Preference Elicitation and Interview Minimization in Stable Matchings

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Abstract

While stable matching problems are widely studied, little work has investigated schemes for effectively eliciting agent preferences using either preference (e.g., comparison) queries or interviews (to form such comparisons); and no work has addressed how to combine both. We develop a new model for representing and assessing agent preferences that accommodates both forms of information and (heuristically) minimizes the number of queries and interviews required to determine a stable matching. Our *Refine-then-Interview (RtI)* scheme uses coarse preference queries to refine knowledge of agent preferences and relies on interviews only to assess comparisons of relatively "close" options. Empirical results show that RtI compares favorably to a recent pure interview minimization algorithm, and that the number of interviews it requires is generally independent of the size of the market.

1 Introduction

Due to its interesting structure and real-world importance, *stable marriage/matching problems (SMPs)* have generated considerable interest since Gale and Shapley's [6] seminal paper. In its most basic form, participants from each of two sides of a matching market (e.g., men and women, hospitals and residents) express preferences—in the form of a ranking—over those on the other side. The aim is to find a bipartite matching (e.g., one-to-one in marriage, many-to-one in resident matching) that is game-theoretically *stable*, i.e., robust to deviation by any couple/pair. The problem has wide applicability, including school choice and matching in residency (or other labor) markets [11]. Since Gale and Shapley's [6] *deferred acceptance algorithm (DFA)*, numerous algorithmic developments have extended the utility of this model.

One impediment to the use of DFA (and its extensions) is the requirement that agents specify their *complete preferences*, in the form of a total order (or preorder) for their counterparts on the other side of the market. This creates two difficulties. First, ranking all options imposes a significant cognitive burden on agents, requiring a large number of (say) pairwise comparisons. Many of these may in fact be unnecessary to the discovery of a stable matching. *Preference elicitation* schemes, including DFA, that incrementally elicit only the preference information deemed to be necessary to construct a stable matching, can help alleviate this burden. In the sequel, we draw heavily on the partial preference representation and minimax-regret-based elicitation scheme devised in our previous work [5] (we refer to this model and method as DB).

A second difficulty is that, even with efficient elicitation schemes, agents may be unable to rank two options without engaging in additional *information gathering* activities. In marriage markets, this might take the form of dating, while in labor markets, this takes the form of *interviews*. Indeed, the National Residency Matching Program (NRMP) has served as a centralized clearing house since 1952 to determine stable matchings of hospitals and residents in the US [13]. However, preference assessment is very costly—residents spend

$1,000 to $5,000 to interview with an average of 11 programs each [2]. Since residents only rank the programs with whom they interview, rankings submitted to NRMP will be strongly influenced their choice of interviews.

Given their costly nature, *interview minimization* is critical. Rastegari et al. [14] address this problem, showing that interview minimization is computationally intractable, but developing a feasible method that is tractable if certain (rather restrictive) preference information is provided *a priori*; we refer to this model as RCIL in the sequel. Two weaknesses of this approach are: the limited circumstances in which it works; and lack of assessment of the elicitation requirements needed to provide the "prior" preferences.

In this work, we provide a unified model that allows for the assessment of preferences using *both direct queries and interviews*. Specifically, we assume that direct comparisons can be made by agents if two options are sufficiently distinct (i.e., well-spaced) in their *true (but partially latent)* underlying ranking; but that such a comparison requires interviews of both options if they are close in their ranking. Our method uses *minimax regret* to determine robust solutions to matching problems given partial information, and generates (a heuristically minimal number of) queries and interviews that are guaranteed to reach a stable matching. In doing so, we: (i) extend the preference representations of both DB and RCIL; (ii) extend the minimax-regret based approach of DB to accommodate interviews; and (iii) provide a tractable (polytime) scheme for generating interviews (and queries) that deals with more general partial preferences that RCIL. Experimental results demonstrate the effectiveness of our scheme.

2 Background

We review *stable matching problems (SMPs)*, focusing on one-to-one matchings (e.g., marriages) for ease of exposition. We also discuss recent work on SMPs with partial preferences, preference elicitation, and interview minimization, explicating key concepts upon which our work is based.

Stable Matching Problems. We describe one-to-one SMPs using stable marriage terminology. Assume a set of men M and women W, each of size n. Each person q provides a strict total ordering $\succ_q$ over the set $R(q)$, where $R(q)$ is the "opposite side" of the market (e.g., if $q \in W$, then $R(q) = M$) and $a \succ_q b$ means q prefers a to b. Let $\succ$ be a *preference profile* consisting of a ranking for each $q \in M \cup W$. A *matching* $\mu : M \cup W \rightarrow M \cup W$ requires $\mu(w) \in M, \forall w \in W$, $\mu(m) \in W, \forall m \in M$, and $\mu(w) = m$ iff $\mu(m) = w$. A *blocking pair* for μ is a pair (m, w) s.t. $w \succ_m \mu(m)$ and $m \succ_w \mu(w)$ (i.e., m, w prefer to be with each other than their partners in μ). We say μ is *stable* if it admits no blocking pairs.

The *deferred acceptance algorithm (DFA)* is a polynomial time algorithm for SMPs [6]. Briefly, the (female-proposing) algorithm proceeds as follows: initially, everyone is unmatched. During each round, all unmatched women propose to the man highest in their ranking to whom they have yet to propose. A man receiving multiple proposals (including any tentative partner) accepts his most preferred proposal (tentative partner). This continues until all women are tentatively matched, and this final tentative match μ is returned. Despite its simplicity, DFA always returns a stable matching, and has many important properties (e.g., it is proposer optimal and strategy-proof). DFA can be applied to many-to-one problems with a variety of real-world applications [15, 1] and extends to handle richer forms of preferences (e.g., indifference, acceptability cutoffs).

Matching and Preference Elicitation. As discussed above, the burden of providing a complete ranking of all potential partners can be considerable in large-scale matching markets. Recent work has considered eliciting *partial information* about agent preferences,

just enough to ensure a stable match is found. In principle, DFA can be used as an elicitation scheme, but rarely done so in practice. Biró and Norman [3] propose a stochastic matching technique that can be interpreted as an elicitation scheme. However, it scales poorly and requires far more rounds than the DB scheme we adopt below [5].

Drummond and Boutilier [5] propose a method for computing robust matchings, using the notion of *minimax regret*, that are approximately stable given partial information about agent preferences in the form of pairwise comparisons. They use this to drive an elicitation scheme that asks "relevant" queries of agents to help determine stable matchings with relatively little preference information—typically $\log_2(n)$ queries per agent on average, much less than elicited by DFA. We describe this model (hereafter dubbed DB) in detail, since we draw on it later.

A *partial preference ranking* P_q for agent q is a consistent set of pairwise comparisons (i.e., partial order) over $R(q)$. We say P_q are *partitioned preferences (PP)* if it partitions $R(q)$ into subsets or *blocks* $G_1, \ldots, G_k$ s.t. $g_i \succ_q g_j$ for any $g_i \in G_i, g_j \in G_j, i < j$, and all $g_i \in G_i$ are uncompared. Let $C(P_q)$ be the set of all consistent completions of P_q. Such partial preferences reflect *incompleteness of the knowledge of the matching mechanism*, hence the set of completions $C(P_q)$ reflect the possible realizations of q's preferences from the perspective of the mechanism. In the DB model, one assumes all agents q have a complete *underlying* ranking $\succ_q$; and that q can compare any two alternatives r and r' (i.e., determine whether $r \succ_q r'$ or $r' \succ_q r$) without gathering additional information. Of course, certain comparisons may be more computationally or cognitively demanding than others; indeed, DB also consider such cognitive costs, as we do below.

If the mechanism must select a matching given a profile $\mathbf{P}$ of *partial* preferences, stability may not be guaranteed. DB use max regret to define the potential *degree of instability* of a matching, assuming a worst-case completion of the profile, and propose *minimax regret* as a robustness criterion for matching. Let $s(r, \succ) = n - rank(r, \succ)$ be the *(inverse Borda) score* of r in ranking $\succ$. DB define:

$$Regret(q, r', r, \succ_q) = s_q(r', \succ_q) - s_q(r, \succ_q) \tag{1}$$

$$PMR(q, r', r, P_q) = max_{\succ_q \in C(P_q)} s_q(r', \succ_q) - s_q(r, \succ_q) \tag{2}$$

$$Inst(m, w, \mu, \succ_m, \succ_w) = \min\{PMR(m, w, \mu(m), \succ_m), PMR(w, m, \mu(w), \succ_w)\} \tag{3}$$

$$Inst(\mu, \succ) = \max_{(m,w)} Inst(m, w, \mu, \succ_m, \succ_w) \tag{4}$$

$$MR(\mu, \mathbf{P}) = \max_{\succ \in C(\mathbf{P})} Inst(\mu, \succ) \tag{5}$$

$$MMR(\mathbf{P}) = \min_{\mu} MR(\mu, \mathbf{P}); \quad \mu^*_{\mathbf{P}} \in \operatorname*{argmin}_{\mu} MR(\mu, \mathbf{P}). \tag{6}$$

The *maximum regret* $MR(\mu, \mathbf{P})$ of matching μ given partial profile $\mathbf{P}$ is the maximum incentive any blocking pair has to defect from μ, under any realization of preferences consistent with the partial profile $\mathbf{P}$. The minimax-optimal matching μ^* minimizes this max regret. If $MMR(\mathbf{P}) = 0$, then μ^* is in fact stable, despite the incompleteness in preferences. While computing μ^* is NP-complete, DB devise polytime methods, known as *PPGS*, that offer excellent results in practice.

DB also describe an elicitation scheme that uses the MMR-solution to guide the querying process. Agent partial preferences are always of the partitioned form, and at each stage, one or more agents is asked to refine one of its blocks by dividing it into a "top half" (more preferred) and "bottom half" (less preferred). They show this scheme to be quite effective, but again it assumes that agents can answer queries without additional information gathering.

Interview Minimization. Rastegari et al.[14] (hereafter RCIL) investigate interview policies that minimize total interview costs for SMPs from a theoretical perspective. Much like the DB elicitation method, their aim is to find an interview policy that, beginning with

a partial profile, will provide enough preference information to ensure a stable matching can be computed. RCIL define an *interview* to be an information gathering step taken by two agents (say m, w), one on each side of the market. After the interview, m (resp. w) is able to rank all options they have interviewed with, including w (resp. m). In other words, if (say) q has interviewed with both r and r', then q can definitively answer whether $r \succ_q r'$ or $r' \succ_q r$. This is the form of interview we assume in this work as well.

RCIL show the general problem of interview minimization is NP-hard, even with the PP model, and propose an MDP formulation to solve the problem (though one that is quite impractical). They also consider the restricted case when the partitioned preferences of agents on one side of the market have *identical structure* (i.e., identical blocks), and provide a poly-time algorithm *Lazy Gale-Shapley (LGS)* to compute an interview-minimizing policy. The method provides a proposer-optimal stable matching. However LGS restricts the true underlying preferences, requires small blocks (i.e., considerable prior information) to ensure small number of interviews, and does not analyze the method or costs needed to assess the "prior" preferences.

3 Combining Elicitation and Interviews

In many settings, the practical assessment of preferences in matching markets requires both direct preference elicitation—asking agents to compare or rank options about which they have adequate information—and interviews—allowing agents to determine the properties of specific options needed to make such comparisons. We outline a framework and elicitation scheme for doing just this. Our scheme is a direct extension of the DB elicitation method, allowing agents to express *uncertainty* in the response to certain queries *prior to interviewing with specific options*. It can also be viewed somewhat loosely as an extension of the RCIL/LGS scheme, since it does not rely on restrictive preference assumptions of LGS; and it provides a means for acquiring the prior preferences needed by the LGS scheme to compute the minimal set of interviews.

We first describe our preference query model and assumptions about agents' ability to answer queries without interviews, as well as the representation of preferences. We then describe a method for approximating minimax regret given a partial profile. Finally, we describe a combined elicitation-interview protocol, *Refine-then-Interview (RtI)*, that in some sense unifies the DB and RCIL models.

3.1 Overlapping Partitioned Preferences

The DB model for representing partial preferences (and computing minimax regret) supports arbitrary pairwise comparisons. However, their elicitation scheme is based on *halving queries*, which ensures that each partial preference is in fact partitioned. Let agent q have true (latent) preference $\succ_q$ over n options, and let the mechanism's knowledge of $\succ_q$ be a PP $G_1, \ldots, G_k$. A halving query asks q to split one of the blocks G_j into a more preferred half G_j^+ and a less preferred half G_j^-, leading to a refined PP $G_1, \ldots, G_{j-1}, G_j^+, G_j^-, \ldots, G_k$. This scheme assumes that q is able to answer arbitrary pairwise comparisons to refine any block of the partition.

While many pairwise comparisons are relatively straightforward, others cannot be assessed without engaging in an interview. To model this, let $w \leq n$ be a *window size*. We assume q is able to state whether option a is preferred to b if $|s(a, \succ_q) - s(b, \succ_q)| \geq w$ in the q's latent ranking $\succ_q$; otherwise we assume that interviewing with both options is needed to distinguish them. For example, if the attributes of a and b are sufficiently distinct (e.g., East Coast vs. West Coast hospital), then q's preference may be obvious; but if they are

similar, then q may need interviews to rank one over the other. Having rank-distance determine the difficulty of comparison (or odds of choosing incorrectly) is commonly assumed in econometrics and psychometrics as well (e.g., as in the Luce-Shepard choice model [8, 16]).

In our query model below, we ask halving queries as in DB. However, we assume that when splitting a block G_k of size $g > w$, q is able to determine the $(g-w)/2$ options she *knows* to lie in the top half G_k^+ of G, and the $(g-w)/2$ options in the bottom half G_k^- but the remaining w options form an uncertain *middle tier*, $G_k^?$, all elements of which could lie somewhere in the top or bottom half of G_k. We assume that distinguishing any two of these options from each other—and from the $w/2$ least-ranked elements of G_k^+ and the $w/2$ top-ranked elements of G_k^-—requires interviews. In particular, prior to interviews, q believes the elements of $G_k^?$ lie in any of the middle $2w$ positions of G_k.

We fix the window size w, and assume that the top and bottom partitions have the same size, i.e., $|G_k^-| = |G_k^+| = (g-w)/2$, for all agents and all queries. This is merely for ease of exposition. Neither of these assumptions impact the algorithms below, and a model where the window size and precise location of the resulting split varies across agents—and for a given individual, across queries—can be addressed using the same techniques.

Our representation, the *overlapping partitioned preferences (OPP) model*, maintains a set of blocks, as in the partitioned case, but also the set of *eligible positions*, $p(G_k) = (t(G_k), b(G_k))$, that elements of any block G_k can occupy in the true ranking $\succ_q$ (where $t(G_k), b(G_k)$ are the top and bottom such positions). A response to a halving query for G_k (with mid-point $m(G_k)$) thus refines that block into three blocks with the following sizes and allowable positions:

$$|G_k^+| = (|G_k| - w)/2, \quad p(G_k^+) = [t(G_k), m(G_k) - 1]$$

$$|G_k^?| = w,\ p(G_k^?) = \begin{cases} [m(G_k) - w, m(G_k) + w], \text{if } |G_k| \geq 2w \\ [t(G_k), b(G_k)], \text{otherwise} \end{cases}$$

$$|G_k^-| = (|G_k| - w)/2, \quad p(G_k^-) = [m(G_k), b(G_k)]$$

We define $t(a) = t(G_k)$ and $b(a) = b(G_k)$ for all $a \in G_k$, i.e., denoting the top and bottom positions an option a can take (given the block it occupies).

These constraints imply that no block of size less than $w + 2$ can be halved. To refine agent preferences further, the mechanism proposes a bidirectional *interview*, which allows agents to determine their relative preference for "close" options. For any agent q on one side of the market, let $I(q)$ be the set of options (from the other side) with who she has interviewed. We assume that q is able to totally order all elements of $I(q)$, so that $\succ_{I(q)} \subseteq \succ_q$. Furthermore, we require $p \in I(q)$ iff $q \in I(p)$. When OPP is complemented by the interview ordering constraints $\succ_{I(q)}$, we call this *overlapping partitioned preferences with interviews (OPPI)*.

Approximating Minimax Regret. As in the DB approach, we use *minimax regret (MMR)* to compute robust matchings given a partial preference profile, where our profiles take the OPPI form. We will use these solutions to drive the querying and interviewing process in the next section. Of course, since DB show computing MMR is NP-hard even for PP, and since OPP and OPPI include PP as a special case, the problem remains NP-hard in our model. Following DB, we adopt the following strategy: given an partial profile $\mathbf{P}$ in OPP or OPPI form, we assign each agent q some complete ranking $\succ_q$ consistent with their partial preference P_q. (We discuss completion functions below.) We then run the GS algorithm on this completion to determine a stable matching μ. The max regret $MR(\mu, \mathbf{P})$ of μ provides an upper bound on $MMR(\mathbf{P})$ (repeating with multiple completions can tighten the bound). DB show this *PPGS* method is tractable by proving that pairwise max regret $PMR(q, r', \mu(q), P_q)$, see Eq. 2, is computable in polytime. These PMR terms can then be used to compute $MR(\mu, \mathbf{P})$ using Eq. 5 over the $O(n^2)$ potential blocking pairs.

Unfortunately, computing $PMR(q, r', r, P_q)$ is somewhat more straightforward in the PP model than in OPP or OPPI. Thus our primary goal is to show that PMR can be

Algorithm 1 Constructing Segments ($|G_k| \geq w,\ \forall k$)

Require: $G_q, I(q)$
for $G_k \in G_q$
  if k is odd
    $\text{mid} = \sum_{i=0}^{k-1} |G_i| + |G_k|/2$
    $t(S_{G_k^-}), b(S_{G_k^-}) = \text{mid} - w, \text{mid} - 1$
    $t(S_{G_k^+}), b(S_{G_k^+}) = \text{mid}, \text{mid} + w - 1$
  else
    $\text{prev\_mid} = \sum_{i=0}^{k-2} |G_i| + |G_{k-1}|/2$
    $\text{next\_mid} = \sum_{i=0}^{k} |G_i| + |G_{k+1}|/2$
    $t(S_{G_k^-}), b(S_{G_k^-}) = \text{prev\_mid}, \text{prev\_mid} + w - 1$
    $t(S_{G_k^+}), b(S_{G_k^+}) = \text{next\_mid} - w, \text{next\_mid} - 1$
  $S_{G_k^-} = \text{generate\_seg}(G_k, t(S_{G_k^-}), b(S_{G_k^-}))$
  $S_{G_k^+} = \text{generate\_seg}(G_k, t(S_{G_k^+}), b(S_{G_k^+}))$
  $S_q[G_k] = S_{G_k^-}, S_{G_k^+}$
def generate_seg(G_k, top, bottom)
  //Generates segment s
  $t(s) = \text{top}$, $b(s) = \text{bottom}$
  $s.\text{dom} = \{G_x \text{ s.t. } t(G_x) \leq b(s) \text{ and } b(G_x) \geq t(s)\}$
  $s.\text{boundaries} = \{(\max(t(s), t(G_x)), \min(b(s), b(G_x)) \text{ s.t. } G_x \in \text{dom}\}$
  $s.\text{required} = \text{populate\_required}(s)$
  $s.\text{ordered\_req} = s.\text{required, s.t. it's consistent with } I(q)$
def populate_required(s)
  $\text{push\_up} = G_{s.\text{dom}[0]}$
  $\text{push\_down} = G_{s.\text{dom}[1]}$
  $s.\text{required}[dom[-1]] = |\{x \in I(q) \cap G_{s.\text{dom}[-1]} \text{ s.t. } \exists y \in \text{push\_up s.t. } x \succ_{I(q)} y\}|$
  $s.\text{required}[dom[0]] = |\{x \in I(q) \cap G_{s.\text{dom}[0]} \text{ s.t. } \exists y \in \text{push\_down s.t. } y \succ_{I(q)} x\}|$

computed efficiently in both models. We begin with OPP. For our purposes, we need accurate calculation only when $PMR \geq 0$ (if $PMR < 0$ we "cut it off" at 0). Using a variant of Hall's Theorem (see Demange, et al. [4]), we show, given OPP G_q, that $PMR(q, r', r, G_q) = \max(b(r) - t(r'), 0)$. Intuitively, we recast PMR computation as an allocation problem, where the options "bid" for a rank position, bidding on all eligible positions given their assigned bounds. Each option must win exactly one spot. The two options used to compute PMR are set by the adversary, and thus are pre-assigned their spots, which is represented by a transformation function that removes the pre-assigned alternatives and now-unavailable spots. We then show that there are no over-demanded sets after the transformation. A full proof can be found in Appendix A.

PMR, hence $MR(\mu, \mathbf{P})$, can also be computed in polynomial time for OPPI, requiring that we account for the interview ranking constraints. We calculate PMR for an agent q with OPPI profile G_q and interview set $I(q)$ via a subtractive counting scheme, where we calculate the maximal possible distance between two options using positional information, and then reduce this distance to account for interview-related information. For clarity, we present only the core of the algorithm, deferring analysis of special cases to the full specification, detailed in Appendix B.

We calculate PMR by computing *segments* S_q, a complementary set to G_q; while each block in G_q is a disjoint set of alternatives (with overlapping positions), each segment in S_q is a disjoint set of positions (with overlapping alternatives). Alg. 1 shows how to calculate these segments. (For ease of exposition, we assume $|G_k| \geq w,\ \forall G_k \in G_q$; see appendix for details for other cases).

Segments are generated by first identifying the extremal, overlapping portions of each block, and then characterizing them. Each block has two extremal positional segments—an upper and lower segment (corresponding to its upper and lower bounds). When $|G_k| \geq w,\ \forall G_k \in G_q$, each segment will be exactly of size w (when $|G_k| < w$, then $|s| > w$ for

Algorithm 2 Calculating *PMR* for OPPI ($|G_k| \geq w,\ \forall k$)

Require: Agent q's OPPI profile G_q, with interviews $I(q)$, segments S_q, reversed input q^{-1}, and alternatives a, a'

1: **if** $|I(q)| < 2$
2: $\quad PMR(q, a', a, G_q) = \max(b(a) - t(a'), 0)$
3: **else**
4: $\quad$ **if** $a \succ_{I(q)} a'$ or $b(a) < t(a')$
5: $\quad\quad PMR(q, a', a, G_q) = 0$
6: $\quad$ **else if** $k < j$
7: $\quad\quad$ Edge case (see appendix)
8: $\quad$ **else**
9: $\quad\quad r(a') = n - 1 -$ find_lower_bound(a', q^{-1})
10: $\quad\quad r(a) =$ find_lower_bound(a, q)
11: $\quad\quad PMR(q, a', a, G_q) = \max(r(a) - r(a'), 0)$
12: **def** find_lower_bound(a, q)
13: $\quad$ **if** $a \notin I_q$
14: $\quad\quad$ seg_up, seg_down $= S_q[G_k]$
15: $\quad\quad$ **if** placing a in seg_down is legal
16: $\quad\quad\quad$ **return** $b(a)$
17: $\quad\quad$ **else** //place in the bottom of upper segment
18: $\quad\quad\quad$ **return** b(seg_up)
19: $\quad$ **else** $//a \in I(q)$
20: $\quad\quad$ **if** a required to be in seg_up
21: $\quad\quad\quad$ lowest_a $= b$(seg_up) $- |\{x \in I(q)$ s.t. $a \succ_{I(q)} x$ and $p(x) \geq b$(seg_up)$\}|$
22: $\quad\quad\quad$ holes $=$ # spaces not already required via $I(q)$ between b(seg_up), b(seg_down)
23: $\quad\quad$ **else**
24: $\quad\quad\quad s =$ seg_down
25: $\quad\quad\quad$ lowest_a $= b(a) - |\{x \in I(q)$ s.t. $a \succ_{I(q)} x$ and $x \in$ seg_down.req$\}|$
26: $\quad\quad\quad$ holes $= 0$
27: $\quad\quad$ free $= |G_{\text{seg\_down.dom}[-1]} - I(q)|$
28: $\quad\quad$ mand $= |$seg_down.req.dom$[-1]|$
29: $\quad\quad$ **return** lowest_a - $\max(0, w/2 -$ (holes + mand + free)

one of its segments s). We identify these extremal positions using the partial preference structure imposed by our elicitation scheme (Lines 2–13), though these upper and lower positional bounds could be given as input. In our setting, the uncertain blocks ($G_j^?$ when halving block G_j) partition the extremal segments. (All uncertain blocks have an odd index; Lines 2–5.) All other blocks have their extremal positions defined by the odd blocks as well (Lines 7–10).

Once the positions are determined (or given) for a segment, then all characteristic information is generated. The *domain* of the segment is every block G_x s.t. G_x overlaps the segment's bounds (Line 17). Each block in the domain has *boundaries* associated with it that are the subset of positions an element in G_x is allowed to take in the segment—some G_x may have tighter bounds than the segment itself (Line 18). Most importantly, some constraints in $I(q)$ *require* that certain elements from some domains must be placed in this segment. This is what allows for us to quickly count the interview constraints when calculating PMR. Lines 21–25 describe the method for finding these required elements. Intuitively, given some x in the less desirable block, and y in the more desirable block, if $x \succ_{I(q)} y$, both x and y *must* be in this segment.

We now examine these extremal segments to calculate $PMR(q, a', a, G_q)$ given chosen option a and adversarial selection a', as shown in Alg. 2 (again, see the paper for the full algorithm). Note that, if $|I(q)| < 2$, calculating PMR for OPPI reduces to calculating PMR on OPP preferences (Lines 1–2). Otherwise, we check if $PMR(q, a', a, G_q)$ is forced to be zero either given available positional information (a' *must* be placed below a) or interview information ($a \succ_{I(q)} a'$) (Lines 4–5). If not, the algorithm consists of two main cases using the segments: a is in a more desirable block than a', and the maximal positions of a and a' must be calculated simultaneously; or, the maximal position of a and a' can be calculated

independently (this is a simple, but tedious, argument by cases to ensure that no options are double-counted). We note that calculating the best position a' can take is equivalent to calculating the worst position a' can take for an agent q^{-1} with "reversed" preferences (Lines 9–11). Thus, the remainder of the PMR algorithm involves calculating this worst position, which uses similar intuitions to those used to calculate a and a''s positions simultaneously (some edge cases must be considered to ensure that no double-counting occurs).

Computing the maximum (worst) position a can take (Lines 12–29) consists of two main cases: $a \notin I(q)$ (i.e., a not interviewed) and $a \in I(q)$ (i.e., a interviewed). If $a \notin I(q)$, there are fewer constraints on a's placement, though we still need to guarantee that a *valid* ranking exists given a's assigned placement. We want to place a as low as possible in its bottom extremal segment. However, if $I(q)$ requires too many options to be placed in this bottom segment, we place a at the bottom of its top segment (Line 18). Otherwise, there is room in the bottom segment, and with no constraints on a's placement imposed by $I(q)$, we place a at its maximal position (Line 16).

When $a \in I(q)$, we first compute the number of options that must lie between a and $b(a)$ dictated by $I(q)$. We then count to check that placing a there still results in a valid ranking; we subtract positions from a's maximal position (i.e., move a higher in the ranking) until a valid ranking is obtained. Since $a \in I(q)$, either a is required to be in the upper extremal segment given interview information $I(q)$ or this is not implied by $I(q)$. The former occurs when there is some option x whose positional constraints require it to be in the upper segment, and $a \succ_{I(q)} x$. In either case, we compute the lowest position a is allowed to take given $I(q)$, a's *maximal* position (Lines 21 and 25). As discussed above in segment generation, $w/2$ options from the least-desirable block whose positional information allows its members to be placed in the bottom segment *must* be placed in that segment (otherwise a valid ranking does not exist). The remainder of the algorithm verifies that enough of the options from this least-desirable block can be placed in the lower segment. Since we need to ensure that there is space in the lowest segment for the $w/2$ options from the least-desirable block, we may need to move a up in the ranking by as many as $w/2$ positions from its maximal position. However, there are different ways in which we could place these $w/2$ options that do not affect a's placement. Counting these ways, as we describe below, allows the accurate computation of the lowest position that a can take while guaranteeing a valid ranking is constructed.

If $a \in I(q)$, the number of positions between a's placement and $b(a)$ may not form a "dense set" (i.e., $I(q)$ may not require that all of these positions be filled; Line 22). Such "holes" allow us to place more of the $w/2$ required elements in the bottom segment without changing a's position. Furthermore, options in the least desirable block that do not have their order fixed by $I(q)$ (i.e., they have not been interviewed) can be placed above a, again not changing a's position (Line 27). All options that are required to be in the segment have already been counted (via the number of "holes" and the lowest legal position of a), and thus we must not double count them (Line 28). If all of these options account for the $w/2$ required positions, the lowest possible position computed for a is valid and is returned. Otherwise, we subtract from this the extra positions required to place the $w/2$ options in the lowest segment, ensuring a valid ranking exists (Line 28). As we show in the appendix, this results in a method that allows $PMR(q, r', r, G_q)$ to be computed in $O(n^3)$ time.

Finally, we require an efficiently computable completion function in order to use PPGS to approximate MMR. Given OPPI profile $\mathbf{P}$, we assume a *reference ranking* σ_M (resp., σ_W) over each side of the market.[1] Our completion function places, for any q, all interviewed options in $I(q)$ in the highest allowable position that results in a valid ranking, and fills remaining positions with uninterviewed options within the relevant blocks (ties are broken

[1]When using Mallows models (see below), we use the reference ranking defining the Mallows model (i.e., the ranking with maximum prior probability). Otherwise, we use an arbitrary ranking.

Algorithm 3 Refine-then-Interview Elicitation Scheme

Require: OPPI profile $\mathbf{G}$, threshold τ

loop

Compute (approximate) $\mu = \mu^*(\mathbf{G})$, compute $MR(\mu, \mathbf{G})$.
if $MR(\mu, \mathbf{G}) \leq \tau$, done.
for each $q \in RI(\mu)$ s.t. q not queried this round
  $Q = \{g_k \in G_r$ s.t. $|g_k| > w + 1 \wedge \exists b \in BP(r) \cup \{\mu(r)\}$ s.t. $b \in g_k\}$
  Query (halve) every block in Q
  if $|Q| = 0$ (i.e., no blocks were halved)
    for $b \in BP(r)$
      if b not queried this round:
        $Q = \{g_k \in G_r$ s.t. $|g_k| > w + 1 \wedge (r \in g_k \vee \mu(b) \in g_k)\}$
        Query (halve) every block in Q
if no blocks were halved this round
  for $r \in RI$
    if r not interviewed with this round
      r interviews with $BP(r) \cup \{\mu(r)\}$
      if r did not interview $(BP(r) \cup \{\mu(r)\} \subseteq I(r))$
        for $b \in BP(r)$
          b interviews with $\mu(b)$ $(r \in I(b))$

using σ). More formally: (i) Sort all blocks according to σ. (ii) For each position p, if no interviewed options can be placed at p, place the next uninterviewed option at p. Otherwise, count the number of uninterviewed options r that must lie before the next block boundary $b(G)$. If $r+1 < b(G)-p$, place an interviewed option at p. Otherwise, place an uninterviewed option at p. This completion takes polynomial time.

3.2 Elicitation Scheme

We now define our *Refine-then-Interview (RtI)* elicitation scheme. Given a current OPPI profile $\mathbf{P}$, RtI computes an (approximately) regret-minimizing matching μ. This solution is used to guide the choice of queries or interviews at each round with the aim or reducing MMR as much as possible. The scheme focuses on *Regret Inducing (RI)* agents in μ, those whose max regret (or instability) dictates the max regret $MR(\mu)$ of the matching. By gathering additional information, through elicitation or interviews, about their preferences or those of their blocking partners, we will reduce the regret of μ, and ideally MMR as well.

At any point we can ask an agent to either halve one of their blocks (which we take to be less costly) or engage in one or more interviews (which we take to be more costly). Of course, halving only works to the point where a block has been refined to size w; after that, interviews are required to make further ranking distinctions. Thus, RtI, see Alg. 3, prioritizes queries over interviews. Let $RI(\mu)$ denoting the set of all regret-inducing individuals in μ. For any $r \in RI(\mu)$, let $BP(r)$ be the set of r's potential blocking partners. Intuitively, RtI uses halving queries to determine roughly where in their preference ranking each agent will be matched. Once their preferences have been refined in the "relevant" regions of their rankings to the extent possible without interviews, RtI proposes interviews for the few options with whom they could still form blocking pairs.

Since halving queries split a block into three smaller blocks, we expect between $\log_3(n)$ and $\log_2(n)$ queries per agent. However, we expect the number of interviews to be *independent* of n (the size of the market) and depend only on w (the degree to which agents can make comparisons without interviews), in fact, to be at most approximately $2w$. This is due to the fact that halving will reduce the "relevant" blocks in any agent q's preference (i.e., blocks containing viable partners) to size no greater than w; and at most two additional blocks (one of which must be of size less than w) can contribute options to such a block.

n	ϕ	RtI interviews	LGS interviews	RtI rounds	LGS rounds
124	0.2	3.93 (0.09)	3.66 (0.07)	7.4 (1.3)	154.3 (6.0)
124	0.6	3.19 (0.18)	2.42 (0.09)	12.2 (2.9)	163.9 (3.4)
124	1.0	3.08 (0.16)	2.37 (0.06)	13.8 (2.7)	130.0 (1.8)
252	0.2	3.92 (0.05)	3.64 (0.05)	8.8 (0.9)	314.6 (0.1)
252	0.6	3.24 (0.16)	2.41 (0.09)	15.1 (3.0)	336.6 (7.8)
252	1.0	3.03 (0.11)	2.31 (0.04)	17.2 (2.1)	258.5 (3.2)

Table 1: RtI vs. LGS (identical input): interviews/person (std.).

4 Empirical Evaluation

We evaluate RtI on a variety of randomly generated matching problems, using several different probabilistic models as well as preferences derived from real-world ratings data. All results are reported over 20 random matching instances.

Comparison to LGS. RtI and LGS solve slightly different problems, since LGS requires very specific (more restrictive) prior preferences, but at the same time finds proposer-optimal matchings (while RtI finds some stable matching). Nevertheless, we show that with the same prior information, RtI generates almost as few interviews as LGS, and that it can more effectively assess "prior" preferences.

Using markets of size $n = 124, 252$, and a window size of $w = 4$, we first compare the two using partitioned preferences of the form needed by LGS: women's (employers') preferences are drawn from a Mallows ϕ-model [9, 10] with dispersion ϕ and reference ranking σ, and assign w options to each of n/w blocks. Men (applicants) must have identical blocks, so we first partition all options into n/w blocks of size w, then create each man's true preference *within each block* by drawing a (smaller) ranking from a Mallows distribution (same ϕ and projected σ). Results for varying dispersion values are shown in Table 1. We see that LGS and RtI generate similar numbers of interviews, with RtI averaging less than one interview per person more than LGS, despite its broader applicability (we note of course that LGS is providing female (employer) optimal matchings and RtI is not). RtI also requires far fewer rounds of interviews, meaning that many interviews can be run in *parallel*; i.e., individual participants can work through their interviews without for additional input from the matching mechanism (e.g., waiting for *other* participants to complete interviews). LGS by contrast must run a large number of interviews sequentially (using prior interview results before setting interviews at the current round).

We also compare RtI to LGS by analyzing the "total information" requirements of the algorithms. To do so, we provide LGS the strict blocks it needs (as above), but provide RtI with no prior preference information. We compare the number/cost of RtI's queries with the queries/cost needed to produce LGS's prior blocks, as well as the number of interviews. Table 2 shows that RtI performs well w.r.t. LGS. While RtI generates roughly twice as many interviews as LGS, the strict block boundaries provided to LGS require pairwise comparisons that in the RtI model cannot be assessed without interviews (hence, this comparison is misleading). Despite the fact that RtI starts with no preference information (in contrast to LGS's blocks), RtI takes significantly fewer rounds of combined elicitation/interviews than LGS needs for only interviews, except when $\phi = 1$ (uniformly distributed preference rankings).

Finally, we compare the (non-interview) elicitation requirements of both. The number of halving queries used by RtI, as well as their *cognitive cost* is show in the table. Here we measure the cognitive cost of a pairwise comparison using a Luce-Shepard model, see [5]. Given preferences $\succ_q$, temperature $\gamma \geq 0$, and threshold τ, the cost for q to compare r with r' is:

$$c(r, r') = e^{\gamma(n - \min(|s_q(r, \succ_q) - s_q(r', \succ_q)|, \tau))} \tag{7}$$

We set $\gamma = 0.5$, $\tau = 5$, and normalize reported cognitive cost by $e^{(\gamma n)}$ (as in DB). To

n	ϕ	RtI interviews	LGS interviews	RtI queries	RtI cog cost	RtI rounds	LGS rounds
124	0.2	8.81 (0.26)	3.66 (0.07)	4.63 (0.17)	17.8 (0.3)	92.3 (11.1)	154.3 (6.0)
124	0.6	7.57 (0.30)	2.42 (0.09)	4.88 (0.19)	18.1 (0.4)	125.9 (12.9)	163.9 (4.1)
124	1.0	4.94 (0.15)	2.37 (0.06)	6.35 (0.07)	21.1 (0.1)	199.1 (22.9)	123.0 (1.8)
252	0.2	8.95 (0.23)	3.64 (0.05)	5.85 (0.22)	38.8 (0.6)	135.3 (9.0)	314.6 (7.1)
252	0.6	7.60 (0.26)	2.41 (0.09)	6.17 (0.3)	39.7 (0.8)	213.4 (22.4)	304.2 (7.8)
252	1.0	4.78 (0.16)	2.31 (0.04)	8.69 (0.1)	48.3 (0.4)	488.4 (48.0)	258.5 (3.2)

Table 2: RtI (with queries) vs. LGS: interviews, etc./person (std.).

	Interviews			Split Queries		
w	$\phi = 0.2$	0.6	1.0	0.2	0.6	1.0
4	9.88	8.00	2.84	6.84	7.42	7.31
6	15.09	13.02	4.23	6.09	6.48	7.10
8	17.82	15.45	5.13	6.27	6.59	7.08

Table 3: RtI performance varying w; $n = 300$.

assess the cost of creating LGS blocks, we assume that agents can engage in partial Quicksort to create blocks and have access to "perfect" pivots at block boundaries. Again, this assumes agents can accurately compare two "close" options without interviews (something not allowed in RtI). RtI's cost is computed similarly, but unanswerable comparisons have cost 0. Eliciting LGS's prior preferences, when $n = 124$ (resp., 252), requires 586 (resp., 1445) comparisons, with a cognitive cost of 107.58 (resp., 241.61). Table 2 shows that the cost of RtI's queries is roughly 16% of that of LGS, and RtI needs about 55% fewer comparisons than LGS. Thus, while LGS requires fewer interviews—though we emphasize that LGS obtains information *without* interviews that force interviews in our model—it needs significantly more preference information overall.

Evaluation of RtI. We now evaluate RtI using random matching problems with preferences generated using several different probabilistic preference models, varying both the market size n and window size w. The models are the same as those used by DB: the Mallows ϕ-model [9]; a riffled independence model [7] derived by riffling two Mallows models; and preferences derived from MovieLens ratings data.[2] In all instances, RtI begins with no preference information.[3]

We first test RtI on a Mallows distributions, varying the degree of preference correlation (or dispersion ϕ). Fig. 1 describes results for $w = 4$. As n increases, the number of interviews (dashed line) remains virtually constant. The exception is when $\phi = 1.0$ (i.e., no preference correlation, or impartial culture), where the number of interviews required *decreases*, which occurs due to more heterogeneous preferences. With $\phi = 0.2$, RtI generates about $2.5w$ interviews/person, consistent with the conjecture that it will require between w and $2w$ interviews. With highly correlated preferences, as expected, more interviews are required (this also occurs with LGS). The average number of halving queries per person (solid lines) increases logarithmically w.r.t. market size, and is unaffected by degree of preference correlation. This mirrors results in the DB query model. Table 3 shows results as we vary w ($n = 300$). As expected, the number of interviews increases with w (and peaks at about $2w$–$2.5w$ interviews/person when preferences are highly correlated), while the number of queries is roughly constant. We note that $w = 4, 6$ results in a number of interviews similar to the 11 that residents average in the NRMP [2].

Since approximately stable matches may be sufficient for many real world problems, we show the anytime performance of RtI in Figs. 2a and 2b, plotting reduction in the max regret (MR) of the induced matching as rounds (either queries or interviews) progress

[2]See http://www.grouplens.org/node/73, the 100K data set.

[3]Error bars are omitted (too small to be seen).

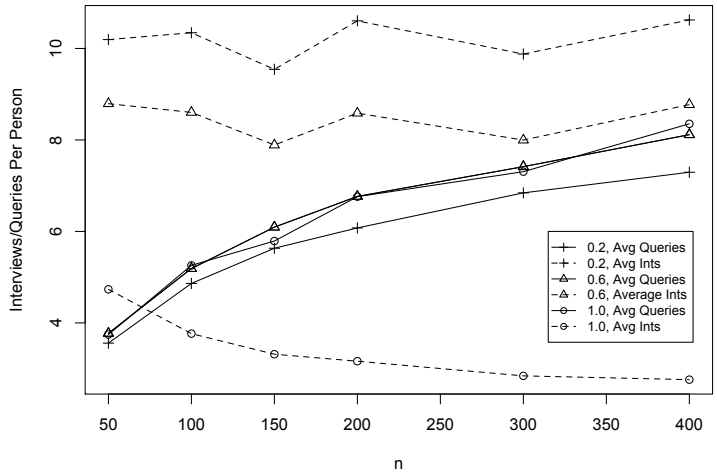
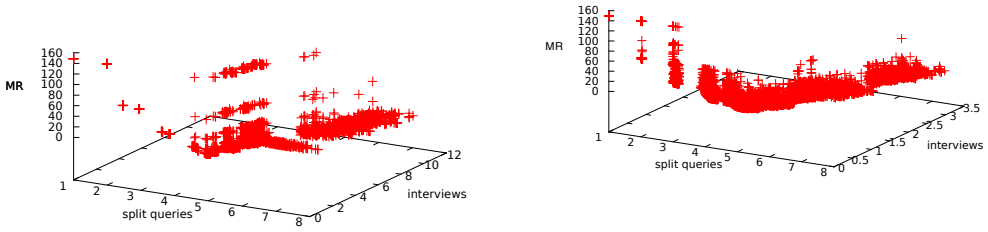


Figure 1: RtI: Avg. queries and interviews/person; varying n, ϕ.

($n = 300$ and $\phi = 0.2, 1.0$). Each point represents MR after a round of RtI, vs. the number of queries/interviews to that point. We see that no costly interviews are generated until MR has been significantly reduced via halving queries. Once MR is sufficiently small, interviews are used drive MR of the matching to 0 (i.e., true stability), though occasionally, new queries are required after some initial interviews (e.g., when the estimated matches are changed significantly after some interviews). These trends are more obvious in Fig. 2b, where few interviews are requested until 5 to 6 split queries per person have been asked, and after interviews begin, relatively few split queries are generated.



(a) $\phi = 0.2$ (b) $\phi = 1.0$

Figure 2: Anytime performance for RtI; $n = 300$.

Results on the riffle model, Fig. 3, exhibit the same trends as above. Comparing Mallows results with $\phi = 0.2$ to riffle results that merge two $\phi = 0.2$ models, we see fewer interviews are required in the riffle case; small perturbations in correlated preferences, combined with two "types" of preferences (as we would expect in real-world data) induces enough heterogeneity to reduce the number of interviews.

Finally, we apply RtI to the MovieLens preference model, with $n = 300$, where agent

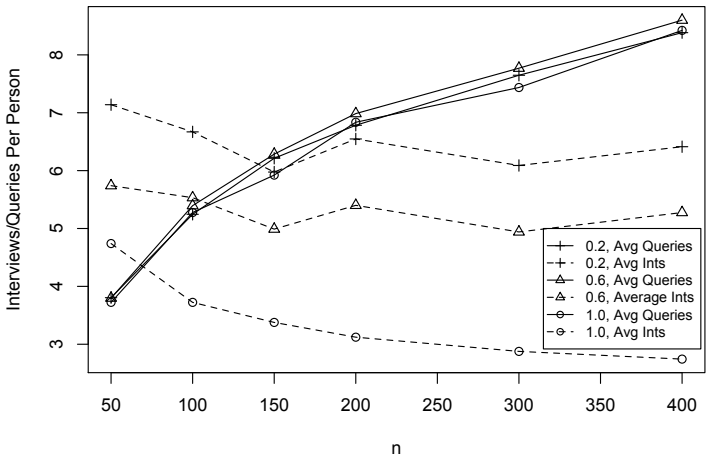


Figure 3: RtI (riffle model): Avg. queries and interviews/person; varying n, ϕ.

preferences are determined using an affinity score based on how similarly they rank movies (see [5] for details). With more correlated affinities, RtI averages 4.95 interviews and 6.76 queries per person, while with less correlated affinities, it averages 2.34 interviews and 6.65 queries per person. These results, on a model based on real-world ratings data, are consistent with the observations above.

5 Conclusions and Future Work

We have developed a new elicitation scheme, which uses both queries and interviews, to assess agent preferences in stable matching problems. When compared with the interview-minimizing LGS algorithm (which works on restricted preference structures), RtI requires a similar number of interviews when given identical input, but generally requires significantly less overall preference information. Our comparison has focused on cases with relatively low preference uncertainty on the part of agents (i.e., small w). We hypothesize this level of uncertainty is realistic in markets of the size studied here, since it generates a number of interviews comparable to those seen in practice [2]. Of course, for markets with greater participant uncertainty, RtI would require more interviews, as would LGS or any other interview-minimizing scheme. RtI also scales well: the number of interviews increases with the agent uncertainty (w), not with market size, and appears to require about $2w$ interviews/person on all probabilistic models tested. The number of (less expensive) halving queries increases logarithmically with market size.

Many interesting questions remain. In many domains, agents naturally express their preferences using option attributes (not ranked lists), requiring an extension of our elicitation method to multi-attribute preferences (e.g., [12]). We also hope to analyze other probabilistic preference models, and account for other objectives (e.g., social-welfare-maximizing stable matchings) and constraints (e.g., matching with couples).

Acknowledgments. We acknowledge the support of NSERC. Drummond was supported by OGS and a Microsoft Research Graduate Women's Scholarship. Thanks to Ettore Damiano for helpful discussions and the reviewers for their suggestions.

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A Expanded Proof for OPP PMR calculation

We claim that, given OPP profile G_q, $PMR(q, a', a, G_q) = \max(b(a) - t(a'), 0)$, for $w \geq 2$, where $p(a) = (t(a), b(a))$ is the set of a's allowable positions. We show this by first showing that the bounds are valid (i.e., given a set of OPP bounds, a complete, valid ranking exists). We then show that these bounds are valid even if, given two alternatives a and a', we fix $r(a) = b(a)$, and $r(a') = t(a')$. Again, OPP bounds are defined recursively in the following manner, with $m(G_k) = (\sum_{G_s \in G_q; s<k} |G_s|) + |G_k|/2$. Given some $G_k \in G_q$, G_k is split as follows:

$$|G_k^+| = (|G_k| - w)/2, \quad p(G_k^+) = [t(G_k), m(G_k) - 1]$$

$$|G_k^?| = w,\ p(G_k^?) = \begin{cases} [m(G_k) - w, m(G_k) + w], \text{if } |G_k| \geq 2w \\ [t(G_k), b(G_k)], \text{otherwise} \end{cases}$$

$$|G_k^-| = (|G_k| - w)/2, \quad p(G_k^-) = [m(G_k), b(G_k)]$$

Let A be the set of all options in G_q. If option $a \in A$ is assigned position y, let $r(a) = y$.

We prove that the bounds are valid in the following manner: we use an extension of Hall's Marriage theorem, due to Demange, Gale, and Sotomayor [4]. We re-frame calculating PMR as an allocation problem, where options "bidding" for their assigned ranking, bidding on all positions they can occupy according to their allowable bounds. Each alternative needs to win exactly one position. When calculating pairwise max regret, by fixing two alternatives' positions, we pre-assign them their positions and thus remove them (and their positions) from the bidding process. If there are no overdemanded sets after this process, then a valid ranking exists. We first show there are no overdemanded sets given OPP's boundaries, and then show there are no overdemanded sets after forcing $r(a') = t(a'), r(a) = b(a)$.

We now define *monotonicity* for the bounds of an OPP profile G_q. We first define an ordering for the blocks in profile G_q, and then define monotonicity for the blocks' bounds. Note that the elicitation scheme we use induces a natural ordering on the blocks of G_q; at each step of the elicitation scheme, a parent block $G_k \in G_q$ is chosen, and split into three children blocks, G_k^+, $G_k^?$, and G_k^-. By definition of the scheme, all elements in G_k^+ are more desirable than the elements in G_k^-, and the elements in $G_k^?$ are the uncertain middle. This gives us a natural ordering of these three children blocks: $G_k^+ < G_k^? < G_k^-$. (Note: 0 is most desirable.) Because this is an iterative elicitation process, all blocks before and after G_k will have been ordered, thus the new full ordering after this elicitation step will be: $G_1, ..., G_{k-1}, G_k^+, G_k^?, G_k^-, G_{k+1}, ..., G_m$ (given m blocks). We thus say that the upper (resp. lower) bounds of a profile G_q are monotonic if $\forall G_j, G_k \in G_q$ s.t. $j < k$, $t(j) < t(k)$ (resp. $b(j) < b(k)$).

Note that, if for all options both the upper and lower bounds are monotonic, and for $\forall a \in A$, $t(a) \leq r(a)$ and $b(a) \geq r(a)$, this is sufficient to show that no overdemanded sets exist. (Again, 0 is the most desirable position.) We call the latter the "45-degree" principle, as graphing these constraints assures all lower bounds are below $f(a) = a$ and all upper bounds are above that line.

We first show that all bounds generated by querying are monotonic, and obey the 45-degree principle. This shows that without allowing the adversary to assign any alternatives' rankings, these bounds are valid. We then provide a transformation function T that transforms G_q into G'_q, a set of bounds with two fewer alternatives, that appropriately adjusts the allowable positions to account for the pre-assignment $r(a') = t(a'), r(a) = b(a)$. We then show that G'_q is also monotonic and obeys the 45-degree principle.

First, we show all bounds in G_q are monotonic. As the elicitation process is iterative, we use an inductive argument. We begin after exactly one split of q's preferences, so G_q

will have exactly 3 blocks, with the following positional information:

$$p(G_k^+) = [0, |A|/2 - 1]$$
$$p(G_k^?) = \begin{cases} [|A|/2 - w, |A|/2 + w], \text{if } |A| \geq 2w \\ [0, |A| - 1], \text{otherwise} \end{cases}$$
$$p(G_k^-) = [|A|/2, |A| - 1]$$

We first show that these bounds obey the 45-degree principle by constructing a ranking r that does so. First order all a s.t., for $a \in G_k^+$, $b \in G_k^?$, $c \in G_k^-$, $r(a) < r(b) < r(c)$ (and ordered arbitrarily within each block). Note that $\forall a \in G_k^+$, $t(a) = 0 \leq r(a)$, and $b(G_k^+) = |A|/2 - 1 \geq (|A| - w)/2 - 1 \geq r(a)$, as required. ($G_k^?$ and G_k^- follow by symmetry.)

We then show monotonicity of these bounds. If $|A| < 2w$, clearly the bounds are monotonic. For $|A| \geq 2w$, starting with the upper bounds, since $|A| \geq 2w$, $|A|/2 - w \geq 2w/2 - w = 0$, thus $t(G_k^?) \geq t(G_k^+)$. $|A|/2 \geq |A|/2 - w$, thus $t(G_k^-) \geq t(G_k^?)$, and thus the upper bounds are monotonic as required. Again, for the lower bounds: $|A|/2 + w - 1 \geq |A|/2 - 1$, thus $b(G_k^?) \geq b(G_k^+)$. Similarly, $b(G_k^-) = |A| - 1 = |A|/2 + A/2 + 1 \geq |A|/2 + w - 1 = b(G_k^?)$, as required. Thus, when $|A| \geq 2w$, and no alternatives have pre-assigned positions, no sets are overdemanded, and thus under these conditions, these bounds are valid. Then, the base case does not have any overdemanded sets.

For the inductive step, we assume that the current set of indistinguishable alternatives G_q generated by the querying scheme does not contain any overdemanded sets, so that all bounds are monotonic and obey the 45-degree principle. We now wish to show that splitting any $G_k \in G_q$ (with $|G_k| > w + 1$) into three segments $G_k^+, G_k^?, G_k^-$ (as in elicitation) creates bounds that are monotonic and obey the 45-degree principle.

Note that $t(G_k^+) \leq t(G_k^?) \leq t(G_k^-)$ (and thus monotonic), and all three obey the 45-degree principle, by the same logic as in the base case (simply translate these indices down to $[0, |G_k|]$). We then simply need to show that all of these new bounds are monotonic w.r.t. the old bounds; we want to show:

$$t(G_{k-1}) \leq t(G_k^+) = t(G_k) \tag{8}$$
$$b(G_k^-) = b(G_k) \leq b(G_{k+1}) \tag{9}$$
$$b(G_{k-1}) \leq m(G_k) - 1 = b(G_k^+) \tag{10}$$
$$m(G_k) = t(G_k^-) \leq t(G_{k+1}) \tag{11}$$

Note that we do not need to further verify that the new bounds for $G_k^?$ violate monotonicity for the following reason: if $|G_k| \geq 2w$, then the bounds $m(G_k) - w, m(G_k) + w$ cannot cause $G_k^?$ to violate monotonicity via the same argument as above; if $|G_k| < 2w$, then $t(G_k^?) = t(G_k)$ and $b(G_k^?) = b(G_k)$, so again, monotonicity is not an issue (as G_k does not violate monotonicity).

Note that Eqs. 8 and 9 are true by construction; if either of these were false, the bounds in G_k before we split would not have been monotonic. To show Eq. 10 is true, note that $b(G_{k-1}) \leq (\sum_{i=0}^{k-1} |G_i|) + w/2 - 1$, because, if we can split G_k, then $|G_k| > w + 1$, which, because of the structure of the elicitation scheme, implies $|G_{k-1}| = w$, and by construction, $b(G_{k-1}) = m(G_{k-1}) + w/2$. Let $s = \sum_{i=0}^{k-1} |G_i|$. So, $b(G_k^+) = m(G_k) - 1 = s + |G_k|/2 - 1 \geq s + w/2 - 1 = b(G_{k-1})$ as required. Similarly, we show that $t(G_k^-) = m(G_k) \leq t(G_{k+1})$. Again, by construction, $t(G_{k+1}) = m(G_{k+1}) - w/2 \geq m(G_k) + |G_k|/2 - w/2$. Since $|G_k| > w + 1$, this implies $t(G_{k+1}) \geq m(G_k) + w/2 - w/2 = t(G_k^-)$ as required. Thus, for all queries that follow the protocol, there are no overdemanded sets.

We now need to show that the transformation T is valid (i.e., it appropriately transforms G_q into G_q' such that G_q with alternatives a, a' pre-assigned to positions $t(a'), b(a)$ is equivalent to G_q'). Intuitively, this transformation removes two alternatives a, a', and positions

$t(a')$, $b(a)$ from G_q. First, note that if $b(a) \leq t(a')$, PMR must be 0. Thus, we assume $t(a') < b(a)$. Let $e(G_k)$ be the set of alternatives OPP block G_k contains, and let $p(G_k)$ be the set of positions OPP block G_k is allowed to take. Transformation T is as follows; given top alternative a', bottom alternative a, and OPP profile G_q over alternative set A, define T:

$$\begin{aligned}
&\text{if } b(G_k) > t(a') : \\
&\quad e(T(G_k)) = e(G_k); \\
&\quad p(T(G_k)) = p(G_k) && (12) \\
&\text{if } a' \in G_k, a \notin G_k : \\
&\quad e(T(G_k)) = e(G_k)/\{a'\}; \\
&\quad p(T(G_k)) = (t(G_k), b(G_k) - 1) && (13) \\
&\text{if } t(a') < t(G_k) \text{ and } b(a) > b(G_k) : \\
&\quad e(T(G_k)) = e(G_k); \\
&\quad p(T(G_k)) = (t(G_k) - 1, b(G_k) - 1) && (14) \\
&\text{if } a', a \in G_k : \\
&\quad e(T(G_k)) = e(G_k)/\{a', a\}; \\
&\quad p(T(G_k)) = (t(G_k), b(G_k) - 2) && (15) \\
&\text{if } a \in G_k, a' \notin G_k : \\
&\quad e(T(G_k)) = e(G_k)/\{a\}; \\
&\quad p(T(G_k)) = (t(G_k) - 1, b(G_k) - 2) && (16) \\
&\text{if } t(G_k) < b(a), b(G_k) > b(a), a \notin G_k : \\
&\quad e(T(G_k)) = e(G_k); \\
&\quad p(T(G_k)) = (t(G_k) - 1, b(G_k) - 2) && (17) \\
&\text{if } t(G_k) > b(a) : \\
&\quad e(T(G_k)) = e(G_k); \\
&\quad p(T(G_k)) = (t(G_k) - 2, b(G_k) - 2) && (18)
\end{aligned}$$

This transformation completely eliminates positions $t(a')$ and $b(a)$, and alternatives a, a', translating $|A|$ positions to $|A| - 2$ positions. Similarly, all $G_k \in G_q$ containing exactly one of $t(a')$ or $b(a)$ as a feasible position are now of size $|G_k| - 1$; those $G_k \in G_q$ containing both $t(a')$ and $b(a)$ are now of size $|G_k| - 2$; those containing neither are their original size. Furthermore, $\forall G_k, G_j \in G_q$, all relative positions of $t(G_k)$ and $t(G_j)$ are the same, accounting for $t(a')$ and $b(a)$ eliminated. (Similarly for $b(g_k)$ and $b(g_j)$.) Thus, T is a valid translation.

Showing that $G'_k = T(G_k)$ is monotonic and obeys the 45-degree principle will conclude the proof. Because T is a valid transformation, if G'_k is monotonic and obeys the 45-degree principle, then there are no over-demanded sets, and thus there exists at least one valid completion of the OPP profile such that $p(a') = t(a')$ and $p(a) = b(a)$, as required. We show that T preserves these properties by going through all cases presented in Eqs. 12–18. For the case stated in Eq. 12, $T(G_k) = G_k$, so monotonicity and the 45-degree principle hold. Given any $G_k \in G_q$, we show that the 45-degree principle holds for G_k, and then show that G_k is monotonic w.r.t. G_{k-1}. Note that the first block ($k = 0$) is a special case, but the only thing that changes is we don't need to test for monotonicity. For Case 13, note that we simply have one less slot and one less item. Thus, the 45-degree principle still holds. Also note that this case occurs at a boundary, where $b(T(G_{k-1})) = b(G_{k-1})$ but $b(T(G_k)) = b(t(G_k)) - 1$. While we get monotonicity for t for free, $b(T(G_{k-1})) = b(G_{k-1}) \leq m(G_k) \leq m(G_{k+1}) - 1 \leq$

$b(T(G_k))$ as required. (Note that some of the special cases require a tighter bound than $b(G_{k-1}) \leq m(G_k)$; these special cases are omitted for clarity, but the tighter bounds are easy to calculate given the OPP structure.) Case 14 is trivial (because everything has only shifted, nothing has changed), unless $t(T(G_{k-1})) = t(G_{k-1})$. Then, we follow a similar argument to prove monotonicity of t as above: $t(G_{k-1}) \leq m(G_{k-1}) - w < m(G_{k-1}) \leq t(T(G_k))$. Case 15, again removes the same number of options in G_k as positions, so the 45-degree principle will still hold, as it held for G_k. Also, t is still monotonic, as t does not change. We use the same argument as in Case 13 for b, but note that the essence of that argument is that there's always a padding of size at least w between $b(G_k)$ and $b(G_{k-1})$. Since $w \geq 2$, then $b(T(G_k)) \geq b(T(G_{k-1}))$. Cases 17 and 18 follow by the same style of argument.

B Full PMR Algorithm for OPPI Preferences

Given some agent q's OPPI profile G_q with interview set $I(q)$, computing *PMR* when $|I(q)| < 2$ reduces to calculating PMR for OPP, and $|I(q)| \geq 2$ requires a counting algorithm. Algorithms 5 and 6 provide the counting method for calculating *PMR* in polynomial time for this setting. For clarity, we do not include the cases when $a, a' \in g_0, g_f$ (where g_f is the last block in G_q); to include these cases, there are simply a few checks required to make sure that placements do not go outside of the $[0, n-1]$ bounds imposed by the fact that we only have n alternatives.

Algorithm 5 describes the counting scheme required for calculating $PMR(q, a', a, G_q)$ when calculating a''s minimal position and a's maximal position can be done separately. (This occurs when a', a are sufficiently far apart in the preferences so that we do not double-count any options.) The counting scheme works by analyzing the extremal portions of the preferences that a', a can occupy, where two blocks overlap. We call these extremal portions *segments*, where a set of segments S_q can be thought of as a complementary set to G_q; while each block in G_q is a disjoint set of alternatives, two blocks in G_q may have overlapping positions they share. Each S_q segment, however, is a disjoint set of positions, but may have overlapping alternatives that may be allowed to fill those positions.

The process for computing PMR is as follows: first, we update all S_q segments; the algorithm for doing so is shown in Algorithm 4. Furthermore, this process is symmetrical; we compute a *reversed* agent q^{-1}, where all interviewed preferences of q are reversed, as are all blocks and their positional information. We also compute the appropriate segments $S_{q^{-1}}$ for q^{-1}, and finally we compute $r(a')$ by computing the worst position for a' in q^{-1}. Note that S_q and q^{-1} only change when the elicitation scheme provides new information regarding q's preferences; thus we only need to update these after new information is gathered.

Algorithm 4 describes how to generate segments for some OPPI profile $G_q, I(q)$. There are two main cases, and two main subcases for generating segments; given some block G_k, if k is odd (i.e., G_k was one of our "uncertain" blocks in elicitation; shown in Lines 1–13) or even (Lines 14–23); and when blocks of size $|G_k| < w$ are involved or not. When $|G_k| < w$, one large segment is generated that encompasses all positions from the previous block of size w's midpoint to the next group of size w's midpoint, thus forming a segment with $w + |G_k|$ positions. All other segments that bridge the overlap of two blocks (which are the segments we're interested in) are of size w.

After determining the top and bottom legal position of the segment, we associate the remaining pertinent information with it. The *domain* of the segment simply is what blocks are allowed to contribute alternatives to this segment (based on its boundaries). The *boundaries* are the top and bottom positions any block in the domain are allowed to take within this segment. For each domain, *required* is the set of elements from that block that must be in this segment because some other alternative is pushing them up or down into this

segment based on $I(q)$; *ordered_req* is simply this as an ordered list that is consistent with $I(q)$.

When calculating a', a's maximal positions separately, to find the worst position a is allowed to occupy in q's preferences, we simply keep trying the lowest positions possible and see if they're legal. If $a \notin I(q)$, a's potential positions are less restricted (Lines 14–22). We first check if placing a as low as possible is available; we must be able to place $w/2$ elements from the least desirable block in the positions available to the lower segment (Line 16). (Note that when $|G_k| < w$ only one segment exists, and thus it is trivially true that we can place in the lower segment.) If $I(q)$ prevents us from doing this by requiring too many elements in the lower segment, we must either place a above the highest position the least desirable block is allowed to take, or at the bottom of the upper segment (Lines 19–22).

When $a \in I(q)$, we first check if a gets pulled up into a higher part of the preferences because of some element o in $I(q)$ (Line 24). We then count how many elements in $I(q)$ are required to go between a and o, according to $I(q)$. This sets the lowest possible position a can legally take: $b(o)$ minus the number of elements between them (Line 25). We will need to guarantee that we can fit $w/2$ elements from the least desirable block whose elements are allowed to be placed in this segment, and since a got pulled up, there may be some empty spaces between a's placement and $b(a)$. We count this in Line 26. If a is not pushed up, we similarly count a's lowest plausible position constrained by $I(q)$ and $b(a)$ (Line 28). This is guaranteed to be a dense set, so there are no holes (Line 29).

We finally check and make sure that we can guarantee that we can fit $w/2$ elements from the least desirable block whose elements are allowed to be placed in this segment (Line 32). There may be some alternatives unconstrained by $I(q)$ that we could put above a (Line 30), and we want to guarantee that we don't count any elements in $I(q)$ twice (Line 31), as these have already been accounted for when calculating a's lowest legal space. Subtracting these finishes the counting algorithm for computing *PMR* separately.

We can compute *PMR* of a', a separately if $a \in G_k, a' \in G_j$ s.t. $j \geq k$. Briefly, if $j > k + 2$, *PMR* can be computed separately because any element that could be placed in the same segment as a cannot be placed in the same segment as a'. The argument for $j = k$ is easily extendable to $k + 2 \geq j \geq k$. This argument is by cases. Upon inspecting all combinations of $a, a' \in I(q)$, $a, a' \notin I(q)$, and all possible S_q segments a, a' could be placed, note that when counting elements for placing a, we never count any elements in $G_j / I(q)$ (likewise for a'). Furthermore, all elements in $G_j \cap I(q)$ that we count are strictly worse than a', otherwise we would have simply returned 0. This guarantees that we do not count a' when calculating the lowest possible position for a (note that this condition does *not* hold when $k > j$, thus we must compute them simultaneously). Finally, placing any other element in such a way that it would help a' would also help a, because the adversary wants the elements as far apart as possible. Thus, we don't double-count any elements by computing $r(a')$ and $r(a)$ separately.

Finally, to compute *PMR* when $k > j$ (i.e., we need to compute *PMR* of a', a at the same time), see Algorithm 6. For brevity, we only present $a, a' \in I(q)$. For all other cases, computing *PMR* is very similar to computing it independently; we simply need to be careful that we don't count a or a' twice. Note that for all other *PMR* calculations, we find the *maximal* position a (resp. a') can take, and subtract to guarantee that we do not violate any conditions on G_q, $I(q)$. For calculating *PMR* when $a, a' \in I(q)$ and $j > k$, we use an *additive* argument, where we find the minimal distance between a' and a (as determined by the number of elements between them in $I(q)$), and we then we place as many other elements as possible in-between them without violating any conditions.

We also briefly discuss correctness. Note that, as our algorithm is a subtractive counting scheme, the only error we could have is that $PMR(q, a', a, G_q)$ is miscalculated as larger than it actually is. However, a simple (though lengthy) analysis shows these errors are not

possible. Given $a \in G_k$, only alternatives in the extremal segments could affect a's placement, because of how the both the segments and OPP profiles are constructed. After that, a simple exploration of all possible cases for $I(q)$'s construction (e.g., number of required elements, $a \in I(q)$, $a \notin I(q)$, etc.), combined with OPP structure properties (e.g., a fixed number of elements are allowed in any segment from each block, etc.) shows that all possible cases where $I(q)$ could affect a's placement are accounted for in the algorithm.

Algorithm 4 Constructing Segments

Require: $G_q, I(q)$

for $G_k \in G_q$
  if k is odd
    $\text{mid} = \sum_{i=0}^{k-1} |G_i| + |G_k|/2$
    if $|G_{k-1}| < w$
      $t(S_{G_k^-}) = \text{mid} - w - |G_{k-1}|$
    else
      $t(S_{G_k^-}) = \text{mid} - w$
    $b(S_{G_k^-}) = \text{mid} - 1$
    $t(S_{G_k^+}) = \text{mid}$
    if $|G_{k+1}| < w$
      $b(S_{G_k^+}) = \text{mid} + w - 1 + |G_{k+1}|$
    else
      $b(S_{G_k^+}) = \text{mid} + w - 1$
  else
    $\text{prev\_mid} = \sum_{i=0}^{k-2} |G_i| + |G_{k-1}|/2$
    $\text{next\_mid} = \sum_{i=0}^{k} |G_i| + |G_{k+1}|/2$
    if $|G_k| < w$
      $t(S_{G_k}) = \text{prev\_mid}$
      $b(S_{G_k}) = \text{next\_mid} - 1$
    else
      $t(S_{G_k^-}) = \text{prev\_mid}$
      $b(S_{G_k^-}) = \text{prev\_mid} + w - 1$
      $t(S_{G_k^+}) = \text{next\_mid} - w$
      $b(S_{G_k^+}) = \text{next\_mid} - 1$
  $S_{G_k^-} = \text{generate\_seg}(G_k, t(S_{G_k^-}), b(S_{G_k^-}))$
  $S_{G_k^+} = \text{generate\_seg}(G_k, t(S_{G_k^+}), b(S_{G_k^+}))$
  $S_q[G_k] = S_{G_k^-}, S_{G_k^+}$

def generate_seg($G_k, \text{top}, \text{bottom}$)
  //Generates segment s
  $t(s) = \text{top}$, $b(s) = \text{bottom}$
  $s.\text{dom} = \{G_x \text{ s.t. } t(G_x) \leq b(s) \text{ and } b(G_x) \geq t(s)\}$
  $s.\text{boundaries} = \{(\max(t(s), t(G_x)), \min(b(s), b(G_x))) \text{ s.t. } G_x \in \text{dom}\}$
  $s.\text{required} = \text{populate\_required}(s)$
  $s.\text{ordered\_req} = s.\text{required}$, s.t. it's consistent with $I(q)$

def populate_required(s)
  if $|s.\text{dom}| == 3$
    $s.\text{required}[s.\text{dom}[1]] = G_{s.\text{dom}[1]}$
    $\text{push\_up} = G_{s.\text{dom}[0]} \cup G_{s.\text{dom}[1]}$
    $\text{push\_down} = G_{s.\text{dom}[1]} \cup G_{s.\text{dom}[2]}$
  else
    $\text{push\_up} = G_{s.\text{dom}[0]}$
    $\text{push\_down} = G_{s.\text{dom}[1]}$
  $s.\text{required}[dom[-1]] = |\{x \in I(q) \cap G_{s.\text{dom}[-1]} \text{ s.t. } \exists y \in \text{push\_up s.t. } x \succ_{I(q)} y\}|$
  $s.\text{required}[dom[0]] = |\{x \in I(q) \cap G_{s.\text{dom}[0]} \text{ s.t. } \exists y \in \text{push\_down s.t. } y \succ_{I(q)} x\}|$

Algorithm 5 Calculating *PMR* for OPPI

Require: Agent q's OPPI profile G_q, with interviews $I(q)$, alternatives $a \in G_k$, $a' \in G_j$, segments S_q, and reversed agent q^{-1}

if $|I(q)| < 2$
  $PMR(q, a', a, G_q) = \max(b(a) - t(a'), 0)$
else
  if $a \succ_{I(q)} a'$ or $b(a) < t(a')$
    $PMR(q, a', a, G_q) = 0$
  else
    if $k < j$
      Compute $PMR(q, a', a, G_q)$ simultaneously (Alg. 6)
    else
      $r(a') = n - 1 - \text{find\_lower\_bound}(a', q^{-1})$
      $r(a) = \text{find\_lower\_bound}(a, q)$
      $PMR(q, a', a, G_q) = \max(r(a) - r(a'), 0)$
def find_lower_bound(a, q)
  if $a \notin I_q$
    seg_up, seg_down $= S_q[G_k]$
    if |seg_down.required.domain$[-1]| < w/2$ or $|G_k| < w$ (i.e., seg_down is not full)
      if placing a doesn't push anyone from seg_down.dom$[-1]$ out of bounds
        return $b(a)$
      else
        return t(seg_down.dom$[-1]) - 1$
    else //place in the bottom of upper segment
      return b(seg_up)
  else $//a \in I(q)$
    if a gets pulled up into seg_up (resp. top of seg if $|G_k| < w$) by some $o \in I(q)$, (thus $o \in G_w$; $w < k$)
      lowest_a $= b$(seg_up) $- |\{x \in I(q)$ s.t. $a \succ_{I(q)} x$ and x must be above $b(o)\}|$
      holes = # spaces not already required via $I(q)$ between lowest_a and b(seg_down)
    else
      lowest_a $= b(a) - |\{x \in I(q)$ s.t. $a \succ_{I(q)} x$ and x must be above $b(a)\}|$
      holes $= 0$
    free $= |G_{\text{seg\_down.dom}[-1]}/I(q)|$
    mand $=$ |seg_down.required.dom$[-1]|$
    return lowest_a - $\max(0, w/2 - (\text{holes} + \text{mand} + \text{free}))$

Algorithm 6 Computing PMR dependently

Require: $G_q, I(q), a, a'$

//When $a, a' \notin I(q)$ or $a \in I(q), a' \notin I(q)$ or $a \notin I(q), a' \in I(q)$, the argument is essentially the same, but we are more careful not to double-count alternatives. Thus we only present when $a, a' \in I(q)$, as this case is the most different.

if $a, a' \in I(q)$
  $\delta = |\{x \in I(q)$ s.t. $a' \succ_{I(q)} x \succ_{I(q)} a\}| + 1$
  $F = \{x \in A$ s.t. x can be legally placed between a' and $a\}$
  while $|F| > 0$ and placing one more item between a', a does not invalidate boundaries
    $F = F/\{x\}$, where x is a valid item to place between a', a
    $\delta = \delta + 1$
  $PMR(q, a', a, G_q) = \delta$