Consensus measures generated by weighted Kemeny distances on linear orders

José Luis GARCÍA-LAPRESTA   David PÉREZ-ROMÁN

PRESAD Research Group

University of Valladolid, Spain

COMSOC 2010, (Düsseldorf, Germany)
September 13, 2010

Motivation

- Each member of a committee arranges a set of alternatives by means of a linear order
- How similar are their opinions?
- Could we measure consensus?
  - Bosch (2005) introduced the notion of consensus measures in the context of linear orders
  - García-Lapresta and Pérez-Román (2008) extended Bosch's concept to the context of weak orders
  - Alcalde-Unzu and Vorsatz (2010) have introduced some consensus measures in the context of linear orders (related to some rank correlation indices)

Proposals

- Since in some decision problems it is not the same to have differences in the top alternatives than in the bottom ones, we introduce weights for distinguishing where these differences occur
- We consider a class of consensus measures generated by weighted Kemeny distances
- We analyze some of their properties

Notation

- $V = \{v_1, \ldots, v_m\}$ set of voters $\quad m \geq 3$
- $X = \{x_1, \ldots, x_n\}$ set of alternatives $\quad n \geq 3$
- $L(X)$ the set of linear orders on X
- $R \in L(X) \longmapsto R^{-1}$ inverse of R

$$x_i \, R^{-1} \, x_j \iff x_j \, R \, x_i$$

- A **profile** is a vector $\boldsymbol{R} = (R_1, \ldots, R_m)$ of linear orders

Codification of linear orders

- Given $R \in L(X)$,
  $o_R : X \longrightarrow \{1, \ldots, n\}$ defines the position of each alternative in R

$$o_R = (o_R(x_1), \ldots, o_R(x_n))$$

$$\begin{matrix} x_2 \\ x_3 \\ x_1 \\ x_4 \end{matrix} \quad \equiv \quad (3, 1, 2, 4)$$

- We can identify $L(X)$ with S_n (the set of permutations on $\{1, \ldots, n\}$)

Distance

- A **distance** on a set $A \neq \emptyset$ is a mapping $d : A \times A \longrightarrow \mathbb{R}$ satisfying the following conditions for all $a, b \in A$:
  1. $d(a, b) \geq 0$ (non-negativity)
  2. $d(a, b) = d(b, a)$ (symmetry)
  3. $d(a, a) = 0$ (reflexivity)

- If d satisfies the following additional conditions for all $a, b \in A$:
  4. $d(a, b) = 0 \Leftrightarrow a = b$ (identity of indescernibles)
  5. $d(a, b) \leq d(a, c) + d(c, b)$ (triangle inequality)

  then we say that d is a **metric**

- M.M. Deza, E. Deza. *Encyclopedia of Distances*. Springer-Verlag, 2009

PRESAD

Distance

- Let $A \subseteq \mathbb{R}^n$ be stable under permutations, i.e., $(a_1^\sigma, \ldots, a_n^\sigma) \in A$ for all $(a_1, \ldots, a_n) \in A$ and $\sigma \in S_n$
A distance (metric) $d : A \times A \longrightarrow \mathbb{R}$ is **neutral** if for every $\sigma \in S_n$, it holds

$$d\left((a_1^\sigma, \ldots, a_n^\sigma), (b_1^\sigma, \ldots, b_n^\sigma)\right) = d\left((a_1, \ldots, a_n), (b_1, \ldots, b_n)\right),$$

for all $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in A$

Typical examples of metrics on $\mathbb{R}^n$ as discrete, Manhattan, Euclidean, Chebyshev and cosine are neutral

Distance on linear orders

- Given $A \subseteq \mathbb{R}^n$ such that $S_n \subseteq A$ and a distance (metric) $d : A \times A \longrightarrow \mathbb{R}$, the **distance** (*metric*) *on* $L(X)$ **induced by** d is the mapping $\bar{d} : L(X) \times L(X) \longrightarrow \mathbb{R}$ defined by

$$\bar{d}(R_1, R_2) = d\big((o_{R_1}(x_1), \ldots, o_{R_1}(x_n)), (o_{R_2}(x_1), \ldots, o_{R_2}(x_n))\big) ,$$

for all $R_1, R_2 \in L(X)$

PRESAD

Kemeny metric

- The **Kemeny metric** on $L(X)$ is the mapping $d^K : L(X) \times L(X) \longrightarrow \mathbb{R}$ defined as the cardinality of the symmetric difference between the linear orders. This metric coincides with the metric on $L(X)$ induced by the distance d_K

$$d^K(R_1, R_2) = \bar{d}_K(R_1, R_2) = d_K\big((a_1, \ldots, a_n), (b_1, \ldots, b_n)\big) =$$

$$\sum_{\substack{i,j=1 \\ i<j}}^{n} \left| \text{sgn } (a_i - a_j) - \text{sgn } (b_i - b_j) \right|$$

$(a_1, \ldots, a_n) \equiv R_1 \in L(X)$ and $(b_1, \ldots, b_n) \equiv R_2 \in L(X)$

Example

Consider four decision makers that rank order the four alternatives of the set $X = \{x_1, x_2, x_3, x_4\}$ through the following linear orders and the corresponding codification vectors

R_1	R_2	R_3	R_4
x_1	x_1	x_2	x_2
x_2	x_2	x_1	x_4
x_3	x_4	x_3	x_3
x_4	x_3	x_4	x_1

R_1	R_2	R_3	R_4
1	1	2	4
2	2	1	1
3	4	3	3
4	3	4	2

Example

Consider four decision makers that rank order the four alternatives of the set $X = \{x_1, x_2, x_3, x_4\}$ through the following linear orders and the corresponding codification vectors

R_1	R_2	R_3	R_4
x_1	x_1	x_2	x_2
x_2	x_2	x_1	x_4
x_3	x_4	x_3	x_3
x_4	x_3	x_4	x_1

R_1	R_2	R_3	R_4
1	1	2	4
2	2	1	1
3	4	3	3
4	3	4	2

$$
\begin{aligned}
d^K(R_1, R_2) = \bar{d}_K(R_1, R_2) = d_K((1,2,3,4),(1,2,4,3)) \quad &= \quad 2 \\
|-1-(-1)| + |-1-(-1)| + |-1-(-1)| + & \\
+|-1-(-1)| + |-1-(-1)| + & \\
+|-1-1| &
\end{aligned}
$$

Example

Consider four decision makers that rank order the four alternatives of the set $X = \{x_1, x_2, x_3, x_4\}$ through the following linear orders and the corresponding codification vectors

R_1	R_2	R_3	R_4
x_1	x_1	x_2	x_2
x_2	x_2	x_1	x_4
x_3	x_4	x_3	x_3
x_4	x_3	x_4	x_1

R_1	R_2	R_3	R_4
1	1	2	4
2	2	1	1
3	4	3	3
4	3	4	2

$$
\begin{aligned}
d^K(R_1, R_3) = \bar{d}_K(R_1, R_3) = d_K((1,2,3,4),(2,1,3,4)) \quad &= \quad 2 \\
|-1-1| + |-1-(-1)| + |-1-(-1)| + & \\
+|-1-(-1)| + |-1-(-1)| + & \\
+|-1-(-1)| &
\end{aligned}
$$

Weighted Kemeny distances

Let $\boldsymbol{w} = (w_1, \ldots, w_{n-1}) \in [0,1]^{n-1}$ be a weighting vector such that $w_1 \geq \cdots \geq w_{n-1}$ and $\sum_{i=1}^{n-1} w_i = 1$. The **weighted Kemeny distance** on $L(X)$ associated with $\boldsymbol{w}$ is the mapping $\bar{d}_{K,\boldsymbol{w}} : L(X) \times L(X) \longrightarrow \mathbb{R}$ defined by

$$\bar{d}_{K,\boldsymbol{w}}(R_1, R_2) = \frac{1}{2}\left[\sum_{\substack{i,j=1 \\ i<j}}^{n} w_i \left| \operatorname{sgn}\left(a_i^{\sigma_1} - a_j^{\sigma_1}\right) - \operatorname{sgn}\left(b_i^{\sigma_1} - b_j^{\sigma_1}\right)\right| + \sum_{\substack{i,j=1 \\ i<j}}^{n} w_i \left| \operatorname{sgn}\left(b_i^{\sigma_2} - b_j^{\sigma_2}\right) - \operatorname{sgn}\left(a_i^{\sigma_2} - a_j^{\sigma_2}\right)\right|\right],$$

where $(a_1, \ldots, a_n) \equiv R_1 \in L(X)$, $(b_1, \ldots, b_n) \equiv R_2 \in L(X)$ and $\sigma_1, \sigma_2 \in S_n$ are such that $R_1^{\sigma_1} = R_2^{\sigma_2} \equiv (1, 2, \ldots, n)$

Example

Consider two linear orders, their corresponding codification vectors, the permutations $\sigma_1 = (3, 1, 2, 4)$ and $\sigma_2 = (3, 2, 4, 1)$ and a weighting vector $\boldsymbol{w} = (w_1, w_2, w_3)$

R_1	R_2	R_1	R_2	$R_1^{\sigma_1}$	$R_2^{\sigma_1}$	$R_2^{\sigma_2}$	$R_1^{\sigma_2}$
x_3	x_3	2	4	1	1	1	1
x_1	x_2	3	2	2	4	2	3
x_2	x_4	1	1	3	2	3	4
x_4	x_1	4	3	4	3	4	2

$$\begin{aligned} \bar{d}_{K,\boldsymbol{w}}(R_1, R_2) &= \frac{1}{2}\left[w_2\big(|-1-1| + |-1-1|\big) + w_2\big(|-1-1|\big) + w_3\big(|-1-1|\big)\right] \\ &= 3\,w_2 + w_3 \end{aligned}$$

Example

Consider two linear orders, their corresponding codification vectors, the permutations $\sigma_1 = (3,1,2,4)$ and $\sigma_2 = (3,2,4,1)$ and a weighting vector $\boldsymbol{w} = (w_1, w_2, w_3)$

R_1	R_2	R_1	R_2	$R_1^{\sigma_1}$	$R_2^{\sigma_1}$	$R_2^{\sigma_2}$	$R_1^{\sigma_2}$
x_3	x_3	2	4	1	1	1	1
x_1	x_2	3	2	2	4	2	3
x_2	x_4	1	1	3	2	3	4
x_4	x_1	4	3	4	3	4	2

$$\begin{aligned}\bar{d}_{K,\boldsymbol{w}}(R_1,R_2) &= \frac{1}{2}\left[w_2\big(|-1-1|+|-1-1|\big)+w_2\big(|-1-1|\big)+w_3\big(|-1-1|\big)\right]\\ &= 3\,w_2+w_3\end{aligned}$$

Example

Consider two linear orders, their corresponding codification vectors, the permutations $\sigma_1 = (3,1,2,4)$ and $\sigma_2 = (3,2,4,1)$ and a weighting vector $\boldsymbol{w} = (w_1, w_2, w_3)$

R_1	R_2	R_1	R_2	$R_1^{\sigma_1}$	$R_2^{\sigma_1}$	$R_2^{\sigma_2}$	$R_1^{\sigma_2}$
x_3	x_3	2	4	1	1	1	1
x_1	x_2	3	2	2	4	2	3
x_2	x_4	1	1	3	2	3	4
x_4	x_1	4	3	4	3	4	2

$$\begin{aligned} \bar{d}_{K,\boldsymbol{w}}(R_1,R_2) &= \frac{1}{2}\left[w_2\big(|-1-1| + |-1-1|\big) + w_2\big(|-1-1|\big) + w_3\big(|-1-1|\big)\right] \\ &= 3\,w_2 + w_3 \end{aligned}$$

Example

Consider again the profile given in the first example, and the weighting vector $\boldsymbol{w} = (\frac{3}{6}, \frac{2}{6}, \frac{1}{6})$

R_1	R_2	R_3	R_4
x_1	x_1	x_2	x_2
x_2	x_2	x_1	x_4
x_3	x_4	x_3	x_3
x_4	x_3	x_4	x_1

R_1	R_2	R_3	R_4
1	1	2	4
2	2	1	1
3	4	3	3
4	3	4	2

Example

Consider again the profile given in the first example, and the weighting vector $\boldsymbol{w} = (\frac{3}{6}, \frac{2}{6}, \frac{1}{6})$

R_1	R_2	R_3	R_4
x_1	x_1	x_2	x_2
x_2	x_2	x_1	x_4
x_3	x_4	x_3	x_3
x_4	x_3	x_4	x_1

R_1	R_2	R_3	R_4
1	1	2	4
2	2	1	1
3	4	3	3
4	3	4	2

$\bar{d}_{K,\boldsymbol{w}}(R_1, R_2) = \frac{1}{3}$

Example

Consider again the profile given in the first example, and the weighting vector $\boldsymbol{w} = (\frac{3}{6}, \frac{2}{6}, \frac{1}{6})$

R_1	R_2	R_3	R_4
x_1	x_1	x_2	x_2
x_2	x_2	x_1	x_4
x_3	x_4	x_3	x_3
x_4	x_3	x_4	x_1

R_1	R_2	R_3	R_4
1	1	2	4
2	2	1	1
3	4	3	3
4	3	4	2

$\bar{d}_{K,\boldsymbol{w}}(R_1, R_3) = 1$

Example

Consider again the profile given in the first example, and the weighting vector $\boldsymbol{w} = (\frac{3}{6}, \frac{2}{6}, \frac{1}{6})$

R_1	R_2	R_3	R_4
x_1	x_1	x_2	x_2
x_2	x_2	x_1	x_4
x_3	x_4	x_3	x_3
x_4	x_3	x_4	x_1

R_1	R_2	R_3	R_4
1	1	2	4
2	2	1	1
3	4	3	3
4	3	4	2

	$R_1,\ R_2$	$R_1,\ R_3$	$R_1,\ R_4$	$R_2,\ R_3$	$R_2,\ R_4$	$R_3,\ R_4$
$\bar{d}_K$	2	2	8	4	6	6
$\bar{d}_{K,\boldsymbol{w}}$	$\frac{1}{3}$	1	3	$\frac{4}{3}$	$\frac{5}{2}$	$\frac{5}{3}$

Table: $\bar{d}_K$ versus $\bar{d}_{K,\boldsymbol{w}}$ outcomes

Properties

- Let $\boldsymbol{w} = (w_1, \ldots, w_{n-1}) \in [0,1]^{n-1}$ be a weighting vector such that $w_1 \geq \cdots \geq w_{n-1}$ and $\sum_{i=1}^{n-1} w_i = 1$
  1. $\bar{d}_{K,\boldsymbol{w}}$ is a neutral distance on $L(X)$
  2. $\bar{d}_{K,\boldsymbol{w}}$ does not always verify the triangle inequality[1]

	R_1, R_2	R_1, R_3	R_1, R_4	R_2, R_3	R_2, R_4	R_3, R_4
$\bar{d}_K$	2	2	8	4	6	6
$\bar{d}_{K,\boldsymbol{w}}$	$\frac{1}{3}$	1	3	$\frac{4}{3}$	$\frac{5}{2}$	$\frac{5}{3}$

  3. $\bar{d}_{K,\boldsymbol{w}}$ verifies the property identity of indiscernibles if and only if $w_{n-1} > 0$.
  4. $\Delta_n = \bar{d}_{K,\boldsymbol{w}}(R_1, R_1^{-1}) = 2\sum_{i=1}^{n-1}(n-i)w_i$

[1] The triangle inequality is not necessarily a natural condition for certain problems (see Barthelemy and Monjardet (1981))

Properties

- Let $\boldsymbol{w} = (w_1, \ldots, w_{n-1}) \in [0,1]^{n-1}$ be a weighting vector such that $w_1 \geq \cdots \geq w_{n-1}$ and $\sum_{i=1}^{n-1} w_i = 1$
  1. $\bar{d}_{K,\boldsymbol{w}}$ is a neutral distance on $L(X)$
  2. $\bar{d}_{K,\boldsymbol{w}}$ does not always verify the triangle inequality[1]

	R_1, R_2	R_1, R_3	R_1, R_4	R_2, R_3	R_2, R_4	R_3, R_4
$\bar{d}_K$	2	2	8	4	6	6
$\bar{d}_{K,\boldsymbol{w}}$	$\frac{1}{3}$	1	3	$\frac{4}{3}$	$\frac{5}{2}$	$\frac{5}{3}$

  3. $\bar{d}_{K,\boldsymbol{w}}$ verifies the property identity of indiscernibles if and only if $w_{n-1} > 0$.
  4. $\Delta_n = \bar{d}_{K,\boldsymbol{w}}(R_1, R_1^{-1}) = 2\sum_{i=1}^{n-1}(n-i)w_i$

[1] The triangle inequality is not necessarily a natural condition for certain problems (see Barthelemy and Monjardet (1981))

Properties

- Let $\boldsymbol{w} = (w_1, \ldots, w_{n-1}) \in [0,1]^{n-1}$ be a weighting vector such that $w_1 \geq \cdots \geq w_{n-1}$ and $\sum_{i=1}^{n-1} w_i = 1$
  1. $\bar{d}_{K,\boldsymbol{w}}$ is a neutral distance on $L(X)$
  2. $\bar{d}_{K,\boldsymbol{w}}$ does not always verify the triangle inequality[1]

	$R_1,\, R_2$	$R_1,\, R_3$	$R_1,\, R_4$	$R_2,\, R_3$	$R_2,\, R_4$	$R_3,\, R_4$
$\bar{d}_K$	2	2	8	4	6	6
$\bar{d}_{K,\boldsymbol{w}}$	$\frac{1}{3}$	1	3	$\frac{4}{3}$	$\frac{5}{2}$	$\frac{5}{3}$

  3. $\bar{d}_{K,\boldsymbol{w}}$ verifies the property identity of indiscernibles if and only if $w_{n-1} > 0$.
  4. $\Delta_n = \bar{d}_{K,\boldsymbol{w}}(R_1, R_1^{-1}) = 2\sum_{i=1}^{n-1}(n-i)w_i$

[1] The triangle inequality is not necessarily a natural condition for certain problems (see Barthelemy and Monjardet (1981))

Properties

- Let $\boldsymbol{w} = (w_1, \ldots, w_{n-1}) \in [0,1]^{n-1}$ be a weighting vector such that $w_1 \geq \cdots \geq w_{n-1}$ and $\sum_{i=1}^{n-1} w_i = 1$
  1. $\bar{d}_{K,\boldsymbol{w}}$ is a neutral distance on $L(X)$
  2. $\bar{d}_{K,\boldsymbol{w}}$ does not always verify the triangle inequality[1]

	$R_1,\, R_2$	$R_1,\, R_3$	$R_1,\, R_4$	$R_2,\, R_3$	$R_2,\, R_4$	$R_3,\, R_4$
$\bar{d}_K$	2	2	8	4	6	6
$\bar{d}_{K,\boldsymbol{w}}$	$\frac{1}{3}$	1	3	$\frac{4}{3}$	$\frac{5}{2}$	$\frac{5}{3}$

  3. $\bar{d}_{K,\boldsymbol{w}}$ verifies the property identity of indiscernibles if and only if $w_{n-1} > 0$.
  4. $\Delta_n = \bar{d}_{K,\boldsymbol{w}}(R_1, R_1^{-1}) = 2\sum_{i=1}^{n-1}(n-i)w_i$

[1] The triangle inequality is not necessarily a natural condition for certain problems (see Barthelemy and Monjardet (1981))

Definition

A **consensus measure** on $L(X)^m$ is a mapping

$$\mathcal{M} : L(X)^m \times \mathcal{P}_2(V) \longrightarrow [0, 1]$$

that satisfies the following conditions:

- *Unanimity*. For all $\boldsymbol{R} \in L(X)^m$ and $I \in \mathcal{P}_2(V)$ it holds
$$\mathcal{M}(\boldsymbol{R}, I) = 1 \Leftrightarrow R_i = R_j \text{ for all } v_i, v_j \in I$$
- *Anonymity*. For all permutation π on $\{1, \ldots, m\}$, $\boldsymbol{R} \in L(X)^m$ and $I \in \mathcal{P}_2(V)$ it holds
$$\mathcal{M}(\boldsymbol{R}_\pi, I_\pi) = \mathcal{M}(\boldsymbol{R}, I)$$
- *Neutrality*. For all permutation σ on $\{1, \ldots, n\}$, $\boldsymbol{R} \in L(X)^m$ and $I \in \mathcal{P}_2(V)$ it holds
$$\mathcal{M}(\boldsymbol{R}^\sigma, I) = \mathcal{M}(\boldsymbol{R}, I)$$

Definition

A **consensus measure** on $L(X)^m$ is a mapping

$$\mathcal{M} : L(X)^m \times \mathcal{P}_2(V) \longrightarrow [0,1]$$

that satisfies the following conditions:

- *Unanimity*. For all $\boldsymbol{R} \in L(X)^m$ and $I \in \mathcal{P}_2(V)$ it holds

$$\mathcal{M}(\boldsymbol{R}, I) = 1 \;\Leftrightarrow\; R_i = R_j \text{ for all } v_i, v_j \in I$$

- *Anonymity*. For all permutation π on $\{1, \ldots, m\}$, $\boldsymbol{R} \in L(X)^m$ and $I \in \mathcal{P}_2(V)$ it holds

$$\mathcal{M}(\boldsymbol{R}_\pi, I_\pi) = \mathcal{M}(\boldsymbol{R}, I)$$

- *Neutrality*. For all permutation σ on $\{1, \ldots, n\}$, $\boldsymbol{R} \in L(X)^m$ and $I \in \mathcal{P}_2(V)$ it holds

$$\mathcal{M}(\boldsymbol{R}^\sigma, I) = \mathcal{M}(\boldsymbol{R}, I)$$

Definition

A **consensus measure** on $L(X)^m$ is a mapping

$$\mathcal{M} : L(X)^m \times \mathcal{P}_2(V) \longrightarrow [0,1]$$

that satisfies the following conditions:

- *Unanimity*. For all $\boldsymbol{R} \in L(X)^m$ and $I \in \mathcal{P}_2(V)$ it holds

$$\mathcal{M}(\boldsymbol{R}, I) = 1 \;\Leftrightarrow\; R_i = R_j \text{ for all } v_i, v_j \in I$$

- *Anonymity*. For all permutation π on $\{1, \ldots, m\}$, $\boldsymbol{R} \in L(X)^m$ and $I \in \mathcal{P}_2(V)$ it holds

$$\mathcal{M}(\boldsymbol{R}_\pi, I_\pi) = \mathcal{M}(\boldsymbol{R}, I)$$

- *Neutrality*. For all permutation σ on $\{1, \ldots, n\}$, $\boldsymbol{R} \in L(X)^m$ and $I \in \mathcal{P}_2(V)$ it holds

$$\mathcal{M}(\boldsymbol{R}^\sigma, I) = \mathcal{M}(\boldsymbol{R}, I)$$

Definition

A **consensus measure** on $L(X)^m$ is a mapping

$$\mathcal{M} : L(X)^m \times \mathcal{P}_2(V) \longrightarrow [0,1]$$

that satisfies the following conditions:

- *Unanimity*. For all $\boldsymbol{R} \in L(X)^m$ and $I \in \mathcal{P}_2(V)$ it holds
$$\mathcal{M}(\boldsymbol{R}, I) = 1 \Leftrightarrow R_i = R_j \text{ for all } v_i, v_j \in I$$
- *Anonymity*. For all permutation π on $\{1, \ldots, m\}$, $\boldsymbol{R} \in L(X)^m$ and $I \in \mathcal{P}_2(V)$ it holds
$$\mathcal{M}(\boldsymbol{R}_\pi, I_\pi) = \mathcal{M}(\boldsymbol{R}, I)$$
- *Neutrality*. For all permutation σ on $\{1, \ldots, n\}$, $\boldsymbol{R} \in L(X)^m$ and $I \in \mathcal{P}_2(V)$ it holds
$$\mathcal{M}(\boldsymbol{R}^\sigma, I) = \mathcal{M}(\boldsymbol{R}, I)$$

Other properties

Other properties that a consensus measure may satisfy

- **Maximum dissension**: For all $\boldsymbol{R} \in L(X)^m$ and $v_i, v_j \in V$ such that $i \neq j$ it holds

$$\mathcal{M}(\boldsymbol{R}, \{v_i, v_j\}) = 0 \;\Leftrightarrow\; R_i, R_j \in L(X) \text{ and } R_j = R_i^{-1}$$

- **Reciprocity**: For all $\boldsymbol{R} \in L(X)^m$ and $I \in \mathcal{P}_2(V)$ it holds

$$\mathcal{M}(\boldsymbol{R}^{-1}, I) = \mathcal{M}(\boldsymbol{R}, I)$$

- Given a distance $\bar{d} : L(X) \times L(X) \longrightarrow [0, \infty)$, the mapping

$$\mathcal{M}_{\bar{d}} : L(X)^m \times \mathcal{P}_2(V) \longrightarrow [0,1]$$

is defined by

$$\mathcal{M}_{\bar{d}}(\boldsymbol{R}, I) = 1 - \frac{\displaystyle\sum_{\substack{v_i, v_j \in I \\ i<j}} \bar{d}(R_i, R_j)}{\displaystyle\binom{|I|}{2} \cdot \Delta_n}$$

where $\Delta_n = \max\{\bar{d}(R_i, R_j) \mid R_i, R_j \in L(X)\}$

- If $\mathcal{M}_{\bar{d}}$ is a consensus measure, then we say that $\mathcal{M}_{\bar{d}}$ is the **consensus measure associated with $\bar{d}$**
- For every distance $\bar{d} : L(X) \times L(X) \longrightarrow \mathbb{R}$, $\mathcal{M}_{\bar{d}}$ satisfies **unanimity** if and only if $\bar{d}$ satisfies the property **identity of indiscernibles**
- For every distance $\bar{d} : L(X) \times L(X) \longrightarrow [0, \infty)$, $\mathcal{M}_{\bar{d}}$ satisfies **anonymity**
- If $\bar{d} : W(X) \times W(X) \longrightarrow \mathbb{R}$ is a **neutral** distance that satisfies **identity of indiscernibles**, then $\mathcal{M}_{\bar{d}}$ is a **consensus measure**

Properties

- Let $\boldsymbol{w} = (w_1, \ldots, w_{n-1}) \in [0,1]^{n-1}$ be a weighting vector such that $w_1 \geq \cdots \geq w_{n-1}$ and $\sum_{i=1}^{n-1} w_i = 1$
  1. $\mathcal{M}_{\bar{d}_{K,\boldsymbol{w}}}$ satisfies **anonymity**.
  2. $\mathcal{M}_{\bar{d}_{K,\boldsymbol{w}}}$ satisfies **unanimity** if and only if $w_{n-1} > 0$.
  3. $\mathcal{M}_{\bar{d}_{K,\boldsymbol{w}}}$ is **reciprocal** if and only if $w_1 = \cdots = w_{n-1} = \frac{1}{n-1}$.
  4. $\mathcal{M}_{\bar{d}_{K,\boldsymbol{w}}}$ satisfies the **maximum dissension** property if and only if $w_{\lfloor \frac{n+1}{2} \rfloor} > 0$.

PRESAD

		Unanimity	Max. diss.	Reciproc.	Consensus measure
$\mathcal{M}_{\bar{d}_K}$		Yes	Yes	Yes	Yes
$\mathcal{M}_{\bar{d}_{K,\boldsymbol{w}}}$	$w_{n-1} > 0$	Yes	Yes		Yes
$\mathcal{M}_{\bar{d}_{K,\boldsymbol{w}}}$	$w_1 = \cdots = w_{n-1} = \frac{1}{n-1}$	Yes	Yes	Yes	Yes
$\mathcal{M}_{\bar{d}_{K,\boldsymbol{w}}}$	$w_1 > w_{n-1} > 0$	Yes	Yes	No	Yes
$\mathcal{M}_{\bar{d}_{K,\boldsymbol{w}}}$	$w_{n-1} = 0$	No		No	No
$\mathcal{M}_{\bar{d}_{K,\boldsymbol{w}}}$	$w_{\lfloor \frac{n+1}{2} \rfloor} = 0$	No	No	No	
$\mathcal{M}_{\bar{d}'}$	(discrete)	Yes	No	Yes	Yes
$\mathcal{M}_{\bar{d}_1}$	(Manhattan)	Yes	No	Yes	Yes
$\mathcal{M}_{\bar{d}_2}$	(Euclidean)	Yes	Yes	Yes	Yes
$\mathcal{M}_{\bar{d}_\infty}$	(Chebyshev)	Yes	No	Yes	Yes
$\mathcal{M}_{\bar{d}_c}$	(cosine)	Yes	Yes	Yes	Yes

Table: Summary

		Unanimity	Max. diss.	Reciproc.	Consensus measure
$\mathcal{M}_{\bar{d}_K}$		Yes	Yes	Yes	Yes
$\mathcal{M}_{\bar{d}_{K,\boldsymbol{w}}}$	$w_{n-1} > 0$	Yes	Yes		Yes
$\mathcal{M}_{\bar{d}_{K,\boldsymbol{w}}}$	$w_1 = \cdots = w_{n-1} = \frac{1}{n-1}$	Yes	Yes	Yes	Yes
$\mathcal{M}_{\bar{d}_{K,\boldsymbol{w}}}$	$w_1 > w_{n-1} > 0$	Yes	Yes	No	Yes
$\mathcal{M}_{\bar{d}_{K,\boldsymbol{w}}}$	$w_{n-1} = 0$	No		No	No
$\mathcal{M}_{\bar{d}_{K,\boldsymbol{w}}}$	$w_{\lfloor \frac{n+1}{2} \rfloor} = 0$	No	No	No	
$\mathcal{M}_{\bar{d}'}$	(discrete)	Yes	No	Yes	Yes
$\mathcal{M}_{\bar{d}_1}$	(Manhattan)	Yes	No	Yes	Yes
$\mathcal{M}_{\bar{d}_2}$	(Euclidean)	Yes	Yes	Yes	Yes
$\mathcal{M}_{\bar{d}_\infty}$	(Chebyshev)	Yes	No	Yes	Yes
$\mathcal{M}_{\bar{d}_c}$	(cosine)	Yes	Yes	Yes	Yes

Table: Summary

Some conclusions

- It is interesting to note that the introduced consensus measures generated by weighted Kemeny distances can be used for designing appropriate decision making processes that require a minimum agreement among decision makers. For instance, in García-Lapresta and Pérez-Román (2008) we propose a voting system where voters' opinions are weighted by the marginal contributions to consensus
- With respect to the computational aspect, we are preparing a computer program to obtain the consensus in real decisions when voters rank order the feasible alternatives
- We are also working in an extension of the weighted consensus measures to the framework of weak orders

Thank for your attention