_L to any bid $a_s > p_L$, because a_s will result in no-transaction with some types of the buyer, while p_L results in a non-losing transaction with all of them (MINthenMAX extends MIN).

- If $v_s > p_L$ and $v_s \leq p_H$: He prefers bidding p_H to any bid $a_s < p_H$, because when facing the types of the buyer that bid $a_b \in [a_s, p_H]$, both bids will result in a transaction, but a_s will result in a lower price, and when facing other types of the buyer (that bid less than a_s), both result in no-transaction (Monotonicity of MINthenMAX).

He prefers bidding p_H to any bid $a_s > p_H$, because a_s results in no-transaction with all types of the buyer, while p_H results in a non-losing transaction with some types of the buyer (Monotonicity of MINthenMAX).

We notice that since in the profiles of the lemma satisfy $\min V_s \leq p_L < \max V_s \leq p_H$, all the types of the seller participate, there are types of the seller that bid p_L , and there are types that bid p_H .

Similarly, consider a buyer of type v_b facing a profile in which all the types of the seller participate, and let p_L be the infimum of the bids chosen by types of the seller and p_H the supremum of the bids.

- If $v_b < p_H$ and $v_b \geq p_L$: He prefers bidding p_L to any bid $a_b < p_L$, because a_b results in no-transaction with all types of the seller, while p_L results in a non-losing transaction with some types of the seller (Monotonicity of MINthenMAX).

He prefers bidding p_L to any bid $a_b > p_L$, because when facing the types of the seller that bid $a_s \in [p_L, a_b]$, both bids will result in a transaction, but a_b will result in a higher price, and when facing other types of the seller (that bid more than a_b), both result in no-transaction (Monotonicity of MINthenMAX).

- If $v_b \geq p_H$: He prefers bidding p_H to any bid $a_b < p_H$, because a_b will result in no-transaction with some types of the seller, while p_H results in a non-losing transaction with all of them (MINthenMAX extends MIN).

He prefers bidding p_H to any bid $a_b > p_H$, because both bids will result in a transaction with all types of the seller, but a_b will result in a higher price with each of them (Monotonicity of MINthenMAX).

We notice that since in the profiles of the lemma satisfy $p_L \leq \min V_b < p_H \leq \max V_b$, all the types of the buyer participate, there are types of the buyer that bid p_L , and there are types that bid p_H .

Next, we prove that any all-participating non single-price profile, other than the above, cannot be an equilibrium.

In an all-participating non single-price profile, all the types of both the buyer and seller participate, and for both players, not all their types choose the same price.

If there is a type of the buyer whose value is lower than any of the prices bid by types of the seller, then this type prefers not to participate, so it cannot be an equilibrium.

Any other type of the buyer bids either the supremum of the infimum of the prices bid by types of the seller. Given that types of the buyer bid one of two prices, if there is a type of the seller whose value is higher than both prices, then this type prefers not to participate, and the profile is not an equilibrium. Otherwise, the types of the seller also bid these two prices and we get the profile is of the type described in the lemma.

- Last, we show that all other profiles are not equilibria of the game.

Assume towards a contradiction that \bar{a} is an equilibrium profile of the game that is

not one of the above profile types.

Clearly, for a match of any pair of types of the seller and the buyer, the utility of neither of them cannot be negative, because he can guarantee himself to get at least zero against all types of the other player by choosing \perp .

Assume there is a type v_s of the seller that chose \perp . Then for any participating type v_b of the buyer, v_b cannot guarantee himself to get a utility greater than zero (his utility from being matched with v_s) in the worst case. Hence, he breaks the tie between all the bids $a_b \leq v_b$, all giving him zero in the worst case, according to the best case which is being matched with the lowest bid of a participating type of the seller. Hence, we get a contradiction by showing the profile is a one-price profile.

Similarly, assume there is a type v_b of the buyer that chose \perp . Then for any participating type v_s of the seller, v_s cannot guarantee himself to get a utility greater than zero (his utility from being matched with v_b) in the worst case. Hence, he breaks the tie between all the bids $a_b \geq v_s$, all giving him zero in the worst case, according to the best case which is being matched with the highest bid of a participating type of the buyer. Hence, we get a contradiction by showing the profile is a one-price profile. \square

F Proof of Lemma 12 (Coordination games)

Lemma. *Let G be a coordination game and let L be a non-empty set of locations. There exists a MINthenMAX-NE profile a s.t. $L(a) = L$ if and only if for all i the mapping $f^i : \mathcal{T}^i \rightarrow L$ that maps a type to his best location in L is onto.*

Moreover, the action profile a in which every type of every player chooses his best location in L is the unique pure MINthenMAX-NE that satisfies $L(a) = L$.

Proof. The case $|L| = 1$ is trivial. Let l be the location in L . The only profile satisfying $L(a) = L$ is when all players choose l regardless of their type, and this is an equilibrium. The mappings f^i are onto, and hence the lemma is proved for this case.

From now on, we assume that $|L| \geq 2$. Given a profile a we define $L^i(a)$ to be the set of locations chosen in a by Player i ,

$$L^i(a) = \{l \mid \exists t^i \in \mathcal{T}^i \text{ s.t. Player } i \text{ of type } t^i \text{ plays } l \text{ in the profile } a\},$$

so $L(a) = \cup_{i \in \mathcal{N}} L^i(a)$.

\Rightarrow : Assume there exists an equilibrium a s.t. $L(a) = L$.

First we claim that for all players, $|L^i(a)| > 1$. Assume towards a contradiction, there is a non-empty set of players S s.t. $i \in S \iff |L^i(a)| = 1$, and let p be a player in S and l the location chosen by p .

If $|S| = 1$, then for any of the other players the only action guaranteeing them to meet another player is choosing l , and hence all players choose l regardless of their type. I.e., $L(a) = \{l\}$ and we get a contradiction.

If $|S| > 1$, then any of the players can guarantee himself to meet another player on the worst case, and hence will choose a location in $\cup_{i \in S} L^i(a)$. Since all players aim to maximize the number of other players they meet on the worst-case, it cannot be that two players in S choose differently (both maximize the number of other players from S that choose like them). Hence, $|\cup_{i \in S} L^i(a)| = 1$ and we get that $|L(a)| = 1$. Contradiction.

Hence, for all players, $|L^i(a)| > 1$. Player i of type $t^i \in \mathcal{T}^i$ cannot guarantee himself to meet other players, and hence he will choose the best location according to his type among the locations in $\cup_{j \neq i} L^j(a)$ that maximize the number of other players he might meet - $m(l) = \#\{j \neq i \mid l \in L^j(a)\}$.

Next, we prove that for all players $L^i(a) = L(a)$. Assume for contradiction there exists Player p s.t. $L^p(a) \subsetneq L(a)$. Let $l \in L^p(a)$ be a location chosen by p , and let l' be a location and p' a player s.t. $l' \in L^{p'}(a) \setminus L^p(a)$. Since Player p (of some type) chose l , we get that

the number of players $j \neq p$ that chose l is at least as large as the number of players $j \neq p$ that chose the location l' . Taking the viewpoint of Player p' we get that the number of players $j \neq p'$ that chose l is strictly larger than the number of players $j \neq p'$ that chose the location l' . But this is a contradiction to $l' \in L^{p'}(a)$.

Hence, we get that in a , Player i of type $t^i \in \mathcal{T}^i$ choose the best location according to his type among the locations in L (in all of them he meets no-one in the worst-case and everyone else in the best case). That is, Player i plays according to the function f^i , and $L = L^i = \text{Im}(f)$, so f is onto. Notice that we also get this profile is the unique equilibrium s.t. $L(a) = L$.

\Leftarrow : Assume the functions f^i are onto. It is easy to verify that in the profile a in which Player i plays according to f^i , Player i best-responds (for all i). - None of his actions guarantees him to meet some other player in the worst case; if he plays an action $a \notin L$, he does not meet another player even in the best-case; if he plays an action $a \in L$, he meets all other players in the best-case; hence, he best-responds playing according to f^i . \square