Ordinal power relations and social rankings[1]

Stefano Moretti and Meltem Öztürk

Abstract

Several real-life complex systems, like human societies or economic networks, are formed by interacting units characterized by patterns of relationships that may generate a group-based social hierarchy. In this paper, we address the problem of how to rank the individuals with respect to their ability to "influence" the relative strength of groups in a society. We also analyse the effect of basic properties in the computation of a social ranking within specific classes of (ordinal) coalitional situations. We show that the pairwise combination of these natural properties yields either to impossibility (i.e., no social ranking exists), or to flattening (i.e., all the individuals are equally ranked), or to dictatorship (i.e., the social ranking is imposed by the relative comparison of coalitions of a given size).

1 Introduction

Ranking is a fundamental ingredient of many real-life situations, like the ranking of candidates applying to a job, the rating of universities around the world, the distribution of power in political institutions, the centrality of different actors in social networks, the accessibility of information on the web, etc. Often, the criterion used to rank the items (e.g., agents, institutions, products, services, etc.) of a set N also depends on the interaction among the items within the subsets of N (for instance, with respect to the users' preferences over bundles of products or services). In this paper we address the following question: given a finite set N of items and a ranking over its subsets, can we derive a "social" ranking over N according to the "overall importance" of its single elements?

For instance, consider a company with three employees $1, 2$ and 3 working in the same department. According to the opinion of the manager of the company, the job performance of the different teams $S \subseteq N = \{1, 2, 3\}$ is as follows: $\{1, 2, 3\} \succcurlyeq \{3\} \succcurlyeq \{1, 3\} \succcurlyeq \{2, 3\} \succcurlyeq \{2\} \succcurlyeq \{1, 2\} \succcurlyeq \{1\} \succcurlyeq \emptyset$ ($S \succcurlyeq T$, for each $S, T \subseteq N$, means that the performance of S is at least as good as the performance of T). Based on this information, the manager asks us to make a ranking over his three employees showing their attitude to work with others as a team or autonomously. Intuitively, 3 seems to be more influential than 1 and 2, as employee 3 belongs to the most successful teams in the above ranking. Can we state more precisely the reasons driving us to this conclusion? And what can we say if we have to decide who between 1 and 2 is more productive and deserves a promotion? In this paper we analyse different properties of ordinal social rankings in order to get some answers to such questions.

The problem studied in this paper can be seen as an ordinal counterpart of the one about how to measure the power of players in *simple games*, which are coalitional games where coalitions may be winning or not [1, 4]. However, our framework is different for at least two reasons: first, we face coalitional situations where only a qualitative (ordinal) comparison of the strength of coalitions is given; second, we look for a ranking over the single objects in N, and we do not require a quantitative assessment of the "power" of the players. As far as we know, the only attempt in the literature to generalize the notions of coalitional game and power index within an ordinal framework has been provided in [10], where, given a total preorder representing the relative strength of coalitions, a social ranking over the player set

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is provided according to a notion of *ordinal influence* and using the Banzhaf index [1] of a "canonical" coalitional game.

In the literature of simple games, related questions deal with the ordinal equivalence of power indices (see, for instance, [3, 6, 9]) and the analysis of the differences between rankings generated by alternative power indices on special classes of simple games (e.g, the papers [13, 8]). Similarly to our work, in [14] the authors investigated alternative notions of ordinal power on different classes of simple games. All the aforementioned papers focus on the notion of simple game, that is a numerical representation of a dichotomous power relation (i.e., winning or loosing coalitions), a much more restricted domain than the one considered in this work, where a power relation can be whatever total preorder over the coalitions.

In a still different context, a model of coalition formation has been introduced in [12], where the relative strength of disjoint coalitions is represented by an exogenous binary relation and the players try to maximize their position in a social ranking. We also notice a connection with some kind of "inverse problems", precisely, how to derive a ranking over the set of all subsets of N in a way that is "compatible" with a primitive ranking over the single elements of N (see, for instance, [2]; see [11] for an approach using coalitional games).

In this paper, a *social ranking* is defined as a map associating to each *power relation* (i.e., a total preorder over the set of all subsets of N) a total preorder over the elements of N. The properties for social rankings that we analyse in this paper have classical interpretations, such as *anonymity* and *symmetry*, saying that the ranking should not depend on the identity of the elements of N, or the *dominance*, saying that an element $i \in N$ should be ranked higher than an element $j \in N$ whenever i dominates j, i.e. a coalition $S \cup \{i\}$ is stronger than $S \cup \{j\}$ for each $S \subset N$ not containing neither i nor j. Another property we study in this paper is the *independence of irrelevant coalitions*, saying that the social ranking between two elements i and j should only depend on their respective contributions when added to coalitions not containing neither i nor j (in other words, the information needed to rank i and j is provided by the relative comparison of coalitions $U, W \subset N$ such that $U \setminus \{i\} = W \setminus \{j\}$).

We use these properties to axiomatically analyse social rankings on particular classes of power relations. We first notice that two natural properties, precisely, dominance and symmetry, are not compatible over the class of all power relations (see Theorem 1 in Section 4), despite the fact that, in some related axiomatic frameworks (see, for instance, [2]), similar axioms have been successfully used in combination. On the other hand, the properties of independence of irrelevant coalitions and symmetry in combination determine a flattening of the social ranking, where all the agents are equivalent (see Proposition 2 in Section 4). Finally, we proved that the property of independence of irrelevant coalitions and dominance property determine a kind of 'dictatorship of the cardinality' when a relation of strong dominance among coalitions of the same size holds: in this case, the only social ranking satisfying those two properties is the one imposed by the relation of dominance of a given cardinality $s \in \{1, \dots, |N|\}$ (see Theorem 2 in Section 5).

The structure of the paper is the following. In the next section, we present some related approaches from the literature and our main contributions. Basic notions and definitions are presented in Section 2. In Section 3 we introduce and discuss some properties for social rankings. In Section 4 we study the compatibility of certain axioms and their effect on some elementary notions of social ranking. In Section 5 we focus on the analysis of social rankings that satisfy both the dominance property and the property of independence of irrelevant coalitions, and that, on particular power relations, are specified by the ordering of coalitions of the same size. Section 6 concludes with some future research directions.

2 Preliminaries and notations

A *binary relation* R on a finite set $N = \{1, \ldots, n\}$ is a collection of ordered pairs of elements of N, i.e. $R \subseteq N \times N$. $\forall x, y \in N$, the more familiar notation xRy will be often used instead of the more formal one $(x, y) \in R$. We provide some standard properties for R. *Reflexivity*: for each $x \in N$, xRx; *transitivity*: for each $x, y, z \in N$, xRy and $yRz \Rightarrow xRz$; *totality*: for each $x, y \in N$, $x \neq y \Rightarrow xRy$ or yRx; *antisymmetry*: for each $x, y \in N$, xRy and yRx $\Rightarrow x = y$. A reflexive and transitive binary relation is called *preorder*. A preorder that is also total is called *total preorder*. A total preorder that also satisfies antisymmetry is called *linear order*. The notation $\neg(xRy)$ means that xRy is not true. We denote by 2^N the power set of N and we use the notations $\mathcal{T}^N$ and $\mathcal{T}^{2^N}$ to denote the set of all total preorders on N and on 2^N, respectively. Moreover, the cardinality of a set $S \in 2^N$ is denoted by $|S|$. In the remaining of the paper, we will also refer to an element $S \in 2^N$ as a *coalition* S.

Consider a total preorder $\succcurlyeq \subseteq 2^N \times 2^N$ over the subsets of N. Often we will use the notation $S \succ T$ to denote the fact that $S \succcurlyeq T$ and $\neg(T \succcurlyeq S)$ (in this case, we also say that the relation between S and T is 'strict'), and the notation $S \sim T$ to denote the fact that $S \succcurlyeq T$ and $T \succcurlyeq S$. For each $i, j \in N$, $i \neq j$, and all $k = 1, \ldots, n-2$, we denote by $\Sigma_{ij}^k = \{S \in 2^{N \setminus \{ij\}} : |S| = k\}$ the set of all subsets of N not containing neither i nor j with k elements. Moreover, for each $i, j \in N$, we define the set $D_{ij}^k(\succcurlyeq) = \{S \in \Sigma_{ij}^k : S \cup \{i\} \succcurlyeq S \cup \{j\}\}$ as the set of coalitions $S \in 2^{N \setminus \{ij\}}$ of cardinality k such that $S \cup \{i\}$ is in relation with $S \cup \{j\}$ (and, changing the ordering of i and j, the set $D_{ji}^k(\succcurlyeq) = \{S \in \Sigma_{ij}^k : S \cup \{j\} \succcurlyeq S \cup \{i\}\}$).

3 Axioms for social rankings

In the following of these notes, we interpret a total preorder $\succcurlyeq$ on 2^N as a *power relation*, that is, for each $S, T \in 2^N$, $S \succcurlyeq T$ stands for 'S is considered at least as strong as T according to the power relation $\succcurlyeq$'.

Given a class $\mathcal{C}^{2^N} \subseteq \mathcal{T}^{2^N}$ of power relations, we call a map $\rho : \mathcal{C}^{2^N} \longrightarrow \mathcal{T}^N$, assigning to each power relation in $\mathcal{C}^{2^N}$ a total preorder on N, a *social ranking solution* or, simply, a *social ranking*. Then, given a power relation $\succcurlyeq$, we will interpret the total binary relation $\rho(\succcurlyeq)$ associated to $\succcurlyeq$ by the social ranking ρ, as the relative power of players in a society under relation $\succcurlyeq$. Precisely, for each $i, j \in N$, $i\rho(\succcurlyeq)j$ stands for 'i is considered at least as influential as j according to the social ranking $\rho(\succcurlyeq)$', where the influence of an agent is intended as her/his ability to join coalitions in the strongest positions of a power relation. Note that we require that $\rho(\succcurlyeq)$ is a total preorder over the elements of N, that is we always want to express the relative comparison of two agents, and such a relation must be transitive.

A social ranking $\rho : \mathcal{C}^{2^N} \longrightarrow \mathcal{T}^N$ such that $i\rho(\succcurlyeq)j \Leftrightarrow \{i\} \succcurlyeq \{j\}$ for each $\succcurlyeq \in \mathcal{C}^{2^N}$ and each $i, j \in N$ is said to be *primitive* (i.e., it neglects any information contained in $\succcurlyeq$ about the comparison of coalitions of cardinality different from 1). A social ranking $\rho : \mathcal{C}^{2^N} \longrightarrow \mathcal{T}^N$ such that $i\rho(\succcurlyeq)j$ and $j\rho(\succcurlyeq)i$ for all $i, j \in N$ is said to be *unanimous* (N is an indifference class with respect to $\rho(\succcurlyeq)$).

Now we introduce some properties for social rankings. The first axiom is the *dominance* one: if each coalition S containing agent i but not j is stronger than coalition S with j in the place of i, then agent i should be ranked higher than agent j in the society, for any $i, j \in N$. Precisely, given a power relation $\succcurlyeq \in \mathcal{T}^{2^N}$ and $i, j \in N$ we say that i *dominates* j in $\succcurlyeq$ if $S \cup \{i\} \succcurlyeq S \cup \{j\}$ for each $S \in 2^{N \setminus \{i,j\}}$ (we also say that i *strictly dominates* j in $\succcurlyeq$ if i dominates j and in addition there exists $S \in 2^{N \setminus \{i,j\}}$ such that $S \cup \{i\} \succ S \cup \{j\}$).

Definition 1 (DOM)**.** *A social ranking $\rho : \mathcal{C}^{2^N} \longrightarrow \mathcal{T}^N$ satisfies the dominance (DOM) property on $\mathcal{C}^{2^N} \subseteq \mathcal{T}^{2^N}$ if and only if for all $\succcurlyeq \in \mathcal{C}^{2^N}$ and $i, j \in N$, if i dominates j in $\succcurlyeq$*

then $i\rho(\succcurlyeq)j$ [*and* $\neg(j\rho(\succcurlyeq)i)$ *if* i *strictly dominates* j *in* $\succcurlyeq$].

The following axiom states that the relative strength of two agents $i, j \in N$ in the social ranking should only depend on their effect when they are added to each possible coalition S not containing neither i nor j, and the relative ranking of the other coalitions is irrelevant. Formally:

Definition 2 (IIC)**.** *A social ranking* $\rho : \mathcal{C}^{2^N} \longrightarrow \mathcal{T}^N$ *satisfies the Independence of Irrelevant Coalitions (IIC) property on* $\mathcal{C}^{2^N} \subseteq \mathcal{T}^{2^N}$ *iff*

$$i\rho(\succcurlyeq)j \Leftrightarrow i\rho(\sqsupseteq)j$$

for all $i, j \in N$ *and all power relations* $\succcurlyeq, \sqsupseteq \in \mathcal{C}^{2^N}$ *such that for each* $S \in 2^{N\setminus\{i,j\}}$

$$S \cup \{i\} \succcurlyeq S \cup \{j\} \Leftrightarrow S \cup \{i\} \sqsupseteq S \cup \{j\}.$$

Let $\succcurlyeq \in \mathcal{T}^{2^N}$ and $i, j, p, q \in N$ be such that $|D^k_{ij}| = |D^k_{pq}|$ and $|D^k_{ji}| = |D^k_{qp}|$ for each $k = 0, \ldots, n-2$. Differently stated, for coalitions S of fixed cardinality, we have that the number of times that $S\cup\{i\}$ is stronger than $S\cup\{j\}$ equals the number of times that $S\cup\{p\}$ is stronger than $S\cup\{q\}$ (and the number of times that $S\cup\{j\}$ is stronger than $S\cup\{i\}$ equals the number of times that $S\cup\{q\}$ is stronger than $S\cup\{p\}$). In this symmetric situation, the following axiom states a principle of equivalence between the pairs i, j and p, q.

Definition 3 (SYM)**.** *A social ranking* $\rho : \mathcal{C}^{2^N} \longrightarrow \mathcal{T}^N$ *satisfies the symmetry (SYM) property on* $\mathcal{C}^{2^N} \subseteq \mathcal{T}^{2^N}$ *iff*

$$i\rho(\succcurlyeq)j \Leftrightarrow p\rho(\succcurlyeq)q$$

for all $i, j, p, q \in N$ *and* $\succcurlyeq \in \mathcal{C}^{2^N}$ *such that* $|D^k_{ij}| = |D^k_{pq}|$ *and* $|D^k_{ji}| = |D^k_{qp}|$ *for each* $k = 0, \ldots, n-2$.

Remark 1. *Note that if a social ranking* ρ *satisfies the SYM axiom on* $\mathcal{C}^{2^N} \subseteq \mathcal{T}^{2^N}$*, then for every* $\succcurlyeq \in \mathcal{C}^{2^N}$ *and* $i, j \in N$*, if* $|D^k_{ij}| = |D^k_{ji}|$ *for each* $k = 0, \ldots, n-2$*, then* $i\rho(\succcurlyeq)j$ *and* $j\rho(\succcurlyeq)i$*, that is* i *and* j *are indifferent in* $\rho(\succcurlyeq)$ *(to see this, simply take* $p = i$ *and* $q = j$ *in Definition 3).*

Remark 2. *If we want to check if a given social ranking rule satisfies DOM, IIC, or SYM only partial information on* $\succcurlyeq$ *is needed. In fact, conditions on the ranking* $\rho(\succcurlyeq)$ *between two elements* $\{i, j\}$ *only depend on the comparisons of subsets having the same cardinality and sharing the same subset* $S \in 2^{N\setminus\{i,j\}}$ *not containing neither* i *nor* j*.*

We conclude this section with an example showing that an apparently natural procedure (namely, the majority rule) to rank the agents of N may fail to provide a transitive social ranking. We first formally introduce such a procedure.

Definition 4 (Majority rule)**.** *A majority rule (denoted by* M*) is a map assigning to each power relation* $\succcurlyeq \in \mathcal{T}^{2^N}$ *a total binary relation* $M(\succcurlyeq)$ *on* N *such that*

$$iM(\succcurlyeq)j \Leftrightarrow d_{ij}(\succcurlyeq) \geq d_{ji}(\succcurlyeq).$$

where $d_{ij}(\succcurlyeq) = |\{S \in 2^{N\setminus\{i,j\}} : S \cup \{i\} \succcurlyeq S \cup \{j\}\}|$ *for each* $i, j \in N$.

Example 1. *One can easily check that the majority rule* M *satisfies the property of DOM, IIC and SYM on the class* $\mathcal{T}^{2^N}$*. On the other hand, it is also easy to find an example of power relation* $\succcurlyeq$ *such that* $M(\succcurlyeq)$ *is not transitive. Consider for instance the power relation* $\succcurlyeq \in \mathcal{T}^{2^N}$ *with* $N = \{1, 2, 3, 4\}$ *such that*

$2 \succ 1 \succ 3$
$23 \succ 13 \succ 12 \succ 14 \succ 34 \succ 24$
$134 \sim 124 \sim 234$

We rewrite the relevant information about $\succcurlyeq$ *by means of Table 1 (From now, we will sometimes omit braces and commas to separate elements, for instance, ij denotes the set* $\{i, j\}$*). Note that* $d_{12}(\succcurlyeq) = 2$, $d_{21}(\succcurlyeq) = 3$, $d_{23}(\succcurlyeq) = 2$, $d_{32}(\succcurlyeq) = 3$, $d_{13}(\succcurlyeq) = 3$ *and*

Table 1: The relevant information about $\succcurlyeq$ of Example 1.

1 vs. 2	2 vs. 3	1 vs. 3
$1 \prec 2$	$2 \succ 3$	$1 \succ 3$
$13 \prec 23$	$12 \prec 13$	$12 \prec 23$
$14 \succ 24$	$24 \prec 34$	$14 \succ 34$
$134 \sim 234$	$124 \sim 134$	$124 \sim 234$

$d_{31}(\succcurlyeq) = 2$. *So, we have that* $2M(\succcurlyeq)1$, $3M(\succcurlyeq)2$ *and* $1M(\succcurlyeq)3$, *but* $\neg(3M(\succcurlyeq)1)$*:* $M(\succcurlyeq)$ *is not a transitive relation.*

4 Primitive and unanimous social rankings

In this section we study the relations between the axioms introduced in the previous section and the social ranking solutions. In the following, we show that DOM and SYM are not compatible in a general case, for $N > 3$ (see Theorem 1), whereas SYM and IIC determine a unanimous social ranking on particular power relations.

We start with showing some consequences of using the axioms introduced in the previous section when the cardinality of the set N is 3 or 4. The analysis for cardinality $|N| = 3$ is easy since we can enumerate all the cases. As we will present in the following, the notion of complementarity plays an important role in this case. We denote by S^* the complement of the subset S ($S^* = N \backslash S$), and we say that a social ranking ρ such that $i\rho(\succcurlyeq)j \Leftrightarrow \{j\}^* \succcurlyeq \{i\}^*$ for each $\succcurlyeq \in \mathcal{T}^{2^N}$ and each $i, j \in N$ is *complement primitive* (i.e., it neglects any information contained in $\succcurlyeq$ about the comparison of coalitions of cardinality different from $n - 1$).

Proposition 1. *If* $|N| = 3$*, then there are only two social ranking solutions satisfying the DOM and SYM conditions: the primitive solution and the complement primitive one.*

Proof. Let $N = \{1, 2, 3\}$ with $1 \succcurlyeq 2 \succcurlyeq 3$. Then six cases may occur in $\succcurlyeq$: case 1) $13 \succcurlyeq 23 \succcurlyeq 12$, case 2) $13 \succcurlyeq 12 \succcurlyeq 23$, case 3) $23 \succcurlyeq 13 \succcurlyeq 12$, case 4) $12 \succcurlyeq 13 \succcurlyeq 23$, case 5) $23 \succcurlyeq 12 \succcurlyeq 13$ and case 6) $12 \succcurlyeq 23 \succcurlyeq 13$.

DOM and SYM impose that:

case 1) by DOM :$1\rho(\succcurlyeq)2$, by SYM ($1\rho(\succcurlyeq)3$ and $2\rho(\succcurlyeq)3$) or ($3\rho(\succcurlyeq)1$ and $3\rho(\succcurlyeq)2$). Hence we have $1\rho(\succcurlyeq)2\rho(\succcurlyeq)3$ (primitive) or $3\rho(\succcurlyeq)1\rho(\succcurlyeq)2$ (complement primitive)

case 2) by DOM :$1\rho(\succcurlyeq)2$ and $1\rho(\succcurlyeq)3$. We can have $2\rho(\succcurlyeq)3$ or $3\rho(\succcurlyeq)2$. Hence we have $1\rho(\succcurlyeq)2\rho(\succcurlyeq)3$ (primitive) or $1\rho(\succcurlyeq)3\rho(\succcurlyeq)2$ (complement primitive)

case 3) by SYM : ($1\rho(\succcurlyeq)2$, $1\rho(\succcurlyeq)3$ and $2\rho(\succcurlyeq)3$) or ($2\rho(\succcurlyeq)1$, $3\rho(\succcurlyeq)1$ and $3\rho(\succcurlyeq)2$).

case 4) by DOM $1\rho(\succcurlyeq)2\rho(\succcurlyeq)3$

case 5) by DOM :$2\rho(\succcurlyeq)3$, by SYM ($1\rho(\succcurlyeq)2$ and $1\rho(\succcurlyeq)3$) or ($2\rho(\succcurlyeq)1$ and $3\rho(\succcurlyeq)1$). Hence we have $1\rho(\succcurlyeq)2\rho(\succcurlyeq)3$ (primitive) or $2\rho(\succcurlyeq)3\rho(\succcurlyeq)1$ (complement primitive) case 6) by DOM :$1\rho(\succcurlyeq)3$ and $2\rho(\succcurlyeq)3$. We can have $1\rho(\succcurlyeq)2$ or $2\rho(\succcurlyeq)1$. Hence we have $1\rho(\succcurlyeq)2\rho(\succcurlyeq)3$ (primitive) or $2\rho(\succcurlyeq)1\rho(\succcurlyeq)3$ (complement primitive)

□

A relation which provides coherent comparisons with respect to the complement of objects is said "self-reflecting" . The notion of "self-reflecting" is introduced in [5]. More formally, if we denote by S^* the complement of the subset S ($S^* = N \setminus S$), we say that the power relation $\succcurlyeq$ is self-reflecting if and only if for all $S, Q \in N$, $S \succcurlyeq Q$ implies $Q^* \succcurlyeq S^*$.

Corollary 1. *If $|N| = 3$ and the power relation is self-reflecting, then the DOM condition is sufficient in order to determine the social ranking and it corresponds to a primitive social rule.*

Proof. Let $N = \{i, j, k\}$. Self-reflecting implies that $\forall i, j \in N\ i \succcurlyeq j \Leftrightarrow j^* \succcurlyeq i^* \Leftrightarrow ik \succcurlyeq jk$. By DOM we get $\forall i, j, k \in N\ i\rho(\succcurlyeq)j \Leftrightarrow i \succcurlyeq j \Leftrightarrow j^* \succcurlyeq i^* \Leftrightarrow ik \succcurlyeq jk$. □

Next theorem shows that on the class $\mathcal{T}^{2^N}$ (all possible total preorders) the properties of DOM and SYM are not compatible.

Theorem 1. *Let $|N| > 3$. There is no social ranking rule $\rho : \mathcal{T}^{2^N} \longrightarrow \mathcal{T}^N$ which satisfies DOM and SYM on $\mathcal{T}^{2^N}$.*

Proof. We first show a particular situation where DOM and SYM are not compatible. Consider a power relation $\succcurlyeq \in \mathcal{T}^{2^N}$ with $N = \{1, 2, 3, 4\}$ and such that

$$\begin{array}{l} 1 \sim 2 \sim 3 \\ 13 \succ 23 \succ 12 \succ 24 \sim 14 \succ 34 \\ 1234 \sim 123 \sim 124 \sim 134 \sim 234 \end{array}$$

We rewrite the relevant informations about $\succcurlyeq$ and the elements $1, 2$ and 3 by means of the following Table 2. By Remark 1, a social ranking rule $\rho : \mathcal{T}^{2^N} \longrightarrow \mathcal{T}^N$ which satisfies SYM

Table 2: The relevant informations about $\succcurlyeq$ and the elements $1, 2$ and 3.

1 vs. 2	2 vs. 3	1 vs. 3
$1 \sim 2$	$2 \sim 3$	$1 \sim 3$
$13 \succ 23$	$12 \prec 13$	$12 \prec 23$
$14 \sim 24$	$24 \succ 34$	$14 \succ 34$
$134 \sim 234$	$124 \sim 134$	$124 \sim 234$

should be such that $2\rho(\succcurlyeq)3$, $3\rho(\succcurlyeq)2$, $1\rho(\succcurlyeq)3$, $3\rho(\succcurlyeq)1$.
By the DOM property, we should have $1\rho(\succcurlyeq)2$, and $\neg(2\rho(\succcurlyeq)1)$, which yields a contradiction with the transitivity of the ranking $\rho(\succcurlyeq)$.

Now, consider power relations in $\mathcal{T}^{2^N}$, with $N \supseteq \{1, 2, 3, 4\}$, that are obtained from the power relation $\succcurlyeq$ defined above and assigning all the additional subsets of N not contained in $\{1, 2, 3, 4\}$ in the same indifference class. More precisely, let $N \supseteq \{1, 2, 3, 4\}$, and take $\succcurlyeq' \in \mathcal{T}^{2^N}$ such that $U \succcurlyeq' W :\Leftrightarrow U \succcurlyeq W$ for all the subsets $U, W \subseteq \{1, 2, 3, 4\}$, and $U \succcurlyeq' W$, $W \succcurlyeq' U$ for all the other subsets of N not included in $\{1, 2, 3, 4\}$. □

The following proposition shows that the adoption of properties IIC and SYM yields a unanimous social ranking over all those power relations $\succcurlyeq \in \mathcal{T}^N$ such that, for some $k \in \{0, \ldots, |N| - 2\}$, $D^t_{ji}(\succcurlyeq) = D^t_{ij}(\succcurlyeq)$ for all cardinalities $t \neq k$ and all $i, j \in N$, and $|D^k_{ji}(\succcurlyeq)|$ is not necessarily equal to $|D^k_{ij}(\succcurlyeq)|$ (provided that $D^k_{ij}(\succcurlyeq) \setminus D^k_{ji}(\succcurlyeq) \neq \emptyset$ and $D^k_{ji}(\succcurlyeq) \setminus D^k_{ij}(\succcurlyeq) \neq \emptyset$).

Proposition 2. *Let $\rho : \mathcal{T}^{2^N} \longrightarrow \mathcal{T}^N$ be a social ranking satisfying IIC and SYM. Let $\succcurlyeq \in \mathcal{T}^{2^N}$ and $k \in \{0, \ldots, |N|-2\}$ be such that $S \cup \{i\} \succcurlyeq S \cup \{j\}$ and $S \cup \{j\} \succcurlyeq S \cup \{i\}$, for all $S \in 2^{N \setminus \{i,j\}}$ with $|S| \neq k$, $D_{ij}^k(\succcurlyeq) \setminus D_{ji}^k(\succcurlyeq) \neq \emptyset$ and $D_{ji}^k(\succcurlyeq) \setminus D_{ij}^k(\succcurlyeq) \neq \emptyset$ for all $i, j \in N$. Then $i\rho(\succcurlyeq)j$ and $j\rho(\succcurlyeq)i$ for each $i, j \in N$.*

Proof. Take $i, j \in N$ such that $|D_{ij}^k(\succcurlyeq)| \geq |D_{ji}^k(\succcurlyeq)|$. Define another power relation $\sqsupseteq \in \mathcal{T}^{2^N}$ such that

$$S \cup \{i\} \succcurlyeq S \cup \{j\} \Leftrightarrow S \cup \{i\} \sqsupseteq S \cup \{j\}$$

for each $S \in 2^{N \setminus \{i,j\}}$ with $|S| = k$, and $S \sqsupseteq T$ and $T \sqsupseteq S$ for all the other coalitions $S, T \in 2^N$ with $|S| = |T| \neq k+1$. We still need to define relation $\sqsupseteq$ on the remaining coalitions of size k.

Take $l \in N \setminus \{i,j\}$. Let $\mathcal{D} \subseteq D_{ij}^k(\succcurlyeq)$ be such that $|\mathcal{D}| = |D_{ji}^k(\succcurlyeq)|$. By Remark 3 (see Section 8 Appendix), define the remaining comparisons in $\sqsupseteq$ as follows (an illustrative example of these cases are given in Table 3):

case 1) for each $S \in D_{ji}^k(\succcurlyeq)$ with $l \in S$, let

$$S \cup \{i,j\} \setminus \{l\} \sqsubseteq S \cup \{i\} \text{ and } S \cup \{i,j\} \setminus \{l\} \sqsubseteq S \cup \{j\};$$

case 2) for each $S \in D_{ji}^k(\succcurlyeq)$ with $l \notin S$, let

$$S \cup \{i\} \sqsubseteq S \cup \{l\} \text{ and } S \cup \{j\} \sqsubseteq S \cup \{l\};$$

case 3) For each $S \in \mathcal{D}$ with $l \in S$, let

$$S \cup \{i,j\} \setminus \{l\} \sqsupseteq S \cup \{i\} \text{ and } S \cup \{i,j\} \setminus \{l\} \sqsubseteq S \cup \{j\};$$

case 4) for each $S \in \mathcal{D}$ with $l \notin S$, let

$$S \cup \{i\} \sqsubseteq S \cup \{l\} \text{ and } S \cup \{j\} \sqsupseteq S \cup \{l\};$$

case 5) for each $S \in D_{ij}^k \setminus \mathcal{D}$ with $l \in S$, let

$$S \cup \{i,j\} \setminus \{l\} \sqsupseteq S \cup \{i\} \text{ and } S \cup \{i,j\} \setminus \{l\} \sqsupseteq S \cup \{j\};$$

case 6) for each $S \in D_{ij}^k \setminus \mathcal{D}$ with $l \notin S$, let

$$S \cup \{i\} \sqsupseteq S \cup \{l\} \text{ and } S \cup \{j\} \sqsupseteq S \cup \{l\}.$$

Note that $|D_{ji}^k(\succcurlyeq)| = |D_{li}^k(\sqsupseteq)| = |D_{jl}^k(\sqsupseteq)|$ and $|D_{ij}^k(\succcurlyeq)| = |D_{il}^k(\sqsupseteq)| = |D_{lj}^k(\sqsupseteq)|$. Suppose now that $i\rho(\succcurlyeq)j$. By IIC, we have $i\rho(\sqsupseteq)j$. By SYM, $j\rho(\sqsupseteq)l$ and $l\rho(\sqsupseteq)i$. By transitivity of $\rho(\sqsupseteq)$, $j\rho(\sqsupseteq)i$. By IIC we conclude that $j\rho(\succcurlyeq)i$ too. In a similar way, if we suppose $j\rho(\succcurlyeq)i$, then we end up with the conclusion that $i\rho(\succcurlyeq)j$ too, and the proof follows. □

An interesting consequence of Proposition 2 is that if the only information making a difference between two objects is given by comparisons of a fixed cardinality, then it is sufficient to have one discordance in order to declare an indifference (with IIC and SYM). Proposition 2 suggests how to deal with situations where coalitions are of a fixed size (such situations are not so eccentric in real life). For instance, let us imagine that we have committees with a given number (k) of persons and that we have a ranking on them (for instance $N = \{1,2,3,4\}$ and $k = 2$, with $12 \succcurlyeq 13 \succcurlyeq 14 \succcurlyeq 34 \succcurlyeq 24 \succcurlyeq 23$). Since committees are always formed by two persons, no information is available on subsets of N with $l \neq k$ elements (or such information is irrelevant). How to define a social ranking in this case? One solution could be to consider all the other comparisons indifferent. Then, by Proposition 2, we know that SYM and IIC properties can be used in order to support a unanimous social ranking.

Table 3: An illustrative example of the six possible cases for a power relation $\sqsupseteq$ as the one considered in Proposition 2 with $N = \{1, 2, 3, i, j, l\}$, $k = 2$ and $\mathcal{D} = \{\{1, 2\}, \{2, l\}\}$.

	i vs j	i vs. l	j vs. l
case 4): $S = \{1, 2\}$	$\{1, 2, i\} \sqsupseteq \{1, 2, j\}$	$\{1, 2, i\} \sqsubseteq \{1, 2, l\}$	$\{1, 2, j\} \sqsupseteq \{1, 2, l\}$
case 6): $S = \{1, 3\}$	$\{1, 3, i\} \sqsupseteq \{1, 3, j\}$	$\{1, 3, i\} \sqsupseteq \{1, 3, l\}$	$\{1, 3, j\} \sqsupseteq \{1, 3, l\}$
case 2): $S = \{2, 3\}$	$\{2, 3, i\} \sqsubseteq \{2, 3, j\}$	$\{2, 3, i\} \sqsubseteq \{2, 3, l\}$	$\{2, 3, j\} \sqsubseteq \{2, 3, l\}$
case 5): $S = \{1, l\}$	$\{1, l, i\} \sqsupseteq \{1, i, j\}$	$\{1, i, j\} \sqsupseteq \{1, j, l\}$	$\{1, i, j\} \sqsupseteq \{1, i, l\}$
case 3): $S = \{2, l\}$	$\{2, l, i\} \sqsupseteq \{2, i, j\}$	$\{2, i, j\} \sqsubseteq \{2, j, l\}$	$\{2, i, j\} \sqsupseteq \{2, i, l\}$
case 1): $S = \{3, l\}$	$\{3, l, i\} \sqsubseteq \{3, i, j\}$	$\{3, i, j\} \sqsubseteq \{3, j, l\}$	$\{3, i, j\} \sqsubseteq \{3, i, l\}$
	$\lvert D_{ij}(\sqsupseteq)\rvert = 4$	$\lvert D_{il}(\sqsupseteq)\rvert = 2$	$\lvert D_{jl}(\sqsupseteq)\rvert = 4$
	$\lvert D_{ji}(\sqsupseteq)\rvert = 2$	$D_{li}(\sqsupseteq)\rvert = 4$	$\lvert D_{lj}(\sqsupseteq)\rvert = 2$

5 Dictatorship of the coalition size

In this section, we define a special class of power relations (namely, the *per size-strong dominant* relations) characterized by the fact that a relation of dominance always exists with respect to coalitions of the same size, but the dominance may change with the cardinality (for instance, an element i could dominate another element j when coalitions of size s are considered, but j could dominate i over coalitions of size $t \neq s$). We first need to introduce the notion of s-strong dominance.

Definition 5. *Let $\succcurlyeq \in \mathcal{T}^{2^N}$, $i, j \in N$ and $s \in \{0, \ldots, n-2\}$. We say that i s-strong dominates j in $\succcurlyeq$, iff*

$$S \cup \{i\} \succ S \cup \{j\} \text{ for each } S \in 2^{N \setminus \{i,j\}} \text{ with } |S| = s. \tag{1}$$

Definition 6. *We say that $\succcurlyeq \in \mathcal{T}^{2^N}$ is per size-strong dominant (shortly, ps-sdom) iff for each $s \in \{0, \ldots, n-2\}$ and all $i, j \in N$, we have either*

$$[i \text{ } s\text{-strong dominates } j \text{ in } \succcurlyeq] \text{ or } [j \text{ } s\text{-strong dominates } i \text{ in } \succcurlyeq].$$

The set of all ps-sdom power relations is denoted by $\mathcal{S}^{2^N} \subseteq \mathcal{T}^{2^N}$.

We first study the effect of the combination of the properties of DOM and IIC on a specific instance of ps-sdom power relations where there exist elements that are always placed at the top or at the bottom in the rankings of coalitions of equal cardinality.

Example 2. *Consider a power relation $\succcurlyeq \in \mathcal{S}^{2^N}$ with $N = \{1, 2, 3, 4\}$ and such that*

$$\begin{aligned} &1 \succ 2 \succ 3 \succ 4 \\ &34 \succ 24 \succ 14 \succ 23 \succ 13 \succ 12 \\ &123 \succ 134 \succ 124 \succ 234. \end{aligned}$$

We rewrite the relevant informations about $\succcurlyeq$ by means of Table 4.

Note that for each $s \in \{0, 2\}$, it holds that either $S \cup \{1\} \succcurlyeq S \cup \{l\}$ for each $S \subseteq N \setminus \{1\}$ with $|S| = s$ and all $l \in N \setminus S$ (i.e., coalitions $S \cup \{1\}$ are ranked above all coalitions $S \cup \{l\}$, with $l \neq i$ and S containing 0 or 2 elements), and $S \cup \{1\} \preccurlyeq S \cup \{l\}$ for each $S \subseteq N \setminus \{1\}$ with $|S| = 1$ and all $l \in N \setminus S$ (i.e., coalitions $S \cup \{1\}$ are ranked below all coalitions $S \cup \{l\}$, with $l \neq i$ and S containing precisely 1 element). Similar considerations can be done for element 4. So, elements 1 and 4 are two "extreme" ones. Let us remark that there can be at most two "extreme" elements of a power relation in $\mathcal{S}^{2^N}$. In Proposition 3 we argue that on this kind of power relations, a social ranking satisfying both DOM and IIC cannot rank "extreme" elements (in this case 1 and 4) in between two others.

Table 4: The relevant informations about $\succcurlyeq$ of Example 2.

1 vs. 2	2 vs. 3	1 vs. 3	1 vs. 4	2 vs. 4	3 vs. 4
$1 \succ 2$	$2 \succ 3$	$1 \succ 3$	$1 \succ 4$	$2 \succ 4$	$3 \succ 4$
$13 \prec 23$	$12 \prec 13$	$12 \prec 23$	$12 \prec 24$	$12 \prec 14$	$13 \prec 14$
$14 \prec 24$	$24 \prec 34$	$14 \prec 34$	$13 \prec 34$	$23 \prec 34$	$23 \prec 24$
$134 \succ 234$	$124 \prec 134$	$124 \succ 234$	$123 \succ 234$	$123 \succ 134$	$123 \succ 124$

The following proposition shows the effect of DOM and IIC on the social position of the "extreme" elements.

Proposition 3. *Let $\rho : \mathcal{S}^{2^N} \longrightarrow \mathcal{T}^N$ be a social ranking satisfying IIC and DOM on $\mathcal{S}^{2^N}$. Let $\succcurlyeq \in \mathcal{S}^{2^N}$ and $i \in N$ be such that for each $s \in \{0, \dots, n-2\}$ either*

$$[S \cup \{i\} \succ S \cup \{j\} \text{ for all } j \in N \setminus \{i\} \text{ and } S \in 2^{N \setminus \{i,j\}} \text{ with } |S| = s] \tag{2}$$

or

$$[S \cup \{j\} \succ S \cup \{i\} \text{ for all } j \in N \setminus \{i\} \text{ and } S \in 2^{N \setminus \{i,j\}} \text{ with } |S| = s]. \tag{3}$$

Then, $[i\rho(\succcurlyeq)j$ for all $j \in N]$ OR $[j\rho(\succcurlyeq)i$ for all $j \in N]$.

Proof. Suppose on the contrary that there exist $j, k \in N \setminus \{i\}$, such that

$$j\rho(\succcurlyeq)i \text{ and } i\rho(\succcurlyeq)k. \tag{4}$$

Define $\sqsupseteq \in \mathcal{T}^{2^N}$ such that

$$S \cup \{i\} \sqsupset S \cup \{j\} \Leftrightarrow S \cup \{i\} \succ S \cup \{j\} \text{ for all } S \subseteq N \setminus \{i, j\}, \tag{5}$$

$$S \cup \{i\} \sqsupset S \cup \{k\} \Leftrightarrow S \cup \{i\} \succ S \cup \{k\} \text{ for all } S \subseteq N \setminus \{i, k\}, \tag{6}$$

and

$$S \cup \{k\} \sqsupset S \cup \{j\} \text{ for all } S \subseteq N \setminus \{j, k\}. \tag{7}$$

[note that each coalition $S \cup \{i\}$, with $S \subseteq N \setminus \{i\}$, by condition (2) and (3), is ranked strictly higher or lower than each other coalition $S \cup \{j\}$, $j \neq i$, so the rearrangement of coalitions in $\succcurlyeq$ to obtain $\sqsupseteq$ is feasible.]

By IIC, we have that

$$i\rho(\succcurlyeq)j \Leftrightarrow i\rho(\sqsupseteq)j \text{ and } i\rho(\succcurlyeq)k \Leftrightarrow i\rho(\sqsupseteq)k.$$

So, by relation (4), $j\rho(\sqsupseteq)i$ and $i\rho(\sqsupseteq)k$. On the other hand, by DOM we have $k\rho(\sqsupseteq)j$ and $\neg(j\rho(\sqsupseteq)k)$, which yields a contradiction with the transitivity of $\rho(\sqsupseteq)$. ☐

Proposition 3 shows that if there is an element $i \in N$ having "contradictory" and "radical" behavior depending on the size of coalitions (very well for size k and very bad for size l), then the social ranking satisfying IIC and DOM can not give him an intermediate position: the element i will be the "best" one or the "worst" one in the social ranking.

In the following we argue that if a power relation is in $\mathcal{S}^{2^N}$ and a social ranking satisfies both DOM and IIC on the set of ps-sdom power realtions $\mathcal{S}^{2^N}$, then it must exist a cardinality $t^* \in \{0, \dots, n-2\}$ whose relation of t^*-strong dominance (dictatorially) determines the social ranking. We first need to introduce the next lemma, where a given element i plays an important role.

Lemma 1. *Let $i \in N$ and $\rho : \mathcal{S}^{2^N} \longrightarrow \mathcal{T}^N$ be a social ranking satisfying IIC and DOM on $\mathcal{S}^{2^N}$. There exists $t^* \in \{0, \dots, n-2\}$ such that*

$$j\rho(\succcurlyeq)k \Leftrightarrow j \ t^*\text{-strong dominates } k \text{ in } \succcurlyeq,$$

for all $j, k \in N \setminus \{i\}$ and $\succcurlyeq \in \mathcal{S}^{2^N}$.

Proof. Given a power relation $\succcurlyeq \in \mathcal{S}^{2^N}$, define another power relation $\succcurlyeq_0 \in \mathcal{S}^{2^N}$ such that for each $S \subseteq N \setminus \{i\}$ we have

$$S \cup \{l\} \succ_0 S \cup \{i\} \text{ for all } l \in N \setminus (S \cup \{i\}), \tag{8}$$

and

$$U \succcurlyeq_0 W :\Leftrightarrow U \succcurlyeq W$$

for all the other possible pairs of coalitions U, W whose comparison is not already considered in (8). Roughly speaking, the only difference between $\succcurlyeq_0$ and $\succcurlyeq$ is that coalitions of size s containing i are placed at the bottom of the ranking induced by $\succcurlyeq$ over the coalitions of the same size. By DOM, it follows that $l\rho(\succcurlyeq_0)i$ for every $l \in N$.

Now, for each $t \in \{0, \dots, n-2\}$, define a power relation $\succcurlyeq_t \in \mathcal{T}^{2^N}$ such that

$$S \cup \{i\} \succ_t S \cup \{l\} \text{ for each } l \in N \text{ and } S \in 2^{N \setminus \{i,l\}} \text{ with } |S| = s, \tag{9}$$

where $s \in \{0, \dots, t\}$, and

$$U \succcurlyeq_t W :\Leftrightarrow U \succcurlyeq_{t-1} W$$

for all the other possible pairs of coalitions U, W whose comparison is not already considered in (9). So, the only difference between $\succcurlyeq_t$ and $\succcurlyeq_{t-1}$, for each $t \in \{1, \dots, n-2\}$, is that in $\succcurlyeq_t$ coalitions of size t containing i are placed at the top of the ranking induced by $\succcurlyeq_{t-1}$ over coalitions of the same size t, and all the remaining comparisons remain the same as in $\succcurlyeq_{t-1}$.

Note that by Proposition 3, we have that either $l\rho(\succcurlyeq_t)i$ for every $l \in N$, or $i\rho(\succcurlyeq_t)l$ for every $l \in N$. Moreover, By DOM, it follows that $i\rho(\succcurlyeq_{n-2})l$ for every $j \in N$.

Let t^* be the smallest number in $\{0, \dots, n-2\}$ such that $l\rho(\succcurlyeq_{t^*-1})i$ for every $l \in N$ and $i\rho(\succcurlyeq_{t^*})l$ for every $l \in N$ (for the considerations above such a t^* must exist, being, at most, $t^* = n-2$).

Next, we argue that for every $j, k \in N \setminus \{i\}$, the social ranking between j and k in $\succcurlyeq$ is imposed by the relation of t^*-strong dominance in $\succcurlyeq$.

W.l.o.g., suppose that $S \cup \{j\} \succcurlyeq S \cup \{k\}$ (and, as a consequence, $S \cup \{j\} \succcurlyeq_{t^*} S \cup \{k\}$) for each $S \in 2^{N \setminus \{j,k\}}$, and $|S| = t^*$. Consider another power relation $\sqsupseteq \in \mathcal{T}^{2^N}$ obtained by $\succcurlyeq_{t^*}$ and such that:

$$S \cup \{j\} \sqsupset S \cup \{i\} \text{ for each } S \in 2^{N \setminus \{i,j\}} \text{ with } |S| = t^*, \tag{10}$$

$$S \cup \{i\} \sqsupset S \cup \{k\} \text{ for each } S \in 2^{N \setminus \{i,k\}} \text{ with } |S| = t^*, \tag{11}$$

$$S \cup \{j\} \sqsupset S \cup \{k\} \text{ for each } S \in 2^{N \setminus \{j,k\}} \setminus \left(2^{N \setminus \{i,j\}} \cup 2^{N \setminus \{i,k\}}\right), \text{ and } |S| = t^*, \tag{12}$$

and, finally,

$$U \sqsupseteq V :\Leftrightarrow U \succcurlyeq_{t^*} V \tag{13}$$

for all the other relevant pairs of coalitions U, W of size $s \neq t^* + 1$. By IIC $j\rho(\sqsupseteq)i$ (since in $\sqsupseteq$ the comparisons between coalitions containing i and j are precisely as in $\succcurlyeq_{t^*-1}$ and, as previously stated, $j\rho(\succcurlyeq_{t^*-1})i$) and $i\rho(\sqsupseteq)k$ (since in $\sqsupseteq$ the comparisons between coalitions containing i and k are precisely as in $\succcurlyeq_{t^*}$ and, as previously stated, $i\rho(\succcurlyeq_{t^*})k$). Then, by transitivity of $\rho(\sqsupseteq)$ we have $j\rho(\sqsupseteq)k$. Note that by IIC, $j\rho(\sqsupseteq)k \Leftrightarrow j\rho(\succcurlyeq_{t^*})k \Leftrightarrow j\rho(\succcurlyeq)k$. We have then proved that whenever j t^*-dominates k, then $j\rho(\succcurlyeq)k$.

□

We can now formulate the following theorem stating the "dictatorship of the coalition's size".

Theorem 2. *Let* $\rho : \mathcal{S}^{2^N} \longrightarrow \mathcal{T}^N$ *be a social ranking satisfying IIC and DOM on* $\mathcal{S}^{2^N}$*. There exists* $t^* \in \{0, \ldots, n-2\}$ *such that*

$$i\rho(\succcurlyeq)j \Leftrightarrow i \ t^*\text{-strong dominates } j \text{ in } \succcurlyeq,$$

for all $i, j \in N$ *and* $\succcurlyeq \in \mathcal{S}^{2^N}$*.*

Proof. Given a power relation $\succcurlyeq \in \mathcal{S}^{2^N}$, let $i \in N$ and define $\succcurlyeq_{t^*}$ starting from $\succcurlyeq$ and i precisely as in the proof of Lemma 1.

Now take $k \in N \setminus \{i\}$ and apply Lemma 1 with k in the role of i. Consequently, we have that there exists $\hat{t} \in \{0, \ldots, n-2\}$ such that

$$h\rho(\succcurlyeq)l \Leftrightarrow h \ \hat{t}\text{-strong dominates } l \text{ in } \succcurlyeq,$$

for each $h, l \in N \setminus \{k\}$, and in particular

$$i\rho(\succcurlyeq)l \Leftrightarrow i \ \hat{t}\text{-strong dominates } l \text{ in } \succcurlyeq,$$

for whatever complete power relation $\succcurlyeq \in \mathcal{S}^{2^N}$.

But in the proof of Lemma 1 we have shown that

$$i\rho(\succcurlyeq)l \Leftrightarrow i \ t^*\text{-strong dominates } l \text{ in } \succcurlyeq_{t^*}$$

(remember that t^* in the proof of Lemma 1 is the smallest number in $\{0, \ldots, n-2\}$ such that $l\rho(\succcurlyeq_{t^*-1})i$ for every $l \in N$ and $i\rho(\succcurlyeq_{t^*})l$ for every $l \in N$). Then it must be $\hat{t} = t^*$, and the proof follows. □

6 Conclusions

In this paper we introduced and studied the problem of how to rank the objects of a set N according to their ability to influence the ranking over the subsets of N. As far as we know, this is the first time that a social ranking is proposed using an axiomatic approach (and without the quantitative notion of power index from cooperative game theory, like in [10]).

A possible direction for future research is the open question about which axioms could be used to characterize a social ranking over the domain of all possible power relations. In view of our results, each combination of the axioms we propose in this paper is not satisfactory. In this respect, it is worth noting that all the properties that we analysed are based on the comparison of subsets having the same number of elements. Therefore, it would be interesting to study properties based on the comparison among subsets with different cardinalities. For instance, if $N = \{1, 2, 3\}$, the information of the type $\{1\} \succ \{2, 3\} \succ \{1, 3\} \succ \{2\}$ could be used to establish that 1 is socially stronger than 2 (note that 1 strictly dominates 2 on coalitions of size 1, and 2 strictly dominates 1 of coalitions of cardinality 2, but the "interval" between $\{2, 3\}$ and $\{1, 3\}$ is smaller than the one between $\{1\}$ and $\{2\}$). Of course, social ranking solutions taking into account the relative comparison of coalitions with different size, would necessarily face the important problem of dealing with a much larger amount of information.

Another interesting direction is to focus on alternative classes of power relations. Similar to the classical approaches adopted in the analysis of power indices for simple games, one could be interested in studying power relations that satisfy a *monotonicity* property, that is to only consider those power relations $\succcurlyeq \in \mathcal{T}^{2^N}$ such that $T \succcurlyeq S$ for each coalitions $S, T \in 2^N$

with $S \subseteq T$. However, as already noticed, it is worth remarking that the constraints imposed by our axioms apply to coalitions of the same size, and therefore a restriction based on the monotonicity property has no relevant impact on the results provided in this paper. Moreover, the property of monotonicity, as many others studied in the literature on ranking sets of objects (see [2]), prevents a kind of interaction among the elements of N that is often observed in practice, i.e., the possibility that the power of a coalition could be deteriorated by the addition of a new individual which is incompatible or redundant with those already contained in it (see [11] for a detailed discussion on this issue).

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8 Appendix

Remark 3. *Note that by transitivity of power relations* $\succcurlyeq \in \mathcal{T}^{2^N}$, *the relations between the elements of the columns of a comparison table must satisfy some constraints, as listed below.*

- *Let* $i, j, k \in N$ *and* $S \in 2^{N \setminus \{i,j,k\}}$ *with* $S \cup \{i\} \succcurlyeq S \cup \{j\}$. *Then, one of the following possibilities may occur:*
  - $S \cup \{i\} \succcurlyeq S \cup \{k\}$ *and* $S \cup \{j\} \succcurlyeq S \cup \{k\}$;
  - $S \cup \{i\} \preccurlyeq S \cup \{k\}$ *and* $S \cup \{j\} \preccurlyeq S \cup \{k\}$;
  - $S \cup \{i\} \succcurlyeq S \cup \{k\}$ *and* $S \cup \{j\} \preccurlyeq S \cup \{k\}$.
- *Let* $i, j, k \in N$ *and* $S \in 2^{N \setminus \{i,j,k\}}$ *with* $S \cup \{i,k\} \succcurlyeq S \cup \{j,k\}$.
  - $S \cup \{i,j\} \succcurlyeq S \cup \{i,k\}$ *and* $S \cup \{i,j\} \succcurlyeq S \cup \{j,k\}$;
  - $S \cup \{i,j\} \preccurlyeq S \cup \{i,k\}$ *and* $S \cup \{i,j\} \preccurlyeq S \cup \{j,k\}$;
  - $S \cup \{i,j\} \succcurlyeq S \cup \{i,k\}$ *and* $S \cup \{i,j\} \preccurlyeq S \cup \{j,k\}$.

Stefano Moretti
Univ. Paris Dauphine, PSL Research University, CNRS, UMR7243, LAMSADE, 75016 Paris, France.
Email: `stefano.moretti@dauphine.fr`

Meltem Öztürk
Univ. Paris Dauphine, PSL Research University, CNRS, UMR7243, LAMSADE, 75016 Paris, France.
Email: `meltem.ozturk@dauphine.fr`