On the Effects of Priors in Weighted Voting Games

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Abstract

We study the Shapley value in weighted voting games. The Shapley value has been used as an index for measuring the power of individual agents in decision-making bodies or political organizations, in which decisions are made by a voting process. Previous studies assume that agent weights (corresponding to the size of a caucus or a political party) are fixed; we analyze new domains in which the weights are generated according to a certain probability distribution, modeling, for example, the effect of elections.

We study how the parameters of the weight generating process affect the power disparity in the resulting game, by showing how different prior beliefs regarding the weight distribution affect the expected Shapley values of the agents.

We examine several natural weight generation processes: the binomial distribution; the Balls and Bins model, with uniform as well as exponentially-decaying probabilities; and i.i.d. weights drawn from a known distribution. In particular, we draw a novel connection between the case of i.i.d. weights and renewal theory. We also analyze weights that admit a super-increasing sequence, answering several open questions pertaining to the Shapley value in such games.

1 Introduction

Consider a large organization with multiple sections given a substantial endowment, e.g., a state that receives annual grants from the federal government, or a university that needs to disburse funds to its various departments. How should funds be distributed?

Similarly, in a parliamentary system, each party receives a number of seats proportional to the number of votes it received in an election. We wish to reason about the effective leverage each party has, in key processes such as forming a coalition, legislation, and budget division.

In all of the examples above, the "size" of participating players (i.e., the sizes of states, or the number of seats a party holds) can be thought of as the result of a *generative process*. In the example of a federal grant, state population can be thought of as the outcome of migration and natural growth; indeed, the stochastic modeling approach to population dynamics has been studied by statisticians and economists (e.g., [6, 18, 7]). The sizes of parties in the parliament can also be naturally modeled via stochastic means; the number of seats each party obtains is determined by an election, in which the votes are cast in a probabilistic manner. For example, one could imagine a scenario where each voter chooses a party to vote for uniformly at random. To conclude, in all of the above scenarios, the weight of each agent can be naturally thought of as the result of a randomized process.

The second common thread is that, once resources are allocated to the agents, they determine the relative influence or *power* of those agents. In other words, the agents are rewarded in a way that reflects their individual contributions, which may or may not be proportional to their actual size.

A well-studied model for analyzing the relative power of weighted agents in cooperative domains is that of *weighted voting games* (WVGs) [15]; WVGs capture agent interactions, where every agent has a non-negative resource (its weight). Resources must be pooled together to achieve a certain goal; a subset of agents (also referred to as a *coalition*) is said to be *winning* if its total weight exceeds some given *quota* (also referred to as a *threshold*). That is, winning sets are those that can achieve the goal, without the use of agents outside the set.

Agent influence is not always proportional to weight; thus, most works on WVGs employ "power indices" that measure an agent's actual power in such settings [15]. The most prominent such power index is the *Shapley value* [19], which measures the average marginal contribution of an agent across all possible orderings (permutations) of the agents, and satisfies important desired axioms. Earlier work has examined many domains, including many political and decision-making bodies, and used the Shapley value to examine the relative power of agents [4].

Our main question is then the following: given the distribution of the weights, what is the expected power disparity between the agents? In the neutral voting example, do we expect large differences in power between the agents? What happens in the case of a more biased process?

From a practical perspective, the advantage of taking a stochastic approach is twofold. First, it allows a system designer to act as a decision maker of sorts; by judiciously choosing a quota, it is possible to obtain a certain power distribution with high probability (given the belief about the distribution of weights). Second, by assuming the existence of a stochastic generative processe, we are able to provide strong characterization results, solving several open problems in [22, 23, 24].

We consider several stochastic processes for generating weights and examine the expected power relations between the agents under these processes. We also characterize the properties of the Shapley value given with the following prior distributions: binomially distributed weights, Balls and Bins processes (both the classical uniform case, and the non-uniform case with exponentially decaying probabilities), and i.i.d. samples from reasonably bounded distributions (e.g., the uniform distribution on the unit interval). We focus on the *expected* differences in power between the weakest agent (of smallest weight) and the strongest agent (of highest weight), i.e. the maximum possible power disparity among agents.

Contributions We initiate our study by considering the case where the n agent weights are drawn from the binomial distribution $\mathrm{B}(m,p)$ (Section 3). This corresponds to *approval voting*, in which each of m voters approves of each of n candidates with probability p each. We show that when the quota is $\Omega(mp)$, the expected gap between any two Shapley values is at most $\frac{1}{n}O_p(\sqrt{\frac{\log n}{n}})$.

Next, we explore the *Balls and Bins* model – a model that has received considerable recent attention in the computer science community [12, 11, 17]. Informally, in this iterative process, in each round, a ball is thrown into one of several bins according to a fixed probability distribution. Each of the bins represents a single agent, and the load of a bin at the end of the process determines the weight of its respective agent. In the election terminology, the interpretation is *plurality* voting: each ball corresponds to a voter, and the bins are the candidates. Furthermore, each ball will be placed in bin i (corresponding to candidate i) independently with probability p_i.

In Section 4, we study the conceptually simplest and most common version of the model, in which each of the m balls lands in one of the n bins uniformly at random ($p_i = 1/n$ for $i \in [n]$). Going back to our motivating scenario of an election, this means that each voter gives his vote to a candidate that was selected uniformly at random. We show that even in this setting, where agent weights are likely to be very similar (assuming a large enough m), the choice of a threshold can be critical. We identify quotas that ensure that power disparity is likely to be low, as well as quotas for which relatively high power disparity is likely to occur.

To complement our findings for the uniform case, in Section 6 we consider the case in which the probabilities decay exponentially, with a decay factor no larger than $1/2$. That is, assuming that the probabilities are given in non-increasing order, than $p_{i-1}/p_i \leq 1/2$. We show that analyzing this case essentially boils down to characterizing the Shapley values in a game in which sorting the weights in ascending order gives a super-increasing sequence. Our results (Section 6) significantly strengthen previous results obtained for this case by Zuckerman et al. [24]. We show that when weights are super-increasing, there is a simple, polynomial-time computable formula for the Shapley value as a function of the quota.

Finally, we explore the case where the weights are drawn i.i.d. from a bounded distribution with a bounded density function. This generalizes our work in Section 3, in which the distribution in question was $\mathrm{B}(m,p)$. Leveraging a novel connection to renewal theory, we provide estimates for both the highest and lowest expected Shapley values, whenever the fractional quota (i.e., the fraction of the total weight) is bounded away from zero and one. We show that in the specified range, these estimates remain stable, up to an exponentially decaying error factor. A particularly intriguing example of such a prior weight distribution is the uniform distribution $U(0,1)$. We demonstrate that whenever the fractional quota is roughly in the range $[\frac{2}{n}, 1-\frac{2}{n}]$, the highest and lowest Shapley values are close to $2/n$ and $2/n^2$, respectively.

1.1 Related Work

Several works have studied the effects of randomization on weighted voting games from a theoretical, computational and empirical perspective. The earliest study of randomization and its effects on voting power is due to Penrose [16], who shows that the Banzhaf power index scales as the square root of players' weight when weights are drawn from bounded distributions.[1] Lindner [10] shows certain convergence results for power indices, when players are sampled from some distributions; Tauman and Jelnov [21] show that when weights are sampled from the uniform distribution, the expected Shapley value of a player is proportional to its weight. Zick [23] considers a model where the quota is sampled from a uniform distribution, and bounds the variance of the Shapley value in this setting, both for general weights and for weights sampled from certain distributions.

[1]The results shown by Penrose predate the work by Banzhaf, but can be applied directly to his work; see Felsenthal and Machover [5] for details.

There is also a growing body of work studying the effect of perturbations on WVGs. Elkind et al. [3] focus on computing solution concepts for WVGs whose weights and quota may change as a function of time. Zuckerman et al. [24] and Zick et al. [22], on the other hand, study the effects of changes to the quota on voting power distribution; however, both works do not consider a randomized weight model, but rather assume that the weights are fixed, while the quota may vary. The effects of changes to the quota have also been studied empirically, mostly in the context of the EU council of members [9, 8, 20].

Both theoretical and empirical works on quota manipulation indicate that even small changes to the quota can have dramatic effects on power distribution; the current paper explores several aspects of this phenomenon.

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2 Preliminaries

General notation Given a vector $\boldsymbol{x} \in \mathbb{R}^n$ and a set $S \subseteq \{1, \ldots, n\}$, let $x(S) = \sum_{i \in S} x_i$. For a random variable X, we let $\mathbb{E}[X]$ be its expectation, and $\mathrm{Var}[X]$ be its variance. For a set S, we denote by $\left[{S \atop k}\right]$ the collection of subsets of S of cardinality k. The notation $T \in_R \left[{S \atop k}\right]$ means that the set T is chosen uniformly at random from $\left[{S \atop k}\right]$. We let $\mathrm{B}(n, p)$ denote the binomial distribution with n trials and success probability p. We let $\mathcal{N}(\mu, \sigma^2)$ denote the normal distribution with mean μ and variance σ^2. We let $U(a, b)$ denote the uniform distribution on the interval $[a, b]$.

We let $O_p(\cdot)$ denote the usual big-O notation, conditioned on a fixed value of p. In other words, having $f(n) = O_p(g(n))$ means that there exist functions $K(\cdot)$, and $N(\cdot)$, such that for $n \geq N(p)$, $f(n) \leq K(p) \cdot g(n)$.

Finally, for a distribution D over $\mathbb{R}$, and some event $\mathcal{E}$, we simplify our notation by letting $\Pr[\mathcal{E}(D)] = \Pr_{x \sim D}[\mathcal{E}(x)]$. For example, for $a > 0$, we can write $\Pr[\mathrm{B}(n, p) \leq a] = \Pr_{x \sim \mathrm{B}(n,p)}[x \leq a]$.

Weighted voting games A *weighted voting game* (WVG) is given by a set of agents $N = \{1, \ldots, n\}$, where each agent $i \in N$ has a positive weight w_i, and a *quota* (or *threshold*) q. Unless otherwise specified, we assume that the weights are arranged in non-decreasing order, $w_1 \leq \cdots \leq w_n$. For a subset of agents $S \subseteq N$, we define $w(S) = \sum_{i \in S} w_i$.

A subset of agents $S \subseteq N$ is called *winning* (has value 1) if $w(S) \geq q$ and is called losing (has value 0) otherwise. WVGs are a subclass of *cooperative games*; a cooperative game $\mathcal{G} = \langle N, v \rangle$ is given by a set of agents N, and $v \colon 2^N \to \mathbb{R}$ assigns a value $v(S)$ to each subset $S \subseteq N$. In the case of WVGs, $v(S) = 1$ if $w(S) \geq q$, and is 0 otherwise.

The Shapley value Let Sym_n be the set of all permutations of N. Given some permutation $\sigma \in \mathrm{Sym}_n$ and an agent $i \in N$, we let $P_i(\sigma) = \{j \in N : \sigma(j) < \sigma(i)\}$; $P_i(\sigma)$ is called the set of i's predecessors in σ. Let us write $m_i(S)$ to be $v(S \cup \{i\}) - v(S)$; in other words, $m_i(S) = 1$ if and only if $v(S) = 0$ but $v(S \cup \{i\}) = 1$. If $m_i(S) = 1$, we say that i is *pivotal* for S; similarly, we write $m_i(\sigma) = m_i(P_i(\sigma))$, and say that i is pivotal for $\sigma \in \mathrm{Sym}_n$ if i is pivotal for $P_i(\sigma)$. The Shapley power index (often referred to as as the Shapley value in the context of WVG's) is simply the probability that i is pivotal for a permutation $\sigma \in \mathrm{Sym}_n$ selected uniformly at random. More explicitly,

$$\varphi_i = \frac{1}{n!} \sum_{\sigma \in \mathrm{Sym}_n} m_i(\sigma).$$

Since $\sigma^{-1}(i)$ is distributed uniformly when σ is chosen at random from Sym_n, we also have the alternative formula

$$\varphi_i = \frac{1}{n} \sum_{\ell=0}^{n-1} \mathop{\mathbb{E}}_{S \in_R \left[{N \setminus \{i\} \atop \ell}\right]} m_i(S). \tag{1}$$

Properties of the Shapley value For weighted voting games, it is not hard to show that $w_i \le w_j$ implies $\varphi_i \le \varphi_j$, and so if the weights are arranged in non-decreasing order, the minimal Shapley value is φ_1 and the maximal one is φ_n. Another useful property that follows immediately from the definitions is that $\sum_{i\in N} \varphi_i = 1$, assuming $0 < q \le \sum_{i\in N} w_i$. When we want to emphasize the role of the quota q, we will think of the Shapley values as functions of q: $\varphi_i(q)$.

Disparity One useful measure in the study of randomized weighted voting games is the expected *disparity*, or difference in voting power, between two agents (either additive or multiplicative, i.e., the ratio). Games in which this difference is high are games that exhibit rather high power imbalance, whereas games with low disparity ensure that in expectation, all agents have an equal share of voting power. We will be interested mostly in the ratio between the largest and smallest Shapley values.

3 Warmup: Power Distribution under the Binomial Distribution

We begin our study by considering the probabilistic model in which the agent weights $w_1, \dots, w_n$ are sampled from the binomial distribution $\mathrm{B}(m, p)$. In the election terminology, this corresponds to the setting of approval voting, in which each of the m voters approve each of the n candidates with probability p, independently. In order to keep the independence between the different weights, in this section we do not assume that the weights are arranged in non-decreasing order.

Our analysis aims to bound the maximum *additive* disparity between Shapley values, $\max_{i\in N} \varphi_i - \min_{i\in N} \varphi_i$. To that end, we prove the following theorem.

Theorem 3.1. Suppose that $m = \Omega_p(n \log n)$ and $q \ge 3mp$. For all agents $i, j \in N$,

$$\mathbb{E}[|\varphi_i - \varphi_j|] = \frac{1}{n} O_p\left(\sqrt{\frac{m \log n}{q}}\right).$$

The theorem shows that we expect the gaps to decrease as q gets larger, reaching order $\frac{1}{n} \cdot \sqrt{\frac{\log n}{n}}$ for $q = \Omega(nmp)$. This is demonstrated in Figure 1. Note that Theorem 3.1 only bounds the difference between *typical* Shapley values, rather than the difference between the maximal and minimal ones. This follows from the methods used: we do not argue about the specific order statistics $\min_{i\in N} \varphi_i$ and $\max_{i\in N} \varphi_i$. In the subsequent sections we will provide more refined bounds that are based on the analysis of order statistics.

We begin by presenting a formula for the difference $\varphi_j - \varphi_i$.

Lemma 3.1. For all agents $i, j \in N$,

$$|\varphi_j - \varphi_i| = \frac{1}{n-1} \sum_{\ell=0}^{n-2} \Pr_{S \in_R \binom{N\setminus\{i,j\}}{\ell}} [q - \max(w_i, w_j) \le w(S) < q - \min(w_i, w_j)].$$

Proof. Assume without loss of generality that $w_j \ge w_i$, and so $\varphi_j \ge \varphi_i$. For $\sigma \in \mathrm{Sym}_n$, let $T_{ij}(\sigma)$ be the permutation obtained by exchanging agents i and j. Then by the definition of the Shapley value and by linearity of expectations:

$$\begin{aligned}
\varphi_j - \varphi_i &= \mathop{\mathbb{E}}_{\sigma\in\mathrm{Sym}_n} (m_j(\sigma) - m_i(\sigma)) \\
&= \mathop{\mathbb{E}}_{\sigma\in\mathrm{Sym}_n} m_j(\sigma) - \mathop{\mathbb{E}}_{\sigma\in\mathrm{Sym}_n} m_i(\sigma) = \mathop{\mathbb{E}}_{\sigma\in\mathrm{Sym}_n} (m_j(T_{ij}(\sigma)) - m_i(\sigma)).
\end{aligned}$$

We proceed to evaluate $m_j(T_{ij}(\sigma)) - m_i(\sigma)$. Suppose first that agent i precedes agent j in σ, so that $\sigma = S\ i\ R\ j\ U$ and $T_{ij}(\sigma) = S\ j\ R\ i\ U$ (where S, R, and U form a partition of $N \setminus \{i, j\}$). In this case $m_j(T_{ij}(\sigma)) - m_i(\sigma) \ne 0$ precisely when $w(S) + w_i < q \le w(S) + w_j$, in which case $m_j(T_{ij}(\sigma)) - m_i(\sigma) = 1$; we can rewrite the condition as $w(S) \in [q - w_j, q - w_i)$.

When agent j precedes agent i in σ, we can write $\sigma = S\ j\ R\ i\ U$ and $T_{ij}(\sigma) = S\ i\ R\ j\ U$. In this case $m_j(T_{ij}(\sigma)) - m_i(\sigma) \ne 0$ precisely when $w(S) + w_i + w(R) < q \le w(S) + w_j + w(R)$, in which case $m_j(T_{ij}(\sigma)) - m_i(\sigma) = 1$; we can rewrite the condition as $w(S \cup R) \in [q - w_j, q - w_i)$.

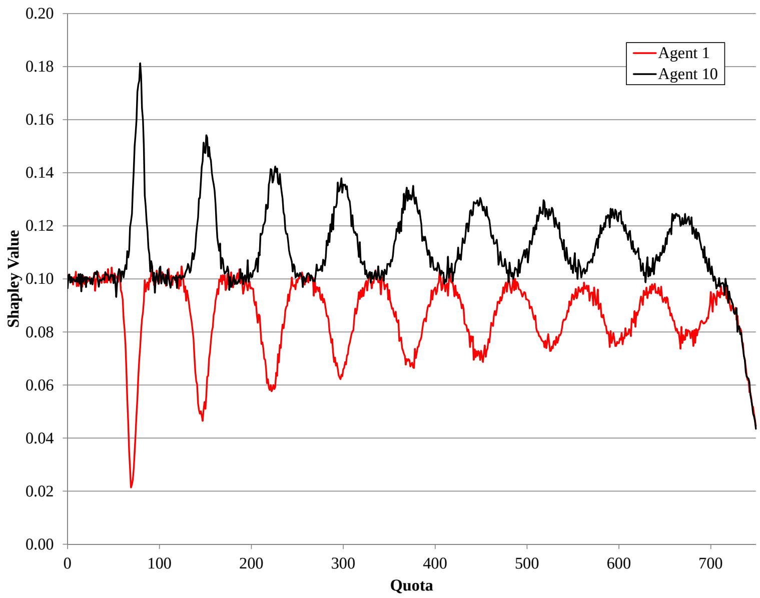


Figure 1: Average values obtained for the binomial process with $n = 10, m = 150$, $p = 1/2$. 250 populations were generated.

In order to unify both conditions together, define $P'_i(\sigma) = P_i(\sigma) \setminus \{j\}$. Using this definition, we see that $m_j(T_{ij}(\sigma)) - m_i(\sigma)$ is the indicator of the event $w(P'_i(\sigma)) \in [q - w_j, q - w_i)$. The cardinality $|P'_i(\sigma)|$ is exactly the position of agent i in the permutation σ' obtained by removing agent j from σ, minus one. Since σ is a uniformly random permutation of N, σ' is a uniformly random permutation of $N \setminus \{j\}$, and so $|P'_i(\sigma)|$ is distributed randomly among $\{0, \dots, n-2\}$. Given $|P'_i(\sigma)|$, the set $P_i(\sigma)$ is chosen randomly among all subsets of $N \setminus \{i, j\}$ of the specified size, yielding our formula. $\square$

In our case, the distribution of each individual $w(S)$ is binomial, and so we get the following simple formula for $\mathbb{E}[|\varphi_j - \varphi_i|]$.

Lemma 3.2. Fix the values of w_i, w_j for some agents $i, j \in N$, and let all other weights be i.i.d. samples of $\mathrm{B}(m, p)$. Then

$$\mathbb{E}[|\varphi_j - \varphi_i|] = \frac{1}{n-1} \sum_{\ell=0}^{n-2} \Pr[q - \max(w_i, w_j) \leq \mathrm{B}(\ell m, p) < q - \min(w_i, w_j)].$$

(Here $\Pr[q - \max(w_i, w_j) \leq \mathrm{B}(\ell m, p) < q - \min(w_i, w_j)] = \Pr_{X \sim \mathrm{B}(\ell m, p)}[q - \max(w_i, w_j) \leq X < q - \min(w_i, w_j)]$.)

Proof. For $S \subseteq N \setminus \{i, j\}$, let I_S be the indicator for the event $q - \max(w_i, w_j) \leq w(S) < q - \min(w_i, w_j)$. Using linearity of expectation, Lemma 3.1 implies that

$$\mathbb{E}[|\varphi_j - \varphi_i|] = \frac{1}{n-1} \sum_{\ell=0}^{n-2} \frac{1}{\binom{n-2}{\ell}} \sum_{S \in \left[N \setminus \{i,j\} \atop \ell \right]} \mathbb{E}[I_S].$$

It is easy to see that $\mathbb{E}[I_S]$ depends only on $|S|$ and is equal to $\Pr[q - \max(w_i, w_j) \leq \mathrm{B}(|S|m, p) < q - \min(w_i, w_j)]$, which implies the lemma. $\square$

The idea behind the proof of Theorem 3.1 is as follows. A simple concentration argument shows that all weights are roughly mp, and furthermore $\max_{i,j} |w_i - w_j| = O(\sqrt{mp \log n})$ with high probability. Since $\mathrm{B}(\ell m, p) \approx \ell mp$ while $q - w_i \approx q - mp$, we see that unless $(\ell + 1)mp \approx q$, the probability $\Pr[q - \max(w_i, w_j) \leq \mathrm{B}(\ell m, p) < q - \min(w_i, w_j)]$ is very small. For the critical values of ℓ, we can bound this probability in terms of $|w_i - w_j|$.

We proceed with the full proof. Recall that we are not assuming that the weights are ordered. We start with a concentration bound on the weights.

Lemma 3.3. Suppose that $mp \geq 9 \log n$. With probability $1 - 2/n^2$, all agents $i \in N$ satisfy $|w_i - mp| \leq \sqrt{9mp \log n} \leq mp$.

Proof. Chernoff's bound shows that

$$\Pr[|\mathrm{B}(m,p) - mp| \geq \delta] \leq 2e^{-\frac{\delta^2}{3mp}}.$$

Choosing $\delta = \sqrt{9mp \log n} \leq mp$ (since $mp \geq 9 \log n$), this probability is at most $2/n^3$. The lemma follows by applying a union bound. □

This bound allows us to show that most terms in Lemma 3.2 are very small.

Lemma 3.4. Let $i, j \in N$ be agents, and suppose that $|w_i - mp|, |w_j - mp| \leq mp$. Suppose that $\ell \leq n$ satisfies $\ell \geq \frac{q}{mp} + 1$ or $\ell \leq \frac{q}{mp} - 2$. For $mp \geq 9n \log n$,

$$\Pr[q - \max(w_i, w_j) \leq \mathrm{B}(\ell m, p) < q - \min(w_i, w_j)] \leq \frac{1}{n^3}.$$

Proof. The assumption on w_i, w_j implies that

$$\Pr[q - \max(w_i, w_j) \leq \mathrm{B}(\ell m, p) < q - \min(w_i, w_j)] \leq \Pr[q - mp \leq \mathrm{B}(\ell m, p) \leq q].$$

Suppose first that $\ell \geq \frac{q}{mp} + 1$. Then $\ell mp \geq q + mp$ and so

$$\Pr[q - mp \leq \mathrm{B}(\ell m, p) \leq q] \leq \Pr[\mathrm{B}(\ell m, p) \leq \ell mp - mp] \leq e^{-\frac{(mp)^2}{3\ell mp}} = e^{-\frac{mp}{3\ell}} \leq \frac{1}{n^3}.$$

Similarly, if $\ell \leq \frac{q}{mp} - 2$ then $\ell mp \leq (q - mp) - mp$, and so

$$\Pr[q - mp \leq \mathrm{B}(\ell m, p) \leq q] \leq \Pr[\mathrm{B}(\ell m, p) \geq \ell mp + mp] \leq e^{-\frac{(mp)^2}{3\ell mp}} = e^{-\frac{mp}{3\ell}} \leq \frac{1}{n^3}.$$

□

In order to estimate the remaining terms, we need the following technical lemma on the binomial distribution.

Lemma 3.5. For all p there exists a constant T_p such that for all $T \geq T_p$ and all x,

$$\Pr[\mathrm{B}(T, p) = x] \leq \frac{e/\pi}{\sqrt{Tp(1-p)}}.$$

Proof. We use Stirling's approximation in the following form:

$$\sqrt{2\pi n}(n/e)^n \leq n! \leq e\sqrt{n}(n/e)^n.$$

Let $x = \alpha T$. Stirling's approximation shows that

$$\begin{aligned}
\Pr[\mathrm{B}(T, p) = x] &= (p^\alpha (1-p)^{1-\alpha})^T \frac{T!}{(\alpha T)!((1-\alpha)T)!} \\
&\leq \left(\left(\frac{p}{\alpha}\right)^\alpha \left(\frac{1-p}{1-\alpha}\right)^{1-\alpha} \right)^T \frac{e}{2\pi} \frac{1}{\sqrt{\alpha(1-\alpha)T}} \\
&= e^{-D(\alpha\|p)T} \frac{e}{2\pi} \frac{1}{\sqrt{\alpha(1-\alpha)T}},
\end{aligned}$$

where $D(\alpha\|p)$ is the Kullback–Leibler distance between two Bernoulli variables. It is well-known that $D(\alpha\|p) \geq 2(\alpha - p)^2$, and so

$$\Pr[\mathrm{B}(T, p) = x] \leq e^{-2(\alpha-p)^2 T} \frac{e}{2\pi} \frac{1}{\sqrt{\alpha(1-\alpha)T}}.$$

This shows that as $T \to \infty$, the minimum of $\sqrt{T}\,\Pr[\mathrm{B}(T,p) = x]$ with respect to α is obtained at a point converging to p. In particular, the minimum tends to $\frac{e}{2\pi}/\sqrt{p(1-p)}$. Therefore for every $\epsilon > 0$ and T large enough (depending on p and ϵ),

$$\Pr[\mathrm{B}(T,p) = x] \leq (1+\epsilon) \cdot \frac{e}{2\pi} \frac{1}{\sqrt{p(1-p)T}}.$$

Choosing $\epsilon = 1$ (arbitrarily), we obtain the desired bound. □

This allows us to bound the terms not covered by Lemma 3.4.

Lemma 3.6. Let $i, j \in N$ be agents. If $mp \geq T_p$, then for any ℓ,

$$\Pr[q - \max(w_i, w_j) \leq \mathrm{B}(\ell m, p) < q - \min(w_i, w_j)] = O_p\left(\frac{|w_i - w_j|}{\sqrt{\ell m}}\right).$$

Proof. Follows directly from Lemma 3.5. □

We can now put everything together.

Proof of Theorem 3.1. Suppose first that all agents $i \in N$ satisfy $|w_i - mp| \leq \sqrt{9mp \log n}$, an event which happens with probability $1 - 2/n^2$ due to Lemma 3.3. Plugging Lemma 3.4 and Lemma 3.6 into Lemma 3.2, we obtain that for all agents $i, j \in N$,

$$\begin{aligned}
\mathbb{E}[|\varphi_i - \varphi_j|] &\leq \frac{1}{n^3} + \frac{1}{n-1} \sum_{\ell \in (\frac{q}{mp}-2, \frac{q}{mp}+1)} O_p\left(\frac{|w_i - w_j|}{\sqrt{\ell m}}\right) \\
&\leq \frac{1}{n^3} + \frac{1}{n-1} O_p\left(\frac{\sqrt{9mp\log n}}{\sqrt{q/p}}\right) = O_p\left(\sqrt{\frac{m \log n}{n^2 q}}\right),
\end{aligned}$$

since there are at most three values of $\ell \in (\frac{q}{mp} - 2, \frac{q}{mp} + 1)$, and all of them are $\Theta(\frac{q}{mp})$. Taking into account the failure of Lemma 3.3, we deduce that for all agents $i, j \in N$,

$$\mathbb{E}[|\varphi_i - \varphi_j|] \leq \left(1 - \frac{2}{n^2}\right) \cdot O_p\left(\sqrt{\frac{m \log n}{n^2 q}}\right) + \frac{2}{n^2} \cdot 1 = O_p\left(\sqrt{\frac{m \log n}{n^2 q}}\right).$$

□

4 The Balls and Bins Distribution: the Uniform Case

We now consider a generative stochastic process called the Balls and Bins process. In its most general form, given a set of m bins and a categorical distribution represented by a vector $\mathbf{p} \in [0,1]^m$ such that $\sum_{i=1}^m p_i = 1$, the process unfolds in m steps. In every step, a ball is thrown into one of the bins based on the probability vector $\mathbf{p}$. The resulting weights are then sorted in non-decreasing order $w_1 \leq \cdots \leq w_n$. We can think of the Balls and Bins setting as a model for the case of a plurality election, where each voter gives her vote to one of the parties according to the distribution $\mathbf{p}$.

This model has been used extensively in recent years in the computer science community, as it provides powerful theoretical tools for many models in areas such as scheduling and load balancing [12, 1], efficient vote elicitation [14] and online matching [13].

We begin our study of the balls and bins process by considering the most commonly studied version of the balls and bins model, in which each ball is thrown into one of the bins with equal probability, i.e., $p_i = 1/n$, for all $i \in N$.

As Figure 2 shows for the case of $n = 10$, the behavior of the Shapley values demonstrates an almost perfect cyclic pattern, with intervals of length m/n. As can be seen in the figure, for quota values that are sufficiently distant from the interval endpoints, all of the Shapley values tend to be equivalent (as the Shapley values of the highest and lowest agents are equal in these regions). Intuitively, this follows from the fact that as the number of balls grows, all of the bins tend to have the same number of balls in them, with very high probability; this low weight discrepancy immediately translates to very low power discrepancy.

We now give a theoretical justification for the above observation.

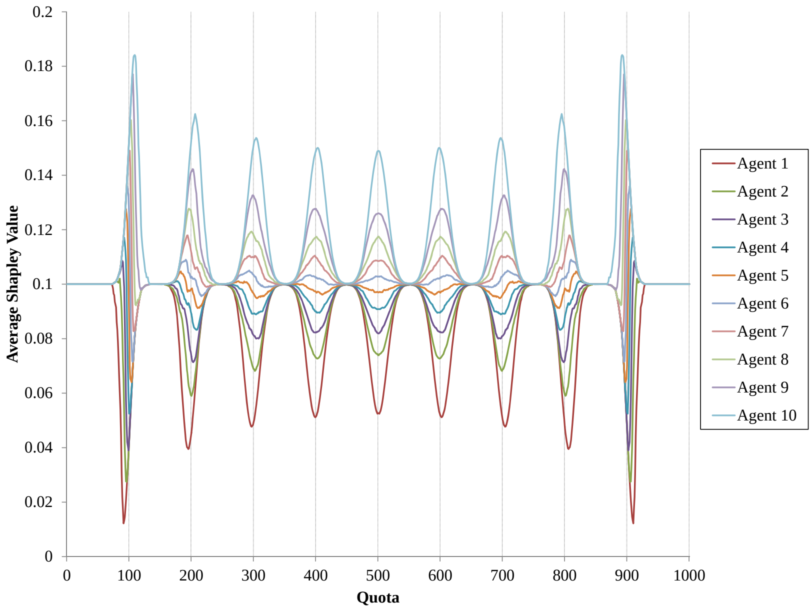


Figure 2: Average values obtained for the uniform balls and bins process with $n = 10, m = 1000$. 200 populations were generated. Note the huge disparity in Shapley values at $q = \ell \cdot \frac{m}{n}$, as opposed to the near equality when q is bounded away from integer multiples of $\frac{m}{n}$.

Theorem 4.1. Let $M = \frac{m}{3n^3}$. Suppose that $|q - \frac{\ell m}{n}| > \frac{1}{\sqrt{M}} \frac{m}{n}$ for all integers ℓ. Then with probability $1 - 2(\frac{2}{e})^n$, all Shapley values are equal to $1/n$.

In this proof, we do not assume that the weights $w_1, \ldots, w_n$ are ordered, in order to maintain the fact that the weights are independent random variables. The idea of the proof is to use the following criterion, which is a consequence of Lemma 3.1:

Proposition 4.1. *Suppose that for all agents $i, j \in N$ and for all subsets $S \subseteq N \setminus \{i, j\}$, we have $q \notin (w(S \cup \{i\}), w(S \cup \{j\})]$. Then all Shapley values are equal to $1/n$.*

Proof. We show that under the assumption on q, all Shapley values are equal, and so all must equal $1/n$. Suppose that for some agents $i \neq j$, we have $\varphi_i < \varphi_j$ (and so $w_i < w_j$). Lemma 3.1 implies the existence of a set $S \subseteq N \setminus \{i, j\}$ satisfying $q - w_j \leq w(S) < q - w_i$, or in other words $w(S) + w_i < q \leq w(S) + w_j$. This is exactly what is ruled out by the assumption on q. $\square$

Next, we show that the weights $w(S)$ are concentrated around points of the form $\ell \frac{m}{n}$.

Lemma 4.1. Suppose that $m > 3n^2$. With probability $1 - 2(\frac{2}{e})^n$, the following holds: for all $S \subseteq N$, $|w(S) - \frac{|S|m}{n}| \leq \sqrt{3nm}$.

Proof. The proof uses a straightforward Chernoff bound. We can assume that $S \neq \emptyset$ (as otherwise the bound is trivial). For each non-empty set $S \subseteq N$, the distribution of $w(S)$ is $\mathrm{B}(m, \frac{|S|}{n})$. Therefore for $0 < \delta < 1$,

$$\Pr\left[\left|w(S) - \frac{|S|m}{n}\right| > \delta \frac{|S|m}{n}\right] \leq 2e^{-\frac{\delta^2 |S| m}{3n}}.$$

Choosing $\delta = \sqrt{\frac{3n^2}{|S|m}} < 1$, we obtain

$$\Pr\left[\left|w(S) - \frac{|S|m}{n}\right| > \sqrt{3|S|m}\right] \leq 2e^{-n}.$$

Since there are 2^n possible sets S, a union bound implies that $|w(S) - \frac{|S|m}{n}| \leq \sqrt{3nm}$ with probability at least $1 - 2(\frac{2}{e})^n$. $\square$

This immediately impplies Theorem 4.1, as we now show.

Proof of Theorem 4.1. First, note that $M < 1$, as otherwise, it would imply that for all $\ell = 1, \ldots, n$, $|q - \ell m/n| \geq m/n$, which is impossible, as every quota in the range $(0, m]$ is within some integral multiple of m/n. Thus, having $M > 1$, implies that $m > 3n^3 \geq 3n^2$, as required by Lemma 4.1.

Lemma 4.1 shows that with probability $1 - 2(\frac{2}{e})^n$, for all sets S we have $|w(S) - \frac{|S|m}{n}| \leq \sqrt{3nm}$. Condition on this event. Suppose, for the sake of obtaining a contradiction, that $\varphi_i < \varphi_j$ for some agents i, j. Then Proposition 4.1 shows that there must exist some $S \subseteq N \setminus \{i, j\}$ such that $q \in (w(S \cup \{i\}), w(S \cup \{j\})]$. Since both $w(S \cup \{i\})$ and $w(S \cup \{j\})$ are $\sqrt{3nm}$-close to $\frac{(|S|+1)m}{n}$, this implies that $|q - \frac{(|S|+1)m}{n}| \leq \sqrt{3nm} = \frac{1}{\sqrt{M}} \cdot \frac{m}{n}$, contradicting our assumption on q. We conclude that all agents have the same Shapley value $1/n$. $\square$

Returning to our voting setting, the interpretation of Theorem 4.1 is that if the voter population is much larger than the number of candidates, and the votes are assumed to be cast uniformly at random (i.e., a totally neutral distribution of preferences), then choosing a quota that is well away from a multiple of $\frac{m}{n}$, will most probably lead to an even distribution of power among the elected representatives (e.g., political parties).

4.1 How weak can the weakest agent get in the uniform case?

As Theorem 4.1 demonstrates, if the quota is sufficiently bounded away from any integral multiple of $\frac{m}{n}$, then the distribution of power tends to be even among the agents. When the quota is close to an integer multiple of $\frac{m}{n}$, it may very well be that the resulting weighted voting game may not display such an even distribution of power, as a result of weight differences, as a result of the intrinsic "noise" of the process. Figure 2 provides an empirical validation of this intuition. Motivated by these observations, we now proceed to study the expected Shapley value of the weakest agent, φ_1 (recall that we assume that the weights are given in non-decreasing order).

We now present two contrasting results. Let $q = \ell \cdot \frac{m}{n}$, for an integer ℓ. When $\ell = o(\log n)$, we show that the expected minimal Shapley value is roughly $\frac{1}{2n}$, and so it is at least half the maximal Shapley value (in expectation).

Theorem 4.2. Let $q = \ell \cdot \frac{m}{n}$ for some integer $\ell = o(\log n)$. For $m = \Omega(n^3 \log n)$, $\mathbb{E}[\varphi_1] = \frac{1}{2n} + o(\frac{1}{n})$.

In contrast, when $\ell = \Omega(n)$, this effect disappears.

Theorem 4.3. Let $q = \ell \cdot \frac{m}{n}$ for $\ell \in \{1, \ldots, n\}$ such that $\gamma \leq \frac{\ell}{n} \leq 1 - \gamma$ for some constant $\gamma > 0$. For $m = \Omega(n^3)$,

$$\mathbb{E}[\varphi_1] \geq \frac{1}{n} - O_\gamma\left(\sqrt{\frac{\log n}{n^3}}\right).$$

The idea behind the proof of both theorems is the following formula for φ_1. In this formula and in the rest of the section, the probabilities are taken over both the displayed variables and the choice of weights.

Lemma 4.2. Let $q = \ell \cdot \frac{m}{n}$, where $\ell \in \{1, \ldots, n-1\}$. For $m = \Omega(n^3 \log n)$,

$$\mathbb{E}[\varphi_1] = \frac{1}{2(n-\ell)} - \frac{\ell}{n(n-\ell)} + \frac{1}{n-\ell} \Pr_{A \in_R \binom{N \setminus \{1\}}{\ell-1}}[w(A) + w_1 \geq q] \pm O\left(\frac{1}{n^2}\right).$$

Proof sketch. Let $p_k = \Pr_{A \in_R \binom{N \setminus \{1\}}{k}}[q - w_1 \leq w(A) < q]$. Then the alternative definition of the Shapley value (Formula (1) in the preliminaries) shows that $\mathbb{E}[\varphi_1] = \frac{1}{n} \sum_{k=0}^{n-1} p_k$. We then consider three cases, corresponding to possible sizes of the set A in the formula for p_k; each of these cases will contribute a term in expression of the lemma. Since $w(A) \approx \frac{|A|m}{n}$, when $|A| \geq \ell + 1$ it is highly unlikely that $w(A) < q$. Similarly, since $w(A) + w_1 \approx \frac{(|A|+1)m}{n}$, when $|A| \leq \ell - 2$ it is highly unlikely that $w(A) \geq q - w_1$. So roughly speaking, $\mathbb{E}[\varphi_1] \approx \frac{p_{\ell-1} + p_\ell}{n}$. Furthermore, when $|A| = \ell - 1$, it is very likely that $w(A) < q$, and when $|A| = \ell$, it is very likely that $w(A) \geq q - w_1$. So roughly speaking,

$$\mathbb{E}[\varphi_1] \approx \frac{1}{n} \Pr_{A \in_R \binom{N \setminus \{1\}}{\ell-1}}[w(A) + w_1 \geq q] + \frac{1}{n} \Pr_{A \in_R \binom{N \setminus \{1\}}{\ell}}[w(A) < q].$$

The trick now is to relate the two terms:

$$\begin{aligned}
\Pr_{A\in_R\binom{N\setminus\{1\}}{\ell}}[w(A)<q] &= \frac{1}{\binom{n-1}{\ell}}\sum_{A\in\binom{N\setminus\{1\}}{\ell}}\Pr[w(A)<q] \\
&= \frac{1}{\binom{n-1}{\ell}}\sum_{A\in\binom{N}{\ell}}\Pr[w(A)<q]-\frac{1}{\binom{n-1}{\ell}}\sum_{A\in\binom{N\setminus\{1\}}{\ell-1}}\Pr[w(A)+w_1<q] \\
&= \frac{n}{n-\ell}\Pr_{A\in_R\binom{N}{\ell}}\Pr[w(A)<q]-\frac{\ell}{n-\ell}\left(1-\Pr_{A\in_R\binom{N\setminus\{1\}}{\ell-1}}[w(A)+w_1\geq q]\right).
\end{aligned}$$

To address the first term in the above expression, note that when $|A|=\ell$, $\mathbb{E}[w(A)]=q$, and so the first probability is roughly $1/2$. Therefore

$$\Pr_{A\in_R\binom{N\setminus\{1\}}{\ell}}[w(A)<q]\approx\frac{n}{2(n-\ell)}-\frac{\ell}{n-\ell}+\frac{\ell}{n-\ell}\Pr_{A\in_R\binom{N\setminus\{1\}}{\ell-1}}\Pr[w(A)+w_1\geq q].$$

Substituting this in our estimate for $\mathbb{E}[\varphi_1]$, we obtain

$$\mathbb{E}[\varphi_1]\approx\frac{1}{n-\ell}\Pr_{A\in_R\binom{N\setminus\{1\}}{\ell-1}}\Pr[w(A)+w_1\geq q]+\frac{1}{2(n-\ell)}-\frac{\ell}{n(n-\ell)}.$$

□

The full details of the proof appear in the subsequent subsection (Subsection 4.2).

In order to estimate the expression $\Pr_{A\in_R\binom{N\setminus\{1\}}{\ell-1}}[w(A)+w_1\geq q]$, we need a good estimate for w_1. Such an estimate is given by the following lemma.

Lemma 4.3. With probability $1-2/n$,

$$\sqrt{\frac{m\log n}{3n}}\leq\frac{m}{n}-w_1\leq\sqrt{\frac{4m\log n}{n}}.$$

We obtain this bound by applying the *Poisson approximation technique* to the Balls and Bins process, which we now roughly describe. Consider the case of a random event, defined with respect to the weight distribution induced by the process. The probability of the event can be well-approximated by the probability of an analogous event, defined with respect to n *i.i.d.* Poisson random variables, assuming the event is monotone in the number of balls.

We can now prove Theorem 4.2.

Proof of Theorem 4.2. Lemma 4.6 (a simple technical result proved in subsection 4.2) shows that

$$\Pr_{A\in_R\binom{N\setminus\{1\}}{\ell-1}}[w(A)+w_1\geq q]\leq\frac{n}{n-\ell+1}\Pr[\mathrm{B}(m,\tfrac{\ell-1}{n})\geq q-w_1].$$

The concentration bound on w_1 (Lemma 4.3) shows that with probability $1-2/n$, $q-w_1\geq\frac{(\ell-1)m}{n}+\sqrt{\frac{m\log n}{3n}}$. Assuming this, a Chernoff bound gives

$$\Pr[\mathrm{B}(m,\tfrac{\ell-1}{n})\geq q-w_1]\leq\Pr[\mathrm{B}(m,\tfrac{\ell-1}{n})\geq\frac{(\ell-1)m}{n}+\sqrt{\frac{m\log n}{3n}}]\leq e^{-\frac{m\log n/(3n)}{3(\ell-1)m/n}}\leq e^{-\frac{\log n}{9\ell}}=o(1),$$

using $\ell=o(\log n)$. Accounting for possible failure of the bound on $q-w_1$, we obtain

$$\Pr_{A\in_R\binom{N\setminus\{1\}}{\ell-1}}[w(A)+w_1\geq q]\leq\left(1-\frac{2}{n}\right)\cdot o\left(\frac{n}{n-\ell}\right)+\frac{2}{n}\cdot 1=o(1),$$

using $\ell=o(\log n)$. Lemma 4.2 therefore shows that

$$\mathbb{E}[\varphi_1]\leq\frac{1}{2(n-\ell)}+o\left(\frac{1}{n-\ell}\right)+O\left(\frac{1}{n^2}\right)=\frac{1}{2n}+o\left(\frac{1}{n}\right),$$

since $\ell = o(\log n)$ implies $\frac{1}{n-\ell} = \frac{1}{n} + \frac{\ell}{n(n-\ell)} = \frac{1}{n} + o(\frac{1}{n})$. Lemma 4.2 also implies a matching lower bound:

$$\mathbb{E}[\varphi_1] \geq \frac{1}{2(n-\ell)} - \frac{\ell}{n-\ell} - O\left(\frac{1}{n^2}\right) \geq \frac{1}{2n} - o\left(\frac{1}{n}\right).$$

□

In the regime of ℓ addressed by Theorem 4.2, $\Pr_{A \in_R \binom{N \setminus \{1\}}{\ell-1}}[w(A) + w_1 \geq q]$ was negligible. In contrast, in the regime of ℓ addressed by Theorem 4.3, $\Pr_{A \in_R \binom{N \setminus \{1\}}{\ell-1}}[w(A) + w_1 \geq q] \approx 1/2$, as the following lemma, which is proved later in Subsection 4.4, using the Berry–Esseen theorem, shows.

Lemma 4.4. Suppose $q = \ell \frac{m}{n}$ for an integer ℓ satisfying $\gamma \leq \frac{\ell-1}{n} \leq 1-\gamma$, and let

$$t_\varepsilon = \Pr_{A \in_R \binom{N \setminus \{1\}}{\ell-1}}\left[w(A) + w_1 \geq q : w_1 = \frac{m}{n} - \varepsilon\sqrt{\frac{m \log n}{n}}\right].$$

Then for $m \geq 4n^3$,

$$t_\varepsilon \geq \frac{1}{2} - \frac{\varepsilon}{2\pi\gamma}\sqrt{\frac{\log n}{n}} - \frac{1}{n}.$$

As Lemma 4.3 shows, $1/3 \leq \varepsilon \leq 4$ with probability $1 - 2/n$, which explains the usefulness of this bound. We can now prove Theorem 4.3.

Proof of Theorem 4.3. Lemma 4.3 shows that with probability $1 - 2/n$, $w_1 = \frac{m}{n} - \varepsilon\sqrt{\frac{m\log n}{n}}$ for some $1/3 \leq \varepsilon \leq 4$, in which regime Lemma 4.4 shows that $t_\varepsilon \geq \frac{1}{2} - \frac{2}{\pi\gamma}\sqrt{\frac{\log n}{n}} - \frac{1}{n}$. Accounting for the case in which ε is out of bounds,

$$\Pr_{A \in_R \binom{N \setminus \{1\}}{\ell-1}}[w(A) + w_1 \geq q] \geq \left(1 - \frac{2}{n}\right)\left(\frac{1}{2} - \frac{2}{\pi\gamma}\sqrt{\frac{\log n}{n}} - \frac{1}{n}\right) \geq \frac{1}{2} - \frac{2}{\pi\gamma}\sqrt{\frac{\log n}{n}} - \frac{3}{n}.$$

Substituting this in Lemma 4.2, we obtain

$$\begin{aligned}
\mathbb{E}[\varphi_1] &\geq \frac{1}{2(n-\ell)} - \frac{\ell}{n(n-\ell)} + \frac{1}{n-\ell}\left(\frac{1}{2} - \frac{2}{\pi\gamma}\sqrt{\frac{\log n}{n}} - \frac{3}{n}\right) - O\left(\frac{1}{n^2}\right) \\
&= \frac{1}{n-\ell} - \frac{\ell}{n(n-\ell)} - \frac{1}{n-\ell}O_\gamma\left(\sqrt{\frac{\log n}{n}}\right) - O\left(\frac{1}{n^2}\right) = \frac{1}{n} - O_\gamma\left(\sqrt{\frac{\log n}{n^3}}\right).
\end{aligned}$$

□

4.2 Proof of Lemma 4.2

We prove the following lemma.

Lemma 4.2. Let $q = \ell \cdot \frac{m}{n}$, where $\ell \in \{1, \ldots, n-1\}$. For $m = \Omega(n^3 \log n)$,

$$\mathbb{E}[\varphi_1] = \frac{1}{2(n-\ell)} - \frac{\ell}{n(n-\ell)} + \frac{1}{n-\ell}\Pr_{A \in_R \binom{N \setminus \{1\}}{\ell-1}}[w(A) + w_1 \geq q] \pm O\left(\frac{1}{n^2}\right).$$

The proof closely follows the proof sketch in Section 4.1.

We will need the fact that with high probability, w_1 is close to m/n.

Lemma 4.5. With probability at least $1 - 1/n$,

$$\frac{m}{n} - \sqrt{\frac{4m\log n}{n}} \leq w_1 \leq \frac{m}{n}.$$

Proof. Clearly $w_1 \le m/n$ always, so we only need to address the lower bound on w_1. Let $w'_1, \dots, w'_n$ be the loads of the bins before sorting them. The loads w'_i are independent random variables with distribution $\mathrm{B}(m, 1/n)$. For each index i, Chernoff's bound shows that

$$\Pr\left[w'_i < \frac{m}{n} - \sqrt{\frac{4m\log n}{n}}\right] \le e^{-\frac{4m\log n/n}{2m/n}} = \frac{1}{n^2}.$$

A union bound shows that with probability $1 - 1/n$, all $i \in N$ satisfy $w'_i \ge \frac{m}{n} - \sqrt{\frac{4m\log n}{n}}$, and so $w_1 \ge \frac{m}{n} - \sqrt{\frac{4m\log n}{n}}$. $\square$

Below we will be interested in bounding probabilities of the form $\Pr_{A \in_R \left[{N\setminus\{1\} \atop k}\right]}[P(w(A))]$ for predicates P. The following lemma shows how to bound these probabilities from above.

Lemma 4.6. For a weight vector $\mathbf{w}$ and $S \subseteq N$, let $\mathcal{E}(w(S))$ be a random event (i.e., some predicate on $w(S)$), and let $0 \le k \le n-1$. Then

$$\Pr_{A \in_R \left[{N\setminus\{1\} \atop k}\right]}[\mathcal{E}(w(A))] \le \frac{n}{n-k} \Pr[\mathcal{E}(\mathrm{B}(m, \tfrac{k}{n}))].$$

Also,

$$\Pr_{A \in_R \left[{N \atop k}\right]}[\mathcal{E}(w(A))] = \Pr[\mathcal{E}(\mathrm{B}(m, \tfrac{k}{n}))].$$

Proof. First, we have

$$\Pr_{A \in_R \left[{N\setminus\{1\} \atop k}\right]}[\mathcal{E}(w(A))] = \frac{1}{\binom{n-1}{k}} \sum_{A \in \left[{N\setminus\{1\} \atop k}\right]} \Pr[\mathcal{E}(w(A))] \le \frac{1}{\binom{n-1}{k}} \sum_{A \in \left[{N \atop k}\right]} \Pr[\mathcal{E}(w(A))] = \frac{n}{n-k} \Pr_{A \in_R \left[{N \atop k}\right]}[\mathcal{E}(w(A))].$$

Consider the last expression. Since the probability is over all subsets of N of size k, the same value is obtained from the *unsorted* Balls and Bins process (without sorting the loads). Under this process, $w(A) \sim \mathrm{B}(m, \frac{k}{n})$ for all $A \in \left[{N \atop k}\right]$, and so

$$\Pr_{A \in_R \left[{N \atop k}\right]}[\mathcal{E}(w(A))] = \Pr_{w \sim \mathrm{B}(m, \frac{k}{n})}[\mathcal{E}(w)].$$

This implies the lemma. $\square$

Let $p_k = \Pr_{A \in_R \left[{N\setminus\{1\} \atop k}\right]}[q - w_1 \le w(A) < q]$, and recall that formula (1) shows that $\varphi_1 = \frac{1}{n}\sum_{k=0}^{n-1} p_k$. We start by showing that the only non-negligible p_k are $p_{\ell-1}$ and p_ℓ, using a Chernoff bound. The idea is that when $k \ge \ell + 1$, it is highly unlikely that $w(A) < q$, and when $k \le \ell - 1$, it is highly unlikely that $w(A) \ge q - w_1$.

Lemma 4.7. Suppose that $m \ge 9n^2 \log n$. Then for $k \in \{1, \dots, n\} \setminus \{\ell - 1, \ell\}$ we have $p_k \le 1/n^2$, and so

$$0 \le \mathbb{E}[\varphi_1] - \frac{p_{\ell-1} + p_\ell}{n} \le \frac{1}{n^2}.$$

Proof. Let $k \in N$. Lemma 4.6 shows that

$$p_k \le n \Pr[q - w_1 \le \mathrm{B}(m, \tfrac{k}{n}) < q].$$

Suppose first that $k \ge \ell + 1$. Chernoff's bound shows that

$$\Pr[q - w_1 \le \mathrm{B}(m, \tfrac{k}{n}) < q] \le \Pr[\mathrm{B}(m, \tfrac{k}{n}) < \tfrac{km}{n} - \tfrac{m}{n}] \le e^{-\frac{(m/n)^2}{3km/n}} = e^{-m/(3nk)} \le \frac{1}{n^3}.$$

Suppose next that $k \le \ell - 2$. Since $w_1 \le m/n$, another application of Chernoff's bound gives

$$\Pr[q - w_1 \le \mathrm{B}(m, \tfrac{k}{n}) < q] \le \Pr[\mathrm{B}(m, \tfrac{k}{n}) \ge \tfrac{(\ell-1)m}{n}] \le \Pr[\mathrm{B}(m, \tfrac{k}{n}) \ge \tfrac{km}{n} + \tfrac{m}{n}] \le e^{-\frac{(m/n)^2}{3km/n}} = e^{-m/(3nk)} \le \frac{1}{n^3}.$$

Therefore $p_k \le 1/n^2$ for all $k \in N \setminus \{\ell - 1, \ell\}$. The estimate for $\mathbb{E}[\varphi_1]$ follows from formula (1). $\square$

The next step is to consider the following estimates for $p_{\ell-1}, p_\ell$:

$$p'_{\ell-1} = \Pr_{A \in_R \binom{N \setminus \{1\}}{\ell-1}}[q - w_1 \le w(A)],$$
$$p'_\ell = \Pr_{A \in_R \binom{N \setminus \{1\}}{\ell}}[w(A) < q].$$

The following lemma shows that $p'_{\ell-1} \approx p_{\ell-1}$ and $p'_\ell \approx p_\ell$.

Lemma 4.8. Suppose that $m \ge 24n^2 \log n$. Then $p_{\ell-1} \le p'_{\ell-1} \le p_{\ell-1} + \frac{1}{n}$ and $p_\ell \le p'_\ell \le p_\ell + \frac{2}{n}$, and so

$$-\frac{3}{n^2} \le \mathbb{E}[\varphi_1] - \frac{p'_{\ell-1} + p'_\ell}{n} \le \frac{1}{n^2}.$$

Proof. Clearly $p_{\ell-1} \le p'_{\ell-1}$ and $p_\ell \le p'_\ell$. First,

$$p'_{\ell-1} - p_{\ell-1} \le \Pr_{A \in_R \binom{N \setminus \{1\}}{\ell-1}}[w(A) \ge q] \le n \Pr[\mathrm{B}(m, \tfrac{\ell-1}{n}) \ge q],$$

using Lemma 4.6. Chernoff's bound shows that

$$\Pr[\mathrm{B}(m, \tfrac{\ell-1}{n}) \ge \tfrac{(\ell-1)m}{n} + \tfrac{m}{n}] \le e^{-\frac{(m/n)^2}{3(\ell-1)m/n}} = e^{-m/(3n(\ell-1))} \le \frac{1}{n^2}.$$

Similarly,

$$p'_\ell - p_\ell \le \Pr_{A \in_R \binom{N \setminus \{1\}}{\ell}}[w(A) < q - w_1] \le n \Pr[\mathrm{B}(m, \tfrac{\ell}{n}) < q - w_1].$$

We now need the lower bound on w_1 given by Lemma 4.5, which holds with probability $1 - 1/n$:

$$q - w_1 \le \frac{\ell m}{n} - \left(\frac{m}{n} - \sqrt{\frac{4m \log n}{n}}\right) \le \frac{\ell m}{n} - \frac{m}{2n},$$

the latter inequality following from $m \ge 24n^2 \log n > 16n \log n$. Assuming the lower bound on w_1,

$$\Pr[\mathrm{B}(m, \tfrac{\ell}{n}) < q - w_1] \le e^{-\frac{(m/(2n))^2}{3(\ell-1)m/n}} = e^{-m/(12n(\ell-1))} \le \frac{1}{n^2}.$$

Therefore

$$p'_\ell - p_\ell \le \left(1 - \frac{1}{n}\right) \cdot \frac{1}{n^2} + \frac{1}{n} \cdot 1 < \frac{2}{n}.$$

The formula for $\mathbb{E}[\varphi_1]$ follows from Lemma 4.7. □

It remains to relate $p'_{\ell-1}$ and p'_ℓ.

Lemma 4.9. Suppose that $m \ge 24n^3 \log n$. Then

$$\left| p'_\ell - \left(\frac{n}{2(n-\ell)} - \frac{\ell}{n-\ell}(1 - p'_{\ell-1})\right) \right| \le \frac{1}{n},$$

and so

$$-\frac{4}{n^2} \le \mathbb{E}[\varphi_1] - \left(\frac{1}{2(n-\ell)} - \frac{\ell}{n(n-\ell)} + \frac{p'_{\ell-1}}{n-\ell}\right) \le \frac{2}{n^2}.$$

Proof. We have

$$\begin{aligned}
p'_\ell &= \Pr_{A \in_R \binom{N\setminus\{1\}}{\ell}}[w(A) < q] \\
&= \frac{1}{\binom{n-1}{\ell}} \sum_{A \in \binom{N\setminus\{1\}}{\ell}} \Pr[w(A) < q] \\
&= \frac{1}{\binom{n-1}{\ell}} \sum_{A \in \binom{N}{\ell}} \Pr[w(A) < q] - \frac{1}{\binom{n-1}{\ell}} \sum_{A \in \binom{N\setminus\{1\}}{\ell-1}} \Pr[w(A) + w_1 < q] \\
&= \frac{n}{n-\ell} \Pr_{A \in_R \binom{N}{\ell}} \Pr[w(A) < q] - \frac{\ell}{n-\ell}\left(1 - \Pr_{A \in_R \binom{N\setminus\{1\}}{\ell-1}}[w(A) + w_1 \geq q]\right) \\
&= \frac{n}{n-\ell} \Pr[\mathrm{B}(m, \frac{\ell}{n}) < q] - \frac{\ell}{n-\ell}(1 - p'_{\ell-1}),
\end{aligned}$$

where the final equality follows from the second part of Lemma 4.6. We proceed to estimate $\Pr[\mathrm{B}(m, \frac{\ell}{n}) < q]$ using the Berry–Esseen theorem. The normalized binomial $\mathrm{B}(m, \frac{\ell}{n}) - q$ is a sum of m independent copies of the random variable X with $\Pr[X = 1 - \frac{\ell}{n}] = \frac{\ell}{n}$ and $\Pr[X = -\frac{\ell}{n}] = 1 - \frac{\ell}{n}$. The Berry–Esseen theorem states that

$$|\Pr[\mathrm{B}(m, \frac{\ell}{n}) - q < 0] - \Pr[\mathcal{N}(0, \sigma^2) < 0]| < \frac{\rho}{\sigma^3 \sqrt{m}},$$

where $\sigma^2 = \mathbb{E}[X^2] = \frac{\ell}{n}(1 - \frac{\ell}{n})^2 + (1 - \frac{\ell}{n})(\frac{\ell}{n})^2 = \frac{\ell}{n}(1 - \frac{\ell}{n})$ and $\rho = \mathbb{E}[|X|^3] = \frac{\ell}{n}(1 - \frac{\ell}{n})^3 + (1 - \frac{\ell}{n})(\frac{\ell}{n})^3 = \frac{\ell}{n}(1 - \frac{\ell}{n})[(\frac{\ell}{n})^2 + (1 - \frac{\ell}{n})^2]$. Since $\Pr[\mathcal{N}(0, \sigma^2) < 0] = 1/2$, we conclude that

$$\left|\Pr[\mathrm{B}(m, \frac{\ell}{n}) - q < 0] - \frac{1}{2}\right| < \frac{1}{\sqrt{m}} \frac{(\frac{\ell}{n})^2 + (1 - \frac{\ell}{n})^2}{\sqrt{\frac{\ell}{n}\left(1 - \frac{\ell}{n}\right)}} \leq 2\sqrt{\frac{n}{m}},$$

since the denominator is at least $\sqrt{\frac{1}{n}(1 - \frac{1}{n})}$, and the numerator is at most $2(1 - \frac{1}{n})^2 \leq 2\sqrt{1 - \frac{1}{n}}$. Since $m \geq 24n^3 \log n \geq 4n^3$, we further have $2\sqrt{\frac{n}{m}} \leq \frac{1}{n}$.

The formula for $\mathbb{E}[\varphi_1]$ follows from Lemma 4.8. $\square$

Lemma 4.9 is simply a reformulation of Lemma 4.2.

4.3 Proof of Lemma 4.3

Let us recall Lemma 4.3.

Lemma 4.3. With probability $1 - 2/n$,

$$\sqrt{\frac{m \log n}{3n}} \leq \frac{m}{n} - w_1 \leq \sqrt{\frac{4m \log n}{n}}.$$

We already proved the upper bound in Lemma 4.5, using a simple union bound. The lower bound (corresponding to an upper bound on w_1) is much more difficult, because of the dependence between the individual bins. One way to overcome this difficulty is to use the Poisson approximation, given by the following theorem.

Theorem 4.4 ([11])**.** Let $w_1, \ldots, w_n$ be sampled according to the Balls and Bins distribution with m balls, and let $X_1, \ldots, X_n$ be n i.i.d. random variables sampled from the distribution $\mathrm{Pois}(\frac{m}{n})$. Let $f\colon \mathbb{R}^n \to \{0, 1\}$ be a boolean function over the weight vector, such that the probability $p(w_1, \ldots, w_n) = \Pr[f(w_1, \ldots, w_n) = 1]$ is monotonically increasing or decreasing with the number of balls. Then $p(w_1, \ldots, w_n) \leq 2p(X_1, \ldots, X_n)$.

The following lemma completes the proof of Lemma 4.3, since calculation shows that for all $n \geq 1$,

$$\frac{m}{n}\sqrt{\frac{\log(n/\log(2n))}{m/n}} = \sqrt{\frac{m \log(n/\log(2n))}{n}} \geq \sqrt{\frac{m \log n}{3n}}.$$

(In fact, the minimum of $\frac{\log(n/\log(2n))}{\log n}$ is obtained for $n = 3$, in which case it is roughly 0.47.)

Lemma 4.10. Let $\lambda = \frac{m}{n}$. For any $\varepsilon \leq \sqrt{\frac{\log\left(\frac{n}{\log(2n)}\right)}{\lambda}}$, $\Pr[w_1 > \lambda(1-\varepsilon)] \leq \frac{1}{n}$.

Proof. We define n i.i.d random variables $X_1, \ldots, X_n$, sampled from the distribution $\mathrm{Pois}(\lambda)$. We first derive a concentration bound on $\min_i X_i$, after which we will make use of Theorem 4.4 to obtain the desired result. By the definition of the Poisson distribution,

$$\Pr[\min_i X_i > t] = \Pr[X_1 > t]^n \leq \Pr[X_1 \neq t]^n \leq \left(1 - e^{-\lambda}\frac{\lambda^t}{t!}\right)^n \leq \left(1 - e^{-\lambda}\left(\frac{e\lambda}{t}\right)^t\right)^n.$$

The last inequality is due to the fact that $t! \geq \left(\frac{t}{e}\right)^t$, by Stirling's approximation. Setting $t = (1-\varepsilon)\lambda$, we get

$$\begin{aligned}
\Pr[\min_i X_i > (1-\varepsilon)\lambda] &\leq \left(1 - e^{-\lambda}\left(\frac{e\lambda}{(1-\varepsilon)\lambda}\right)^{(1-\varepsilon)\lambda}\right)^n \\
&= \left(1 - e^{-\lambda}\left(\frac{e}{1-\varepsilon}\right)^{(1-\varepsilon)\lambda}\right)^n \\
&\leq \left(1 - e^{-\varepsilon\lambda}e^{(1-\varepsilon)\varepsilon\lambda}\right)^n \\
&= \left(1 - e^{-\varepsilon^2\lambda}\right)^n \leq e^{-ne^{-\varepsilon^2\lambda}}.
\end{aligned}$$

The second inequality follows from the inequality $\frac{1}{1-x} \geq e^x$, for $|x| < 1$. The third inequality follows from the inequality $1 - x \leq e^{-x}$.

Now, for any $\varepsilon \leq \sqrt{\frac{\log\left(\frac{n}{\log(2n)}\right)}{\lambda}}$, we have

$$e^{-ne^{-\varepsilon^2\lambda}} \leq e^{-ne^{-\log\left(\frac{n}{\log(2n)}\right)}} = e^{-\log(2n)} = \frac{1}{2n}.$$

A simple coupling argument shows that $\Pr[\min_i w_i > (1-\varepsilon)\lambda]$ is monotone increasing in the number of balls (here, $f(w_1, \ldots, w_n)$ is 1 if and only if $\min_i w_i > (1-\varepsilon)\lambda$). Therefore Theorem 4.4 holds, and we have

$$\Pr[\min_i w_i > (1-\varepsilon)\lambda] \leq 2\Pr[\min_i X_i > (1-\varepsilon)\lambda] \leq \frac{1}{n},$$

which concludes the proof. □

4.4 Proof of Lemma 4.4

Lemma 4.4. Suppose $q = \ell\frac{m}{n}$ for an integer ℓ satisfying $\gamma \leq \frac{\ell-1}{n} \leq 1-\gamma$, and let

$$t_\varepsilon = \Pr_{A \in_R \binom{N\setminus\{1\}}{\ell-1}}\left[w(A) + w_1 \geq q : w_1 = \frac{m}{n} - \varepsilon\sqrt{\frac{m\log n}{n}}\right].$$

Then for $m \geq 4n^3$,

$$t_\varepsilon \geq \frac{1}{2} - \frac{\varepsilon}{2\pi\gamma}\sqrt{\frac{\log n}{n}} - \frac{1}{n}.$$

Proof. The idea of the proof is to replace $w(A)$ by the weight of a random set of size $\ell - 1$. A simple coupling argument shows that

$$t_\varepsilon \geq \Pr_{A \in_R \binom{N}{\ell-1}}\left[w(A) + w_1 \geq q : w_1 = \frac{m}{n} - \varepsilon\sqrt{\frac{m\log n}{n}}\right] = \Pr\left[\mathrm{B}(m, \tfrac{\ell-1}{n}) \geq \frac{(\ell-1)m}{n} + \varepsilon\sqrt{\frac{m\log n}{n}}\right],$$

using the second part of Lemma 4.6.

As in the proof of Lemma 4.9, since $m \geq 4n^3$, we can use the Berry–Esseen theorem to estimate the latter expression up to an additive error of $\frac{1}{n}$:

$$t_\varepsilon \geq \Pr\left[\mathrm{B}(m, \tfrac{\ell-1}{n}) \geq \frac{(\ell-1)m}{n} + \varepsilon\sqrt{\frac{m\log n}{n}}\right] \geq \Pr\left[\mathcal{N}(\tfrac{(\ell-1)m}{n}, \tfrac{(\ell-1)m}{n}(1 - \tfrac{\ell-1}{n})) \geq \frac{(\ell-1)m}{n} + \varepsilon\sqrt{\frac{m\log n}{n}}\right] - \frac{1}{n}.$$

In order to estimate the latter probability, we use the bound $\Pr[\mathcal{N}(0,1) \geq x] \geq 1/2 - \frac{x}{\sqrt{2\pi}}$ (for $x \geq 0$), which follows from $\Pr[\mathcal{N}(0,1) \geq 0] = 1/2$ and the fact that the density of $\mathcal{N}(0,1)$ is bounded by $1/\sqrt{2\pi}$. In our case,

$$x = \varepsilon\sqrt{\frac{m\log n}{n}}\Big/\sqrt{\tfrac{(\ell-1)m}{n}(1-\tfrac{\ell-1}{n})} \leq \varepsilon\sqrt{\frac{m\log n}{n}}\Big/\sqrt{\gamma^2 m} = \varepsilon\sqrt{\frac{\log n}{\gamma^2 n}}.$$

Therefore

$$t_\varepsilon \geq \frac{1}{2} - \frac{\varepsilon}{2\pi\gamma}\sqrt{\frac{\log n}{n}} - \frac{1}{n}.$$

□

5 The Balls and Bins Distribution: the Exponential Case

In the previous section, we showed that even in the case where the distribution is not inherently biased towards any of the agents, substantial inequalities may arise due to random noise. We now turn to study the case in which the distribution is biased in a way that exacerbates the inequality among the agents. Returning to our formal definition of the general balls and bins process, we assume that the probabilities in the vector $\boldsymbol{p}$ are ordered in increasing order and $\frac{p_i}{p_{i+1}} = \rho$, for some $\rho < 1/2$. We observe that as m approaches ∞, the weight vector follows a power law with probability 1, where for each $i = 1, \dots, n-1$, $\frac{w_i}{w_{i+1}} = \rho$. A closely related family of weight vectors that we will refer to is the family of *super-increasing* weight vectors:

Definition 5.1 (Super-increasing weights)**.** A series of positive weights $\boldsymbol{w} = (w_1, \dots, w_n)$ is said to be super-increasing if for every $i = 1, \dots, n$, $\sum_{j=1}^{i-1} w_j < w_i$.

The following three results (Lemma 5.1, Lemma 5.2 and Theorem 5.1) show that for a sufficiently large value of m, estimating the Shapley values in a weighted voting game where the weights are sampled from an exponential distribution can be reduced to the study of Shapley values in a game with a prescribed (fixed) super-increasing weight vector; Section 6 studies power distribution in WVGs with super-increasing weights. The following lemma gives a characterization of the necessary size of the voter population, so as to make the weight vector super-increasing, if the voters vote according to the above exponential distribution.

Lemma 5.1. Assume that m voters submit the votes according to the exponential distribution over candidates, such that for $\rho \in (0, \frac{1}{2})$, $\Pr[\text{voter } j \text{ votes for candidate } i] \propto \rho^{n-i}$. There is a constant $C > 0$ such that if $m \geq C\rho^{-n}(2\rho-1)^{-2}\log n$ then the resulting weight vector is super-increasing with probability $1 - O(\frac{1}{n})$. Furthermore, as $m \to \infty$, the probability approaches 1.

Proof. The proof uses Bernstein's inequality with a subsequent application of the union bound. Consider a sequence $w_1 \leq w_2 \leq \cdots \leq w_n$. The sequence is clearly super-increasing if for every $i = 2, \dots, n$, $w_i/w_{i-1} \geq 2$, and $w_1 > 0$. We now lower bound the probability of this event, by upper-bounding the probability of the following bad events: E_i is the event that $w_i < 2w_{i-1}$ (for $i = 2, \dots, n$), and E_1 is the event that $w_1 = 0$. A union bound shows that the sequence $\mathbf{w}$ is super-increasing with probability at least $1 - \sum_{i=1}^n \Pr[E_i]$.

First note that the probability that voter j votes for candidate i is equal to

$$p_i = \frac{\rho^{n-i}}{\sum_{i=1}^n \rho^{n-i}} = \frac{\rho^{n-i}(1-\rho)}{1-\rho^n} = \Theta(\rho^{n-i}).$$

Bounding the probability of E_1 is easy:

$$\Pr[E_1] = (1-p_1)^m \leq e^{-p_1 m} = e^{-\Theta(\rho^{n-1}m)}.$$

In order to bound the probability of E_i for $i \neq 1$, consider the random variable $X = 2w_{i-1} - w_i$. This random variable is a sum of m i.i.d. random variables $X^{(1)}, \dots, X^{(m)}$ corresponding to the different voters with the following distribution:

$$X^{(j)} = \begin{cases} 2 & \text{w.p. } p_{i-1}, \\ -1 & \text{w.p. } p_i, \\ 0 & \text{w.p. } 1 - p_{i-1} - p_i. \end{cases}$$

Using the identity $p_{i-1} = \rho p_i$, the moments of X are

$$\begin{aligned}\mathbb{E}[X] &= m\,\mathbb{E}[X^{(j)}] = (2\rho-1)p_i m = \Theta((2\rho-1)\rho^{n-i}m),\\ \mathrm{Var}[X] &= m(\mathbb{E}[X^{(j)2}] - \mathbb{E}[X^{(j)}]^2) = (4\rho+1)p_i m - (2\rho-1)^2 p_i^2 m = O(\rho^{n-i}m).\end{aligned}$$

Since $|X^{(j)} - \mathbb{E}[X^{(j)}]| = O(1)$, Bernstein's equality gives

$$\begin{aligned}\Pr[E_i] &= \Pr[X > 0]\\ &\le \exp -\frac{\frac{1}{2}\,\mathbb{E}[X]^2}{\mathrm{Var}[X] + O(\mathbb{E}[X])}\\ &= \exp -\frac{\Theta((2\rho-1)^2\rho^{2(n-i)}m^2)}{O(\rho^{n-i}m)}\\ &= \exp -\Omega((2\rho-1)^2\rho^{n-i}m).\end{aligned}$$

Summarizing,

$$\sum_{i=1}^{n} \Pr[E_i] \le e^{-\Theta(\rho^{n-1}m)} + \sum_{i=2}^{n} e^{-\Omega((2\rho-1)^2\rho^{n-i}m)}.$$

When $m \ge C\rho^{-n}(2\rho-1)^{-2}\log n$ for an appropriate C, all the terms are $O(1/n^2)$, and so the total error probability is $O(1/n)$, proving the first part of lemma. As $m \to \infty$, all the terms tend to 0, and so the total error probability tends to 0, proving the second part of the lemma. □

Before we proceed, it would be helpful to provide some intuition about the behavior of the Shapley values. Assuming that agent weights are given by an n-length (increasing) sequence $\boldsymbol{w}$ of real-values, consider the set of all distinct subset sums of the weights $\mathcal{S}(\boldsymbol{w}) = \{s : \exists P \subseteq [n] \text{ s.t. } s = \sum_{i\in P} w_{n+1-i}\}$ (we use w_{n+1-i} instead of w_i to make some formulas below nicer). Furthermore, suppose that the subset sums are ordered in increasing order; i.e., $\mathcal{S}(\boldsymbol{w}) = \{s_j\}_{j=1}^{t}$, such that $s_j < s_{j+1}$ for $1 \le j < t$. It is easy to show, using the definition of the Shapley value, that for any quota $q \in (s_j, s_{j+1}]$, for $1 \le j < t$, the Shapley values of every agent $i \in N$ remain constant at some value $\varphi_i(j)$, defined for the j'th interval. We formalize this intuition in Section 6, where we give a formula for $\varphi_i(j)$.

Before we state the formula (Proposition 5.2 below), we need some notation. For each $P \subseteq N$, let $\tilde{w}(P) = \sum_{i\in P} w_{n+1-i}$. For some j, $\tilde{w}(P) = s_j$, where $s_j \in \mathcal{S}$. If $P \ne N$ then $j < t$ and so $s_{j+1} = \tilde{w}(P^+)$ for some $P^+ \subseteq N$. We define $I_P^{\boldsymbol{w}} = (\tilde{w}(P), \tilde{w}(P^+)]$. From the definition it follows that the intervals I_P^w partition the interval $(0, w(N)]$. We can now state the formula for $\varphi_i(j)$. Given a weight vector $\boldsymbol{w}$, let $\varphi_i^{\boldsymbol{w}}(q)$ denote the Shapley value of player i when the quota is q and the weights are given by $\boldsymbol{w}$.

Proposition 5.2. *Suppose that $\boldsymbol{w} = (w_1, \dots, w_n)$ is a super-increasing sequence of weights, and suppose that $q \in (0, w(N)]$, say $q \in I_P^{\boldsymbol{w}}$ for some $P \subseteq N$. Write $P = \{j_0, \dots, j_r\}$ in increasing order. If $i \notin P$ then*

$$\varphi_{n+1-i}^{w}(q) = \sum_{\substack{t\in\{0,\dots,r\}:\\ j_t > i}} \frac{1}{j_t\binom{j_t-1}{t}}.$$

If $i \in P$, say $i = j_s$, then

$$\varphi_{n+1-i}^{\boldsymbol{w}}(q) = \frac{1}{j_s\binom{j_s-1}{s}} - \sum_{\substack{t\in\{0,\dots,r\}:\\ j_t > i}} \frac{1}{j_t\binom{j_t-1}{t-1}}.$$

Suppose that $\boldsymbol{w}$ is generated using a Balls and Bins process with probabilities $\boldsymbol{p}$, where $\boldsymbol{p}$ is a super-increasing sequence; then it stands to reason that if a sufficiently large number of balls is tossed (i.e., m is large enough), then voting power distribution under $\boldsymbol{w}$ will be very close to power distribution under the weight vector $\boldsymbol{p}$. This intuition is captured in the following lemma.

Lemma 5.2. Suppose that $\boldsymbol{p} = (p_1, \dots, p_n)$ is a super-increasing sequence summing to 1, and let $w_1, \dots, w_n$ be obtained by sampling m times from the distribution $p_1, \dots, p_n$.

Suppose that $T \in (0, 1]$, say $T \in I^{\boldsymbol{p}}(P)$ for some $P \subseteq \{1, \dots, n\}$. If the distance of T from the endpoints $\tilde{p}(P), \tilde{p}(P^+)$ of $I^{\boldsymbol{p}}(P)$ is at least $\Delta = \sqrt{\log(nm)/m}$ then with probability $1 - \frac{2}{(nm)^2}$ it holds that if $\boldsymbol{w}$ is super-increasing then for all $i \in N$, $\varphi_i^{\boldsymbol{w}}(mT) = \varphi_i^{\boldsymbol{p}}(T)$.

Proof. Suppose that $\mathbf{w}$ is super-increasing. Lemma 6.1 implies that $I^{\mathbf{w}}(P) = (\tilde{w}(P), \tilde{w}(P^+)]$, since both $\mathbf{p}$ and $\mathbf{w}$ are super-increasing (a priori, it could be that P^+ would have different values when defined with respect to $\mathbf{p}$ and to $\mathbf{w}$). The idea of the proof is to show that with high probability, $mT \in I^{\mathbf{w}}(P)$, and then the lemma follows from Proposition 5.2. We do that by upper-bounding the probability of the following two bad events: $\tilde{w}(P) \geq mT$ and $\tilde{w}(P^+) < mT$.

The random variable $\tilde{w}(P)$ is a sum of m i.i.d. indicator random variables which are 1 with probability $\tilde{p}(P)$. Therefore $\mathbb{E}[\tilde{w}(P)] = m\tilde{p}(P)$. Hoeffding's inequality shows that

$$\Pr[\tilde{w}(P) \geq mT] \leq \Pr[\tilde{w}(P) \geq \mathbb{E}[\tilde{w}(P)] + m\Delta] \leq e^{-2\Delta^2 m}.$$

Similarly $\Pr[\tilde{w}(P^+) < mT] \leq e^{-2\Delta^2 m}$. When $\Delta \geq \sqrt{\log n/m}$, both error probabilities are at most $1/(nm)^2$. $\square$

Combining both lemmas, we obtain our main result on the exponential case of the Balls and Bins distribution.

Theorem 5.1. Assume that m voters submit the votes according to the exponential distribution over candidates, such that for $\rho \in (0, \frac{1}{2})$, $p_i = \Pr[\text{voter } j \text{ votes for candidate } i] \propto \rho^{n-i}$. Assume further that $m \geq C\rho^{-n}(2\rho - 1)^{-2}\log n$, where $C > 0$ is some global constant.

Suppose that $T \in (0, 1]$, say $T \in I^{\boldsymbol{p}}(P)$ for some $P \subseteq \{1, \dots, n\}$. If the distance of T from the endpoints $\tilde{p}(P), \tilde{p}(P^+)$ of $I^{\boldsymbol{p}}(P)$ is at least $\Delta = \sqrt{\log(nm)/m}$ then with probability $1 - O(1/n)$ it holds that for all $i \in \{1, \dots, n\}$, $\varphi_i^{\boldsymbol{w}}(mT) = \varphi_i^{\boldsymbol{p}}(T)$.

Furthermore, for all but finitely many values of $T \in (0, 1]$, the probability that $\varphi_i^{\boldsymbol{w}}(mT) = \varphi_i^{\boldsymbol{p}}(T)$ tends to 1 as $m \to \infty$.

Proof. Lemma 5.1 gives a constant $C > 0$ such that if $m \geq C\rho^{-n}(2\rho - 1)^{-2}\log n$ then $\boldsymbol{w}$ is super-increasing with probability $1 - O(1/n)$. Hence the first part of the theorem follows from Lemma 5.2.

For the second part, Lemma 5.1 shows that as $m \to \infty$, the probability that $\boldsymbol{w}$ is super-increasing approaches 1. Suppose now that T is *not* of the form $\tilde{p}(P)$ (these are the finitely many exceptions). When m is large enough, the conditions of Lemma 5.2 are satisfied, and so as $m \to \infty$, the error probability in that lemma goes to 0. The second part of the theorem follows. $\square$

The theorem shows that in the case of the exponential distribution, if the number of balls is large enough then we can calculate with high probability the Shapley values of the resulting distribution based on the Shapley values of the original exponential distribution (without sampling). It therefore behooves us to study the Shapley values of an exponential distribution, or indeed any super-increasing sequence, a study which we undertake in Section 6.

6 Super-increasing sequences

In the previous section, we discussed the case where the weights are distributed according to a discrete exponential distribution, as modeled by a balls and bins process. As we have shown, for a long enough process, studying the distribution of Shapley values boils down to the study of the Shapley values for the case where the weights are given by a super-increasing sequence. This section constitutes a thorough analysis of power distribution in a setting where weights are super-increasing; in particular, we provide strong generalizations of the results by [22] and [24].

First, we give an explicit formula for φ_i for any super-increasing sequence of weights.

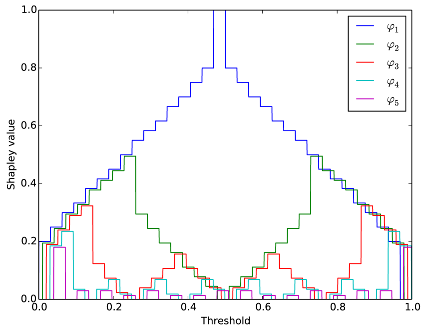
Second, we show that when agent weights are super-increasing, the Shapley value is extremely well-behaved: Lemma 6.5 shows that it is possible to easily determine when $\varphi_i(q) = \varphi_{i+1}(q)$—a problem that is computationally intractable even for the player with the smallest weight [22]— as well as bounds on the rate of increase/decrease in $\varphi_i(q)$ as q changes (Lemma 6.8).

Finally, suppose that agent weights are the prefix of an infinite sequence of weights; for example, if the weights are given by $(1, 1/2, 1/4, \dots, 2^{-n})$, then they are the prefix of the sequence $(2^{-n})_{n=0}^{\infty}$. Fixing an agent i, we observe $\varphi_i(q)$ as we keep adding weights; continuing our previous example, we observe $\varphi_i(q)$ when the weights are $(1, \dots, 2^{-n})$, $(1, \dots, 2^{-n}, 2^{-n-1})$, and so on. We show that this sequence of Shapley values is convergent (Theorem 6.3), and its limit is a continuous function of q (Theorem 6.4).

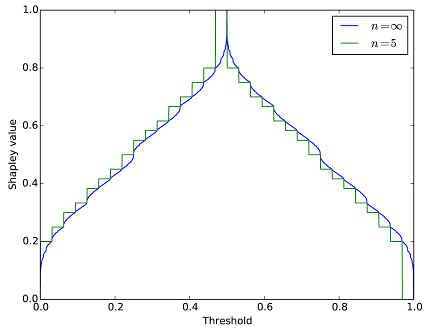
Up to this point, we assumed that the weights are arranged in non-decreasing order. In order to simplify our formulas, we will somewhat abuse our definitions by assuming that the weights are rather ordered in non-increasing order, $w_1 > w_2 > \cdots > w_n > 0$. We also assume that $\boldsymbol{w}$ is a super-increasing sequence; that is, a sequence satisfying $w_i > \sum_{j=i+1}^{n} w_j$ for all $i \in N$.

When considering different weight vectors, we will use $\varphi_i^{\boldsymbol{w}}(q)$ for the Shapley value of agent i under weight vector $\boldsymbol{w}$ and quota q.

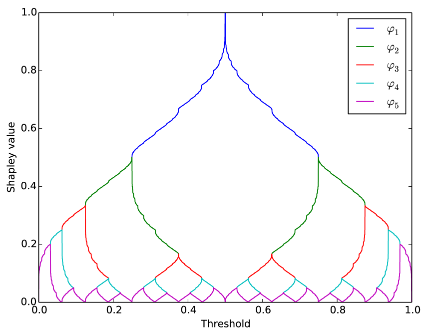
Figure 3 illustrates the behavior of the Shapley value under super-increasing weights, for various configurations of the game.



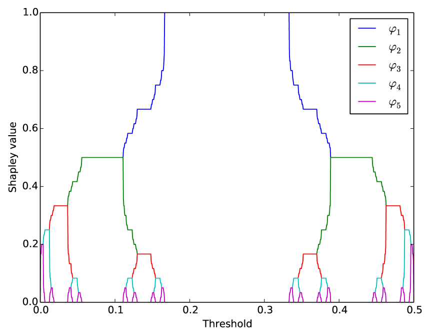
(a) Shapley values for $n = 5$, $w_i = 2^{-i}$. Values $\varphi_i(q)$ for different i are slightly nudged to show the effects of Lemma 6.5.



(b) Shapley values $\varphi_1(q)$ for $n = 5$, $w_i = 2^{-i}$ compared to the limiting case $n = \infty$.



(c) Shapley values in the limiting case, $w_i = 2^{-i}$.



(d) Shapley values in the limiting case, $w_i = 3^{-i}$.

Figure 3: Examples of several Shapley values corresponding to super-increasing sequences.

6.1 Reducing super-increasing weights to the case of a power law of 2

While not every quota in the range $(0, w(N)]$ can be expanded as a sum of members of $\{w_1, \dots, w_n\}$, there are certain naturally defined intervals that partition $(0, w(N)]$. For a subset $C \subseteq N$, define $\beta(C) = \sum_{i \in C} 2^{n-i}$. Intuitively, we think of $\beta(C)$ as the value resulting from the binary characteristic vector of the set of agents C. The purpose of the following two lemmas is to reduce every super-increasing weight vector to the case where the weights obey a power-law distribution, with a power of 2.

Lemma 6.1. Let $C_1, C_2 \subseteq N$. Then $\beta(C_1) < \beta(C_2)$ if and only if $w(C_1) < w(C_2)$.

Proof. In order to prove the claim, it suffices to observe adjacent sets $C_1, C_2 \subseteq N$, i.e., ones satisfying $\beta(C_2) = \beta(C_1) + 1$. Let $\ell = \max(N \setminus C_1)$, and define $C = C_1 \cap \{1, \dots, \ell - 1\}$. Then $C_1 = C \cup \{\ell + 1, \dots, n\}$ and $C_2 = C \cup \{\ell\}$. Therefore $w(C_2) - w(C_1) = w_\ell - w(\{\ell + 1, \dots, n\}) > 0$, since $w_1, \dots, w_n$ is super-increasing. □

For example, when $n = 3$ the intervals are

$$\begin{aligned}(0, w_1 + w_2 + w_3] = (0, w_3] \cup (w_3, w_2] \cup (w_2, w_2 + w_3] \cup (w_2 + w_3, w_1] \\ \cup (w_1, w_1 + w_3] \cup (w_1 + w_3, w_1 + w_2] \cup (w_1 + w_2, w_1 + w_2 + w_3].\end{aligned}$$

For a non-empty set of agents $C \subseteq N$, we let $P^- \subseteq N$ be the unique subset of agents satisfying $\beta(P^-) = \beta(P) - 1$. Lemma 6.1 shows that every quota $q \in (0, w(N)]$ belongs to a unique interval $(w(P^-), w(P)]$; we denote P by $A(q)$. We think of $A(q)$ as an increasing sequence $a_0, \ldots, a_r$ depending on q, for some value of r which also depends on q. Whenever we write $P = \{a_0, \ldots, a_r\}$, we will always assume that $a_0 < \cdots < a_r$.

Lemma 6.2. For all agents $i \in N$ and quotas $q \in (0, w(N)]$, $\varphi_i^{\boldsymbol{w}}(q) = \varphi_i^{\boldsymbol{b}}(\beta(A(q)))$, where $\boldsymbol{b} = (2^{n-1}, \ldots, 1)$.

Proof. Let σ be a random permutation in Sym_n, and recall that $P_i(\sigma)$ is the set of agents appearing before agent i in σ. The Shapley value $\varphi_i^{\boldsymbol{w}}(q)$ is the probability that $w(P_i(\sigma)) \in [q - w_i, q)$, or equivalently, that $q \in (w(P_i(\sigma)), w(P_i(\sigma)) + w_i]$. Since the intervals $(w(C^-), w(C)]$ partition $(0, w(N)]$, q is in $(w(P_i(\sigma)), w(P_i(\sigma)) + w_i]$ if and only if $w(P_i(\sigma)) \leq w(A(q)^-)$ and $w(A(q)) \leq w(P_i(\sigma) \cup \{i\})$. Lemma 6.1 shows that this is equivalent to checking whether $\beta(P_i(\sigma)) \leq \beta(A(q)^-)$ and $\beta(A(q)) \leq \beta(P_i(\sigma) \cup \{i\})$. Now, note that $\beta(A(q)^-) = \beta(A(q)) - 1$, so the above condition simply states that i is pivotal for σ under $\boldsymbol{b}$ when the quota is $\beta(A(q))$. □

Lemma 6.2 implies that for any super-increasing $\boldsymbol{w}$, if one wishes to compute $\varphi_i^{\boldsymbol{w}}(q)$, it is only necessary to find $A(q)$. However, finding $A(q)$ is easy; as the following claim shows, a simple greedy algorithm can find $A(q)$ in polynomial time.

Lemma 6.3. Given a point $q \in (0, w(N)]$ and a vector of super-increasing weights $\boldsymbol{w}$, it is possible to find $A(q)$ in time $O(n)$.

Proof. Algorithm 1 calculates $A(q)$, as we show below. While the algorithm as stated does not run in linear time, it is easy to modify it so that it does.

Let $A(q) = a_0, \ldots, a_r$, so that $A(q)^- = a_0, \ldots, a_{r-1}, a_r + 1, \ldots, n$. Denote by A_i the value of A in the algorithm after i iterations of the loop. We prove by induction on i that $A_i = A(q) \cap \{1, \ldots, i\}$, which shows that the algorithm returns $A(q)$. The inductive claim trivially holds for $i = 0$. Assuming that $A_{i-1} = A(q) \cap \{1, \ldots, i-1\}$, we now prove that $A_i = A(q) \cap \{1, \ldots, i\}$. We consider two cases: $i \notin A(q)$ and $i \in A(q)$. If $i \notin A(q)$ then $q \leq w(A(q)) = w(A_{i-1}) + w(A(q) \cap \{i, \ldots, n\}) \leq w(A_{i-1}) + w(\{i+1, \ldots, n\})$, and so line 5 does not get executed. Suppose now that $i \in A(q)$. If $a_r = i$ then $q > w(A(q)^-) = w(A_{i-1}) + w(\{i+1, \ldots, n\})$, and so line 5 gets executed. If $a_r > i$ then $q > w(A(q)^-) \geq w(A_{i-1}) + w_i > w(A_{i-1}) + w(\{i+1, \ldots, n\})$, since $\boldsymbol{w}$ is super-increasing, and so line 5 gets executed in this case as well. □

Algorithm 1 An algorithm for finding $A(q)$

```
1: procedure FIND-SET(w, q)
2:     A ← ∅
3:     for i ← 1 to n do
4:         if q > w(A ∪ {i+1, ..., n}) then
5:             A ← A ∪ {i}
6:         end if
7:     end forreturn A
8: end procedure
```

In the case where agent weights follow a simple exponential increase, $A(q)$ can be characterized in a very simple manner, as stated in the following lemma.

Lemma 6.4. Suppose $w_i = d^{n-i}$ for some integer $d \geq 2$, and let $q \in (0, w(N)]$. Write $\lceil q \rceil$ in base d: $\lceil q \rceil = (t_1 \ldots t_n)_d$. If the base d representation only consists of the digits 0 and 1 then $A(q) = \{i \in N : t_i = 1\}$. Otherwise, let ℓ be the minimal index such that $t_\ell > 1$, and let $k < \ell$ be the maximal index less than ℓ satisfying $t_k = 0$ (the proof shows that such an index exists). Then $A(q) = \{i \in \{1, \ldots, k-1\} : t_i = 1\} \cup \{k\}$.

Proof. Suppose first that $t_i \in \{0, 1\}$ for all $i \in N$, and let $Q(q) = \{i \in N : t_i = 1\}$. Since $\lceil q \rceil \geq 1$, $Q(q) \neq \emptyset$. Lemma 6.1 shows that $w(Q(q)^-) < w(Q(q))$ and so $q = w(Q(q)) \in (w(Q(q)^-), w(Q(q))]$, showing that $A(q) = Q(q)$.

Suppose next that ℓ is the minimal index such that $t_\ell > 1$. If $t_k = 1$ for all $k < \ell$ then

$$q > \lceil q \rceil - 1 \geq \sum_{j=1}^{\ell-1} w_j + 2w_\ell - 1 \geq w(N),$$

since the fact that the w_i are integral and super-increasing implies that

$$w_\ell \geq \sum_{j=\ell+1}^{n} w_j + 1.$$

We conclude that the maximal index $k < \ell$ satisfying $t_k = 0$ exists. Let $Q(q) = \{i \in \{1, \ldots, k-1\} : t_i = 1\} \cup \{k\}$. On the one hand,

$$q \leq \lceil q \rceil \leq \sum_{j \in Q(q) \setminus \{k\}} w_j + (d-1) \sum_{j=k+1}^{n} w_j < w(Q(q)).$$

On the other hand,

$$\begin{aligned} q > \lceil q \rceil - 1 &\geq \sum_{j \in Q(q) \setminus \{k\}} w_j + \sum_{j=k+1}^{\ell-1} w_j + 2w_\ell - 1 \\ &\geq \sum_{j \in Q(q) \setminus \{k\}} w_j + \sum_{j=k+1}^{n} w_j = w(Q(q)^-). \end{aligned}$$

Therefore $A(q) = Q(q)$. □

We now present a closed-form formula for the Shapley values. The resulting Shapley values are illustrated in Figure 3.

Theorem 6.1. Consider an agent $i \in N$ and a prescribed quota value $q \in (0, w(N)]$. Let $A(q) = \{a_0, \ldots, a_r\}$. If $i \notin A(q)$ then

$$\varphi_i(q) = \sum_{\substack{t \in \{0,\ldots,r\}: \\ a_t > i}} \frac{1}{a_t \binom{a_t - 1}{t}}.$$

If $i \in A(q)$, say $i = a_s$, then

$$\varphi_i(q) = \frac{1}{a_s \binom{a_s - 1}{s}} - \sum_{\substack{t \in \{0,\ldots,r\}: \\ a_t > i}} \frac{1}{a_t \binom{a_t - 1}{t-1}}.$$

Proof. Lemma 6.2 shows that $\varphi_i^{\boldsymbol{w}}(q) = \varphi_i^{\boldsymbol{b}}(\beta(A(q)))$, where $\boldsymbol{b} = 2^{n-1}, \ldots, 1$. Therefore we can assume without loss generality that $\boldsymbol{w} = 2^{n-1}, \ldots, 1$, i.e., $w_i = 2^{n-i}$, and that $q = \sum_{j \in A(q)} w_j$.

Recall that $\varphi_i(q)$ is the probability that $w(P_i(\pi)) \in [q - w_i, q)$, where π is chosen randomly from Sym_n, and $P_i(\pi)$ is the set of predecessors of i in π. The idea of the proof is to consider the maximal $\tau \in \{1, \ldots, r+1\}$ such that $a_t \in P_i(\pi)$ for all $t < \tau$. We will show that when $i \notin A(q)$, each possible value of $\tau(\pi)$ corresponds to one summand in the expression for $\varphi_i(q)$. When $i \in A(q)$, say $i = a_s$, we will show that the events that i is pivotal with respect to q and that i is pivotal with respect to $q - w_i$ are disjoint, and their union is an event having probability $1/a_s\binom{a_s-1}{s}$.

Suppose that i is pivotal for π and $\tau(\pi) = \tau$. We start by showing that $\tau \leq r$, ruling out the case $\tau = r+1$. If $\tau = r+1$ then by definition

$$w(P_i(\pi)) \geq \sum_{j \in A(q)} w_j = q,$$

contradicting the assumption $w(P_i(\pi)) < q$. Therefore $\tau \leq r$, and so a_τ is well-defined. We claim that if $k \in P_i(\pi)$ for some agent $k < a_\tau$ then $k \in A(q)$. Indeed, otherwise

$$w(P_i(\pi)) \geq \sum_{t=0}^{\tau-1} w_{a_t} + w_k \geq \sum_{t=0}^{\tau-1} w_{a_t} + w_{a_\tau - 1} > \sum_{t=0}^{\tau-1} w_{a_t} + \sum_{j=a_\tau}^{n} w_j \geq q,$$

again contradicting $w(P_i(\pi)) < q$ (the third inequality made use of the fact that $\boldsymbol{w}$ is super-increasing).

Furthermore, we claim that $a_\tau \geq i$. Otherwise,

$$w(P_i(\pi)) \leq \sum_{t=0}^{\tau-1} w_{a_t} + \sum_{j=a_\tau+1}^{n} w_j - w_i < \sum_{t=0}^{\tau} w_{a_t} - w_i \leq q - w_i,$$

contradicting the assumption $w(P_i(\pi)) \geq q - w_i$.

Summarizing, we have shown that $\tau \leq r$, $a_\tau \geq i$ and

$$P_i(\pi) \cap \{1, \ldots, a_\tau\} = \{a_0, \ldots, a_{\tau-1}\}. \tag{2}$$

Denote this event E_τ, and call a $\tau \leq r$ satisfying $a_\tau \geq i$ *legal*.

Suppose first that $i \notin A(q)$. We have shown above that if i is pivotal then E_τ happens for some legal τ. We claim that the converse is also true. Indeed, given E_τ defined with respect to a permutation π, and for some legal τ, the weight of $P_i(\pi)$ can be bounded as follows.

$$\sum_{t=0}^{\tau-1} w_{a_t} \leq w(P_i(\pi)) \leq \sum_{t=0}^{\tau-1} w_{a_t} + \sum_{j=a_\tau+1}^{n} w_j < \sum_{t=0}^{\tau} w_{a_t}.$$

The second inequality follows from the definition of τ, whereas the third inequality follows as before from the definition of a super-increasing sequence. The upper bound is clearly at most q, and the lower bound satisfies

$$\sum_{t=0}^{\tau-1} w_{a_t} \geq q - \sum_{j=a_\tau}^{n} w_j > q - w_{a_\tau - 1} \geq q - w_i,$$

since $i < a_\tau$.

It remains to calculate $\Pr[E_\tau]$. The event E_τ states that the restriction of π to $\{1, \ldots, a_\tau\}$ consists of the elements $\{a_0, \ldots, a_{\tau-1}\}$ in some order, followed by i (recall that $i \leq a_\tau$). For each of the $\tau!$ possible orders, the probability of this is $1/a_\tau \cdots (a_\tau - \tau) = (a_\tau - \tau - 1)!/a_\tau!$, and so

$$\Pr[E_\tau] = \frac{\tau!(a_\tau - \tau - 1)!}{a_\tau!} = \frac{1}{a_\tau \binom{a_\tau - 1}{\tau}}. \tag{3}$$

Summing over all legal τ, we obtain the formula in the statement of the theorem. This completes the proof in the case $i \notin A(q)$.

Suppose next that $i \in A(q)$, say $i = a_s$. Since $a_\tau \geq a_s = i$ while $i \notin P_i(\pi)$, we deduce that $\tau = s$. Therefore the event E_s happens. Conversely, when E_s happens,

$$w(P_i(\pi)) \leq \sum_{t=0}^{s-1} w_{a_t} + \sum_{j=a_s+1}^{n} w_j < \sum_{t=0}^{s} w_{a_t} \leq q.$$

Therefore i is pivotal (with respect to q) if and only if E_s happens and $w(P_i(\pi)) \geq q - w_i$.

It is easy to check that $A(q - w_i) = A(q) \setminus \{i\} = a_0, \ldots, a_{s-1}, a_{s+1}, \ldots, a_r$. The argument above shows that if i is pivotal with respect to $q - w_i$ then for some $\tau' \geq s + 1$,

$$P_i(\pi) \cap \{1, \ldots, a_{\tau'}\} = \{a_0, \ldots, a_{s-1}, a_{s+1}, \ldots, a_{\tau'-1}\}.$$

In particular, the event E_s happens. Conversely, when E_s happens,

$$w(P_i(\pi)) \geq \sum_{t=0}^{s-1} w_{a_t} \geq q - w_{a_s} - \sum_{j=a_s+1}^{n} w_j > (q - w_{a_s}) - w_{a_s}.$$

Therefore i is pivotal with respect to $q - w_i$ if and only if E_s happens and $w(P_i(\pi)) < q - w_i$. We conclude that

$$\Pr[w_i \text{ is pivotal with respect to } q] = \Pr[E_s] - \Pr[w_i \text{ is pivotal with respect to } q - w_i].$$

Above we have calculated $\Pr[E_s] = 1/a_s \binom{a_s - 1}{s}$, and we obtain the formula in the statement of the theorem. $\square$

In Section 6.3, we further provide a characterization of the Shapley values for the limiting case where n, the number of agents, goes to infinity.

6.2 Properties of the Shapley values

Zuckerman et al. [24] provide a nice characterization of super-increasing sets:

Theorem 6.2 ([24])**.** If the weights $\boldsymbol{w}$ are super-increasing then for every quota $q \in (0, w(N)]$, either $\varphi_1(q) = \varphi_2(q)$ or $\varphi_2(q) = \varphi_3(q)$.

In this section, we further generalize this result, using Theorem 6.1. Specifically, as a consequence of the theorem, we can determine in which cases $\varphi_i(q) = \varphi_{i+1}(q)$. The results are summarized in the following lemma.

Lemma 6.5. Let a quota $q \in (0, w(N)]$ be given, and let $A(q) = \{a_0, \ldots, a_r\}$. For each $i \in N \setminus \{n\}$:

(a) If $i, i+1 \notin A(q)$ then $\varphi_i(q) = \varphi_{i+1}(q)$.

(b) If $i \notin A(q)$ and $i+1 \in A(q)$ then $\varphi_i(q) \geq \varphi_{i+1}(q)$, with equality if and only if $i+1 = a_r$.

(c) If $i \in A(q)$ and $i+1 \notin A(q)$ then $\varphi_i(q) > \varphi_{i+1}(q)$.

(d) If $i, i+1 \in A(q)$ then $\varphi_i(q) = \varphi_{i+1}(q)$.

For each $i \in N$, let Ψ_i be the truth value of $i \in A(q)$. Lemma 6.5 shows that if $\Psi_i = \Psi_{i+1}$ then $\varphi_i(q) = \varphi_{i+1}(q)$. Since there are only two possible truth values, for each $i \in N \setminus \{1, n\}$, either $\varphi_{i-1}(q) = \varphi_i(q)$ or $\varphi_i(q) = \varphi_{i+1}(q)$. This generalizes Theorem 6.2.

To prove Lemma 6.5, we will need some combinatorial identities.

Lemma 6.6. Let p, t be integers satisfying $p > t \geq 1$. Then

$$\frac{1}{p\binom{p-1}{t}} + \frac{1}{p\binom{p-1}{t-1}} = \frac{1}{(p-1)\binom{p-2}{t-1}}.$$

Proof. The proof is a simple calculation:

$$\begin{aligned} \frac{1}{p\binom{p-1}{t}} + \frac{1}{p\binom{p-1}{t-1}} &= \frac{t!(p-t-1)! + (t-1)!(p-t)!}{p!} \\ &= \frac{(t-1)!(p-t-1)![t+(p-t)]}{p!} = \frac{(t-1)!(p-t-1)!}{(p-1)!} = \frac{1}{(p-1)\binom{p-2}{t-1}}. \end{aligned}$$

□

Lemma 6.7. Let p, t, k be integers satisfying $p > t \geq 0$ and $k \geq 0$. Then

$$\frac{1}{p\binom{p-1}{t}} - \sum_{\ell=1}^{k} \frac{1}{(p+\ell)\binom{p+\ell-1}{t+\ell-1}} = \frac{1}{(p+k)\binom{p+k-1}{t+k}}.$$

In particular,

$$\frac{1}{p\binom{p-1}{t}} = \sum_{\ell=1}^{\infty} \frac{1}{(p+\ell)\binom{p+\ell-1}{t+\ell-1}}.$$

Proof. The proof is by induction on k. If $k = 0$ then there is nothing to prove. For $k > 0$ we have

$$\frac{1}{p\binom{p-1}{t}} - \sum_{\ell=1}^{k} \frac{1}{(p+\ell)\binom{p+\ell-1}{t+\ell-1}} = \frac{1}{(p+k-1)\binom{p+k-2}{t+k-1}} - \frac{1}{(p+k)\binom{p+k-1}{t+k-1}} = \frac{1}{(p+k)\binom{p+k-1}{t+k}},$$

using Lemma 6.6. The second expression of the lemma follows from rearranging the first formula and taking the limit $k \to \infty$. □

We are now ready to prove Lemma 6.5.

Proof of Lemma 6.5. For the first item, since $i+1 \notin A(q)$ then $a_t > i$ iff $a_t > i+1$, and so

$$\varphi_i(q) = \sum_{\substack{t \in \{0,\ldots,r\}: \\ a_t > i}} \frac{1}{a_t \binom{a_t - 1}{t}} = \sum_{\substack{t \in \{0,\ldots,r\}: \\ a_t > i+1}} \frac{1}{a_t \binom{a_t - 1}{t}} = \varphi_{i+1}(q).$$

For the second item, suppose that $i+1 = a_s$. We have

$$\begin{aligned}
\varphi_i(q) - \varphi_{i+1}(q) &= \sum_{t=s}^{r} \frac{1}{a_t \binom{a_t-1}{t}} - \left[\frac{1}{a_s \binom{a_s-1}{s}} - \sum_{t=s+1}^{r} \frac{1}{a_t \binom{a_t-1}{t-1}} \right] \\
&= \sum_{t=s+1}^{r} \left[\frac{1}{a_t \binom{a_t-1}{t}} + \frac{1}{a_t \binom{a_t-1}{t-1}} \right] = \sum_{t=s+1}^{r} \frac{1}{(a_t-1)\binom{a_t-2}{t-1}},
\end{aligned}$$

using Lemma 6.6. Therefore $\varphi_i(q) \geq \varphi_{i+1}(q)$, with equality if and only if $s = r$.

For the third item, suppose that $i = a_s$. We have

$$\begin{aligned}
\varphi_i(q) - \varphi_{i+1}(q) &= \frac{1}{a_s \binom{a_s-1}{s}} - \sum_{t=s+1}^{r} \frac{1}{a_t \binom{a_t-1}{t-1}} - \sum_{t=s+1}^{r} \frac{1}{a_t \binom{a_t-1}{t}} \\
&= \frac{1}{a_s \binom{a_s-1}{s}} - \sum_{t=s+1}^{r} \frac{1}{(a_t-1)\binom{a_t-2}{t-1}},
\end{aligned}$$

using Lemma 6.6. The same lemma also implies that the expression $1/p\binom{p}{t-1}$ is decreasing in p. Since $i+1 \notin A(q)$, if a_{s+1} exists then $a_{s+1} \geq a_s + 2$, and in general $a_{s+\ell} \geq a_s + \ell + 1$. Therefore

$$\varphi_i(q) - \varphi_{i+1}(q) \geq \frac{1}{a_s \binom{a_s-1}{s}} - \sum_{\ell=1}^{r-s} \frac{1}{(a_s+\ell)\binom{a_s+\ell-1}{s+\ell-1}} = \frac{1}{(a_s+r-s)\binom{a_s+r-s-1}{r}} > 0,$$

using Lemma 6.7.

For the fourth item, suppose that $i = a_s$. We have

$$\begin{aligned}
\varphi_i(q) - \varphi_{i+1}(q) &= \left[\frac{1}{a_s \binom{a_s-1}{s}} - \sum_{t=s+1}^{r} \frac{1}{a_t \binom{a_t-1}{t-1}} \right] - \left[\frac{1}{a_{s+1} \binom{a_{s+1}-1}{s+1}} - \sum_{t=s+2}^{r} \frac{1}{a_t \binom{a_t-1}{t-1}} \right] \\
&= \frac{1}{a_s \binom{a_s-1}{s}} - \frac{1}{a_{s+1} \binom{a_{s+1}-1}{s+1}} - \frac{1}{a_{s+1} \binom{a_{s+1}-1}{s}} = 0,
\end{aligned}$$

using Lemma 6.6 together with $a_{s+1} = a_s + 1$. $\square$

Since the Shapley values are constant in the interval $(w(P^-), w(P)]$, it follows that in order to analyze the behavior of $\varphi_i(q)$, one need only determine the rate of increase or decrease at quotas of the form $w(P)$ for $P \subseteq N$. These are given by the following lemma.

Lemma 6.8. Let $P \subseteq N$ be a non-empty set of agents, and let $i \in N$ be an agent. If $i \notin P^-$ then $\varphi_i(w(P^-)) < \varphi_i(w(P))$. If $i \in P^-$ then $\varphi_i(w(P^-)) > \varphi_i(w(P))$.

Moreover, $|\varphi_i(w(P)) - \varphi_i(w(P^-))| \leq \frac{1}{n}$. Furthermore, this inequality is tight only in one of the following cases:

(a) $P = \{n\}$.

(b) $i < n$ and $P = \{1, \ldots, i\}$ or $P = \{i, n\}$.

(c) $i = n$ and $P = \{n-1\}$.

Otherwise, $|\varphi_i(w(P)) - \varphi_i(w(P^-))| \leq \frac{1}{n(n-1)}$.

Proof. Define $\varphi_+ = \varphi_i(w(P))$ and $\varphi_- = \varphi_i(w(P^-))$. Let $P = a_0, \ldots, a_r$. We have $P^- = a_0, \ldots, a_{r-1}, a_r + 1, \ldots, n$.

Suppose first that $i > a_r$, and let s be the index of i in the sequence P^-. According to Theorem 6.1, $\varphi_+ = 0$ and

$$\varphi_- = \frac{1}{i\binom{i-1}{s}} - \sum_{\ell=1}^{n-i} \frac{1}{(i+\ell)\binom{i+\ell-1}{s+\ell-1}} = \frac{1}{n\binom{n-1}{s+n-i}}.$$

We see that $i \in P^-$ and $\varphi_- > \varphi_+$. Furthermore, $|\varphi_+ - \varphi_-| \leq \frac{1}{n(n-1)}$ unless $s + n - i \in \{0, n-1\}$. If $s + n - i = 0$ then $s = 0$ and $i = n$, implying $P^- = \{n\}$ and so $P = \{n-1\}$. If $s + n - i = n - 1$ then $s = i - 1$ and so $P^- = 1, \ldots, n$, which is impossible.

Suppose next that $i = a_r$. According to the theorem,

$$\varphi_+ - \varphi_- = \frac{1}{i\binom{i-1}{r}} - \sum_{\ell=1}^{n-i} \frac{1}{(i+\ell)\binom{i+\ell-1}{r+\ell-1}} = \frac{1}{n\binom{n-1}{r+n-i}}.$$

We see that $i \notin P^-$ and $\varphi_+ > \varphi_-$. Furthermore, $|\varphi_+ - \varphi_-| \leq \frac{1}{n(n-1)}$ unless $r + n - i \in \{0, n-1\}$. If $r + n - i = 0$ then $r = 0$ and $i = n$, and so $P = \{n\}$. If $r + n - i = n$ then $r = i - 1$ and so $P = 1, \ldots, i$.

Finally, suppose that $i < a_r$. If $i \notin P$ then

$$\varphi_+ - \varphi_- = \frac{1}{a_r\binom{a_r-1}{r}} - \sum_{\ell=1}^{n-a_r} \frac{1}{(a_r+\ell)\binom{a_r+\ell-1}{r+\ell-1}} = \frac{1}{n\binom{n-1}{r+n-a_r}}.$$

We see that $i \notin P^-$ and $\varphi_+ > \varphi_-$. Furthermore, $|\varphi_+ - \varphi_-| \leq \frac{1}{n(n-1)}$ unless $r + n - a_r \in \{0, n-1\}$. If $r + n - a_r = 0$ then $r = 0$ and $a_r = n$, and so $P = \{n\}$. If $r + n - a_r = n - 1$ then $a_r = r + 1$, which implies $P = \{1, \ldots, r+1\}$. However, this contradicts the assumption $i \notin P$.

If $i < a_r$ and $i \in P$ then

$$\varphi_- - \varphi_+ = \frac{1}{a_r\binom{a_r-1}{r-1}} - \sum_{\ell=1}^{n-a_r} \frac{1}{(a_r+\ell)\binom{a_r+\ell-1}{r+\ell-2}} = \frac{1}{n\binom{n-1}{r+n-a_r-1}}.$$

We see that $i \in P^-$ and $\varphi_- > \varphi_+$. Furthermore, $|\varphi_+ - \varphi_-| \leq \frac{1}{n(n-1)}$ unless $r + n - a_r - 1 \in \{0, n-1\}$. If $r + n - a_r - 1 = 0$ then $r = 1$ and $a_r = n$, and so $P = \{i, n\}$. If $r + n - a_r - 1 = n - 1$ then $a_r = r$, which is impossible. ☐

6.3 Limiting case

Given a super-increasing sequence $w_1, \ldots, w_n$ (where again, $w_1 > w_2 > \cdots > w_n$) and some $m \in N$, let us write $\boldsymbol{w}|_m$ for $(w_1, \ldots, w_m)$ and $[m]$ for $\{1, \ldots, m\}$. We write $\varphi_i^{\boldsymbol{w}|_m}(q)$ for the Shapley value of agent $i \in [m]$ in the weighted voting game in which the set of agents is $[m]$, the weights are $\boldsymbol{w}|_m$, and the quota is q. We also write $A|_m(q)$ for the set $P \subseteq [m]$ such that $q \in (w|_m(P^-), w|_m(P)]$.

The following lemma relates $\varphi_i^{\boldsymbol{w}}(q)$ and $\varphi_i^{\boldsymbol{w}|_m}(q)$.

Lemma 6.9. Let $m \in N$ and $i \in [m]$, and let $q \in (0, w([m])]$. Then

$$\varphi_i^{\boldsymbol{w}|_m}(q) = \varphi_i^{\boldsymbol{w}}(w(A|_m(q))).$$

Proof. Theorem 6.1 provides a function Φ such that $\varphi_i^{\boldsymbol{w}|_m}(q) = \Phi(A|_m(q))$ and $\varphi_i^{\boldsymbol{w}}(w(A|_m(q))) = \Phi(A(w(A|_m(q)))) = \Phi(A|_m(q))$. We conclude that the Shapley values coincide. ☐

Therefore the plot of $\varphi_i^{\boldsymbol{w}|_m}$ can be readily obtained from that of $\varphi_i^{\boldsymbol{w}}$. This suggests looking at the limiting case of an *infinite* super-increasing sequence $(w_i)_{i=1}^{\infty}$, which is a sequence satisfying $w_i > 0$ and $w_i \geq \sum_{j=i+1}^{\infty} w_j$ for all $i \geq 1$. The super-increasing condition implies that the sequence sums to some value $w(\infty) \leq 2w_1$. Lemma 6.9 suggests how to define φ_i in this case: for $q \in (0, w(\infty))$ and $i \geq 1$, let

$$\varphi_i^{(\infty)}(q) = \lim_{n\to\infty} \varphi_i^{\boldsymbol{w}|_n}(q).$$

We show that the limit exists by providing an explicit formula for it, as given in Theorem 6.3. The theorem is proved in the following subsection. In the theorem, we consider possibly infinite subsets $P = \{a_0, \dots, a_r\}$ of the positive integers, ordered in increasing order; when $r = \infty$, the subset is infinite. Also, the notation $\{a, \dots, \infty\}$ (or $\{a, \dots, r\}$ when $r = \infty$) means all integers larger than or equal to a.

Theorem 6.3. Let $q \in (0, w(\infty))$ and let i be a positive integer.

(a) There exists a non-empty subset of the positive integers $P = \{a_0, \dots, a_r\}$ such that either $q = w(P)$ or P is finite and $q \in (w(P^-), w(P)]$, where $P^- = \{a_0, \dots, a_{r-1}\} \cup \{a_r + 1, \dots, \infty\}$.

(b) The limit $\varphi_i^{(\infty)}(q) = \lim_{n\to\infty} \varphi_i^{\boldsymbol{w}|_n}(q)$ exists. When $i \notin P$,

$$\varphi_i^{(\infty)}(q) = \sum_{\substack{t\in\{0,\dots,r\}:\\ a_t>i}} \frac{1}{a_t\binom{a_t-1}{t}},$$

and when $i \in P$, say $i = a_s$, then

$$\varphi_i^{(\infty)}(q) = \frac{1}{a_s\binom{a_s-1}{s}} - \sum_{\substack{t\in\{0,\dots,r\}:\\ a_t>i}} \frac{1}{a_t\binom{a_t-1}{t-1}}.$$

Lemma 6.9 easily extends to the case $n = \infty$.

Lemma 6.10. Let $m \geq 1$ be an integer, let $i \in [m]$, and let $q \in (0, w([m])]$. Then $\varphi_i^{\boldsymbol{w}|_m}(q) = \varphi_i^{(\infty)}(w(A|_m(q)))$.

Proof. Lemma 6.9 shows that for $n \geq m$, $\varphi_i^{\boldsymbol{w}|_m}(q) = \varphi_i^{\boldsymbol{w}|_n}(w(A|_m(q)))$, and therefore $\varphi_i^{\boldsymbol{w}|_m}(q) = \lim_{n\to\infty} \varphi_i^{\boldsymbol{w}|_n}(w(A|_m(q))) = \varphi_i^{(\infty)}(w(A|_m(q)))$. □

We conclude by showing that the limiting functions $\varphi_i^{(\infty)}$ are continuous.

Theorem 6.4. Let i be a positive integer. The function $\varphi_i^{(\infty)}$ is continuous on $(0, w(\infty))$, and $\lim_{q\to 0} \varphi_i^{(\infty)}(q) = \lim_{q\to w(\infty)} \varphi_i^{(\infty)}(q) = 0$.

Proof. Let $q \in (0, w(\infty))$. We start by showing that $\varphi_i^{(\infty)}$ is continuous from the right at q. Lemma 6.12 shows that we can find a subset P such that either $q = w(P)$ or $q \in (w(P^-), w(P)]$. If $q < w(P)$ then since $\varphi_i^{(\infty)}$ is constant on $(w(P^-), w(P)]$ according to Theorem 6.3, clearly $\varphi_i^{(\infty)}$ is continuous from the right at q. Therefore we can assume that $q = w(P)$. Since $q < w(\infty)$, we can further assume that there are infinitely many $n \notin P$.

Suppose that we have a sequence q_j tending to q strictly from the right. For each j we can find a subset P_j such that either $q_j = w(P_j)$ or $q_j \in (w(P_j^-), w(P_j)]$. We can assume that the second case doesn't happen by replacing q_j with $w(P_j^-)$; the new sequence still tends to q strictly from the right. So we can assume that $q_j = w(P_j) > w(P)$. Let $k(j) = \min(P_j \setminus P)$, and let $l(j) > k(j)$ be the smallest index larger than $k(j)$ such that $l(j) \notin P$. Then

$$q_j - q = w(P_j) - w(P) \geq w_{k(j)} - \left(\sum_{t=k(j)+1}^{\infty} w_t - w_{l(j)}\right) \geq w_{l(j)}.$$

As $j \to \infty$, $l(j) \to \infty$ and so $k(j) \to \infty$. Therefore we can assume without loss of generality that $k(j) > i$ for all j. Theorem 6.3 then implies that

$$|\varphi_i^{(\infty)}(q_j) - \varphi_i^{(\infty)}(q)| \leq \sum_{s=0}^{\infty} \frac{1}{(k(j)+s)\binom{k(j)+s-1}{s}} = \frac{1}{k(j)-1},$$

using Lemma 6.7. Since $k(j) \to \infty$, $\varphi_i^{(\infty)}(q_j) \to \varphi_i^{(\infty)}(q)$.

We proceed to show that $\varphi_i^{(\infty)}$ is continuous from the left at q. Lemma 6.12 shows that we can find a subset P such that either $q = w(P)$ or $q \in (w(P^-), w(P)]$. In the second case, since $\varphi_i^{(\infty)}$ is constant on $(w(P^-), w(P)]$

according to Theorem 6.3, clearly $\varphi_i^{(\infty)}$ is continuous from the left at q. Therefore we can assume that $q = w(P)$. Since $q > 0$, we can further assume that there are infinitely many $n \in P$.

Suppose that we have a sequence q_j tending to q strictly from the left. For each j we can find a subset P_j such that either $q_j = w(P_j)$ or $q_j \in (w(P_j^-), w(P_j)]$, and in both cases $q_j \le w(P_j) < w(P)$. Let $k(j) = \min(P \setminus P_j)$, and let $l(j) > k(j)$ be the smallest index larger than $k(j)$ such that $l(j) \in P$. Then

$$q - q_j \ge w(P) - w(P_j) \ge w_{k(j)} + w_{l(j)} - \sum_{t=k(j)+1}^{\infty} w_t \ge w_{l(j)}.$$

At this point we can prove that $\varphi_i^{(\infty)}(q_j) \to \varphi_i^{(\infty)}(q)$ as in the preceding case.

It remains to show that $\lim_{q \to 0} \varphi_i^{(\infty)}(q) = \lim_{q \to w(\infty)} \varphi_i^{(\infty)}(q) = 0$. We start by showing that $\lim_{q \to 0} \varphi_i^{(\infty)}(q) = 0$. Let q_j be a sequence tending to 0 strictly from the right. As before, we can assume that $q_j = w(P_j)$ for each j. Let $k(j) = \min P_j$. Since $q_j \ge w_{k(j)}$, $k(j) \to \infty$. Therefore we can assume without loss of generality that $k(j) > i$ for all j. Theorem 6.3 then implies that

$$\varphi_i^{(\infty)}(q_j) \le \sum_{s=0}^{\infty} \frac{1}{(k(j)+s)\binom{k(j)+s-1}{s}} = \frac{1}{k(j)-1},$$

using Lemma 6.7. Since $k(j) \to \infty$, $\varphi_i^{(\infty)}(q_j) \to 0$.

We finish the proof by showing that $\lim_{q \to w(\infty)} \varphi_i^{(\infty)}(q) = 0$. Let q_j be a sequence tending to M strictly from the left. As before, we can find subsets P_j such that $q_j \le w(P_j)$ and $\varphi_i^{(\infty)}(q_j) = \varphi_i^{(\infty)}(w(P_j))$. Let $k(j)$ be the minimal $k \notin P_j$. Since $q_j \le w(\infty) - w_{k(j)}$, $k(j) \to \infty$. Therefore we can assume without loss of generality that $k(j) > i$ for all j. Theorem 6.3 implies that

$$\varphi_i^{(\infty)}(q_j) \le \frac{1}{i\binom{i-1}{i-1}} - \sum_{\ell=1}^{k(j)-1-i} \frac{1}{(i+\ell)\binom{i+\ell-1}{i+\ell-2}} = \frac{1}{k(j)-1},$$

using Lemma 6.7. Since $k(j) \to \infty$, $\varphi_i^{(\infty)}(q_j) \to 0$. $\square$

Summarizing, we can extend the functions $\varphi_i^{\boldsymbol{w}|_n}$ to a continuous function $\varphi_i^{(\infty)}$ which agrees with $\varphi_i^{\boldsymbol{w}|_n}$ on the points $w(P)$ for $P \subseteq \{1, \dots, n\}$. When $w_i = 2^{-i}$ then the plot of $\varphi^{(\infty)}$ has no flat areas, but when $w_i = d^{-i}$ for $d > 2$, the limiting function is constant on intervals $(w(P^-), w(P)]$. This is reflected in Figure 3.

6.4 Proof of Theorem 6.3

We start with some preliminary lemmas. For a (possibly infinite) subset P of the positive integers, define

$$\beta_\infty(P) = \sum_{i \in P} 2^{-i}.$$

We have the following analog of Lemma 6.1.

Lemma 6.11. Suppose P_1, P_2 are two subsets of the positive integers. Then $\beta_\infty(P_1) \le \beta_\infty(P_2)$ if and only if $w(P_1) \le w(P_2)$. Furthermore, if $\beta_\infty(P_1) < \beta_\infty(P_2)$ then $w(P_1) < w(P_2)$.

Proof. Suppose that $\beta_\infty(P_1) \le \beta_\infty(P_2)$ and $P_1 \ne P_2$. Let $i = \min(P_2 \setminus P_1)$. Then

$$w(P_2) - w(P_1) \ge w_i - \sum_{j=i+1}^{\infty} w_j \ge 0.$$

Equality is only possible if $\max P_2 = i$ and $P_1 = P_2 \setminus \{i\} \cup \{i+1, \dots, \infty\}$. However, in that case $\beta_\infty(P_1) = \beta_\infty(P_2)$. $\square$

There is a subtlety involved here: we can have $\beta_\infty(P_1) = \beta_\infty(P_2)$ for $P_1 \neq P_2$. This is because dyadic rationals (numbers of the form $\frac{A}{2^B}$) have two different binary expansions. For example, $\frac{1}{2} = (0.1000\ldots)_2 = (0.0111\ldots)_2$. The lemma states (in this case) that $w(\{1\}) \geq w(\{2,3,4,\ldots\})$, but there need not be equality.

In the sequel, we will use the fact that each real $r \in (0,1)$ has a binary expansion with infinitely many 0s (alternatively, a set P such that $\beta_\infty(P) = r$ and there are infinitely many $n \notin P$), and a binary expansion with infinitely many 1s (alternatively, a set P such that $\beta_\infty(P) = r$ and there are infinitely many $n \in P$). If r is not dyadic, then it has a unique binary expansion which has infinitely many 0s and 1s. If r is dyadic, say $r = \frac{1}{2}$, then it has one expansion $(0.1000\ldots)_2$ with infinitely many 0s and another expansion $(0.0111\ldots)_2$ with infinitely many 1s.

The following lemma, which forms the first part of Theorem 6.3, describes the analog of the intervals $(w(P^-), w(P)]$ in the infinite case.

Lemma 6.12. Let $q \in (0, w(\infty))$. There exists a non-empty subset P of the positive integers such that either $q = w(P)$ or $P = \{a_0, \ldots, a_r\}$ is finite and $q \in (w(P^-), w(P)]$, where $P^- = \{a_0, \ldots, a_{r-1}\} \cup \{a_r + 1, \ldots, \infty\}$.

Proof. Since $q < w(\infty)$, for some m we have $q \leq w([m])$. For $n \geq m$, let $A|_n = A|_n(q)$. Let $Q|_n$ be the subset of $[n]$ preceding $A|_n$, and let $R|_n$ be the subset of $[n+1]$ preceding $A|_n$; here "preceding" is in the sense of $X \mapsto X^-$. The interval $(w(Q|_n), w(A|_n)]$ splits into $(w(Q|_n), w(R|_n)] \cup (w(R|_n), w(A|_n)]$, and so $A|_{n+1} \in \{R|_n, A|_n\}$. Also $\beta_\infty(A|_{n+1}) \leq \beta_\infty(A|_n)$, with equality only if $A|_{n+1} = A|_n$.

We consider two cases. The first case is when for some integer M, for all $n \geq M$ we have $A|_n = A = \{a_0, \ldots, a_r\}$. In that case for all $n \geq M$,

$$\sum_{t=0}^{r-1} w_{a_t} + \sum_{t=a_r+1}^{n} w_t < q \leq \sum_{t=0}^{r} w_{a_t},$$

and taking the limit $n \to \infty$ we obtain $q \in (w(A^-), w(A)]$.

The other case is when $A|_n$ never stabilizes. The sequence $\beta_\infty(A|_n)$ is monotonically decreasing, and reaches a limit b satisfying $b < \beta_\infty(A|_n)$ for all n. Since $w(A|_m) \in (w(Q|_n), w(A|_n)]$ for all integers $m \geq n \geq 1$, Lemma 6.11 implies that $b \in [\beta_\infty(Q|_n), \beta_\infty(A|_n))$.

Let L be a subset such that $b = \beta_\infty(L)$ and there are infinitely many $i \notin L$, and define $L|_n = L \cap [n]$. We have $b \in [\beta_\infty(L|_n), \beta_\infty(L|_n) + 2^{-n})$. Therefore $Q|_n = L|_n$, and so $q > w(Q|_n) = w(L|_n)$. Taking the limit $n \to \infty$, we deduce that $q \geq w(L)$.

If $n \notin L$ then $A|_n = Q|_n \cup \{n\}$, and so $q \leq w(A|_n) = w(L|_n) + w_n$. Since there are infinitely many such n, taking the limit $n \to \infty$ we conclude that $q \leq w(L)$ and so $q = w(L)$. ☐

We can now give an explicit formula for $\varphi_i^{(\infty)}$.

Theorem 6.3. Let $q \in (0, w(\infty))$ and let i be a positive integer.

(a) There exists a non-empty subset of the positive integers $P = \{a_0, \ldots, a_r\}$ such that either $q = w(P)$ or P is finite and $q \in (w(P^-), w(P)]$, where $P^- = \{a_0, \ldots, a_{r-1}\} \cup \{a_r + 1, \ldots, \infty\}$.

(b) The limit $\varphi_i^{(\infty)}(q) = \lim_{n\to\infty} \varphi_i^{\boldsymbol{w}|_n}(q)$ exists. When $i \notin P$,

$$\varphi_i^{(\infty)}(q) = \sum_{\substack{t \in \{0,\ldots,r\}:\\ a_t > i}} \frac{1}{a_t \binom{a_t - 1}{t}},$$

and when $i \in P$, say $i = a_s$, then

$$\varphi_i^{(\infty)}(q) = \frac{1}{a_s \binom{a_s - 1}{s}} - \sum_{\substack{t \in \{0,\ldots,r\}:\\ a_t > i}} \frac{1}{a_t \binom{a_t - 1}{t-1}}.$$

We comment that the convergence of the sums in the theorem is guaranteed by Lemma 6.7.

Proof. The first part has been proved as Lemma 6.12, and it remains to prove the second part.

Suppose first that P is finite P and either $q = w(P)$ or $q \in (w(P^-), w(P)]$. For all $n \geq \max P$, $P|_n(q) = P$, and so Lemma 6.9 shows that $\varphi_i^{\boldsymbol{w}|_n}(q) = \varphi_i^{\boldsymbol{w}|_{\max P}}(q)$. Therefore the limit exists and equals the stated formula, which is the same as the one given by Theorem 6.1.

Suppose next that P is infinite and $q = w(P)$. Consider first the case in which we can also write $q = w(Q)$ for some finite Q, say $Q = \{q_0, \ldots, q_u\}$. Then $P = \{q_0, \ldots, q_{u-1}\} \cup \{q_u + 1, q_u + 2, \ldots, \infty\}$. We now consider several cases.

If $i < q_u$ and $i \notin P$ then $i \notin Q$ and

$$\varphi_i^{(\infty)}(q) = \sum_{\substack{t \in \{0,\ldots,u\}: \\ q_t > i}} \frac{1}{q_t \binom{q_t - 1}{t}} = \sum_{\substack{t \in \{0,\ldots,u-1\}: \\ q_t > i}} \frac{1}{q_t \binom{q_t - 1}{t}} + \sum_{\ell=1}^{\infty} \frac{1}{(q_u + \ell)\binom{q_u + \ell - 1}{t + \ell - 1}},$$

using Lemma 6.7. The right-hand side is the expression we gave for $\varphi_i^{(\infty)}(w(P))$.

If $i < q_u$ and $i \in P$, say $i = q_s$, then $i \in Q$ and

$$\varphi_i^{(\infty)}(q) = \frac{1}{i\binom{i-1}{s}} - \sum_{\substack{t \in \{0,\ldots,u\}: \\ q_t > i}} \frac{1}{q_t \binom{q_t - 1}{t}} = \frac{1}{i\binom{i-1}{s}} - \sum_{\substack{t \in \{0,\ldots,u-1\}: \\ q_t > i}} \frac{1}{q_t \binom{q_t - 1}{t}} - \sum_{\ell=1}^{\infty} \frac{1}{(q_u + \ell)\binom{q_u + \ell - 1}{t + \ell - 1}},$$

using Lemma 6.7. The right-hand side is the expression we gave for $\varphi_i^{(\infty)}(w(P))$.

If $i = q_u$ then $i \in Q$ and $i \notin P$. In that case

$$\varphi_i^{(\infty)}(q) = \frac{1}{i\binom{i-1}{u}} = \sum_{\ell=1}^{\infty} \frac{1}{(i + \ell)\binom{i + \ell - 1}{u + \ell - 1}},$$

using Lemma 6.7. The right-hand side is the expression we gave for $\varphi_i^{(\infty)}(w(P))$.

Finally, if $i > q_u$ then $i \notin Q$ and $i \in P$. Suppose that i is the vth member in P. In that case

$$\varphi_i^{(\infty)}(q) = 0 = \frac{1}{i\binom{i-1}{v}} - \sum_{\ell=1}^{\infty} \frac{1}{(i + \ell)\binom{i + \ell - 1}{v + \ell - 1}},$$

using Lemma 6.7. The right-hand side is the expression we gave for $\varphi_i^{(\infty)}(w(P))$.

It remains to consider the case in which q cannot be written as $q = w(Q)$ for finite Q. In that case, there are infinitely many positive integers n such that $n \in P$ and infinitely many such that $n \notin P$. This implies that for every positive integer n, $q \in (w(P \cap [n]), w(P \cap [n]) + w_n)$, and so $P|_n^-(q) = P \cap [n]$. Lemma 6.8 shows that $|\varphi_n(q) - \varphi_n(w(P \cap [n]))| \leq \frac{1}{n}$. On the other hand, Theorem 6.1 readily implies that $\varphi_n(w(P \cap [n]))$ tends to the expression we gave for $\varphi_i^{(\infty)}(w(P))$. We conclude that $\varphi_n(q)$ tends to the same expression. ☐

7 Independent and Identical Samples

Up to this point, we have been studying cases where the agent weights are sampled from discrete distributions. In this section, we slightly change our direction, by considering the case where the agent weights are independently sampled from an identical continuous distribution $\mathcal{D}$ that is reasonably bounded in a way that would be made precise momentarily. An especially interesting example of such a distribution is the case where all of the weights are uniformly sampled from the interval $[0, 1]$. We remind the reader that $U(a, b)$ signifies the uniform distribution over the interval $[a, b]$.

In order to reason about the distribution of the Shapley values, we again take a probabilistic approach by estimating the probability that a given agent is pivotal. In order to estimate the Shapley value of the highest-weighted agent n, it is not hard to see that an answer to the following question would prove instrumental:

Conditioning on them being less than or equal to the highest weight w_n, let $Y_1, \ldots, Y_{n-1} \sim \mathcal{D}$, and define $S_i = \sum_{i=1}^{i} Y_i$. What is the expected number of points from $\{S_1, \ldots, S_{n-1}\}$ that lie in the interval $[q - w_n, q)$?

A symmetric question can be phrased for the lowest-weighted agent as well. Put in these terms, we can think of the process that generates the $n - 1$ weights (conditioning on the highest weight) as a renewal process, where the

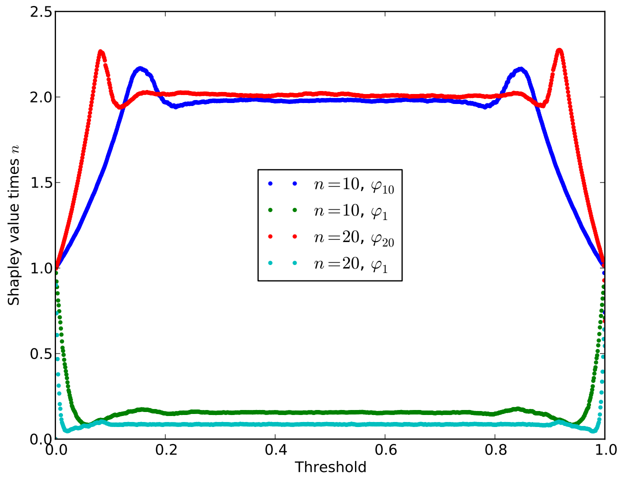


Figure 4: Shapley values for $X = U([0, 1])$ and $n = 10, 20$ of both minimal and maximal player, normalized by n. Results of 10^6 experiments.

inter-arrival times are given by the agent weights, and the measure in question is the expected number of arrivals within the specified interval. Our analysis gives a powerful characterization of the Shapley values of both the highest- and lowest-weighted agents.

Before we begin to analyze the Shapley values, we consider the special case of the uniform distribution, for which our simulation results are depicted in Figure 4. Intuitively, we can see that apart from two relatively short intervals at the two extremes of the interval $[0, 1]$, the Shapley values are stable at $2/n$ for the highest Shapley value, and roughly $\Theta(1/n^2)$ (this asymptotic term will be justified momentarily) for the lowest Shapley value. From a more practical point of view, this means that as the number of players increases, the ratio of the highest to lowest Shapley values grows at a *linear* rate.

Given the above results, we proceed to a rigorous analysis of the two extreme Shapley values. We use the following notation in this section: for a random variable Z and real number t, $Z_{\leq t}$ is the distribution of Z conditioned on being at most t, and $Z_{\geq t}$ is the distribution of Z conditioned on being at least t. Furthermore, given an integer n, we let the random variable $X^n_{\max}$ ($X^n_{\min}$) denote the n'th (first) order statistic in an experiment involving n i.i.d. samples of X.

Let X be a continuous random variable supported on $[\chi_{\min}, \chi_{\max}]$, where $\chi_{\min} \geq 0$, with bounded density. We further assume that for small enough ε, $\Pr[X > \chi_{\min} + \varepsilon] > 0$ and $\Pr[X < \chi_{\max} - \varepsilon] < 0$, which just means that we chose the "tight" $\chi_{\min}, \chi_{\max}$.

Consider the following random process for generating weights: sample $x_1, \ldots, x_n$ from X independently, and let $S_n = \sum_{i=1}^n x_i$. We define the weights $w_1, \ldots, w_n$ to be the sequence obtained from $x_1/S_n, \ldots, x_n/S_n$ by sorting them in non-decreasing order, i.e., w_1 is the smallest weight and w_n is the largest weight.

The following is our main theorem. We defer its proof to Section 8 in favor of discussing a couple of its implications.

Theorem 7.1. For all $m \in (\chi_{\min}, \chi_{\max})$ there exist $\psi < 1$ and $C > 0$ such that the following holds.
For all $q \in [(1 + Cn^{-1/3})\chi_{\max}/(n\,\mathbb{E}[X]), (1 - Cn^{-1/3})\,\mathbb{E}[X_{\leq m}]/\,\mathbb{E}[X]]$,

$$\mathbb{E}[\varphi_n(q)] = \frac{1}{n} \mathop{\mathbb{E}}_{x \sim (X^n_{\max})_{\geq m}} \left[\frac{x}{\mathbb{E}[X_{\leq x}]} \right] \pm O(n\psi^{n^{1/3}} + \psi^{nq}).$$

Similarly, for all $q \in [(1 + Cn^{-1/3})\chi_{\max}/(n\,\mathbb{E}[X]), 1 - Cn^{-1/3}]$,

$$\mathbb{E}[\varphi_1(q)] = \frac{1}{n} \mathop{\mathbb{E}}_{x \sim X^n_{\min}} \left[\frac{x}{\mathbb{E}[X_{\geq x}]} \right] \pm O(n\psi^{n^{1/3}} + \psi^{nq}).$$

In particular, we can determine the limiting values of $n\,\mathbb{E}[\varphi_n(q)]$ and $n\,\mathbb{E}[\varphi_1(q)]$.

Corollary 7.1. *Suppose that $q \in (0,1)$. Then*

$$\lim_{n\to\infty} n\,\mathbb{E}[\varphi_n(q)] = \frac{\chi_{\max}}{\mathbb{E}[X]},$$
$$\lim_{n\to\infty} n\,\mathbb{E}[\varphi_1(q)] = \frac{\chi_{\min}}{\mathbb{E}[X]}.$$

Proof. Take $m = (\chi_{\max} + q)/2 > q$ in the theorem. The function $x/\mathbb{E}[X_{\le x}]$ is continuous in x, and tends to the limit $\chi_{\max}/\mathbb{E}[X]$ as x tends to $\chi_{\max}$. Since $X^n_{\max}$ tends to the constant distribution $\chi_{\max}$ as n grows, we obtain the first formula. The proof of the second formula is similar (this time any m would do). □

Corollary 7.1 tells us that the uniform case, i.e., $X \sim U(0,1)$, for any $q \in (0,1)$, $n\varphi_n(q) \to 2$ (using the fact that $\mathbb{E}[X] = 1/2$), while $n\varphi_1(q) \to 0$. Moreover, using Theorem 7.1, we can obtain the following result for the case of a uniform distribution:

Theorem 7.2. For all $m \in (0,1)$ there exist $\psi < 1$ and $C > 0$ such that the following holds.
For all $q \in [(1 + Cn^{-1/3})(2/n), (1 - Cn^{-1/3})m]$,

$$\mathbb{E}[\varphi_n(q)] = \frac{2}{n} \pm O(n\psi^{n^{1/3}} + \psi^{nt}).$$

Similarly, for all $q \in [(1 + Cn^{-1/3})(2/n), 1 - Cn^{-1/3}]$,

$$\mathbb{E}[\varphi_1(q)] = 2\int_0^1 \frac{x(1-x)^{n-1}}{x+1}\,\mathrm{d}x \pm O(n\psi^{n^{1/3}} + \psi^{nt}),$$

where the integral lies in the range

$$\frac{2}{(n+1)(n+2)} < 2\int_0^1 \frac{x(1-x)^{n-1}}{x+1} < \frac{2}{n(n+1)}.$$

The proof is given in Section 8.2. As remarked above, the proof can be adjusted to extend the range of q in the formula for $\varphi_n(q)$. Figure 4 illustrates Theorem 7.2. The (rather technical) proof can be found in Section 8.

8 Proving Theorem 7.1

Recall the statement of Theorem 7.1:

Theorem 7.1. For all $m \in (\chi_{\min}, \chi_{\max})$ there exist $\psi < 1$ and $C > 0$ such that the following holds.
For all $q \in [(1 + Cn^{-1/3})\chi_{\max}/(n\,\mathbb{E}[X]), (1 - Cn^{-1/3})\,\mathbb{E}[X_{\le m}]/\mathbb{E}[X]]$,

$$\mathbb{E}[\varphi_n(q)] = \frac{1}{n} \mathop{\mathbb{E}}_{x\sim(X^n_{\max})_{\ge m}} \left[\frac{x}{\mathbb{E}[X_{\le x}]}\right] \pm O(n\psi^{n^{1/3}} + \psi^{nq}).$$

Similarly, for all $q \in [(1 + Cn^{-1/3})\chi_{\max}/(n\,\mathbb{E}[X]), 1 - Cn^{-1/3}]$,

$$\mathbb{E}[\varphi_1(q)] = \frac{1}{n} \mathop{\mathbb{E}}_{x\sim X^n_{\min}} \left[\frac{x}{\mathbb{E}[X_{\ge x}]}\right] \pm O(n\psi^{n^{1/3}} + \psi^{nq}).$$

The crux of the proof is the following formula for the Shapley values of the original sequence $x_1, \ldots, x_n$. We start with several definitions, which depend on an implicit parameter n:

- $x_{\max} = \max(x_1, \ldots, x_n)$, the corresponding distribution is $X^n_{\max}$, and the corresponding Shapley value (with respect to $x_1, \ldots, x_n$) is $\varphi^{[\mathrm{x}]}_{\max}$.
- $x_{\min} = \min(x_1, \ldots, x_n)$, the corresponding distribution is $X^n_{\min}$, and the corresponding Shapley value (with respect to $x_1, \ldots, x_n$) is $\varphi^{[\mathrm{x}]}_{\min}$.

Lemma 8.1. For any quota value Q,

$$\mathbb{E}[\varphi^{[x]}_{\max}(Q)] = \mathop{\mathbb{E}}_{x\sim X^n_{\max}}\left[\frac{1}{n}\sum_{i=1}^{n}\Pr_{y_1,\ldots,y_{n-1}\sim X_{\le x}}\left[\sum_{j=1}^{i-1} y_j \in [Q-x,Q)\right]\right], \tag{4}$$

$$\mathbb{E}[\varphi^{[x]}_{\min}(Q)] = \mathop{\mathbb{E}}_{x\sim X^n_{\min}}\left[\frac{1}{n}\sum_{i=1}^{n}\Pr_{y_1,\ldots,y_{n-1}\sim X_{\ge x}}\left[\sum_{j=1}^{i-1} y_j \in [Q-x,Q)\right]\right]. \tag{5}$$

Proof. The proofs of both formulas are similar, so we only prove the first one. We show that conditioned on $x_{\max} = x$,

$$\mathbb{E}[\varphi^{[x]}_{\max}(Q)] = \frac{1}{n}\Pr_{y_1,\ldots,y_{n-1}\sim X_{\le x}}\left[\sum_{j=1}^{i-1} y_j \in [Q-x,Q)\right].$$

We can assume without loss of generality that $x_{\max} = x_n$. Given only this data, the variables $x_1,\ldots,x_{n-1}$ are distributed independently according to $X_{\le x_n}$. Therefore

$$\begin{aligned}
\mathbb{E}[\varphi^{[x]}_{n}(Q)] &= \mathop{\mathbb{E}}_{\pi\in S_n}[x_n \text{ is pivotal in } x_{\pi_1},\ldots,x_{\pi_n}] \\
&= \frac{1}{n}\sum_{i=1}^{n}\mathop{\mathbb{E}}_{\substack{\pi\in S_n:\\ \pi_i=n}} \Pr[x_n \text{ is pivotal in } x_{\pi_1},\ldots,x_{\pi_n}] \\
&= \frac{1}{n}\sum_{i=1}^{n}\mathop{\mathbb{E}}_{\substack{\pi\in S_n:\\ \pi_i=n}} \Pr\left[\sum_{j=1}^{i-1} x_{\pi_j} \in [Q-x,Q)\right].
\end{aligned}$$

Here *pivotal* is always with respect to the threshold Q. Since $x_1,\ldots,x_{n-1}$ are independent and identically distributed, $x_{\pi_1},\ldots,x_{\pi_{i-1}}$ are distributed identically to $y_1,\ldots,y_{i-1}$, proving the first part of the theorem. ☐

8.1 Estimating the formulas

Recall that our main approach is to use an analogy to renewal processes, in which each of the agent weights can be thought of as renewal 'steps' and that furthermore, estimating the expected number of points that land within the interval $[q-w_n,q)$ will be used for proving the formulas for the highest Shapley value (and similarly for the lowest Shapley value).

The first step towards achieving this goal is to extend the sums in Lemma 8.1 to infinite sums. Estimating these infinite sums will be done using the following Lemma, which is relevant to renewal processes with general, bounded renewal time distributions.

Proposition 8.1 (Soundararajan)**.** *Suppose Y is a continuous distribution supported on $[\chi_{\min},\chi_{\max}]$ (where $\chi_{\min}\ge 0$) whose density is bounded by C. Then for $Q\ge 0$,*

$$\sum_{i=1}^{\infty}\Pr_{y_1,\ldots,y_{i-1}\sim Y}\left[\sum_{j=1}^{i-1} y_j < Q\right] = \frac{Q}{\mathbb{E}\,Y} + \frac{\mathbb{E}(Y^2)}{2(\mathbb{E}\,Y)^2} \pm O(e^{-\gamma Q}),$$

where $\gamma > 0$ and the constant in $O(\cdot)$ depend only on the parameters $\chi_{\min},\chi_{\max},C$.

For completeness, we provide the proof of Proposition 8.1 in Section 9.

In order to utilize this proposition for the estimation the sums in Lemma 8.1, we need to restrict the value of x in $X_{\ge x}$ and $X_{\le x}$. For $m\in(\chi_{\min},\chi_{\max})$, we say that Y is an *m-reasonable* random variable if either $Y = X_{\ge x}$ for $x\le m$ or $Y = X_{\le x}$ for $x \ge m$.

Corollary 8.2. *Let Y be an m-reasonable random variable, for some $m\in(\chi_{\min},\chi_{\max})$. Then for some $\gamma<1$ depending only on m and for all $Q\ge 0$,*

$$\sum_{i=1}^{\infty}\Pr_{y_1,\ldots,y_{i-1}\sim Y}\left[\sum_{j=1}^{i-1} y_j < Q\right] = \frac{Q}{\mathbb{E}\,Y} + \frac{\mathbb{E}(Y^2)}{2(\mathbb{E}\,Y)^2} \pm O(\gamma^{Q}).$$

In particular, for all $x \in [\chi_{\min}, \chi_{\max}]$ *and* $Q \geq x$,

$$\sum_{i=1}^{\infty} \Pr_{y_1,\dots,y_{i-1}\sim Y}\left[\sum_{j=1}^{i-1} y_j \in [Q-x, Q)\right] = \frac{x}{\mathbb{E}\, Y} \pm O(\gamma^Q).$$

Proof. Let $\mu = \min(\Pr[X \leq m], \Pr[X \geq m]) > 0$. Denote the density of X by f and the density of Y by g. By assumption f is bounded by some constant D. Clearly Y is supported on $[\chi_{\min}, \chi_{\max}]$. We claim that furthermore, g is bounded by $C = D/\mu$. Indeed, if $Y = X_{\geq x}$ then for $y \in [x, \chi_{\max}]$ we have $g(y) = f(y)/\Pr[X \geq x] \leq f(y)/\mu$ since $x \leq m$. A similar argument applies for $Y = X_{\leq x}$. Since m is fixed, so is μ, and therefore also $C = D/\mu$. The first statement of the corollary thus follows directly from the proposition.

The first statement implies that for all $x \in [\chi_{\min}, \chi_{\max}]$ and $Q \geq x$,

$$\sum_{i=1}^{\infty} \Pr_{y_1,\dots,y_{i-1}\sim Y}\left[\sum_{j=1}^{i-1} y_j \in [Q-x, Q)\right] = \frac{x}{\mathbb{E}\, Y} \pm O(\gamma^Q + \gamma^{Q-x}).$$

Since $\gamma^{Q-x} \leq \gamma^{Q-\chi_{\max}}$, and χ_{max} is fixed, we obtain the corollary. □

The corollary affords us with a good estimate of the sums in Lemma 8.1, when extended from n to ∞. In order to estimate the actual sums, we estimate the tail from $n+1$ to ∞.

Lemma 8.2. Let Y be an m-reasonable random variable, for some $m \in (\chi_{\min}, \chi_{\max})$. For some $\delta < 1$ depending only on m and for all $Q \leq (n - n^{2/3})\, \mathbb{E}\, Y$,

$$\sum_{i=n+1}^{\infty} \Pr_{y_1,\dots,y_{i-1}\sim Y}\left[\sum_{j=1}^{i-1} y_j < Q\right] = O(n\delta^{n^{1/3}}),$$

where the constant in $O(\cdot)$ depends only on m.

Proof. Since Y is m-reasonable, it is bounded by $\chi_{\max}$, and its variance is bounded by $\chi^2_{\max}$. Bernstein's inequality states that

$$\Pr\left[\sum_{j=1}^{i} y_j < i\,\mathbb{E}[Y] - t\right] \leq \exp\Big(-\frac{t^2/2}{i\mathrm{Var}[Y]^2 + t\chi_{\max}/3}\Big) \leq \exp\Big(-\frac{t^2/2}{i\chi^2_{\max} + t\chi_{\max}/3}\Big).$$

As we will later plug $i\,\mathbb{E}[Y] - t = Q$, we will assume that $i\,\mathbb{E}[Y] - t \geq 0$,and so $t \leq i\,\mathbb{E}[Y] \leq i\chi_{\max}$. [2] Under assumption,

$$\Pr\left[\sum_{j=1}^{i} y_j < i\,\mathbb{E}[Y] - t\right] \leq c^{t^2/i}, \quad c = \exp\Big(-\frac{3/8}{\chi^2_{\max}} < 1\Big).$$

Therefore

$$\begin{aligned}\sum_{i=n+1}^{\infty} \Pr_{y_1,\dots,y_{i-1}\sim Y}\left[\sum_{j=1}^{i-1} y_j < Q\right] &= \sum_{k=0}^{\infty} \Pr_{y_1,\dots,y_{n+k}\sim Y}\left[\sum_{j=1}^{n+k} y_j < Q\right] \\ &\leq \sum_{k=0}^{\infty} c^{((n+k)\,\mathbb{E}[Y]-Q)^2/(n+k)} \leq \sum_{k=0}^{\infty} (c^{(\mathbb{E}[Y])^2})^{(k+n^{2/3})^2/(n+k)}.\end{aligned}$$

where the last inequality follows from the upper-bound $Q \leq (n - n^{2/3})\,\mathbb{E}[Y]$.

Since Y is m-reasonable, $\mathbb{E}[Y] \geq \mathbb{E}[X_{\leq m}] > 0$, and so $c^{(\mathbb{E}[Y])^2} \leq d$ for $d = c^{(\mathbb{E}[X_{\leq m}])^2} < 1$. We can crudely bound the infinite series:

$$\begin{aligned}\sum_{k=0}^{\infty} d^{(n^{2/3}+k)^2/(n+k)} &\leq \sum_{k=0}^{n-1} d^{n^{4/3}/(n+k)} + \sum_{k=n}^{\infty} d^{k^2/(n+k)} \\ &\leq nd^{n^{1/3}/2} + \sum_{k=n}^{\infty} d^{k/2} = nd^{n^{1/3}/2} + \frac{d^{n/2}}{1-\sqrt{d}}.\end{aligned}$$

[2] If $Q < 0$ the events in question happen with zero probability.

This implies the lemma with $\delta = \sqrt{d}$. $\square$

Combining this with Corollary 8.2, we obtain the following estimate.

Corollary 8.3. *Let Y be an m-reasonable random variable, for some $m \in (\chi_{\min}, \chi_{\max})$. Then for some $\zeta < 1$ depending only on m, for all $x \in [\chi_{\min}, \chi_{\max}]$ and for all $Q \in [x, (n - n^{2/3}) \mathbb{E}[Y]]$,*

$$\sum_{i=1}^{n} \Pr_{y_1,\ldots,y_{n-1} \sim X_{\le x}} \left[\sum_{j=1}^{i-1} y_j \in [Q-x, Q) \right] = \frac{x}{\mathbb{E}[Y]} \pm O(n\zeta^{n^{1/3}} + \zeta^Q).$$

Proof. Clearly

$$\sum_{i=n+1}^{\infty} \Pr_{y_1,\ldots,y_{i-1} \sim X_{\le x}} \left[\sum_{j=1}^{i-1} y_j \in [Q-x, Q) \right] \le \sum_{i=n+1}^{\infty} \Pr_{y_1,\ldots,y_{i-1} \sim X_{\le x}} \left[\sum_{j=1}^{i-1} y_j < Q \right] = O(n\delta^{n^{1/3}}),$$

using Lemma 8.2. Therefore Corollary 8.2 implies that

$$\sum_{i=1}^{n} \Pr_{y_1,\ldots,y_{n-1} \sim X_{\le x}} \left[\sum_{j=1}^{i-1} y_j \in [Q-x, Q) \right] = \frac{x}{\mathbb{E}[Y]} \pm O(n\delta^{n^{1/3}} + \gamma^Q).$$

The Corollary follows by taking $\zeta = \max(\delta, \gamma)$. $\square$

Using this estimate, we can estimate the sums in Lemma 8.1. The idea is to focus on the case in which the variable $X_{\le x}$ or $X_{\ge x}$ is m-reasonable.

Lemma 8.3. Let $m \in (\chi_{\min}, \chi_{\max})$. For some $\xi < 1$ depending on m and for all $Q \in [\chi_{\max}, (n-n^{2/3}) \mathbb{E}[X_{\le m}]]$,

$$\varphi^{[x]}_{\max}(Q) = \frac{1}{n} \mathop{\mathbb{E}}_{x \sim (X^n_{\max})_{\ge m}} \left[\frac{x}{\mathbb{E}[X_{\le x}]}\right] + O(n\xi^{n^{1/3}} + \xi^Q).$$

Similarly, for all $Q \in [\chi_{\max}, (n - n^{2/3}) \mathbb{E}[X]]$,

$$\varphi^{[x]}_{\min}(Q) = \frac{1}{n} \mathop{\mathbb{E}}_{x \sim X^n_{\min}} \left[\frac{x}{\mathbb{E}[X_{\ge x}]}\right] + O(n\xi^{n^{1/3}} + \xi^Q).$$

Proof. We start with the first formula. Let $\mu_{\le} = \Pr[X \le m]$ and $\mu_{\ge} = \Pr[X \ge m] = 1 - \mu_{\le}$. Clearly $\Pr[X^n_{\max} \le m] = \mu^n_{\le}$. Therefore (using Lemma 8.1)

$$\varphi^{[x]}_{\max}(Q) = \mathop{\mathbb{E}}_{x \sim (X^n_{\max})_{\ge m}} \left[\frac{1}{n} \sum_{i=1}^{n} \Pr_{y_1,\ldots,y_{n-1} \sim X_{\le x}} \left[\sum_{j=1}^{i-1} y_j \in [Q-x, Q) \right] \right] \pm O(\mu^n_{\le}).$$

Corollary 8.3 implies that for all $Q \in [\chi_{\max}, (n - n^{2/3}) \mathbb{E}[X_{\le m}]]$,

$$\varphi^{[x]}_{\max}(Q) = \frac{1}{n} \mathop{\mathbb{E}}_{x \sim (X^n_{\max})_{\ge m}} \left[\frac{x}{\mathbb{E}[X_{\le x}]}\right] \pm O(n\zeta^{n^{1/3}} + \zeta^Q + \mu^n_{\le}).$$

This implies the formula in the statement of the lemma, with $\xi = \max(\zeta, \mu_{\le}, \mu_{\ge})$ (we need $\mu_{\ge}$ for the other part of the lemma).

We continue with the second formula. As before, we have

$$\varphi^{[x]}_{\min}(Q) = \mathop{\mathbb{E}}_{x \sim (X^n_{\min})_{\le m}} \left[\frac{1}{n} \sum_{i=1}^{n} \Pr_{y_1,\ldots,y_{n-1} \sim X_{\ge x}} \left[\sum_{j=1}^{i-1} y_j \in [Q-x, Q) \right] \right] \pm O(\mu^n_{\ge}).$$

Corollary 8.3 implies that for all $Q \in [\chi_{\max}, (n - n^{2/3}) \mathbb{E}[X]]$,

$$\varphi^{[x]}_{\min}(Q) = \frac{1}{n} \mathop{\mathbb{E}}_{x \sim (X^n_{\min})_{\le m}} \left[\frac{x}{\mathbb{E}[X_{\ge x}]}\right] \pm O(n\zeta^{n^{1/3}} + \zeta^Q + \mu^n_{\ge}) \tag{6}$$

$$\tag{7}$$

Now, as by definition $Pr[X^n_{\min} \geq m] = \mu^n_{\geq}$, we have that

$$\begin{aligned}
\mathop{\mathbb{E}}_{x\sim(X^n_{\min})}\big[\frac{x}{\mathbb{E}[X_{\geq x}]}\big] &= \mu^n_{\geq}\cdot \mathop{\mathbb{E}}_{x\sim(X^n_{\min})_{\geq m}}\big[\frac{x}{\mathbb{E}[X_{\geq x}]}\big] + (1-\mu^n_{\geq})\cdot \mathop{\mathbb{E}}_{x\sim(X^n_{\min})_x}\big[\frac{x}{\mathbb{E}[X_{\geq x}]}\big] \\
&= \mathop{\mathbb{E}}_{x\sim(X^n_{\min})_{\leq m}} \frac{x}{\mathbb{E}[X_{\geq x}]} + \mu^n_{\geq x}\cdot\big(\mathop{\mathbb{E}}_{x\sim(X^n_{\min})_{\geq}}\big[\frac{x}{\mathbb{E}[X_{\geq x}]}\big] - \mathop{\mathbb{E}}_{x\sim(X^n_{\min})_{\leq m}}\big[\frac{x}{\mathbb{E}[X_{\geq x}]}\big]\big) \\
&= \mathop{\mathbb{E}}_{x\sim(X^n_{\min})_{\leq m}} \frac{x}{\mathbb{E}[X_{\geq x}]} + O(\mu^n_{\geq x}) \qquad (8)
\end{aligned}$$

where the last equality follows from the fact that $x/(\mathbb{E}\,X_{\geq x}) \leq 1$. Combining this with Eq. 6, gives the second formula in the statement of the lemma. □

We can now prove the main theorem of this section, using a concentration bound on S.

Proof of Theorem 7.1. Recall that $S = \sum_{i=1}^n x_i$. Bernstein's inequality shows that

$$\Pr[|S - n\,\mathbb{E}\,X| \geq n^{2/3}\,\mathbb{E}\,X] \leq \exp -\frac{n^{4/3}(\mathbb{E}\,X)^2/2}{n\mathrm{Var}X + n^{2/3}\chi_{\max}\,\mathbb{E}\,X/3} = e^{-O(n^{1/3})}.$$

Suppose first that S is within these bounds, and let $Q = qS$. Using $(1/t)' = -1/t^2$, we deduce

$$q = \frac{Q}{n\,\mathbb{E}\,X} \pm O\left(\frac{n^{2/3}\,\mathbb{E}\,X}{n^2(\mathbb{E}\,X)^2}\right) = \frac{Q}{n\,\mathbb{E}\,X} \pm O\left(\frac{n^{-1/3}}{n\,\mathbb{E}\,X}\right).$$

An elementary calculation shows that the conditions on q imply the conditions stated in Lemma 8.3 for Q. Applying the lemma, we obtain the estimates appearing in the statement of the theorem; the error terms are

$$\pm O(n\xi^{n^{1/3}} + \xi^Q) = O(n\xi^{n^{1/3}} + \xi^{\Omega(\mathbb{E}\,Xnq)}).$$

Setting $\psi = \max(\xi, \xi^{\Omega(\mathbb{E}\,X)})$ (the same constant in the $\Omega(\cdot)$ as in the display), we obtain the error terms in the theorem. These estimates, however, are only true as long as S is within the stated bounds. Accounting for the small probability $e^{-O(n^{1/3})}$ that S is out of bounds results in an additional error term $O(e^{-O(n^{1/3})})$, which can be swallowed into the error term $O(n\psi^{n^{1/3}})$ by possibly increasing ψ. □

8.2 Proving Theorem 7.2

Recall the statement of the theorem:

Theorem 7.2. For all $m \in (0,1)$ there exist $\psi < 1$ and $C > 0$ such that the following holds.
For all $q \in [(1 + Cn^{-1/3})(2/n), (1 - Cn^{-1/3})m]$,

$$\mathbb{E}[\varphi_n(q)] = \frac{2}{n} \pm O(n\psi^{n^{1/3}} + \psi^{nt}).$$

Similarly, for all $q \in [(1 + Cn^{-1/3})(2/n), 1 - Cn^{-1/3}]$,

$$\mathbb{E}[\varphi_1(q)] = 2\int_0^1 \frac{x(1-x)^{n-1}}{x+1}\,\mathrm{d}x \pm O(n\psi^{n^{1/3}} + \psi^{nt}),$$

where the integral lies in the range

$$\frac{2}{(n+1)(n+2)} < 2\int_0^1 \frac{x(1-x)^{n-1}}{x+1} < \frac{2}{n(n+1)}.$$

Proof. We have $\chi_{\min} = 0$, $\chi_{\max} = 1$, $\mathbb{E}[X] = 1/2$, and $\mathbb{E}[X_{\leq m}] = m/2$. This explains the ranges of q. The formula $\varphi_n(q)$ follows from

$$\mathop{\mathbb{E}}_{x\sim(X^n_{\max})_{\geq m}}[\frac{x}{\mathbb{E}[X_{\leq x}]}] = \mathop{\mathbb{E}}_{x\sim(X^n_{\max})_{\geq m}}[2] = 2.$$

It remains to compute the formula for $\varphi_1(q)$. Since $\Pr[X^n_{\min} \geq x] = (1-x)^n$, the density of $X^n_{\min}$ is $n(1-x)^{n-1}$. Therefore

$$\mathop{\mathbb{E}}_{x\sim X^n_{\min}}[\frac{x}{\mathbb{E}[X_{\geq x}]}] = \mathop{\mathbb{E}}_{x\sim X^n_{\min}}[\frac{2x}{1+x}] = \int_0^1 \frac{2nx(1-x)^{n-1}}{1+x}\,\mathrm{d}x.$$

We deduce the given formula for $\varphi_1(q)$. In order to estimate the integral, we use the well-known formula $\int_0^1 x^a(1-x)^b\,\mathrm{d}x = 1/(a+b+1)\binom{a+b}{a}$. Using this formula,

$$\int_0^1 \frac{2x(1-x)^{n-1}}{1+x}\,\mathrm{d}x = \sum_{k=0}^{\infty}(-1)^k \int_0^1 2x^{k+1}(1-x)^{n-1}\,\mathrm{d}x = \sum_{k=0}^{\infty}(-1)^k \frac{2}{(n+k+1)\binom{n+k}{k+1}}.$$

where the first equality follows from the expansion: $\frac{1}{1+x} = \sum_{i=0}^{\infty}(-1)^k x^k$, for $x \in [0,1)$.

Since $\binom{n+k}{k+1} = \binom{n+k}{n-1}$, the expressions in the denominators are increasing, and so we have

$$\frac{2}{(n+2)(n+1)} = \frac{2}{(n+1)n} - \frac{4}{(n+2)(n+1)n} < \int_0^1 \frac{2x(1-x)^{n-1}}{1+x}\,\mathrm{d}x < \frac{2}{(n+1)n}.$$

This completes the proof. □

9 Proving Proposition 8.1

In this section, we complete the proof of Theorem 7.1 by proving Proposition 8.1.

The idea of the proof is to use the Mellin transform to write

$$m(Q) := \sum_{i=1}^{\infty} \Pr_{y_1,\dots,y_{i-1}\sim Y}\left[\sum_{j=1}^{i-1} y_j < Q\right] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{sQ}}{s(1-\mathbb{E}[e^{-sY}])}\,ds,$$

where $c > 0$ is arbitrary. The integrand has a double pole at $s = 0$ with residue $Q/\mathbb{E}\,Y + \mathbb{E}(Y^2)/2(\mathbb{E}\,Y)^2$, which gives rise to the main term in the proposition. The conditions on the distribution Y imply that apart from the pole at $s = 0$, the integrand has no poles in some halfspace $\Re s > -\gamma$. Therefore we can move the line of integration to the left toward $-\gamma$, and then read off the error term. The exponential dependence comes from the numerator e^{sQ}.

In the rest of this section, whenever we use the term “constant”, we mean a constant depending on the parameters $\chi_{\min}, \chi_{\max}, C$.

We start by proving the formula for $m(Q)$.

Lemma 9.1. For all $c > 0$,

$$m(Q) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{sQ}}{s(1-\mathbb{E}[e^{-sY}])}\,ds.$$

Proof. It is well-known that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{sx}}{s}\,ds = \begin{cases} 1 & x > 0, \\ 0 & x < 0. \end{cases}$$

Therefore, letting $(y_i)_1^\infty \sim Y$,

$$\begin{aligned}
m(Q) &= \sum_{i=1}^{\infty} \Pr_{y_1,\dots,y_{i-1}\sim Y}\left[\sum_{j=1}^{i-1} y_j < Q\right] \\
&= \sum_{i=1}^{\infty} \mathbb{E}[1_{Q-y_1+\dots+y_{i-1}>0}] \\
&= \sum_{i=1}^{\infty} \mathbb{E}_{y_1,\dots,y_{i-1}\sim Y} \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{s(Q-y_1-\dots-y_{i-1})}}{s}\, ds \\
&= \sum_{i=1}^{\infty} \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{sQ}}{s}\, \mathbb{E}[e^{-sy_1-\dots-sy_{i-1}}]\, ds \\
&= \sum_{i=1}^{\infty} \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{sQ}}{s}\, \mathbb{E}[e^{-sY}]^{i-1}\, ds \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{sQ}}{s(1-\mathbb{E}[e^{-sY}])}\, ds.
\end{aligned}$$

(Since Y is bounded, Fubini's theorem can be applied to switch the order of integration twice.) $\square$

We proceed to show that in some halfspace $\Re s > -\gamma$, the integrand has no poles other than the double pole at $s = 0$. In fact, we will show more: in this halfspace, excepting a fixed neighborhood of zero, $|1-\mathbb{E}[e^{-sY}]| = \Omega(1)$.

Lemma 9.2. For every $R_2 > 0$ there are constants $\eta, R_1 > 0$ such that

$$\Re\, \mathbb{E}[e^{-sY}] \le 1 - \eta$$

whenever $\Re s \ge -R_1$ and $|\Im s| \ge R_2$.

Proof. Let $s = -\alpha + i\beta$, and let f be the density of Y. Then

$$\begin{aligned}
\Re\, \mathbb{E}[e^{-sY}] &= \int_{\chi_{\min}}^{\chi_{\max}} f(u)e^{\alpha u}\cos(\beta u)\, du \\
&= \int_{\chi_{\min}}^{\chi_{\max}} f(u)[\cos(\beta u)(e^{\alpha u}-1) + \cos(\beta u)]\, du \\
&\le e^{\alpha\chi_{\max}} - 1 + \int_{\chi_{\min}}^{\chi_{\max}} f(u)\cos(\beta u)\, du \\
&= e^{\alpha\chi_{\max}} - \int_{\chi_{\min}}^{\chi_{\max}} f(u)(1-\cos(\beta u))\, du.
\end{aligned}$$

For every ϵ, $|1-\cos x| \le \epsilon$ only for intervals of length $O(\sqrt{\epsilon})$ around $2\pi\mathbb{Z}$. In particular, $|1-\cos(\beta u)| \le \epsilon$ for an $O(\sqrt{\epsilon}/|\beta|)$-fraction of $[\chi_{\min}, \chi_{\max}]$. Therefore

$$\begin{aligned}
&\int_{\chi_{\min}}^{\chi_{\max}} f(u)(1-\cos(\beta u))\, du \\
\ge&\epsilon\int_{\chi_{\min}}^{\chi_{\max}} f(u)\, du - \epsilon O(\sqrt{\epsilon}/|\beta|)(\chi_{\max}-\chi_{\min})C \\
=&\epsilon - O(\epsilon^{3/2}/|\beta|).
\end{aligned}$$

Since $|\beta| \ge R_2$, for small enough $\epsilon > 0$ this expression is at least 2η for some $\eta > 0$. An appropriate choice of R_1 guarantees that $e^{\alpha\chi_{\max}} \le 1 + \eta$, and we deduce that

$$\Re\, \mathbb{E}[e^{-sY}] \le (1+\eta) - 2\eta \le 1 - \eta.$$

$\square$

The constant R_2 arises from the following result.

Lemma 9.3. There is a constant $R_m > 0$ such that for all $R_3 \in (0, R_m)$ there exists a constant $\delta > 0$ such that

$$|\mathbb{E}[e^{-sY}] - 1| \geq \delta$$

whenever $R_3 \leq |s| \leq R_m$.

Proof. We have

$$\mathbb{E}[e^{-sY}] = 1 - \mathbb{E}[Y]s + \sum_{k=2}^{\infty} \frac{\mathbb{E}[Y^k]}{k!} s^k.$$

Since $\mathbb{E}[Y^k] \leq \chi_{\max}^k$, there exists an absolute constant $K > 0$ such that

$$|\mathbb{E}[e^{-sY}] - 1 + \mathbb{E}[Y]s| \leq K|s|^2$$

for $|s| \leq 1/\chi_{\max}$. On the other hand, it is not hard to check that $\mathbb{E}[Y] \geq \chi_{\min} + 1/(2C)$ (the minimal expectation is obtained for $U(\chi_{\min}, \chi_{\min} + 1/C)$). Therefore when $|s| \leq \mathbb{E}[Y]/(2K)$, $K|s|^2 \leq |\mathbb{E}[Y]s|/2$ and so

$$|\mathbb{E}[e^{-sY}] - 1| \geq \frac{|\mathbb{E}[Y]s|}{2} \geq \frac{|s|}{4C}.$$

The lemma easily follows. $\square$

Combining the two lemmas, we obtain the following information on $\mathbb{E}[e^{-sY}]$.

Lemma 9.4. There are constants $\epsilon, \gamma, R > 0$ such that $\mathbb{E}[e^{-sY}] \neq 1$ whenever $\Re s \geq -\gamma$ and $s \neq 0$, and furthermore

$$|\mathbb{E}[e^{-sY}] - 1| \geq \epsilon$$

whenever $\Re s = -\gamma$, or $\Re s \geq -\gamma$ and $|\Im s| \geq R$.

Proof. Let R_m be the constant in Lemma 9.3. Choose $R_3 = R_m/\sqrt{2}$ in Lemma 9.2. The lemma shows that for some $\eta, R_1 > 0$, $|\mathbb{E}[e^{-sY}] - 1| \geq \eta$ whenever $\Re s \geq -R_1$ and $|\Im s| \geq R_m/\sqrt{2}$. When $|\Im s| \leq R_m/\sqrt{2}$, either $|\Re s| \geq R_m/\sqrt{2}$ or $|s| \leq R_m$. In the latter case, Lemma 9.3 shows that $\mathbb{E}[e^{-sY}] \neq 1$ unless $s = 0$. This shows that the only solution to $\mathbb{E}[e^{-sY}] = 1$ for $|\Re s| \leq \gamma := \min(R_1, R_m/\sqrt{2})$ is $s = 0$. Since $|\mathbb{E}[e^{-sY}]| < 1$ for $\Re s > \gamma$, we conclude that the only solution to $\mathbb{E}[e^{-sY}] = 1$ for $\Re s \geq -\gamma$ is $s = 0$.

Next, invoke Lemma 9.3 with $R_3 = \gamma$ to obtain $\delta > 0$. If $\Re s = -\gamma$ and $|\Im s| \geq R_m/\sqrt{2}$ then $|\mathbb{E}[e^{-sY}] - 1| \geq \eta$, as noted before. If $|\Im s| \leq R_m/\sqrt{2}$ then $R_3 \leq |s| \leq R_m$, and so $|\mathbb{E}[e^{-sY}] - 1| \geq \delta$. This completes the proof of the lemma, with $\epsilon = \min(\eta, \delta)$. $\square$

Next, we move the line of integration to the left in order to separate the main term $Q/\mathbb{E}\,Y + \mathbb{E}(Y^2)/2(\mathbb{E}\,Y)^2$ from the error term.

Lemma 9.5. For all $Q > 0$,

$$m(Q) = \frac{Q}{\mathbb{E}\,Y} + \frac{\mathbb{E}(Y^2)}{(\mathbb{E}\,Y)^2} + \frac{1}{2\pi i} \int_{-\gamma - i\infty}^{-\gamma + i\infty} \frac{e^{sQ}\,\mathbb{E}[e^{-sY}]}{s(1 - \mathbb{E}[e^{-sY}])}\, ds,$$

where $\gamma > 0$ is the constant from Lemma 9.4.

Proof. Our starting point is the formula of Lemma 9.1, for $c = \gamma$. Lemma 9.4 shows that the only pole of the integrand in the strip $|s| \leq \gamma$ is at $s = 0$. Standard arguments (using the bound $|\mathbb{E}[e^{-sY}] - 1| \geq \epsilon$ given by Lemma 9.4) show that

$$\begin{aligned} m(Q) &= \frac{1}{2\pi i} \int_{-\gamma - i\infty}^{-\gamma + i\infty} \frac{e^{sQ}}{s(1 - \mathbb{E}[e^{-sY}])}\, ds + \operatorname{Res}\left[\frac{e^{sQ}}{s(1 - \mathbb{E}[e^{-sY}])}, s = 0\right] \\ &= \frac{1}{2\pi i} \int_{-\gamma - i\infty}^{-\gamma + i\infty} \frac{e^{sQ}\,\mathbb{E}[e^{-sY}]}{s(1 - \mathbb{E}[e^{-sY}])}\, ds + \operatorname{Res}\left[\frac{e^{sQ}}{s(1 - \mathbb{E}[e^{-sY}])}, s = 0\right]; \end{aligned}$$

the two integrals differ by the quantity

$$\int_{-\gamma-i\infty}^{-\gamma+i\infty} \frac{e^{sQ}}{s}\, ds = 0.$$

In order to compute the residue, write

$$\begin{aligned}\frac{e^{sQ}}{s(1-\mathbb{E}[e^{-sY}])} &= \frac{1+sQ+O(s^2)}{s^2(\mathbb{E}\,Y - \frac{1}{2}\,\mathbb{E}(Y^2)s + O(s^2))} \\ &= \frac{(1+sQ+O(s^2))\left(1+\frac{\mathbb{E}(Y^2)}{2\,\mathbb{E}\,Y}s + O(s^2)\right)}{s^2\,\mathbb{E}\,Y}.\end{aligned}$$

Calculating the coefficient of s^{-1} in this expression completes the proof. □

In order to estimate the error term, we need to understand the behavior of $\mathbb{E}[e^{-sY}]$ as $|s| \to \infty$.

Lemma 9.6. Suppose $\alpha = -\Re s > 0$. Then

$$|\,\mathbb{E}[e^{-sY}]| \le \frac{e^{\alpha\chi_{\max}} 3C}{|s|}.$$

Proof. Let $f(u)$ be the density function of Y. Integration by parts gives

$$\begin{aligned}\mathbb{E}[e^{-sY}] &= \int_{\chi_{\min}}^{\chi_{\max}} f(u)e^{-su}\, du \\ &= -\left.\frac{f(u)e^{-su}}{s}\right|_{\chi_{\min}}^{\chi_{\max}} + \int_{\chi_{\min}}^{\chi_{\max}} f'(u)\frac{e^{-su}}{s}\, du \\ &= \frac{f(\chi_{\min})e^{-s\chi_{\min}} - f(\chi_{\max})e^{-s\chi_{\max}}}{s} + \int_{\chi_{\min}}^{\chi_{\max}} f'(u)\frac{e^{-su}}{s}\, du.\end{aligned}$$

Taking absolute values,

$$|\,\mathbb{E}[e^{-sY}]| \le \frac{2Ce^{\alpha\chi_{\max}}}{|s|} + \frac{e^{\alpha\chi_{\max}}}{|s|}\int_{\chi_{\min}}^{\chi_{\max}} f'(u)\, du = \frac{e^{\alpha\chi_{\max}}3C}{|s|}.$$

□

Finally, we estimate the error term.

Lemma 9.7. Let $\gamma > 0$ be the constant from Lemma 9.4. We have

$$\left|\int_{-\gamma-i\infty}^{-\gamma+i\infty} \frac{e^{sQ}\,\mathbb{E}[e^{-sY}]}{s(1-\mathbb{E}[e^{-sY}])}\, ds\right| = O(e^{-\gamma Q}).$$

Proof. The triangle inequality shows that

$$\left|\int_{-\gamma-i\infty}^{-\gamma+i\infty} \frac{e^{sQ}\,\mathbb{E}[e^{-sY}]}{s(1-\mathbb{E}[e^{-sY}])}\, ds\right| \le e^{-\gamma Q}\int_{-\gamma-i\infty}^{-\gamma+i\infty} \left|\frac{\mathbb{E}[e^{-sY}]}{s(1-\mathbb{E}[e^{-sY}])}\right| ds.$$

Using Lemma 9.6 and Lemma 9.4, we can estimate the integrand:

$$\left|\frac{\mathbb{E}[e^{-sY}]}{s(1-\mathbb{E}[e^{-sY}])}\right| \le e^{\gamma\chi_{\max}}3C\epsilon^{-1}\frac{1}{|s|^2}.$$

We conclude that the integral converges, and the lemma follows. □

Proposition 8.1 follows by combining Lemma 9.5 and Lemma 9.7.

10 Conclusions and Future Work

We have presented a study of the distribution of the Shapley values in a number of natural weight distributions. In contrast to the general case of a weighted voting game, we were able to reason about the distribution of Shapley values. Along the way, we were also able to reason about cases where the agent weights are super-increasing, and strongly characterize the distribution of the Shapley in a way that generalizes previous results.

Our results demonstrate a stark contrast between the different weight distributions in terms of an egalitarian objective: whereas in some distributions (e.g., uniform) the multiplicative ratio of the highest to lowest Shapley values was roughly stable at 2, in others, this ratio fluctuates, while mostly concentrated around values that are much closer to 1.

We believe that there are other, interested cases of distributions that are worth exploring. In particular, we believe that models of preferential attachment (a.k.a. the generalized, multiple-urn Pólya urn model[2]), in which the probability that a ball enters a specified urn at a given step is proportional to the number of balls currently contained in that urn.

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