Different approaches to influence in social networks

Agnieszka RUSINOWSKA

Centre d'Economie de la Sorbonne
CNRS - Université Paris I Panthéon-Sorbonne

106-112 Bd de l'Hôpital, 75647 Paris, France
agnieszka.rusinowska@univ-paris1.fr

Aim and context of the tutorial

- The aim of this tutorial is to deliver a short overview of different approaches to influence applied in several scientific domains, e.g., in the economics and game-theoretic literature, with a particular focus on studying influence in networks.

Aim and context of the tutorial

- The aim of this tutorial is to deliver a short overview of different approaches to influence applied in several scientific domains, e.g., in the economics and game-theoretic literature, with a particular focus on studying influence in networks.
- Context of the tutorial:

Aim and context of the tutorial

- The aim of this tutorial is to deliver a short overview of different approaches to influence applied in several scientific domains, e.g., in the economics and game-theoretic literature, with a particular focus on studying influence in networks.
- Context of the tutorial:
  - Selected works on influence in sociology, social psychology, political science, marketing (short notes)

Aim and context of the tutorial

- The aim of this tutorial is to deliver a short overview of different approaches to influence applied in several scientific domains, e.g., in the economics and game-theoretic literature, with a particular focus on studying influence in networks.
- Context of the tutorial:
  - Selected works on influence in sociology, social psychology, political science, marketing (short notes)
  - Studying influence in economics

Aim and context of the tutorial

- The aim of this tutorial is to deliver a short overview of different approaches to influence applied in several scientific domains, e.g., in the economics and game-theoretic literature, with a particular focus on studying influence in networks.
- Context of the tutorial:
  - Selected works on influence in sociology, social psychology, political science, marketing (short notes)
  - Studying influence in economics
    - Leadership in economics

Aim and context of the tutorial

- The aim of this tutorial is to deliver a short overview of different approaches to influence applied in several scientific domains, e.g., in the economics and game-theoretic literature, with a particular focus on studying influence in networks.
- Context of the tutorial:
  - Selected works on influence in sociology, social psychology, political science, marketing (short notes)
  - Studying influence in economics
    - Leadership in economics
    - Game-theoretic (cooperative and non-cooperative) approaches to influence - influence relation in simple games and in voting games with abstention, command games

Aim and context of the tutorial

- The aim of this tutorial is to deliver a short overview of different approaches to influence applied in several scientific domains, e.g., in the economics and game-theoretic literature, with a particular focus on studying influence in networks.
- Context of the tutorial:
  - Selected works on influence in sociology, social psychology, political science, marketing (short notes)
  - Studying influence in economics
    - Leadership in economics
    - Game-theoretic (cooperative and non-cooperative) approaches to influence - influence relation in simple games and in voting games with abstention, command games
  - Research on influence in social networks

Aim and context of the tutorial

- The aim of this tutorial is to deliver a short overview of different approaches to influence applied in several scientific domains, e.g., in the economics and game-theoretic literature, with a particular focus on studying influence in networks.
- Context of the tutorial:
  - Selected works on influence in sociology, social psychology, political science, marketing (short notes)
  - Studying influence in economics
    - Leadership in economics
    - Game-theoretic (cooperative and non-cooperative) approaches to influence - influence relation in simple games and in voting games with abstention, command games
  - Research on influence in social networks
    - The Hoede-Bakker framework of influence and model of influence functions

Aim and context of the tutorial

- The aim of this tutorial is to deliver a short overview of different approaches to influence applied in several scientific domains, e.g., in the economics and game-theoretic literature, with a particular focus on studying influence in networks.
- Context of the tutorial:
  - Selected works on influence in sociology, social psychology, political science, marketing (short notes)
  - Studying influence in economics
    - Leadership in economics
    - Game-theoretic (cooperative and non-cooperative) approaches to influence - influence relation in simple games and in voting games with abstention, command games
  - Research on influence in social networks
    - The Hoede-Bakker framework of influence and model of influence functions
    - Relation-algebraic approach and applications of RelView to the influence concepts

Aim and context of the tutorial

- The aim of this tutorial is to deliver a short overview of different approaches to influence applied in several scientific domains, e.g., in the economics and game-theoretic literature, with a particular focus on studying influence in networks.
- Context of the tutorial:
  - Selected works on influence in sociology, social psychology, political science, marketing (short notes)
  - Studying influence in economics
    - Leadership in economics
    - Game-theoretic (cooperative and non-cooperative) approaches to influence - influence relation in simple games and in voting games with abstention, command games
  - Research on influence in social networks
    - The Hoede-Bakker framework of influence and model of influence functions
    - Relation-algebraic approach and applications of RelView to the influence concepts
    - The DeGroot model and its variations

Aim and context of the tutorial

- The aim of this tutorial is to deliver a short overview of different approaches to influence applied in several scientific domains, e.g., in the economics and game-theoretic literature, with a particular focus on studying influence in networks.
- Context of the tutorial:
  - Selected works on influence in sociology, social psychology, political science, marketing (short notes)
  - Studying influence in economics
    - Leadership in economics
    - Game-theoretic (cooperative and non-cooperative) approaches to influence - influence relation in simple games and in voting games with abstention, command games
  - Research on influence in social networks
    - The Hoede-Bakker framework of influence and model of influence functions
    - Relation-algebraic approach and applications of RelView to the influence concepts
    - The DeGroot model and its variations
    - Social learning, "Herd behavior", "Informational cascades"

Different approaches to influence

- Influence is faced in all kinds of real life situations, and as a consequence it has been studied in many scientific areas: in sociology and social psychology, in political science, in economics, in management and business science, etc.

Different approaches to influence

- Influence is faced in all kinds of real life situations, and as a consequence it has been studied in many scientific areas: in sociology and social psychology, in political science, in economics, in management and business science, etc.
- Different approaches are applied to study influence concepts: theoretical investigations, empirical study, experiments.

Social influence in social psychology

- *Influence* = force of change that comes from the outside of the person or thing that is influenced by it

Social influence in social psychology

- *Influence* = force of change that comes from the outside of the person or thing that is influenced by it
- *Social influence* occurs when an individual's thoughts or actions are affected by other people, and it may take many forms, e.g., it can be seen in *conformity* and *leadership.*

Social influence in social psychology

- *Influence* = force of change that comes from the outside of the person or thing that is influenced by it
- *Social influence* occurs when an individual's thoughts or actions are affected by other people, and it may take many forms, e.g., it can be seen in *conformity* and *leadership.*
- Deutsch & Gerard (1955) describe two psychological needs that lead humans to conform:

Social influence in social psychology

- *Influence* = force of change that comes from the outside of the person or thing that is influenced by it
- *Social influence* occurs when an individual's thoughts or actions are affected by other people, and it may take many forms, e.g., it can be seen in *conformity* and *leadership.*
- Deutsch & Gerard (1955) describe two psychological needs that lead humans to conform:
  - "our need to be right" (informational social influence)

Social influence in social psychology

- *Influence* = force of change that comes from the outside of the person or thing that is influenced by it
- *Social influence* occurs when an individual's thoughts or actions are affected by other people, and it may take many forms, e.g., it can be seen in *conformity* and *leadership.*
- Deutsch & Gerard (1955) describe two psychological needs that lead humans to conform:
  - "our need to be right" (informational social influence)
  - "our need to be liked" (normative social influence).

Social influence in social psychology

- *Influence* = force of change that comes from the outside of the person or thing that is influenced by it
- *Social influence* occurs when an individual's thoughts or actions are affected by other people, and it may take many forms, e.g., it can be seen in *conformity* and *leadership*.
- Deutsch & Gerard (1955) describe two psychological needs that lead humans to conform:
  - "our need to be right" (informational social influence)
  - "our need to be liked" (normative social influence).
- Kelman (1958) identifies three varieties of social influence:

Social influence in social psychology

- *Influence* = force of change that comes from the outside of the person or thing that is influenced by it
- *Social influence* occurs when an individual's thoughts or actions are affected by other people, and it may take many forms, e.g., it can be seen in *conformity* and *leadership.*
- Deutsch & Gerard (1955) describe two psychological needs that lead humans to conform:
  - "our need to be right" (informational social influence)
  - "our need to be liked" (normative social influence).
- Kelman (1958) identifies three varieties of social influence:
  - *Compliance* = people appear to agree with others, but actually keep their dissenting opinions private (normative conformity)

Social influence in social psychology

- *Influence* = force of change that comes from the outside of the person or thing that is influenced by it
- *Social influence* occurs when an individual's thoughts or actions are affected by other people, and it may take many forms, e.g., it can be seen in *conformity* and *leadership*.
- Deutsch & Gerard (1955) describe two psychological needs that lead humans to conform:
  - "our need to be right" (informational social influence)
  - "our need to be liked" (normative social influence).
- Kelman (1958) identifies three varieties of social influence:
  - *Compliance* = people appear to agree with others, but actually keep their dissenting opinions private (normative conformity)
  - *Identification* = people are influenced by someone who is liked and respected

Social influence in social psychology

- *Influence* = force of change that comes from the outside of the person or thing that is influenced by it
- *Social influence* occurs when an individual's thoughts or actions are affected by other people, and it may take many forms, e.g., it can be seen in *conformity* and *leadership.*
- Deutsch & Gerard (1955) describe two psychological needs that lead humans to conform:
  - "our need to be right" (informational social influence)
  - "our need to be liked" (normative social influence).
- Kelman (1958) identifies three varieties of social influence:
  - *Compliance* = people appear to agree with others, but actually keep their dissenting opinions private (normative conformity)
  - *Identification* = people are influenced by someone who is liked and respected
  - *Internalization* = people accept a belief or behavior and agree both publicly and privately (informational conformity).

Experiments in sociology and social psychology

- Sherif's autokinetic experiment (Sherif, 1936) - first experiment on informational social influence. Participants placed in a dark room are asked to estimate the amount a small dot of light moved. *How many people change their opinions to bring them in line with the opinion of a group?*

Experiments in sociology and social psychology

- Sherif's autokinetic experiment (Sherif, 1936) - first experiment on informational social influence. Participants placed in a dark room are asked to estimate the amount a small dot of light moved. *How many people change their opinions to bring them in line with the opinion of a group?*
- Asch (1955) - first experiment on normative social influence. A modification of Sherif's study (when the situation is very clear, conformity will be drastically reduced). But, his results show a surprisingly high degree of conformity.

Experiments in sociology and social psychology

- Sherif's autokinetic experiment (Sherif, 1936) - first experiment on informational social influence. Participants placed in a dark room are asked to estimate the amount a small dot of light moved. *How many people change their opinions to bring them in line with the opinion of a group?*
- Asch (1955) - first experiment on normative social influence. A modification of Sherif's study (when the situation is very clear, conformity will be drastically reduced). But, his results show a surprisingly high degree of conformity.
- McKelvey & Kerr (1988) - using similar procedures they find significantly less conformity in groups of friends as compared to groups of strangers.

Other examples in sociology and social psychology

- Latane (1981) - normative influence is a function of social impact theory with three components:

Other examples in sociology and social psychology

- Latane (1981) - normative influence is a function of social impact theory with three components:
  - number of people in the group (the higher the number, the less of an impact each person has)

Other examples in sociology and social psychology

- Latane (1981) - normative influence is a function of social impact theory with three components:
  - number of people in the group (the higher the number, the less of an impact each person has)
  - importance of the group to a person (groups valued higher have more social influence)

Other examples in sociology and social psychology

- Latane (1981) - normative influence is a function of social impact theory with three components:
  - number of people in the group (the higher the number, the less of an impact each person has)
  - importance of the group to a person (groups valued higher have more social influence)
  - immediacy (how close the group is in time and space when the influence is taking place).

Other examples in sociology and social psychology

- Latane (1981) - normative influence is a function of social impact theory with three components:
  - number of people in the group (the higher the number, the less of an impact each person has)
  - importance of the group to a person (groups valued higher have more social influence)
  - immediacy (how close the group is in time and space when the influence is taking place).
- Latane & Bourgeois (2001) - using these three factors, they construct a mathematical model to predict the amount of conformity that occurs with some degree of accuracy.

Theoretical models in sociology (1/2)

- Models of social influence and persuasion - French (1956), Harary (1959)

Theoretical models in sociology (1/2)

- Models of social influence and persuasion - French (1956), Harary (1959)
- Measures of prestige and centrality in a network - Katz (1953), Bonacich (1972, 1987), Bonacich & Lloyd (2001)

Theoretical models in sociology (1/2)

- Models of social influence and persuasion - French (1956), Harary (1959)
- Measures of prestige and centrality in a network - Katz (1953), Bonacich (1972, 1987), Bonacich & Lloyd (2001)
  - How important, central, or influential a node's neighbors are.

Theoretical models in sociology (1/2)

- Models of social influence and persuasion - French (1956), Harary (1959)
- Measures of prestige and centrality in a network - Katz (1953), Bonacich (1972, 1987), Bonacich & Lloyd (2001)
  - How important, central, or influential a node's neighbors are.
  - The prestige of a node is a sum of the prestige of its neighbors divided by their respective degrees.

Theoretical models in sociology (1/2)

- Models of social influence and persuasion - French (1956), Harary (1959)
- Measures of prestige and centrality in a network - Katz (1953), Bonacich (1972, 1987), Bonacich & Lloyd (2001)
  - How important, central, or influential a node's neighbors are.
  - The prestige of a node is a sum of the prestige of its neighbors divided by their respective degrees.
  - The centrality of a node is proportional to the sum of the centrality of its neighbors.

Theoretical models in sociology (1/2)

- Models of social influence and persuasion - French (1956), Harary (1959)
- Measures of prestige and centrality in a network - Katz (1953), Bonacich (1972, 1987), Bonacich & Lloyd (2001)
  - How important, central, or influential a node's neighbors are.
  - The prestige of a node is a sum of the prestige of its neighbors divided by their respective degrees.
  - The centrality of a node is proportional to the sum of the centrality of its neighbors.
- Threshold models of collective behavior - Granovetter (1978)

Theoretical models in sociology (1/2)

- Models of social influence and persuasion - French (1956), Harary (1959)
- Measures of prestige and centrality in a network - Katz (1953), Bonacich (1972, 1987), Bonacich & Lloyd (2001)
  - How important, central, or influential a node's neighbors are.
  - The prestige of a node is a sum of the prestige of its neighbors divided by their respective degrees.
  - The centrality of a node is proportional to the sum of the centrality of its neighbors.
- Threshold models of collective behavior - Granovetter (1978)
  - Agents have two alternatives and the costs and benefits of each depend on how many other agents choose which alternative.

Theoretical models in sociology (1/2)

- Models of social influence and persuasion - French (1956), Harary (1959)
- Measures of prestige and centrality in a network - Katz (1953), Bonacich (1972, 1987), Bonacich & Lloyd (2001)
  - How important, central, or influential a node's neighbors are.
  - The prestige of a node is a sum of the prestige of its neighbors divided by their respective degrees.
  - The centrality of a node is proportional to the sum of the centrality of its neighbors.
- Threshold models of collective behavior - Granovetter (1978)
  - Agents have two alternatives and the costs and benefits of each depend on how many other agents choose which alternative.
  - The author focuses on the effect of the individual thresholds (i.e., the proportion or number of others that make their decision before a given agent) on the collective behavior, he discusses an equilibrium in a process occurring over time and the stability of equilibrium outcomes.

Theoretical models in sociology (2/2)

- A sociological model of interactions on networks by Conlisk (1976, 1978, 1992)

Theoretical models in sociology (2/2)

- A sociological model of interactions on networks by Conlisk (1976, 1978, 1992)
  - The author introduces the interactive Markov chain, a certain deterministic and discrete-time dynamical system, in which each entry in a state vector at each time represents the fraction of the population with some attribute. The matrix depends on the current state vector which means that the current social structure is taken into account for evolution in sociological dynamics.

Theoretical models in sociology (2/2)

- A sociological model of interactions on networks by Conlisk (1976, 1978, 1992)
  - The author introduces the interactive Markov chain, a certain deterministic and discrete-time dynamical system, in which each entry in a state vector at each time represents the fraction of the population with some attribute. The matrix depends on the current state vector which means that the current social structure is taken into account for evolution in sociological dynamics.
- Interpersonal influence model - Friedkin (1999), Friedkin & Johnsen (1990, 1999), Friedkin & Cook (1990).

Theoretical models in sociology (2/2)

- A sociological model of interactions on networks by Conlisk (1976, 1978, 1992)
  - The author introduces the interactive Markov chain, a certain deterministic and discrete-time dynamical system, in which each entry in a state vector at each time represents the fraction of the population with some attribute. The matrix depends on the current state vector which means that the current social structure is taken into account for evolution in sociological dynamics.
- Interpersonal influence model - Friedkin (1999), Friedkin & Johnsen (1990, 1999), Friedkin & Cook (1990).
  - The authors study a framework, in which social attitudes depend on the attitudes of neighbors and evolve over time. In their model, agents start with initial attitudes and then mix in some of their neighbors' recent attitudes with their starting attitudes.

Studying influence in political science

- van Winden (2004) - nice survey of interest group behavior and influence. In particular, the author investigates two influence channels of affecting policies by interest groups:

Studying influence in political science

- van Winden (2004) - nice survey of interest group behavior and influence. In particular, the author investigates two influence channels of affecting policies by interest groups:
  - directly by influencing the behavior of policymakers (the influence function models)

Studying influence in political science

- van Winden (2004) - nice survey of interest group behavior and influence. In particular, the author investigates two influence channels of affecting policies by interest groups:
  - directly by influencing the behavior of policymakers (the influence function models)
  - indirectly by influencing the behavior of voters (the vote function models).

Studying influence in political science

- van Winden (2004) - nice survey of interest group behavior and influence. In particular, the author investigates two influence channels of affecting policies by interest groups:
  - directly by influencing the behavior of policymakers (the influence function models)
  - indirectly by influencing the behavior of voters (the vote function models).
- For more surveys of theoretical and empirical literature on this issue, see, e.g., Sloof (1998), Drazen (2000), Persson & Tabellini (2000), Grossman & Helpman (2001); see also Potters & Sloof (1996), and Austen-Smith (1997).

Leadership in sociology and marketing (1/2)

- One of the concepts closely related to influence is the concept of leadership; see, e.g., Burns (1978) and Bass (1985) for books on the behavioral theory of leadership.

- One of the concepts closely related to influence is the concept of leadership; see, e.g., Burns (1978) and Bass (1985) for books on the behavioral theory of leadership.
- Opinion leaders form an attractive group for marketing and policy purposes, because their existence (or non-existence) in a society and their relations to their followers may have a considerable impact on market behavior.

Leadership in sociology and marketing (2/2)

- The two-step flow of communication theory - the communication process is a two-step process, in which information distributed by mass media first reaches the opinion leaders; see Lazarsfeld et al. (1944), Katz & Lazarsfeld (1955).

Leadership in sociology and marketing (2/2)

- The two-step flow of communication theory - the communication process is a two-step process, in which information distributed by mass media first reaches the opinion leaders; see Lazarsfeld et al. (1944), Katz & Lazarsfeld (1955).
  - Lazarsfeld et al. (1944) investigate the influence of mass communication on the 1940 presidential election campaign in the US. They find that the voters' choices were more influenced by the opinion leaders than by mass communication.

Leadership in sociology and marketing (2/2)

- The two-step flow of communication theory - the communication process is a two-step process, in which information distributed by mass media first reaches the opinion leaders; see Lazarsfeld et al. (1944), Katz & Lazarsfeld (1955).
  - Lazarsfeld et al. (1944) investigate the influence of mass communication on the 1940 presidential election campaign in the US. They find that the voters' choices were more influenced by the opinion leaders than by mass communication.
- Troldahl (1966) introduces a modified version called the two-cycle flow of communication model which distinguishes between two phases in the communication process:

Leadership in sociology and marketing (2/2)

- The two-step flow of communication theory - the communication process is a two-step process, in which information distributed by mass media first reaches the opinion leaders; see Lazarsfeld et al. (1944), Katz & Lazarsfeld (1955).
  - Lazarsfeld et al. (1944) investigate the influence of mass communication on the 1940 presidential election campaign in the US. They find that the voters' choices were more influenced by the opinion leaders than by mass communication.
- Troldahl (1966) introduces a modified version called the two-cycle flow of communication model which distinguishes between two phases in the communication process:
  - *flow of information* from the mass media to the members of the society (one-step process);

Leadership in sociology and marketing (2/2)

- The two-step flow of communication theory - the communication process is a two-step process, in which information distributed by mass media first reaches the opinion leaders; see Lazarsfeld et al. (1944), Katz & Lazarsfeld (1955).
  - Lazarsfeld et al. (1944) investigate the influence of mass communication on the 1940 presidential election campaign in the US. They find that the voters' choices were more influenced by the opinion leaders than by mass communication.
- Troldahl (1966) introduces a modified version called the two-cycle flow of communication model which distinguishes between two phases in the communication process:
  - *flow of information* from the mass media to the members of the society (one-step process);
  - *flow of influence* on beliefs and behavior (two-step process) - opinion leaders form their own opinion based on additional information provided by experts, and then they try to influence the behavior of their followers.

Studying influence in economics

- Studying influence can find its place in different fields of economics, like, e.g., in labor economics, political and public economics, experimental economics, industrial organization, game and contract theory ...

Studying influence in economics

- Studying influence can find its place in different fields of economics, like, e.g., in labor economics, political and public economics, experimental economics, industrial organization, game and contract theory ...
- "Yes Men" Theory (Prendergast, 1993) - one of the leading works on conformity. A trade-off between inducing workers to tell the truth and inducing them to exert effort is shown. The author illustrates an incentive for subordinates to conform to the opinion of their superiors' opinions when firms use subjective performance evaluation, and shows that such an incentive implies inefficiencies, even if workers are risk-neutral.

Studying influence in economics

- Studying influence can find its place in different fields of economics, like, e.g., in labor economics, political and public economics, experimental economics, industrial organization, game and contract theory ...
- "Yes Men" Theory (Prendergast, 1993) - one of the leading works on conformity. A trade-off between inducing workers to tell the truth and inducing them to exert effort is shown. The author illustrates an incentive for subordinates to conform to the opinion of their superiors' opinions when firms use subjective performance evaluation, and shows that such an incentive implies inefficiencies, even if workers are risk-neutral.
- Similar approach is applied in works on:

Studying influence in economics

- Studying influence can find its place in different fields of economics, like, e.g., in labor economics, political and public economics, experimental economics, industrial organization, game and contract theory ...
- "Yes Men" Theory (Prendergast, 1993) - one of the leading works on conformity. A trade-off between inducing workers to tell the truth and inducing them to exert effort is shown. The author illustrates an incentive for subordinates to conform to the opinion of their superiors' opinions when firms use subjective performance evaluation, and shows that such an incentive implies inefficiencies, even if workers are risk-neutral.
- Similar approach is applied in works on:
  - influence activities by Milgrom and Roberts (1988)

Studying influence in economics

- Studying influence can find its place in different fields of economics, like, e.g., in labor economics, political and public economics, experimental economics, industrial organization, game and contract theory ...
- "Yes Men" Theory (Prendergast, 1993) - one of the leading works on conformity. A trade-off between inducing workers to tell the truth and inducing them to exert effort is shown. The author illustrates an incentive for subordinates to conform to the opinion of their superiors' opinions when firms use subjective performance evaluation, and shows that such an incentive implies inefficiencies, even if workers are risk-neutral.
- Similar approach is applied in works on:
  - influence activities by Milgrom and Roberts (1988)
  - conformity by Bernheim (1994) - a model of social interaction in which individuals are assumed to care about status (popularity, esteem, respect) and about actions (consumption); see also Akerlof (1980) and Jones (1984).

Leadership in economics (1/5)

- Calvert (1992) and Wilson & Rhodes (1997) - studies of the leader's ability to solve social dilemmas and coordination games

- Calvert (1992) and Wilson & Rhodes (1997) - studies of the leader's ability to solve social dilemmas and coordination games
- DeMarzo (1992) - examines the set of outcomes sustainable by a leader with the power to make suggestions which are important even if players can communicate and form coalitions. The author considers both finite-horizon games and infinite-horizon two-player repeated games.

- Different scores and measures for analyzing collective decision-making situations with an influence between the actors:

Leadership in economics (2/5)

- Different scores and measures for analyzing collective decision-making situations with an influence between the actors:
  - Some measures for arbitrary digraphs are studied in van den Brink & Borm (2002) and van den Brink & Gilles (2000)

Leadership in economics (2/5)

- Different scores and measures for analyzing collective decision-making situations with an influence between the actors:
  - Some measures for arbitrary digraphs are studied in van den Brink & Borm (2002) and van den Brink & Gilles (2000)
  - Van den Brink et al. (2009) define the satisfaction and power scores for opinion leaders - followers structures and examine common properties of these scores.

- Different scores and measures for analyzing collective decision-making situations with an influence between the actors:
  - Some measures for arbitrary digraphs are studied in van den Brink & Borm (2002) and van den Brink & Gilles (2000)
  - Van den Brink et al. (2009) define the satisfaction and power scores for opinion leaders - followers structures and examine common properties of these scores.
  - This research is in some respect also related to work on opinion leaders and the Condorcet Jury Theorem (Estlund, 1994) and to models on organizational hierarchies based on subordinates and their superiors, where an organizational choice is to be made; see, e.g., Hammond & Thomas (1990).

Leadership in economics (3/5)

- Hermalin (1998) - an important contribution to the literature on leadership, which has been tested in several experiments. The author presents a model of leadership which captures the feature that following is a voluntary activity. He considers:

Leadership in economics (3/5)

- Hermalin (1998) - an important contribution to the literature on leadership, which has been tested in several experiments. The author presents a model of leadership which captures the feature that following is a voluntary activity. He considers:
  - "leading-by-example" - the leader's effort is observable by the followers (the leader works first publicly on an activity and convinces the followers that the activity is indeed worthwhile)

Leadership in economics (3/5)

- Hermalin (1998) - an important contribution to the literature on leadership, which has been tested in several experiments. The author presents a model of leadership which captures the feature that following is a voluntary activity. He considers:
  - "leading-by-example" - the leader's effort is observable by the followers (the leader works first publicly on an activity and convinces the followers that the activity is indeed worthwhile)
  - "leading by sacrifice" - the leader gives up a part of his payoff.

Leadership in economics (3/5)

- Hermalin (1998) - an important contribution to the literature on leadership, which has been tested in several experiments. The author presents a model of leadership which captures the feature that following is a voluntary activity. He considers:
  - "leading-by-example" - the leader's effort is observable by the followers (the leader works first publicly on an activity and convinces the followers that the activity is indeed worthwhile)
  - "leading by sacrifice" - the leader gives up a part of his payoff.
  - Hermalin (1998) proves that leading by example can support more efficient outcomes; see also, e.g., Arce (2001).

Leadership in economics (3/5)

- Hermalin (1998) - an important contribution to the literature on leadership, which has been tested in several experiments. The author presents a model of leadership which captures the feature that following is a voluntary activity. He considers:
  - "leading-by-example" - the leader's effort is observable by the followers (the leader works first publicly on an activity and convinces the followers that the activity is indeed worthwhile)
  - "leading by sacrifice" - the leader gives up a part of his payoff.
  - Hermalin (1998) proves that leading by example can support more efficient outcomes; see also, e.g., Arce (2001).
- Meidinger & Villeval (2002) test these two signaling devices: leadership-by-example and leadership-by-sacrifice.

Leadership in economics (4/5)

- Potters et al. (2007) conduct an experiment on the effect of leadership in a voluntary contribution game both in an asymmetric and full information environment. They find that leading by example increases contributions if the leader has private information about the returns from contributing.

Leadership in economics (4/5)

- Potters et al. (2007) conduct an experiment on the effect of leadership in a voluntary contribution game both in an asymmetric and full information environment. They find that leading by example increases contributions if the leader has private information about the returns from contributing.
- Potters et al. (2005) show that the followers choose to contribute sequentially and the contributions are larger in the sequential-move then in the simultaneous-move game.

- Potters et al. (2007) conduct an experiment on the effect of leadership in a voluntary contribution game both in an asymmetric and full information environment. They find that leading by example increases contributions if the leader has private information about the returns from contributing.
- Potters et al. (2005) show that the followers choose to contribute sequentially and the contributions are larger in the sequential-move then in the simultaneous-move game.
- Andreoni (1998), List & Lucking-Reiley (2002), Vesterlund (2003), Shang & Croson (2007) - an asymmetric information with informed leaders and uninformed followers, and a positive effect of early contributions on later contributions.

Leadership in economics (5/5)

- Moxnes & van der Heijden (2003) observe in a public bad experiment that the presence of a leader improves the overall outcome.

Leadership in economics (5/5)

- Moxnes & van der Heijden (2003) observe in a public bad experiment that the presence of a leader improves the overall outcome.
- Gächter & Renner (2006) test a sequential public good game and show that leaders exert a long-lasting influence on followers' beliefs.

Research on influence in cooperative game theory (CGT)

- A *simple game* is an ordered pair $(N, \mathcal{W})$, where $N = \{1, 2, \ldots, n\}$ denotes the set of players and $\mathcal{W}$ is a subset of the powerset 2^N.

- A *simple game* is an ordered pair $(N, \mathcal{W})$, where $N = \{1, 2, \ldots, n\}$ denotes the set of players and $\mathcal{W}$ is a subset of the powerset 2^N.
- Any element of 2^N is called a *coalition*.

Research on influence in cooperative game theory (CGT)

- A *simple game* is an ordered pair $(N, \mathcal{W})$, where $N = \{1, 2, \ldots, n\}$ denotes the set of players and $\mathcal{W}$ is a subset of the powerset 2^N.
- Any element of 2^N is called a *coalition*.
- A coalition S with $S \in \mathcal{W}$ is called *winning*, while those with $S \notin \mathcal{W}$ are called *losing*.

Research on influence in cooperative game theory (CGT)

- A *simple game* is an ordered pair $(N, \mathcal{W})$, where $N = \{1, 2, \ldots, n\}$ denotes the set of players and $\mathcal{W}$ is a subset of the powerset 2^N.
- Any element of 2^N is called a *coalition*.
- A coalition S with $S \in \mathcal{W}$ is called *winning*, while those with $S \notin \mathcal{W}$ are called *losing*.
- A simple game $(N, \mathcal{W})$ is *monotone* if $\mathcal{W}$ is an up-set in the order $(2^N, \subseteq)$, i.e., for all $S, T \in 2^N$

$$\text{if } S \subseteq T \text{ and } S \in \mathcal{W}, \text{ then } T \in \mathcal{W}.$$

Research on influence in cooperative game theory (CGT)

- A *simple game* is an ordered pair $(N, \mathcal{W})$, where $N = \{1, 2, \ldots, n\}$ denotes the set of players and $\mathcal{W}$ is a subset of the powerset 2^N.
- Any element of 2^N is called a *coalition*.
- A coalition S with $S \in \mathcal{W}$ is called *winning*, while those with $S \notin \mathcal{W}$ are called *losing*.
- A simple game $(N, \mathcal{W})$ is *monotone* if $\mathcal{W}$ is an up-set in the order $(2^N, \subseteq)$, i.e., for all $S, T \in 2^N$

$$\text{if } S \subseteq T \text{ and } S \in \mathcal{W}, \text{ then } T \in \mathcal{W}.$$

- A *voting game* is a monotone simple game $(N, \mathcal{W})$ with $\mathcal{W} \neq \emptyset$ and $\emptyset \notin \mathcal{W}$.

Research on influence in CGT - Influence relation in simple games

- Isbell (1958) introduces the concept of influence relation to qualitatively compare the a priori influence of voters in a simple game, where players can vote either ‘yes’ or ‘no’.

Research on influence in CGT - Influence relation in simple games

- Isbell (1958) introduces the concept of influence relation to qualitatively compare the a priori influence of voters in a simple game, where players can vote either ‘yes’ or ‘no’.
- This influence relation is defined as follows:

Research on influence in CGT - Influence relation in simple games

- Isbell (1958) introduces the concept of influence relation to qualitatively compare the a priori influence of voters in a simple game, where players can vote either 'yes' or 'no'.
- This influence relation is defined as follows:
  - Voter *i* is said to be at least as influential as voter *j*, if whenever *j* can transform a loosing coalition into a winning one by joining it, *i* can achieve the same *ceteris paribus*.

Research on influence in CGT - Influence relation in VGAs

- Tchantcho et al. (2008) extend this influence relation to voting games with abstention (VGAs).

- Tchantcho et al. (2008) extend this influence relation to voting games with abstention (VGAs).
  - A VGA consists of a non-empty set $\mathcal{W}$ of tripartitions of a set of voters, and $(S_1, S_2, S_3) \in \mathcal{W}$ means that if the players of S_1 vote in favor of a social alternative, the players of S_2 abstain or are neutral, and the members of S_3 vote against it, then this alternative will be adopted as the social choice.

Research on influence in CGT - Influence relation in VGAs

- Tchantcho et al. (2008) extend this influence relation to voting games with abstention (VGAs).
  - A VGA consists of a non-empty set $\mathcal{W}$ of tripartitions of a set of voters, and $(S_1, S_2, S_3) \in \mathcal{W}$ means that if the players of S_1 vote in favor of a social alternative, the players of S_2 abstain or are neutral, and the members of S_3 vote against it, then this alternative will be adopted as the social choice.
  - $(S_1, S_2, S_3) \in \mathcal{W}$ is called a winning partition or a majority.

- Tchantcho et al. (2008) extend this influence relation to voting games with abstention (VGAs).
  - A VGA consists of a non-empty set $\mathcal{W}$ of tripartitions of a set of voters, and $(S_1, S_2, S_3) \in \mathcal{W}$ means that if the players of S_1 vote in favor of a social alternative, the players of S_2 abstain or are neutral, and the members of S_3 vote against it, then this alternative will be adopted as the social choice.
  - $(S_1, S_2, S_3) \in \mathcal{W}$ is called a winning partition or a majority.
- The influence relation of a VGA is defined as follows:

- Tchantcho et al. (2008) extend this influence relation to voting games with abstention (VGAs).
  - A VGA consists of a non-empty set $\mathcal{W}$ of tripartitions of a set of voters, and $(S_1, S_2, S_3) \in \mathcal{W}$ means that if the players of S_1 vote in favor of a social alternative, the players of S_2 abstain or are neutral, and the members of S_3 vote against it, then this alternative will be adopted as the social choice.
  - $(S_1, S_2, S_3) \in \mathcal{W}$ is called a winning partition or a majority.
- The influence relation of a VGA is defined as follows:
  - Assuming that voters i and j have the same initial degree of approval, i is said to be at least as influential as j if whenever j can transform a losing partition into a winning one by an upward shift in her level of approval, i can achieve the same by the identical shift *ceteris paribus*.

Research on influence in CGT - Command games (1/3)

- Hu & Shapley (2003a, 2003b) use the command structure to model players' interaction relations by simple games.

Research on influence in CGT - Command games (1/3)

- Hu & Shapley (2003a, 2003b) use the command structure to model players' interaction relations by simple games.
- Let $N = \{1, ..., n\}$ be the set of players (voters). For $k \in N$ and $S \subseteq N \setminus k$:

Research on influence in CGT - Command games (1/3)

- Hu & Shapley (2003a, 2003b) use the command structure to model players' interaction relations by simple games.
- Let $N = \{1, ..., n\}$ be the set of players (voters). For $k \in N$ and $S \subseteq N \setminus k$:
  - S is a boss set for k if S determines the choice of k;

Research on influence in CGT - Command games (1/3)

- Hu & Shapley (2003a, 2003b) use the command structure to model players' interaction relations by simple games.
- Let $N = \{1, ..., n\}$ be the set of players (voters). For $k \in N$ and $S \subseteq N \setminus k$:
  - S is a boss set for k if S determines the choice of k;
  - S is an approval set for k if k can act with an approval of S.

Research on influence in CGT - Command games (1/3)

- Hu & Shapley (2003a, 2003b) use the command structure to model players' interaction relations by simple games.
- Let $N = \{1, ..., n\}$ be the set of players (voters). For $k \in N$ and $S \subseteq N \setminus k$:
  - S is a boss set for k if S determines the choice of k;
  - S is an approval set for k if k can act with an approval of S.
- For each $k \in N$, where N is the set of players, a simple game $(N, \mathcal{W}_k)$ is built, called the command game for k, with the set of winning coalitions defined by

$$\mathcal{W}_k := \{S \mid S \text{ is a boss set for } k\} \cup \{S \cup k \mid S \text{ is a boss or approval set for}$$

Research on influence in CGT - Command games (1/3)

- Hu & Shapley (2003a, 2003b) use the command structure to model players' interaction relations by simple games.
- Let $N = \{1, ..., n\}$ be the set of players (voters). For $k \in N$ and $S \subseteq N \setminus k$:
  - S is a boss set for k if S determines the choice of k;
  - S is an approval set for k if k can act with an approval of S.
- For each $k \in N$, where N is the set of players, a simple game $(N, \mathcal{W}_k)$ is built, called the command game for k, with the set of winning coalitions defined by

$$\mathcal{W}_k := \{S \mid S \text{ is a boss set for } k\} \cup \{S \cup k \mid S \text{ is a boss or approval set for}$$

- We can recover the boss and approval sets for k

$$Boss_k = \{S \subseteq N \setminus k \mid S \in \mathcal{W}_k\} = \mathcal{W}_k \cap 2^{N \setminus k}$$

$$App_k = \{S \subseteq N \setminus k \mid S \cup k \in \mathcal{W}_k \text{ but } S \notin \mathcal{W}_k\}.$$

We have $Boss_k \cap App_k = \emptyset$.

Research on influence in CGT - Command games (2/3)

- Given $\{(N, \mathcal{W}_k), k \in N\}$, the command function $\omega : 2^N \to 2^N$ is defined as

$$\omega(S) := \{k \in N \mid S \in \mathcal{W}_k\}, \ \forall S \subseteq N.$$

$\omega(S)$ is the *set of all members that are 'commandable' by* S, and $\omega(\emptyset) = \emptyset$, $\omega(N) = N$, and $\omega(S) \subseteq \omega(S')$ if $S \subset S'$.

Research on influence in CGT - Command games (2/3)

- Given $\{(N, \mathcal{W}_k), k \in N\}$, the command function $\omega : 2^N \to 2^N$ is defined as

$$\omega(S) := \{k \in N \mid S \in \mathcal{W}_k\},\ \forall S \subseteq N.$$

  $\omega(S)$ is the *set of all members that are 'commandable' by S*, and $\omega(\emptyset) = \emptyset$, $\omega(N) = N$, and $\omega(S) \subseteq \omega(S')$ if $S \subset S'$.
- How to define a fair distribution of "power" in an organization $(N, \{(N, \mathcal{W}_k) \mid k \in N\})$?

Research on influence in CGT - Command games (3/3)

- The authors define an *authority distribution* $\pi = (\pi_1, ..., \pi_n)$, where $\pi_i \geq 0$ and $\sum_{i \in N} \pi_i = 1$, and create the *power transition matrix* of the organization, which is the stochastic matrix $P = [P(j,k)]_{j,k=1}^{n}$ such that

$$P(j,k) := Sh_k(N, \mathcal{W}_j)$$

and $Sh_k(N, \mathcal{W}_j)$ is the Shapley-Shubik index of k in the command game for j. If $P(j,k) > 0$, then some of j's "power" transfers to k. $P(j,j)$ is j's personal discretion.

Research on influence in CGT - Command games (3/3)

- The authors define an *authority distribution* $\pi = (\pi_1, ..., \pi_n)$, where $\pi_i \geq 0$ and $\sum_{i \in N} \pi_i = 1$, and create the *power transition matrix* of the organization, which is the stochastic matrix $P = [P(j,k)]_{j,k=1}^{n}$ such that

$$P(j,k) := Sh_k(N, \mathcal{W}_j)$$

and $Sh_k(N, \mathcal{W}_j)$ is the Shapley-Shubik index of k in the command game for j. If $P(j,k) > 0$, then some of j's "power" transfers to k. $P(j,j)$ is j's personal discretion.
- They use a Markov chain to describe the organization's long-run authority π. The authority distribution π is assumed to satisfy the authority equilibrium equation given by

$$\pi = \pi P, \; i.e., \; \pi_k = \sum_{j \in N} \pi_j P(j,k), \; \forall k \in N.$$

$\pi_j P(j,k)$ is the authority flowing from j to k. The existence of π is known from the Markovian theory.

Research on influence in CGT - Confucian model (1/2)

Four players in the society, i.e., $N = \{1, 2, 3, 4\}$, with the king (1), the man (2), the wife (3), and the child (4). The rules are:

(i) The man follows the king;

(ii) The wife and the child follow the man;

(iii) The king should respect his people.

By virtue of the rules (i) and (ii), we have:

$$\mathcal{W}_2 = \{1, 12, 13, 14, 123, 124, 134, 1234\}$$

$$\mathcal{W}_3 = \mathcal{W}_4 = \{2, 12, 23, 24, 123, 124, 234, 1234\}$$

$$Boss_2 = \{1, 13, 14, 134\},\ Boss_3 = \{2, 12, 24, 124\}$$

$$Boss_4 = \{2, 12, 23, 123\},\ App_2 = App_3 = App_4 = \emptyset.$$

How can we translate the rule (iii) into the set $\mathcal{W}_1$ of winning coalitions in the command game for player 1?

Research on influence in CGT - Confucian model (2/2)

If $\mathcal{W}_1 = \{1234\}$, then $Boss_1 = \emptyset$, $App_1 = \{234\}$, and

$$\omega(1) = \omega(13) = \omega(14) = \omega(134) = \{2\}$$

$$\omega(2) = \omega(23) = \omega(24) = \omega(234) = \{3,4\}$$

$$\omega(3) = \omega(4) = \omega(34) = \emptyset,\ \omega(12) = \omega(123) = \omega(124) = \{2,3,4\}.$$

$$P = [Sh_k(N, \mathcal{W}_j)]_{j,k=1}^{n} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} \pi_1 = \frac{1}{4}\pi_1 + \pi_2 \\ \pi_2 = \frac{1}{4}\pi_1 + \pi_3 + \pi_4 \\ \pi_3 = \frac{1}{4}\pi_1 \\ \pi_4 = \frac{1}{4}\pi_1 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \end{cases}$$

Hence, the authority distribution $\pi = \frac{1}{9}(4,3,1,1)$.

Research on influence in social networks

- Koller & Milch (2003) - a different approach; the authors propose a graphical representation for non-cooperative games called multi-agent influence diagrams (MAIDs), which represent decision problems involving multiple agents.

Research on influence in social networks

- Koller & Milch (2003) - a different approach; the authors propose a graphical representation for non-cooperative games called multi-agent influence diagrams (MAIDs), which represent decision problems involving multiple agents.
- Game theoretic approach to influence based on using social networks:

Research on influence in social networks

- Koller & Milch (2003) - a different approach; the authors propose a graphical representation for non-cooperative games called multi-agent influence diagrams (MAIDs), which represent decision problems involving multiple agents.
- Game theoretic approach to influence based on using social networks:
  - Social networks play a central role in the formation of opinions.

Research on influence in social networks

- Koller & Milch (2003) - a different approach; the authors propose a graphical representation for non-cooperative games called multi-agent influence diagrams (MAIDs), which represent decision problems involving multiple agents.
- Game theoretic approach to influence based on using social networks:
  - Social networks play a central role in the formation of opinions.
  - It is therefore critical to have a good understanding of how the structure of such networks affects the diffusion of information.

The Hoede-Bakker index (1/3)

SOCIAL NETWORK, PLAYERS, INFLUENCE

inclinations i	$\rightarrow \rightarrow$	decisions Bi	$\rightarrow$	group decision
('yes' or 'no')		influence function B		$gd(Bi)$

- A social network with the set of players $N := \{1, ..., n\}$

The Hoede-Bakker index (1/3)

SOCIAL NETWORK, PLAYERS, INFLUENCE

inclinations i	$\rightarrow \rightarrow$	decisions Bi	$\rightarrow$	group decision
('yes' or 'no')		influence function B		$gd(Bi)$

- A social network with the set of players $N := \{1, ..., n\}$
- The players (agents, actors, voters) make a YES-NO decision.

The Hoede-Bakker index (1/3)

SOCIAL NETWORK, PLAYERS, INFLUENCE

inclinations i	$\rightarrow\rightarrow$	decisions Bi	$\rightarrow$	group decision
('yes' or 'no')		influence function B		$gd(Bi)$

- A social network with the set of players $N := \{1, ..., n\}$
- The players (agents, actors, voters) make a YES-NO decision.
- Each agent has an inclination either to say YES (+1) or NO (-1).

The Hoede-Bakker index (1/3)

SOCIAL NETWORK, PLAYERS, INFLUENCE

inclinations i	$\rightarrow \rightarrow$	decisions Bi	$\rightarrow$	group decision
('yes' or 'no')		influence function B		$gd(Bi)$

- A social network with the set of players $N := \{1, ..., n\}$
- The players (agents, actors, voters) make a YES-NO decision.
- Each agent has an inclination either to say YES (+1) or NO (-1).
- $i = (i_1, ..., i_n)$ inclination vector, where $i_k \in \{0, 1\}$, $k \in N$

The Hoede-Bakker index (1/3)

SOCIAL NETWORK, PLAYERS, INFLUENCE

inclinations i	$\rightarrow \rightarrow$	decisions Bi	$\rightarrow$	group decision
('yes' or 'no')		influence function B		$gd(Bi)$

- A social network with the set of players $N := \{1, ..., n\}$
- The players (agents, actors, voters) make a YES-NO decision.
- Each agent has an inclination either to say YES (+1) or NO (-1).
- $i = (i_1, ..., i_n)$ inclination vector, where $i_k \in \{0, 1\}$, $k \in N$
- $I = \{+1, -1\}^n$ the set of all inclination vectors

The Hoede-Bakker index (1/3)

SOCIAL NETWORK, PLAYERS, INFLUENCE

inclinations i	$\rightarrow \rightarrow$	decisions Bi	$\rightarrow$	group decision
('yes' or 'no')		influence function B		$gd(Bi)$

- A social network with the set of players $N := \{1, ..., n\}$
- The players (agents, actors, voters) make a YES-NO decision.
- Each agent has an inclination either to say YES (+1) or NO (-1).
- $i = (i_1, ..., i_n)$ inclination vector, where $i_k \in \{0, 1\}$, $k \in N$
- $I = \{+1, -1\}^n$ the set of all inclination vectors
- Due to influence in the network, the decision of a player may be different from his inclination.

The Hoede-Bakker index (1/3)

SOCIAL NETWORK, PLAYERS, INFLUENCE

inclinations i	$\to \to$	decisions Bi	$\to$	group decision
('yes' or 'no')		influence function B		$gd(Bi)$

- A social network with the set of players $N := \{1, ..., n\}$
- The players (agents, actors, voters) make a YES-NO decision.
- Each agent has an inclination either to say YES (+1) or NO (-1).
- $i = (i_1, ..., i_n)$ inclination vector, where $i_k \in \{0, 1\}$, $k \in N$
- $I = \{+1, -1\}^n$ the set of all inclination vectors
- Due to influence in the network, the decision of a player may be different from his inclination.
- $B : I \to I$ influence function Bi - decision vector

The Hoede-Bakker index (1/3)

SOCIAL NETWORK, PLAYERS, INFLUENCE

inclinations i	$\rightarrow \rightarrow$	decisions Bi	$\rightarrow$	group decision
('yes' or 'no')		influence function B		$gd(Bi)$

- A social network with the set of players $N := \{1, ..., n\}$
- The players (agents, actors, voters) make a YES-NO decision.
- Each agent has an inclination either to say YES (+1) or NO (-1).
- $i = (i_1, ..., i_n)$ inclination vector, where $i_k \in \{0, 1\}$, $k \in N$
- $I = \{+1, -1\}^n$ the set of all inclination vectors
- Due to influence in the network, the decision of a player may be different from his inclination.
- $B : I \rightarrow I$ influence function Bi - decision vector
- $gd : B(I) \rightarrow \{+1, -1\}$ group decision function, where $B(I)$ - the set of all decision vectors under B.

The Hoede-Bakker index (2/3)

- The Hoede-Bakker index (Hoede and Bakker, 1982)

$$HB(k) = \frac{1}{2^{n-1}} \cdot \sum_{\{i:\ i_k=+1\}} gd(Bi)$$

where for each $i \in I \in \{-1,+1\}^n$, $gd(B(-i)) = -gd(Bi)$.

The Hoede-Bakker index (2/3)

- The Hoede-Bakker index (Hoede and Bakker, 1982)

$$HB(k) = \frac{1}{2^{n-1}} \cdot \sum_{\{i:\ i_k=+1\}} gd(Bi)$$

where for each $i \in I \in \{-1, +1\}^n$, $gd(B(-i)) = -gd(Bi)$.

- Rusinowska & de Swart (2007) - Investigating properties of the index (postulates for power indices and paradoxes)

The Hoede-Bakker index (2/3)

- The Hoede-Bakker index (Hoede and Bakker, 1982)

$$HB(k) = \frac{1}{2^{n-1}} \cdot \sum_{\{i:\ i_k=+1\}} gd(Bi)$$

  where for each $i \in I \in \{-1,+1\}^n$, $gd(B(-i)) = -gd(Bi)$.
- Rusinowska & de Swart (2007) - Investigating properties of the index (postulates for power indices and paradoxes)
- Rusinowska & de Swart (2006) - Generalization and modifications of the index that coincide with other power indices

$$GHB(k) = \frac{1}{2^n} \cdot \left(\sum_{\{i:\ i_k=+1\}} gd(Bi) - \sum_{\{i:\ i_k=-1\}} gd(Bi) \right)$$

The Hoede-Bakker index (2/3)

- The Hoede-Bakker index (Hoede and Bakker, 1982)

$$HB(k) = \frac{1}{2^{n-1}} \cdot \sum_{\{i:\ i_k=+1\}} gd(Bi)$$

where for each $i \in I \in \{-1, +1\}^n$, $gd(B(-i)) = -gd(Bi)$.
- Rusinowska & de Swart (2007) - Investigating properties of the index (postulates for power indices and paradoxes)
- Rusinowska & de Swart (2006) - Generalization and modifications of the index that coincide with other power indices

$$GHB(k) = \frac{1}{2^n} \cdot \left(\sum_{\{i:\ i_k=+1\}} gd(Bi) - \sum_{\{i:\ i_k=-1\}} gd(Bi) \right)$$

- Rusinowska (2008) - the not-preference-based GHB

The Hoede-Bakker index (2/3)

- The Hoede-Bakker index (Hoede and Bakker, 1982)

$$HB(k) = \frac{1}{2^{n-1}} \cdot \sum_{\{i:\ i_k=+1\}} gd(Bi)$$

where for each $i \in I \in \{-1,+1\}^n$, $gd(B(-i)) = -gd(Bi)$.

- Rusinowska & de Swart (2007) - Investigating properties of the index (postulates for power indices and paradoxes)
- Rusinowska & de Swart (2006) - Generalization and modifications of the index that coincide with other power indices

$$GHB(k) = \frac{1}{2^n} \cdot \left(\sum_{\{i:\ i_k=+1\}} gd(Bi) - \sum_{\{i:\ i_k=-1\}} gd(Bi) \right)$$

- Rusinowska (2008) - the not-preference-based GHB
- Rusinowska (2009) - other modifications of the GHB.

The Hoede-Bakker index (3/3)

- The Hoede-Bakker index does NOT measure influence!

The Hoede-Bakker index (3/3)

- The Hoede-Bakker index does NOT measure influence!
- $N = \{1 = \text{child}, 2 = \text{mother}, 3 = \text{father}\}$

The Hoede-Bakker index (3/3)

- The Hoede-Bakker index does NOT measure influence!
- $N = \{1 = \text{child}, 2 = \text{mother}, 3 = \text{father}\}$
- decision to be made : to go or not to go for a bicycle trip

The Hoede-Bakker index (3/3)

- The Hoede-Bakker index does NOT measure influence!
- $N = \{1 = \text{child}, 2 = \text{mother}, 3 = \text{father}\}$
- decision to be made : to go or not to go for a bicycle trip
- rules:

The Hoede-Bakker index (3/3)

- The Hoede-Bakker index does NOT measure influence!
- $N = \{1 = \text{child}, 2 = \text{mother}, 3 = \text{father}\}$
- decision to be made : to go or not to go for a bicycle trip
- rules:
  - influence: if the parents have the same opinion, the child follows them, otherwise he follows his own inclination

The Hoede-Bakker index (3/3)

- The Hoede-Bakker index does NOT measure influence!
- $N = \{1 = \text{child}, 2 = \text{mother}, 3 = \text{father}\}$
- decision to be made : to go or not to go for a bicycle trip
- rules:
  - influence: if the parents have the same opinion, the child follows them, otherwise he follows his own inclination
  - group decision: majority vote

The Hoede-Bakker index (3/3)

- The Hoede-Bakker index does NOT measure influence!
- $N = \{1 = \text{child}, 2 = \text{mother}, 3 = \text{father}\}$
- decision to be made : to go or not to go for a bicycle trip
- rules:
  - influence: if the parents have the same opinion, the child follows them, otherwise he follows his own inclination
  - group decision: majority vote
- this gives:

i	$(1,1,1)$	$(1,1,-1)$	$(1,-1,1)$	$(-1,1,1)$	$(1,-1,-1)$	$(-1,1,-1)$	$(-1,-1,1)$	$(-1,-1,-1)$
$B(i)$	$(1,1,1)$	$(1,1,-1)$	$(1,-1,1)$	$(1,1,1)$	$(-1,-1,-1)$	$(-1,1,-1)$	$(-1,-1,1)$	$(-1,-1,-1)$
$gd(Bi)$	1	1	1	1	-1	-1	-1	-1

- The Hoede-Bakker index does NOT measure influence!
- $N = \{1 = \text{child}, 2 = \text{mother}, 3 = \text{father}\}$
- decision to be made : to go or not to go for a bicycle trip
- rules:
  - influence: if the parents have the same opinion, the child follows them, otherwise he follows his own inclination
  - group decision: majority vote
- this gives:

i	$(1,1,1)$	$(1,1,-1)$	$(1,-1,1)$	$(-1,1,1)$	$(1,-1,-1)$	$(-1,1,-1)$	$(-1,-1,1)$	$(-1,-1,-1)$
$B(i)$	$(1,1,1)$	$(1,1,-1)$	$(1,-1,1)$	$(1,1,1)$	$(-1,-1,-1)$	$(-1,1,-1)$	$(-1,-1,1)$	$(-1,-1,-1)$
$gd(Bi)$	1	1	1	1	-1	-1	-1	-1

- computation of GHB index gives:

$$GHB_1 = GHB_2 = GHB_3 = \frac{1}{2} \quad (!)$$

The Hoede-Bakker index (3/3)

- The Hoede-Bakker index does NOT measure influence!
- $N = \{1 = \text{child}, 2 = \text{mother}, 3 = \text{father}\}$
- decision to be made : to go or not to go for a bicycle trip
- rules:
  - influence: if the parents have the same opinion, the child follows them, otherwise he follows his own inclination
  - group decision: majority vote
- this gives:

i	$(1,1,1)$	$(1,1,-1)$	$(1,-1,1)$	$(-1,1,1)$	$(1,-1,-1)$	$(-1,1,-1)$	$(-1,-1,1)$	$(-1,-1,-1)$
$B(i)$	$(1,1,1)$	$(1,1,-1)$	$(1,-1,1)$	$(1,1,1)$	$(-1,-1,-1)$	$(-1,1,-1)$	$(-1,-1,1)$	$(-1,-1,-1)$
$gd(Bi)$	1	1	1	1	-1	-1	-1	-1

- computation of GHB index gives:

$$GHB_1 = GHB_2 = GHB_3 = \frac{1}{2} \quad (!)$$

- What does the index really measure?
  'net Success' = Success − Failure, not always Decisiveness!

Measuring influence

- Grabisch & Rusinowska (2009, 2010a - 2010f)

Measuring influence

- Grabisch & Rusinowska (2009, 2010a - 2010f)
- Introducing influence indices and tools to analyze the influence function, studying properties of some influence functions

Measuring influence

- Grabisch & Rusinowska (2009, 2010a - 2010f)
- Introducing influence indices and tools to analyze the influence function, studying properties of some influence functions
- Generalization of the yes-no model to multi-choice games

Measuring influence

- Grabisch & Rusinowska (2009, 2010a - 2010f)
- Introducing influence indices and tools to analyze the influence function, studying properties of some influence functions
- Generalization of the yes-no model to multi-choice games
- Comparing the influence model to command games

Measuring influence

- Grabisch & Rusinowska (2009, 2010a - 2010f)
- Introducing influence indices and tools to analyze the influence function, studying properties of some influence functions
- Generalization of the yes-no model to multi-choice games
- Comparing the influence model to command games
- Studying the exact relations between:

Measuring influence

- Grabisch & Rusinowska (2009, 2010a - 2010f)
- Introducing influence indices and tools to analyze the influence function, studying properties of some influence functions
- Generalization of the yes-no model to multi-choice games
- Comparing the influence model to command games
- Studying the exact relations between:
    - an influence function and a follower function (sufficient and necessary conditions for a function to be the follower function of some influence function)

Measuring influence

- Grabisch & Rusinowska (2009, 2010a - 2010f)
- Introducing influence indices and tools to analyze the influence function, studying properties of some influence functions
- Generalization of the yes-no model to multi-choice games
- Comparing the influence model to command games
- Studying the exact relations between:
  - an influence function and a follower function (sufficient and necessary conditions for a function to be the follower function of some influence function)
  - a command game and a command function (sufficient and necessary conditions for a function to be the command function of some command game)

Measuring influence

- Grabisch & Rusinowska (2009, 2010a - 2010f)
- Introducing influence indices and tools to analyze the influence function, studying properties of some influence functions
- Generalization of the yes-no model to multi-choice games
- Comparing the influence model to command games
- Studying the exact relations between:
  - an influence function and a follower function (sufficient and necessary conditions for a function to be the follower function of some influence function)
  - a command game and a command function (sufficient and necessary conditions for a function to be the command function of some command game)
  - a command game and an influence function (sufficient and necessary condition for the equivalence between an influence function and a normal command game)

Measuring influence

- Grabisch & Rusinowska (2009, 2010a - 2010f)
- Introducing influence indices and tools to analyze the influence function, studying properties of some influence functions
- Generalization of the yes-no model to multi-choice games
- Comparing the influence model to command games
- Studying the exact relations between:
  - an influence function and a follower function (sufficient and necessary conditions for a function to be the follower function of some influence function)
  - a command game and a command function (sufficient and necessary conditions for a function to be the command function of some command game)
  - a command game and an influence function (sufficient and necessary condition for the equivalence between an influence function and a normal command game)
- A model of influence with a continuum of actions

Measuring influence

- Grabisch & Rusinowska (2009, 2010a - 2010f)
- Introducing influence indices and tools to analyze the influence function, studying properties of some influence functions
- Generalization of the yes-no model to multi-choice games
- Comparing the influence model to command games
- Studying the exact relations between:
  - an influence function and a follower function (sufficient and necessary conditions for a function to be the follower function of some influence function)
  - a command game and a command function (sufficient and necessary conditions for a function to be the command function of some command game)
  - a command game and an influence function (sufficient and necessary condition for the equivalence between an influence function and a normal command game)
- A model of influence with a continuum of actions
- Work in progress: iterating influence.

Basic influence index of a coalition on a player

- For $S \subseteq P$ such that $|S| \geq 2$:

$$I_S := \{i \in I \mid \forall\, k, j \in S : i_k = i_j\}$$

For each $i \in I_S$, let $i_S := i_k$ for some $k \in S$.

Basic influence index of a coalition on a player

- For $S \subseteq P$ such that $|S| \geq 2$:

$$I_S := \{i \in I \mid \forall\, k, j \in S : i_k = i_j\}$$

For each $i \in I_S$, let $i_S := i_k$ for some $k \in S$.

- For each $S \subseteq N$ and $j \in P$:

$$\begin{aligned} I_{S \to j} &:= \{i \in I_S \mid i_j = \neg i_S\} \\ I^*_{S \to j}(B) &:= \{i \in I_{S \to j} \mid (Bi)_j = i_S\} \end{aligned}$$

$I_{S \to j}$ = set of inclination vectors of potential influence of S on j

$I^*_{S \to j}(B)$ = set of inclination vectors of influence of S on j under the given B

Basic influence index of a coalition on a player

- For $S \subseteq P$ such that $|S| \geq 2$:

$$I_S := \{i \in I \mid \forall\, k, j \in S : i_k = i_j\}$$

For each $i \in I_S$, let $i_S := i_k$ for some $k \in S$.

- For each $S \subseteq N$ and $j \in P$:

$$\begin{aligned} I_{S \to j} &:= \{i \in I_S \mid i_j = \neg i_S\} \\ I^*_{S \to j}(B) &:= \{i \in I_{S \to j} \mid (Bi)_j = i_S\} \end{aligned}$$

$I_{S \to j}$ = set of inclination vectors of potential influence of S on j
$I^*_{S \to j}(B)$ = set of inclination vectors of influence of S on j under the given B

- The possibility influence index of coalition S on player j:

$$\overline{d}(B, S \to j) := \frac{|I^*_{S \to j}(B)|}{|I_{S \to j}|}$$

Basic influence index of a coalition on a player

- For $S \subseteq P$ such that $|S| \geq 2$:

$$I_S := \{i \in I \mid \forall\, k, j \in S : i_k = i_j\}$$

For each $i \in I_S$, let $i_S := i_k$ for some $k \in S$.

- For each $S \subseteq N$ and $j \in P$:

$$I_{S \to j} := \{i \in I_S \mid i_j = \neg i_S\}$$

$$I^*_{S \to j}(B) := \{i \in I_{S \to j} \mid (Bi)_j = i_S\}$$

$I_{S \to j}$ = set of inclination vectors of potential influence of S on j
$I^*_{S \to j}(B)$ = set of inclination vectors of influence of S on j under the given B

- The possibility influence index of coalition S on player j:

$$\overline{d}(B, S \to j) := \frac{|I^*_{S \to j}(B)|}{|I_{S \to j}|}$$

- Weighted influence indices - definitions and properties.

Applying relational algebra & RelView to networks

- Relation algebra is used successfully to formal problem specification, prototyping, and algorithm development - Brink et al. (1997), Schmidt and Ströhlein (1993), de Swart et al. (2003, 2006).

Applying relational algebra & RelView to networks

- Relation algebra is used successfully to formal problem specification, prototyping, and algorithm development - Brink et al. (1997), Schmidt and Ströhlein (1993), de Swart et al. (2003, 2006).
- Relations are well suited for modeling and reasoning about many discrete structures (graphs, games, Petri nets, orders and lattices) and, due to the easy and/or efficient mechanization using, e.g., Boolean matrices, successor lists or binary decision diagrams (BDDs), for computations on them.

Applying relational algebra & RelView to networks

- Relation algebra is used successfully to formal problem specification, prototyping, and algorithm development - Brink et al. (1997), Schmidt and Ströhlein (1993), de Swart et al. (2003, 2006).
- Relations are well suited for modeling and reasoning about many discrete structures (graphs, games, Petri nets, orders and lattices) and, due to the easy and/or efficient mechanization using, e.g., Boolean matrices, successor lists or binary decision diagrams (BDDs), for computations on them.
- RelView (University of Kiel, 1993) - BDD-based tool for the visualization and manipulation of relations and for relational programming

Applying relational algebra & RelView to networks

- Relation algebra is used successfully to formal problem specification, prototyping, and algorithm development - Brink et al. (1997), Schmidt and Ströhlein (1993), de Swart et al. (2003, 2006).
- Relations are well suited for modeling and reasoning about many discrete structures (graphs, games, Petri nets, orders and lattices) and, due to the easy and/or efficient mechanization using, e.g., Boolean matrices, successor lists or binary decision diagrams (BDDs), for computations on them.
- RelView (University of Kiel, 1993) - BDD-based tool for the visualization and manipulation of relations and for relational programming
- Berghammer, Rusinowska, de Swart - applying relational algebra and RelView to:

Applying relational algebra & RelView to networks

- Relation algebra is used successfully to formal problem specification, prototyping, and algorithm development - Brink et al. (1997), Schmidt and Ströhlein (1993), de Swart et al. (2003, 2006).
- Relations are well suited for modeling and reasoning about many discrete structures (graphs, games, Petri nets, orders and lattices) and, due to the easy and/or efficient mechanization using, e.g., Boolean matrices, successor lists or binary decision diagrams (BDDs), for computations on them.
- RelView (University of Kiel, 1993) - BDD-based tool for the visualization and manipulation of relations and for relational programming
- Berghammer, Rusinowska, de Swart - applying relational algebra and RelView to:
  - coalition formation (2007, 2009a, 2009b)

Applying relational algebra & RelView to networks

- Relation algebra is used successfully to formal problem specification, prototyping, and algorithm development - Brink et al. (1997), Schmidt and Ströhlein (1993), de Swart et al. (2003, 2006).
- Relations are well suited for modeling and reasoning about many discrete structures (graphs, games, Petri nets, orders and lattices) and, due to the easy and/or efficient mechanization using, e.g., Boolean matrices, successor lists or binary decision diagrams (BDDs), for computations on them.
- RelView (University of Kiel, 1993) - BDD-based tool for the visualization and manipulation of relations and for relational programming
- Berghammer, Rusinowska, de Swart - applying relational algebra and RelView to:
  - coalition formation (2007, 2009a, 2009b)
  - measures in networks (2010a)

Applying relational algebra & RelView to networks

- Relation algebra is used successfully to formal problem specification, prototyping, and algorithm development - Brink et al. (1997), Schmidt and Ströhlein (1993), de Swart et al. (2003, 2006).
- Relations are well suited for modeling and reasoning about many discrete structures (graphs, games, Petri nets, orders and lattices) and, due to the easy and/or efficient mechanization using, e.g., Boolean matrices, successor lists or binary decision diagrams (BDDs), for computations on them.
- RelView (University of Kiel, 1993) - BDD-based tool for the visualization and manipulation of relations and for relational programming
- Berghammer, Rusinowska, de Swart - applying relational algebra and RelView to:
  - coalition formation (2007, 2009a, 2009b)
  - measures in networks (2010a)
  - simple games (2010b, with Stefan Bolus).

Relation Algebra (1/2)

- R is a relation with domain X and range Y:

$$\boxed{R : X \leftrightarrow Y}$$

  $X \leftrightarrow Y$ is the type of R.

- Instead of $(x, y) \in R$ we use Boolean matrix notation:

  $\boxed{R_{x,y}}$ (or R_x if the range is a singleton set)

- Signature of relation algebra:
  - Constants: $\mathsf{O}, \mathsf{L}, \mathsf{I}$.
  - Operations: $R \cup S, R \cap S, R\,S, \overline{R}, R^{\mathsf{T}}$.
  - Tests: $R \subseteq S, R = S$.
- A relation $v : X \leftrightarrow Y$ is a column vector if $v = v\,\mathsf{L}$.
- The normal case is $v : X \leftrightarrow \mathbf{1}$, where $\mathbf{1} := \{\perp\}$ is a singleton set. Then we define for subsets Y of X:

$$v \text{ represents } Y \iff Y = \{x \in X : v_x\}$$

Relation Algebra (2/2)

- A non-empty column vector v is a column point if $vv^{\mathsf{T}} \subseteq \mathsf{I}$.
- Further relational modeling of sets via membership relation $\mathsf{M} : X \leftrightarrow 2^X$, such that for all $x \in X$ and subsets Y of X:

$$\mathsf{M}_{x,Y} \iff x \in Y$$

- Projection relations $\pi : X \times Y \leftrightarrow X$ and $\rho : X \times Y \leftrightarrow X$:

$$\pi_{u,x} \iff u_1 = x \qquad \rho_{u,y} \iff u_2 = x$$

- Given a column vector $v : X \leftrightarrow \mathbf{1}$, one can compute the injective embedding mapping

$$inj(v) : Y \leftrightarrow X$$

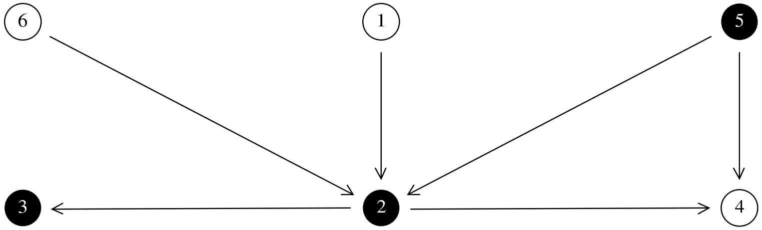
which describes Y as a subset of X in such a way that $inj(v)_{y,x}$ holds iff $y = x$ for all $y \in Y$ and $x \in X$.

Dependency graph

N - set of players
$D : N \leftrightarrow N$ the dependency relation, where

$D_{j,k}$ holds iff there is an arc from j to k
(k is dependent on j)

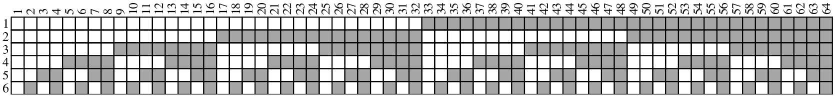
$N = \{1, 2, 3, 4, 5, 6\}$, D_{62}, D_{12}, D_{52}, D_{54}, D_{23}, D_{24}



Membership Relation $\mathsf{M} : N \leftrightarrow 2^N$:

$$\forall\, k \in N, X \in 2^N : k \in X \leftrightarrow \mathsf{M}_{k,X}$$

If we consider inclination vectors as relational column vectors, then the membership relation $\mathsf{M} : N \leftrightarrow 2^N$ column-wisely enumerates the set I of all inclination vectors.



Computing decision vectors

Let N be the set of players and $D : N \leftrightarrow N$ be the relation of the dependency graph. Then the set of the dependent players relation-algebraically is described by the column vector

$$depend(D) := D^{\mathsf{T}}\mathsf{L}$$

of type $[N \leftrightarrow \mathbf{1}]$, where the used L has type $[N \leftrightarrow \mathbf{1}]$ too.

Theorem

Let $d := depend(D)$. Given the influence rule 'following only unanimous trend-setters', the relation

$$Dvec(D) = (\mathsf{M} \cap (\overline{d\mathsf{L}} \cup (d\mathsf{L} \cap D^{\mathsf{T}}\mathsf{M} \cap D^{\mathsf{T}}\overline{\mathsf{M}}))) \cup (d\mathsf{L} \cap \overline{D^{\mathsf{T}}\overline{\mathsf{M}}})$$

of type $[N \leftrightarrow 2^N]$ column-wisely enumerates the set $B(I)$.

For all $X \in 2^N$, if the X-column of M equals $i : N \leftrightarrow \mathbf{1}$ then, under the assumed rule, the X-column of $Dvec(D)$ equals $Bi : N \leftrightarrow \mathbf{1}$.

Computing group decisions (1/2)

For the group decisions under majority, we need a row vector $m : \mathbf{1} \leftrightarrow 2^N$ such that for all $X \in 2^N$ we have

$$m_{\perp,X} \text{ iff } |X| \geq [\frac{|N|}{2}] + 1.$$

In RelView such a vector can be easily obtained with the help of the base operation cardfilter

$$m := \overline{cardfilter(\mathsf{L}, w)}^{\mathsf{T}}$$

where the first argument $\mathsf{L} : 2^N \leftrightarrow \mathbf{1}$ describes the entire powerset 2^N, and the second argument $w : W \leftrightarrow \mathbf{1}$ determines the threshold for majority by its length, i.e., fulfills $|W| = [\frac{|N|}{2}] + 1$.

Theorem

Let the row vector $gdv(D)$ of type $[\mathbf{1} \leftrightarrow 2^N]$ be defined by

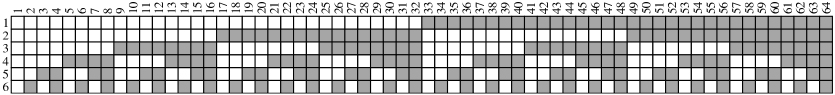
$$gdv(D) := m\,syq(\mathrm{M}, Dvec(D))$$

where $syq(R,S) := \overline{R^{\mathsf{T}}\,\overline{S}} \cap \overline{\overline{R}^{\mathsf{T}}\,S}$ is by definition the symmetric quotient of R and S. Then for all $X \in 2^N$:

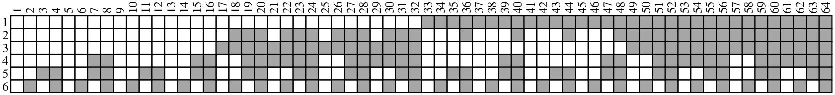
If the decision vector $Bi : N \leftrightarrow \mathbf{1}$ equals the X-column of $Dvec(D)$, then $gdv(D)_{\perp,X}$ holds iff the number of 1-entries in Bi is at least $[\frac{|N|}{2}] + 1$ (group decision by majority).

Example - Decision vectors and group decisions

Inclination vectors:



D_{62}, D_{12}, D_{52}, D_{54}, D_{23}, D_{24}. Applied to the relation D, we get the following representation of $Dvec(D)$ in RelView:



Applied to the relation D and a column vector w of length 4 (the threshold of majority), we get the following row vector $gdv(D)$:

The generalized Hoede-Bakker index

Here YES $= 1$, NO $= 0$.
Given B and gd, we define:

$$I^+(B, gd) := \{i \in I \mid gd(Bi) = 1\}$$

$$I^-(B, gd) := \{i \in I \mid gd(Bi) = 0\}$$

and for each $k \in N$:

$$I_k^{++}(B, gd) := \{i \in I \mid i_k = 1 \ \wedge \ gd(Bi) = 1\}$$
$$I_k^{+-}(B, gd) := \{i \in I \mid i_k = 1 \ \wedge \ gd(Bi) = 0\}$$
$$I_k^{-+}(B, gd) := \{i \in I \mid i_k = 0 \ \wedge \ gd(Bi) = 1\}$$
$$I_k^{--}(B, gd) := \{i \in I \mid i_k = 0 \ \wedge \ gd(Bi) = 0\}$$

The generalized Hoede-Bakker index of a player $k \in N$:

$$\mathrm{GHB}_k(B, gd) := \frac{|I_k^{++}| - |I_k^{+-}| + |I_k^{--}| - |I_k^{-+}|}{2^n}$$

Modifications of the Hoede-Bakker index

$$\mathrm{M_1GHB}_k(B, gd) := \frac{|I_k^{++}| - |I_k^{-+}|}{|I^+|}$$

$$\mathrm{M_2GHB}_k(B, gd) := \frac{|I_k^{--}| - |I_k^{+-}|}{|I^-|}$$

$$\mathrm{M_3GHB}_k(B, gd) := \frac{|I_k^{++}| + |I_k^{--}|}{2^n}$$

$$\mathrm{M_4GHB}_k(B, gd) := \frac{|I_k^{++}|}{|I^+|} \qquad \mathrm{MGHB}(B, gd) := \frac{|I^+|}{2^n}$$

$\mathrm{M_1GHB}$ - Coleman's index 'to prevent action'
$\mathrm{M_2GHB}$ - Coleman's index 'to initiate action'
$\mathrm{M_3GHB}$ - Rae index $\mathrm{M_4GHB}$ - König-Bräuninger index
MGHB - Coleman's 'power of a collectivity to act'

Theorem

Let a player $k \in N$ be described by a column point $p : N \leftrightarrow \mathbf{1}$ in the relational sense. Let $g := gdv(D)$ be the group decision row vector. Let the four vectors $ipp(p,g)$, $ipm(p,g)$, $imp(p,g)$ and $imm(p,g)$ of type $[\mathbf{1} \leftrightarrow 2^P]$ be defined as follows:

$$ipp(p,g) := p^{\mathsf{T}}\mathrm{M} \cap g \qquad ipm(p,g) := p^{\mathsf{T}}\mathrm{M} \cap \overline{g}$$

$$imp(p,g) := p^{\mathsf{T}}\,\overline{\mathrm{M}} \cap g \qquad imm(p,g) := p^{\mathsf{T}}\,\overline{\mathrm{M}} \cap \overline{g}$$

Then we have for all $X \in 2^N$: If the X-column of M equals the inclination vector $i : N \leftrightarrow \mathbf{1}$, then we have that

$ipp(p,g)_{\perp,X}$ holds iff $i \in I_k^{++}$ $\qquad$ $ipm(p,g)_{\perp,X}$ holds iff $i \in I_k^{+-}$

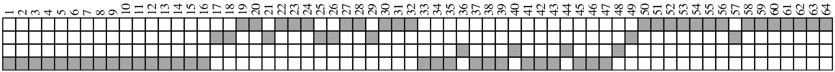
$imp(p,g)_{\perp,X}$ holds iff $i \in I_k^{-+}$ $\qquad$ $imm(p,g)_{\perp,X}$ holds iff $i \in I_k^{--}$

(i.e., the row vector $ipp(p,g)$ precisely designates those columns of the membership relation M which belong to the set I_k^{++}, etc.)

Example - Computing power indices

Player 2, 'following only unanimous trend-setters' as influence rule, *gd* given by simple majority

In the following 4×64 RELVIEW-matrix:



the first row depicts the row vector $ipp(p, g)$, i.e., precisely designates those columns of the membership relation $M : N \leftrightarrow 2^N$ that belong to the set $I_2^{++}(B, gd)$.
The second, third and fourth rows of the matrix do the same for $I_2^{+-}(B, gd)$, $I_2^{-+}(B, gd)$ and $I_2^{--}(B, gd)$ respectively.

Counting the 1-entries of the single rows, one can easily obtain

$$\mathrm{GHB}_2(B, gd) = \frac{5}{8}.$$

Computing influence indices (1/2)

Theorem

Assume $s : N \leftrightarrow \mathbf{1}$ as description of the coalition $S \subseteq N$ and the row vector $is(s)$ of type $[\mathbf{1} \leftrightarrow 2^N]$ to be defined as

$$is(s) := [s^{\mathsf{T}}, s^{\mathsf{T}}]\ \overline{(\overline{\pi \mathsf{M}} \cup \rho \mathsf{M}) \cap (\overline{\rho \mathsf{M}} \cup \pi \mathsf{M})}$$

where $\mathsf{M} : N \leftrightarrow 2^N$ is the membership relation, and $\pi : N \times N \leftrightarrow P$ and $\rho : N \times N \leftrightarrow P$ are the projection relations.
Then we have for all $X \in 2^N$: If the X-column of M equals the inclination vector $i : N \leftrightarrow \mathbf{1}$, then $is(s)_{\perp,X}$ holds iff $i \in I_S$.

Hence, the row vector $is(s)$ precisely designates those columns of the membership relation M which belong to the set I_S.

Computing influence indices (2/2)

Theorem

Assume $s : N \leftrightarrow \mathbf{1}$ to describe the coalition $S \subseteq N$, the column point $p : N \leftrightarrow \mathbf{1}$ to describe the player $j \in N$, the column point $q \subseteq s$ to describe some player $k \in S$, and the row vector $\mathit{potinf}(s, p)$ of type $[\mathbf{1} \leftrightarrow 2^N]$ to be defined as

$$\mathit{potinf}(s,p) := ((r \cup r') \cap \overline{r \cap r'})\, \mathit{inj}(\mathit{is}(s)^{\mathsf{T}})$$

where $r := p^{\mathsf{T}} \mathrm{M}\, \mathit{inj}(\mathit{is}(s)^{\mathsf{T}})^{\mathsf{T}}$, $r' := q^{\mathsf{T}} \mathrm{M}\, \mathit{inj}(\mathit{is}(s)^{\mathsf{T}})^{\mathsf{T}}$.
Then we have for all $X \in 2^N$: If the X-column of M equals the inclination vector $i : N \leftrightarrow \mathbf{1}$, then $\mathit{potinf}(s,p)_{\perp,X}$ holds iff $i \in I_{S \to j}$.

Consequently, the set $I^*_{S \to j}(B)$ is described by the row vector

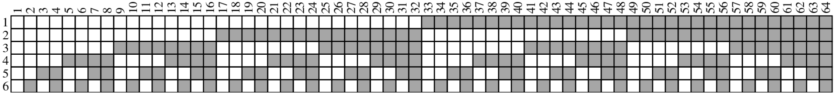
$$\mathit{inf}(s,p,D) := \mathit{potinf}(s,p) \cap \overline{(r \cup r') \cap \overline{r \cap r'}}\; \mathit{inj}(\mathit{is}(s)^{\mathsf{T}})$$

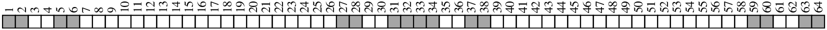
now with $r := p^{\mathsf{T}} \mathit{Dvec}(D)\, \mathit{inj}(\mathit{is}(s)^{\mathsf{T}})^{\mathsf{T}}$ and $r' := q^{\mathsf{T}} \mathrm{M}\, \mathit{inj}(\mathit{is}(s)^{\mathsf{T}})^{\mathsf{T}}$.

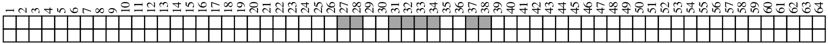
Example - Influence indices (1/2)

The RELVIEW-representations of I, I_S, $I_{S\to j}$ and $I^*_{S\to j}(B)$, for $S = \{2,3,5\}$, $j = 1$ and B = 'following only unanimous trend-setters'.

The first picture is $\mathsf{M} : N \leftrightarrow 2^N$, the second one is the row vector $is(s) : \mathbf{1} \leftrightarrow 2^N$, where the column vector $s : N \leftrightarrow \mathbf{1}$ describes S. The row vector precisely designates those columns of the matrix where the entries 2, 3 and 5 have the same color.



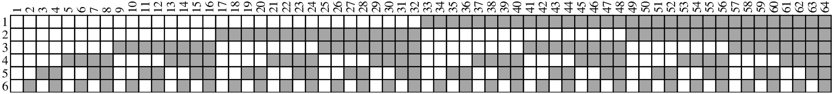


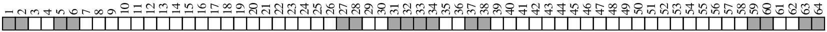


Because $I^*_{S\to 1}(B) = \emptyset$, it follows that $\overline{d}(B, 235 \to 1) = 0$.

Example - Influence indices (2/2)

For $j = 4$, $S = \{2, 3, 5\}$ and $B =$ 'following only unanimous trend-setters', the RELVIEW-representations of the sets I, I_S, $I_{S \to j}$ and $I^*_{S \to j}(B)$ are respectively:





Hence,

$$\overline{d}(B, 235 \to 4) = 1.$$

Sets of followers

The set of followers of coalition $\emptyset \neq S \subseteq N$ under the influence function B is defined as

$$F_B(S) := \{j \in N \mid \forall\, i \in I_S : (Bi)_j = i_S\}.$$

Theorem

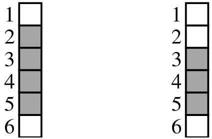
Assume $s : N \leftrightarrow \mathbf{1}$ to describe the coalition $S \subseteq N$, and the column point $q \subseteq s$ to describe some player $k \in S$. Furthermore, let $\mathrm{M} : P \leftrightarrow 2^N$ be the membership relation. If the column vector $follow(D, s)$ of type $[N \leftrightarrow \mathbf{1}]$ is defined as

$$follow(D, s) := syq(Q^{\mathsf{T}}, R^{\mathsf{T}} q)$$

with relations $R := \mathrm{M}\, inj(is(s)^{\mathsf{T}})^{\mathsf{T}}$ and $Q := Dvec(D)\, inj(is(s)^{\mathsf{T}})^{\mathsf{T}}$, then for all $j \in P$ we have $follow(D, s)_{j,\perp}$ iff $j \in F_B(S)$.

Example - Sets of followers

The left column vector describes the set of followers of $S = \{2, 3, 5\}$ under the influence rule ‘following only unanimous trend-setters’ and the right column vector does the same with ‘following the majority of the trend-setters’.



The Dutch Parliament Example (1/2)

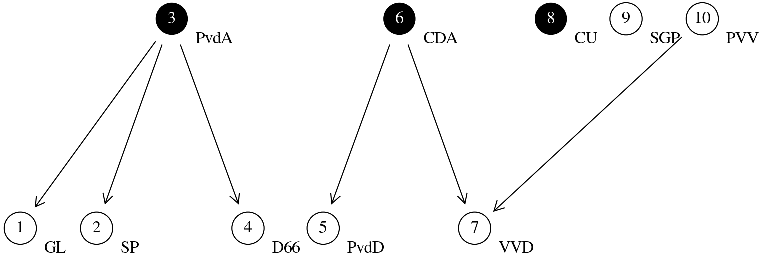
$N := \{\mathrm{CDA}, \mathrm{CU}, \mathrm{D66}, \mathrm{GL}, \mathrm{PvdA}, \mathrm{PvdD}, \mathrm{PVV}, \mathrm{SGP}, \mathrm{SP}, \mathrm{VVD}\}$

GL	SP	PvdA	D66	PvdD	CDA	VVD	CU	SGP	PVV
7	25	33	3	2	41	22	6	2	9

CDA - Christen-Democratisch Appel (Christian Democrats)
CU - Christen Unie (Christian Union)
D66 - Democraten66 (Democrats 66)
GL - GroenLinks (Green Left)
PvdA - Partij van de Arbeid (Labor Party)
PvdD - Partij voor de Dieren (Animal Party)
PVV - Partij voor de Vrijheid (Party for Freedom)
SGP - Staatkundig Gereformeerde Partij (Political Reformed Party)
SP - Socialistische Partij (Socialist Party)
VVD - Volkspartij voor Vrijheid en Democratie (People's Party for Freedom and Democracy)

The Dutch Parliament Example (2/2)

The RelView-representation of the dependency relation and the coalition $S = \{\mathrm{CDA}, \mathrm{CU}, \mathrm{PvdA}\}$



Advantages of applying the RelView approach to networks

- The algorithms used are expressed by the extremely short RelView programs.

Advantages of applying the RelView approach to networks

- The algorithms used are expressed by the extremely short RelView programs.
- Usefulness of our approach with respect to applying the tools to organizations and trend-setter structures with a larger number of players.

Advantages of applying the RelView approach to networks

- The algorithms used are expressed by the extremely short RelView programs.
- Usefulness of our approach with respect to applying the tools to organizations and trend-setter structures with a larger number of players.
- Although all problems are hard since they deal with sets of exponential size, the BDD based implementation of RelView is of immense help.

Advantages of applying the RelView approach to networks

- The algorithms used are expressed by the extremely short RelView programs.
- Usefulness of our approach with respect to applying the tools to organizations and trend-setter structures with a larger number of players.
- Although all problems are hard since they deal with sets of exponential size, the BDD based implementation of RelView is of immense help.
- Example of the very efficient BDD-implementation of relations - RelView needs on a Sun Fire-280R workstation (750 MHz, 4 GByte main memory, running Solaris) only 0.04 seconds to compute the group decision vector in the case of the Dutch parliament. Note the symmetric quotient $syq(\mathrm{M}, WDvec(D))$ used here has type $[2^N \leftrightarrow 2^N]$. Regarded as a Boolean matrix, this means that it has 2^{150} rows and columns.

The DeGroot model

- The seminar network interaction model of information transmission, opinion formation, and consensus formation is presented in DeGroot (1974), see also Jackson (2008).

The DeGroot model

- The seminar network interaction model of information transmission, opinion formation, and consensus formation is presented in DeGroot (1974), see also Jackson (2008).
- Each individual in a society has an initial opinion on a subject. The opinions are represented by a vector $p(0) = (p_1(0), ..., p_n(0))^T$ of probabilities, and each $p_i(0)$ can be seen as the probability that a given statement is true, or the quality of a given product, etc.

The DeGroot model

- The seminar network interaction model of information transmission, opinion formation, and consensus formation is presented in DeGroot (1974), see also Jackson (2008).
- Each individual in a society has an initial opinion on a subject. The opinions are represented by a vector $p(0) = (p_1(0), ..., p_n(0))^T$ of probabilities, and each $p_i(0)$ can be seen as the probability that a given statement is true, or the quality of a given product, etc.
- The interactions are determined by a stochastic matrix T, where T_{ij} represents the weight or trust that agent i places on the current belief of agent j in forming i's belief for the next period.

The DeGroot model

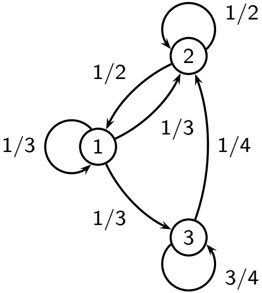
- The seminar network interaction model of information transmission, opinion formation, and consensus formation is presented in DeGroot (1974), see also Jackson (2008).
- Each individual in a society has an initial opinion on a subject. The opinions are represented by a vector $p(0) = (p_1(0), ..., p_n(0))^T$ of probabilities, and each $p_i(0)$ can be seen as the probability that a given statement is true, or the quality of a given product, etc.
- The interactions are determined by a stochastic matrix T, where T_{ij} represents the weight or trust that agent i places on the current belief of agent j in forming i's belief for the next period.
- The beliefs are updated over time so that

$$p(t) = Tp(t-1) = T^t p(0)$$

Updating process in the DeGroot model (1/2)

Example - Updating in the DeGroot model (Jackson, 2008)

$$T = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{bmatrix}$$



Let the vector of beliefs be initially $p(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Updating process in the DeGroot model (2/2)

$$p(1) = Tp(0) = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/2 \\ 0 \end{bmatrix}$$

$$p(2) = Tp(1) = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 5/18 \\ 5/12 \\ 1/8 \end{bmatrix}$$

Iterating the process leads to beliefs that converge

$$p(t) = Tp(t-1) = T^t p(0) \to \begin{bmatrix} 3/11 \\ 3/11 \\ 3/11 \end{bmatrix}.$$

Updating process in the DeGroot model (2/2)

$$p(1) = Tp(0) = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/2 \\ 0 \end{bmatrix}$$

$$p(2) = Tp(1) = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 5/18 \\ 5/12 \\ 1/8 \end{bmatrix}$$

Iterating the process leads to beliefs that converge

$$p(t) = Tp(t-1) = T^t p(0) \to \begin{bmatrix} 3/11 \\ 3/11 \\ 3/11 \end{bmatrix}.$$

- Under what conditions does the updating process converge to a well-defined limit?

Updating process in the DeGroot model (2/2)

$$p(1) = Tp(0) = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/2 \\ 0 \end{bmatrix}$$

$$p(2) = Tp(1) = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 5/18 \\ 5/12 \\ 1/8 \end{bmatrix}$$

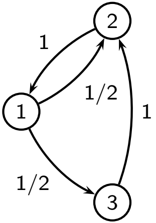
Iterating the process leads to beliefs that converge

$$p(t) = Tp(t-1) = T^t p(0) \rightarrow \begin{bmatrix} 3/11 \\ 3/11 \\ 3/11 \end{bmatrix}.$$

- Under what conditions does the updating process converge to a well-defined limit?
- A social influence matrix T is convergent if $\lim_t T^t p$ exists for all initial vectors of beliefs p.

Example - Convergence (Jackson, 2008)

$$T = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$



$$T^2 = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \end{bmatrix}, \quad T^3 = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

Convergence in the DeGroot model (2/5)

$$T^t \to \begin{bmatrix} 2/5 & 2/5 & 1/5 \\ 2/5 & 2/5 & 1/5 \\ 2/5 & 2/5 & 1/5 \end{bmatrix}$$

No matter what initial beliefs $p(0)$ are, the agents end up with limiting beliefs corresponding to the entries of

$$p(\infty) = \lim_t T^t p(0)$$

where

$$p_1(\infty) = p_2(\infty) = p_3(\infty) = \frac{2}{5} p_1(0) + \frac{2}{5} p_2(0) + \frac{1}{5} p_3(0).$$

Convergence in the DeGroot model (2/5)

$$T^t \to \begin{bmatrix} 2/5 & 2/5 & 1/5 \\ 2/5 & 2/5 & 1/5 \\ 2/5 & 2/5 & 1/5 \end{bmatrix}$$

No matter what initial beliefs $p(0)$ are, the agents end up with limiting beliefs corresponding to the entries of

$$p(\infty) = \lim_t T^t p(0)$$

where

$$p_1(\infty) = p_2(\infty) = p_3(\infty) = \frac{2}{5}p_1(0) + \frac{2}{5}p_2(0) + \frac{1}{5}p_3(0).$$

The example

- shows that beliefs converge over time

Convergence in the DeGroot model (2/5)

$$T^t \to \begin{bmatrix} 2/5 & 2/5 & 1/5 \\ 2/5 & 2/5 & 1/5 \\ 2/5 & 2/5 & 1/5 \end{bmatrix}$$

No matter what initial beliefs $p(0)$ are, the agents end up with limiting beliefs corresponding to the entries of

$$p(\infty) = \lim_t T^t p(0)$$

where

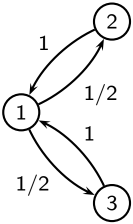
$$p_1(\infty) = p_2(\infty) = p_3(\infty) = \frac{2}{5}p_1(0) + \frac{2}{5}p_2(0) + \frac{1}{5}p_3(0).$$

The example

- shows that beliefs converge over time
- illustrates that the agents reach a consensus and that agents 1 and 2 have twice as much influence over the limiting beliefs as agent 3 does.

Example - Nonconvergence (Jackson, 2008)

$$T = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$



$$T^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}, \quad T^3 = \begin{bmatrix} 1/2 & 1/2 & 1/0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad T^4 = T^2, \ldots$$

Convergence in the DeGroot model (4/5)

- A *directed graph (network)* of the updating (interaction) matrix T is the directed graph, where a directed link (i, j) exists from i to j if and only if $T_{ij} > 0$.

Convergence in the DeGroot model (4/5)

- A *directed graph (network)* of the updating (interaction) matrix T is the directed graph, where a directed link (i, j) exists from i to j if and only if $T_{ij} > 0$.
- A *walk* is a sequence of nodes $(j_1, j_2, ..., j_K)$, not necessarily distinct, such that link (j_k, j_{k+1}) exists for all $1 \leq k < K$, and the length of the walk is $K - 1$.

Convergence in the DeGroot model (4/5)

- A *directed graph (network)* of the updating (interaction) matrix T is the directed graph, where a directed link (i, j) exists from i to j if and only if $T_{ij} > 0$.
- A *walk* is a sequence of nodes $(j_1, j_2, ..., j_K)$, not necessarily distinct, such that link (j_k, j_{k+1}) exists for all $1 \leq k < K$, and the length of the walk is $K - 1$.
- A *path* is a walk consisting of distinct nodes.

Convergence in the DeGroot model (4/5)

- A *directed graph (network)* of the updating (interaction) matrix T is the directed graph, where a directed link (i, j) exists from i to j if and only if $T_{ij} > 0$.
- A *walk* is a sequence of nodes $(j_1, j_2, ..., j_K)$, not necessarily distinct, such that link (j_k, j_{k+1}) exists for all $1 \leq k < K$, and the length of the walk is $K - 1$.
- A *path* is a walk consisting of distinct nodes.
- A *cycle* is a walk $(j_1, j_2, ..., j_K)$ such that $j_1 = j_K$.

Convergence in the DeGroot model (4/5)

- A *directed graph (network)* of the updating (interaction) matrix T is the directed graph, where a directed link (i, j) exists from i to j if and only if $T_{ij} > 0$.
- A *walk* is a sequence of nodes $(j_1, j_2, ..., j_K)$, not necessarily distinct, such that link (j_k, j_{k+1}) exists for all $1 \leq k < K$, and the length of the walk is $K - 1$.
- A *path* is a walk consisting of distinct nodes.
- A *cycle* is a walk $(j_1, j_2, ..., j_K)$ such that $j_1 = j_K$.
- A cycle is *simple* if the only node appearing twice in the sequence is the starting (ending) node.

Convergence in the DeGroot model (4/5)

- A *directed graph (network)* of the updating (interaction) matrix T is the directed graph, where a directed link (i, j) exists from i to j if and only if $T_{ij} > 0$.
- A *walk* is a sequence of nodes $(j_1, j_2, ..., j_K)$, not necessarily distinct, such that link (j_k, j_{k+1}) exists for all $1 \leq k < K$, and the length of the walk is $K - 1$.
- A *path* is a walk consisting of distinct nodes.
- A *cycle* is a walk $(j_1, j_2, ..., j_K)$ such that $j_1 = j_K$.
- A cycle is *simple* if the only node appearing twice in the sequence is the starting (ending) node.
- The matrix T is *aperiodic* if the greatest common divisor of the lengths of its simple cycles is 1.

Convergence in the DeGroot model (4/5)

- A *directed graph (network)* of the updating (interaction) matrix T is the directed graph, where a directed link (i, j) exists from i to j if and only if $T_{ij} > 0$.
- A *walk* is a sequence of nodes $(j_1, j_2, ..., j_K)$, not necessarily distinct, such that link (j_k, j_{k+1}) exists for all $1 \leq k < K$, and the length of the walk is $K - 1$.
- A *path* is a walk consisting of distinct nodes.
- A *cycle* is a walk $(j_1, j_2, ..., j_K)$ such that $j_1 = j_K$.
- A cycle is *simple* if the only node appearing twice in the sequence is the starting (ending) node.
- The matrix T is *aperiodic* if the greatest common divisor of the lengths of its simple cycles is 1.
- The matrix T is *strongly connected (irreducible)* if there is path relative to T from any node to any other node.

Convergence in the DeGroot model (4/5)

- A *directed graph (network)* of the updating (interaction) matrix T is the directed graph, where a directed link (i, j) exists from i to j if and only if $T_{ij} > 0$.
- A *walk* is a sequence of nodes $(j_1, j_2, ..., j_K)$, not necessarily distinct, such that link (j_k, j_{k+1}) exists for all $1 \leq k < K$, and the length of the walk is $K - 1$.
- A *path* is a walk consisting of distinct nodes.
- A *cycle* is a walk $(j_1, j_2, ..., j_K)$ such that $j_1 = j_K$.
- A cycle is *simple* if the only node appearing twice in the sequence is the starting (ending) node.
- The matrix T is *aperiodic* if the greatest common divisor of the lengths of its simple cycles is 1.
- The matrix T is *strongly connected (irreducible)* if there is path relative to T from any node to any other node.
- If T is strongly connected and aperiodic, then it is convergent.

Convergence in the DeGroot model (5/5)

- Frequently it is assumed that T is strongly connected and $T_{ii} > 0$ for some i, which implies that T is aperiodic, and hence convergent.

Convergence in the DeGroot model (5/5)

- Frequently it is assumed that T is strongly connected and $T_{ii} > 0$ for some i, which implies that T is aperiodic, and hence convergent.
- However, to ensure convergence it is NOT necessary to have $T_{ii} > 0$ for even one i. (see Example)

Convergence in the DeGroot model (5/5)

- Frequently it is assumed that T is strongly connected and $T_{ii} > 0$ for some i, which implies that T is aperiodic, and hence convergent.
- However, to ensure convergence it is NOT necessary to have $T_{ii} > 0$ for even one i. (see Example)
- If a stochastic matrix is strongly connected, then it is convergent if and only if it is aperiodic. (Golub & Jackson, 2010)

Convergence in the DeGroot model (5/5)

- Frequently it is assumed that T is strongly connected and $T_{ii} > 0$ for some i, which implies that T is aperiodic, and hence convergent.
- However, to ensure convergence it is NOT necessary to have $T_{ii} > 0$ for even one i. (see Example)
- If a stochastic matrix is strongly connected, then it is convergent if and only if it is aperiodic. (Golub & Jackson, 2010)
- A group of nodes N' is *closed* relative to T if $i \in N'$ and $T_{ij} > 0$ implies that $j \in N'$, i.e., there is no directed link from a node in N' to a node outside N'.

Convergence in the DeGroot model (5/5)

- Frequently it is assumed that T is strongly connected and $T_{ii} > 0$ for some i, which implies that T is aperiodic, and hence convergent.
- However, to ensure convergence it is NOT necessary to have $T_{ii} > 0$ for even one i. (see Example)
- If a stochastic matrix is strongly connected, then it is convergent if and only if it is aperiodic. (Golub & Jackson, 2010)
- A group of nodes N' is *closed* relative to T if $i \in N'$ and $T_{ij} > 0$ implies that $j \in N'$, i.e., there is no directed link from a node in N' to a node outside N'.
- The matrix T is convergent if and only if every set of nodes that is strongly connected and closed is aperiodic. (Golub & Jackson, 2010)

Consensus in the DeGroot model

- A group of agents A *reaches a consensus* under T for an initial vector of beliefs $p(0)$ if

$$\lim_t p_i(t) = \lim_t p_j(t) \text{ for each } i, j \in A.$$

Consensus in the DeGroot model

- A group of agents A *reaches a consensus* under T for an initial vector of beliefs $p(0)$ if

$$\lim_t p_i(t) = \lim_t p_j(t) \text{ for each } i, j \in A.$$

- Under T, any strongly connected and closed group of individuals reaches a consensus for every initial vector of beliefs if and only if it is aperiodic. (Jackson, 2008)

Consensus in the DeGroot model

- A group of agents A *reaches a consensus* under T for an initial vector of beliefs $p(0)$ if

$$\lim_t p_i(t) = \lim_t p_j(t) \text{ for each } i, j \in A.$$

- Under T, any strongly connected and closed group of individuals reaches a consensus for every initial vector of beliefs if and only if it is aperiodic. (Jackson, 2008)
- A consensus is reached in the DeGroot model if and only if there is exactly one strongly connected and closed group of agents and T is aperiodic on that group.

Consensus in the DeGroot model

- A group of agents A *reaches a consensus* under T for an initial vector of beliefs $p(0)$ if

$$\lim_t p_i(t) = \lim_t p_j(t) \text{ for each } i, j \in A.$$

- Under T, any strongly connected and closed group of individuals reaches a consensus for every initial vector of beliefs if and only if it is aperiodic. (Jackson, 2008)
- A consensus is reached in the DeGroot model if and only if there is exactly one strongly connected and closed group of agents and T is aperiodic on that group.
- see also Berger (1981).

Generalizations of the DeGroot model (1/2)

- The updating can vary with time and circumstances.

Generalizations of the DeGroot model (1/2)

- The updating can vary with time and circumstances.
- DeMarzo et al. (2003) (Time-Varying Weight on Own Beliefs)
  The updating rule is

$$p(t) = [(1 - \lambda_t)I + \lambda_t \hat{T}]p(t - 1)$$

  I = identity matrix, $\lambda_t \in (0, 1]$ = adjustment factor, $\hat{T}$ = stochastic matrix

Generalizations of the DeGroot model (1/2)

- The updating can vary with time and circumstances.
- DeMarzo et al. (2003) (Time-Varying Weight on Own Beliefs)
  The updating rule is

$$p(t) = [(1 - \lambda_t)I + \lambda_t \hat{T}]p(t-1)$$

I = identity matrix, $\lambda_t \in (0,1]$ = adjustment factor, $\hat{T}$ = stochastic matrix

  - For λ_t constant over time, this corresponds to the DeGroot model;

Generalizations of the DeGroot model (1/2)

- The updating can vary with time and circumstances.
- DeMarzo et al. (2003) (Time-Varying Weight on Own Beliefs)
  The updating rule is

$$p(t) = [(1-\lambda_t)I + \lambda_t \hat{T}]p(t-1)$$

I = identity matrix, $\lambda_t \in (0,1]$ = adjustment factor, $\hat{T}$ = stochastic matrix

  - For λ_t constant over time, this corresponds to the DeGroot model;
  - Otherwise the updating varies over time and an agent places more (or less) weight on his own belief over time.

Generalizations of the DeGroot model (1/2)

- The updating can vary with time and circumstances.
- DeMarzo et al. (2003) (Time-Varying Weight on Own Beliefs)
  The updating rule is

  $$p(t) = [(1 - \lambda_t)I + \lambda_t \hat{T}]p(t-1)$$

  I = identity matrix, $\lambda_t \in (0, 1]$ = adjustment factor, $\hat{T}$ = stochastic matrix
  - For λ_t constant over time, this corresponds to the DeGroot model;
  - Otherwise the updating varies over time and an agent places more (or less) weight on his own belief over time.
- Krause (2000) (Only Weighting Those with Similar Beliefs)
  An agent pays attention only to other agents whose beliefs do not differ much from his own, i.e., he places equal weight on all opinions that are within some distance of his own current opinion, and weight zero otherwise.
  For consensus reaching, see Lorenz (2005), Jackson (2008).

Generalizations of the DeGroot model (2/2)

Friedkin and Johnsen (1990, 1997) (Time-Varying Weight on Own Beliefs)

- The updating always mixes in some weight on an agent's initial beliefs. The rule is

$$p(t) = D\hat{T}p(t-1) + (I - D)p(0)$$

D is an $n \times n$ matrix with positive entries only along the diagonal, $D_{ii} \in (0, 1)$ indicates the extent to which i pays attention to others' attitudes.

Friedkin and Johnsen (1990, 1997) (Time-Varying Weight on Own Beliefs)

- The updating always mixes in some weight on an agent's initial beliefs. The rule is

$$p(t) = D\hat{T}p(t-1) + (I - D)p(0)$$

D is an $n \times n$ matrix with positive entries only along the diagonal, $D_{ii} \in (0,1)$ indicates the extent to which i pays attention to others' attitudes.

- Consensus may never be reached (e.g., $n = 2$, $D_{ii} = 1/2$, $\hat{T}_{12} = \hat{T}_{21} = 1$)
An agent is always averaging his original belief with the latest belief of the other agent:

$$p_i(t) = \frac{p_j(t-1)}{2} + \frac{p_i(0)}{2}.$$

If $p_1(0) = 1$, $p_2(0) = 0$, then $p_1(t) \to 2/3$, $p_2(t) \to 1/3$.

Social influence in the DeGroot model (1/3)

- How does each agent in the social network influence the limiting belief? (Jackson, 2008)

Social influence in the DeGroot model (1/3)

- How does each agent in the social network influence the limiting belief? (Jackson, 2008)
- Consider a closed and strongly connected group of agents. Let T be aperiodic. Hence, all beliefs converge and a consensus is reached. Let $p(0)$ be an arbitrary starting belief vector and $p(\infty) = (p^\infty, ..., p^\infty)$ be the vector of limiting consensus beliefs. We search for an influence vector $s \in [0, 1]^n$ such that $\sum_i s_i = 1$ and

$$p^\infty = s \cdot p(0) = \sum_i s_i p_i(0).$$

If such an s exists, then the limiting beliefs would be weighted averages of the initial beliefs, and the relative weights would be the influences of the agents on the final consensus beliefs.

Social influence in the DeGroot model (2/3)

- Suppose that an influence vector exists. Since starting with $p(0)$ or with $p(1) = Tp(0)$ yields the same limit, we have $s \cdot p(1) = s \cdot p(0)$, and therefore $s \cdot (Tp(0)) = s \cdot p(0)$, which has to hold for every $p(0)$. Hence,

$$sT = s.$$

Thus s is a left-hand unit eigenvector of T (eigenvector with eigenvalue 1).

Social influence in the DeGroot model (2/3)

- Suppose that an influence vector exists. Since starting with $p(0)$ or with $p(1) = Tp(0)$ yields the same limit, we have $s \cdot p(1) = s \cdot p(0)$, and therefore $s \cdot (Tp(0)) = s \cdot p(0)$, which has to hold for every $p(0)$. Hence,

$$sT = s.$$

  Thus s is a left-hand unit eigenvector of T (eigenvector with eigenvalue 1).
- When T is strongly connected, aperiodic, and row stochastic, there is a unique such unit eigenvector that has all positive values.

Social influence in the DeGroot model (2/3)

- Suppose that an influence vector exists. Since starting with $p(0)$ or with $p(1) = Tp(0)$ yields the same limit, we have $s \cdot p(1) = s \cdot p(0)$, and therefore $s \cdot (Tp(0)) = s \cdot p(0)$, which has to hold for every $p(0)$. Hence,

$$sT = s.$$

  Thus s is a left-hand unit eigenvector of T (eigenvector with eigenvalue 1).
- When T is strongly connected, aperiodic, and row stochastic, there is a unique such unit eigenvector that has all positive values.
- Since $s \cdot p(0)$ must lead to the same belief as any entry of $p(\infty) = (p^{\infty}, \ldots, p^{\infty}) = T^{\infty} p(0)$, each row of T^{∞} must converge to s.

Social influence in the DeGroot model (3/3)

- Example (contd) (Jackson, 2008)

$$T = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad T^t \rightarrow \begin{bmatrix} 2/5 & 2/5 & 1/5 \\ 2/5 & 2/5 & 1/5 \\ 2/5 & 2/5 & 1/5 \end{bmatrix}$$

$s = (2/5, 2/5, 1/5)$ is a unit eigenvector of T, i.e.,

$$sT = (2/5, 2/5, 1/5) \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = (2/5, 2/5, 1/5) = s.$$

Social influence in the DeGroot model (3/3)

- Example (contd) (Jackson, 2008)

$$T = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad T^t \to \begin{bmatrix} 2/5 & 2/5 & 1/5 \\ 2/5 & 2/5 & 1/5 \\ 2/5 & 2/5 & 1/5 \end{bmatrix}$$

$s = (2/5, 2/5, 1/5)$ is a unit eigenvector of T, i.e.,

$$sT = (2/5, 2/5, 1/5) \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = (2/5, 2/5, 1/5) = s.$$

- This social influence measure is related to the eigenvector-based centrality measures - Katz's prestige measure (1953), eigenvector centrality of Bonacich (1972, 1987), Bonacich & Lloyd (2001).

Research on influence in social networks (contd 1)

- Other questions in the context of the DeGroot model, e.g.,

Research on influence in social networks (contd 1)

- Other questions in the context of the DeGroot model, e.g.,
  - Convergence speed - How quickly beliefs reach their limit

Research on influence in social networks (contd 1)

- Other questions in the context of the DeGroot model, e.g.,
  - Convergence speed - How quickly beliefs reach their limit
  - Are consensus beliefs "correct"- Do beliefs converge to the right probability?

Research on influence in social networks (contd 1)

- Other questions in the context of the DeGroot model, e.g.,
  - Convergence speed - How quickly beliefs reach their limit
  - Are consensus beliefs "correct"- Do beliefs converge to the right probability?
- For related work see, e.g., Seneta (1973), DeMarzo et al. (2003), Jackson (2008), Golub & Jackson (2010).

Research on influence in social networks (contd 1)

- Other questions in the context of the DeGroot model, e.g.,
  - Convergence speed - How quickly beliefs reach their limit
  - Are consensus beliefs "correct"- Do beliefs converge to the right probability?
- For related work see, e.g., Seneta (1973), DeMarzo et al. (2003), Jackson (2008), Golub & Jackson (2010).
- DeMarzo et al. (2003) - the agents in a network try to estimate some unknown parameter, which allows updating to vary over time, i.e., an agent may place more or less weight on his own belief over time. Moreover, the authors show the phenomenon of unidimensional opinions: they study the case of multidimensional opinions, in which each agent has a vector of beliefs, and they show that often the individuals' opinions can be well approximated by a one-dimensional line, where an agent's position on the line determines his position on all issues.

Research on influence in social networks (contd 2)

- Krackhardt (1987) collected data regarding a small (100 employees, 21 managers) manufacturing firm in the US. He collected information from the managers about who sought advice from whom.

Research on influence in social networks (contd 2)

- Krackhardt (1987) collected data regarding a small (100 employees, 21 managers) manufacturing firm in the US. He collected information from the managers about who sought advice from whom.
- For an application of the DeGroot model to Krackhardt's advice network; see Jackson (2008).

Research on influence in social networks (contd 2)

- Krackhardt (1987) collected data regarding a small (100 employees, 21 managers) manufacturing firm in the US. He collected information from the managers about who sought advice from whom.
- For an application of the DeGroot model to Krackhardt's advice network; see Jackson (2008).
- A model of influence by Asavathiratham (2000) - it consists of a network of nodes, each with a status evolving over time. The evolution of the status is according to an internal Markov chain, but transition probabilities depend not only on the current status of the node, but also on the statuses of the neighboring nodes.

Research on influence in social networks (contd 2)

- Krackhardt (1987) collected data regarding a small (100 employees, 21 managers) manufacturing firm in the US. He collected information from the managers about who sought advice from whom.
- For an application of the DeGroot model to Krackhardt's advice network; see Jackson (2008).
- A model of influence by Asavathiratham (2000) - it consists of a network of nodes, each with a status evolving over time. The evolution of the status is according to an internal Markov chain, but transition probabilities depend not only on the current status of the node, but also on the statuses of the neighboring nodes.
- Koster, Lindner, Napel (2010)

Social learning models

- Literature on social learning in the context of social networks - Banerjee (1992), Ellison (1993), Ellison & Fudenberg (1993, 1995), Bala & Goyal (1998, 2001), Gale & Kariv (2003), Celen & Kariv (2004), Banerjee & Fudenberg (2004).

Social learning models

- Literature on social learning in the context of social networks - Banerjee (1992), Ellison (1993), Ellison & Fudenberg (1993, 1995), Bala & Goyal (1998, 2001), Gale & Kariv (2003), Celen & Kariv (2004), Banerjee & Fudenberg (2004).
- In social learning models agents observe choices over time and update their beliefs accordingly (different from the models in which the choices depend on the influence of others).

Social learning models

- Literature on social learning in the context of social networks - Banerjee (1992), Ellison (1993), Ellison & Fudenberg (1993, 1995), Bala & Goyal (1998, 2001), Gale & Kariv (2003), Celen & Kariv (2004), Banerjee & Fudenberg (2004).
- In social learning models agents observe choices over time and update their beliefs accordingly (different from the models in which the choices depend on the influence of others).
- Bayesian learning model - Bala & Goyal (1998), also Jackson (2008) - Agents are connected in an undirected social network and in each period they simultaneously choose among a finite set of actions. The payoffs to the actions are random, and their distribution depends on an unknown state of nature. The agents have identical tastes and face the same uncertainty about the actions. In each period, besides observing his own outcome, an agent also observes choices and outcomes of the neighbors.

Herd behavior, Informational cascades

- Lopez-Pintado (2008) studies a network of interacting agents whose actions are determined by the actions of their neighbors.

Herd behavior, Informational cascades

- Lopez-Pintado (2008) studies a network of interacting agents whose actions are determined by the actions of their neighbors.
- The "herd behavior" literature (Banerjee, 1992; Scharfstein & Stein, 1990) or the "informational cascades" study (Bikhchandani et al., 1992) - it is assumed that people get information by observing others' actions and are inclined to imitate those who are supposed to be better informed.

Herd behavior, Informational cascades

- Lopez-Pintado (2008) studies a network of interacting agents whose actions are determined by the actions of their neighbors.
- The "herd behavior" literature (Banerjee, 1992; Scharfstein & Stein, 1990) or the "informational cascades" study (Bikhchandani et al., 1992) - it is assumed that people get information by observing others' actions and are inclined to imitate those who are supposed to be better informed.
  - Informational cascades form quickly as people decide to ignore their internal signals and follow what other people are doing (Bikhchandani et al., 1992; Anderson & Holt, 1997).

Herd behavior, Informational cascades

- Lopez-Pintado (2008) studies a network of interacting agents whose actions are determined by the actions of their neighbors.
- The "herd behavior" literature (Banerjee, 1992; Scharfstein & Stein, 1990) or the "informational cascades" study (Bikhchandani et al., 1992) - it is assumed that people get information by observing others' actions and are inclined to imitate those who are supposed to be better informed.
  - Informational cascades form quickly as people decide to ignore their internal signals and follow what other people are doing (Bikhchandani et al., 1992; Anderson & Holt, 1997).
  - Grabisch & Rusinowska (2010a) formalize a similar phenomenon as the mass psychology function.

Long time ago ...

Hamlet: *Do you see yonder cloud that's almost in shape of a camel?*
Polonius: *By th'mass, and 'tis: like a camel, indeed.*
Hamlet: *Methinks it is like a weasel.*
Polonius: *It is back'd like a weasel.*
Hamlet: *Or like a whale.*
Polonius: *Very like a whale.*

William Shakespeare (1600) *Hamlet*, Act 3, Scene 2

Akerlof GA (1980) A theory of social custom, of which unemployment may be one consequence, *Quarterly Journal of Economics* 94, 749-775

Anderson LR, Holt CA (1997) Information cascades in the laboratory, *American Economic Review* 87(5), 847-862

Andreoni J (1998) Toward a theory of charitable fund-raising, *Journal of Political Economy* 106, 1186-1213

Arce D (2001) Leadership and the aggregation of international collective action, *Oxford Economic Papers* 53, 114-137

Aronson E, Wilson TD, Akert AM (2007) *Social Psychology* (6th Ed.), Upper Saddle River, NJ: Pearson Prentice Hall

Asavathiratham C (2000) *Influence model: a tractable representation of networked Markov chains*, PhD thesis, Massachusetts Institute of Technology, Cambridge, MA

Asavathiratham C, Roy S, Lesieutre B, Verghese G (2001) The influence model, *IEEE Control Systems Magazine* 21, 5264

Asch SE (1955) Opinions and social pressure, *Scientific American* 193, 31-35

Austen-Smith D (1997) Interest groups: money, information and influence, In Mueller DC (Ed.) *Perspectives of Public Choice*, Cambridge: Cambridge University Press

Bala V, Goyal S (1998) Learning from neighbors, *Review of Economic Studies* 65, 595-621

Bala V, Goyal S (2001) Conformism and diversity under social learning, *Economic Theory* 17, 101-120

Banerjee AV (1992) A simple model of herd behavior, *Quarterly Journal of Economics* 107, 797-819

Banerjee AV, Fudenberg D (2004) Word-of-mouth learning, *Games and Economic Behavior* 46, 1-22

Baron RS, Vandello JA, Brunsman B (1996) The forgotten variable in conformity research: Impact of task importance on social influence, *Journal of Personality and Social Psychology* 71, 915-927

Bass BM (1985) *Leadership and Performance Beyond Expectations*, New-York: Free Press

Behnke R, Berghammer R, Meyer E, Schneider P (1998) RelView — A system for calculation with relations and relational programming, In Astesiano E (Ed.) *Fundamental Approaches to Software Engineering*, LNCS 1382, pages 318-321, Springer-Verlag

Berger RL (1981) A necessary and sufficient condition for reaching a consensus using DeGroot's method, *Journal of the American Statistical Association* 76, 415-419

Berghammer R, Rusinowska A, de Swart H (2007) Applying relational algebra and RelView to coalition formation, *European Journal of Operational Research* 178/2, 530-542

Berghammer R, Rusinowska A, de Swart H (2009) An interdisciplinary approach to coalition formation, *European Journal of Operational Research* 195, 487-496

Berghammer R, Rusinowska A, de Swart H (2010a) Applying relational algebra and RelView to measures in a social network, *European Journal of Operational Research* 202, 182-195

Berghammer R, Bolus S, Rusinowska A, de Swart H (2010b) A relation - algebraic approach to simple games, *European Journal of Operational Research*, forthcoming

Berghammer R, Schmidt G, Winter M (2003) RelView and Rath – Two systems for dealing with relations, In de Swart H, Orlowska E, Schmidt G, Roubens M (Eds.), *Theory and Applications of Relational Structures as Knowledge Instruments*, LNCS 2929, pages 1–16, Springer

Bernheim BD (1994) A theory of conformity, *Journal of Political Economy* 102(5), 841-877

Bikhchandani S, Hirshleifer D, Welch I (1992) A theory of fads, fashion, custom, and cultural change as informational cascades, *Journal of Political Economy* 100, 992-1026

Bonacich PB (1972) Factoring and weighting approaches to status scores and clique identification, *Journal of Mathematical Sociology* 2, 113-120

Bonacich PB (1987) Power and centrality: a family of measures, *American Journal of Sociology* 92, 1170–1182

Bonacich PB, Lloyd P (2001) Eigenvector-like measures of centrality for asymmetric relations, *Social Networks* 23(3), 191-201

Brink C, Kahl W, Schmidt G (1997) (Eds.) *Relational Methods in Computer Science, Advances in Computing Science*, Springer

van den Brink R, Borm P (2002) Digraph competitions and cooperative games, *Theory and Decision* 53, 327-342

van den Brink R, Gilles RP (2000) Measuring domination in directed networks, *Social Networks* 22, 141-157

van den Brink R, Rusinowska A, Steffen F (2009) Measuring power and satisfaction in societies with opinion leaders: Dictator and opinion leader properties, In Holler MJ & Widgren M (Eds.), Essays in Honor of Hannu Nurmi, *Homo Oeconomicus* 26, in print

van den Brink R, Rusinowska A, Steffen F (2010) Measuring power and satisfaction in societies with opinion leaders: An axiomatization, Mimeo

Burns JM (1978) *Leadership* New York: Harper & Row

Buss DM, Gomes M, Higgins DS, Lauterbach K (1987) Tactics of manipulation, *Journal of Personality and Social Psychology* 52, 1219-1229

Burnstein E (1966) Book review: Ingratiation: A social psychological analysis by Edward E. Jones, *The American Journal of Psychology* 79 (1), 159-161

Calvert RL (1992) Leadership and its basis in problems of social coordination, *International Political Science Review* 13, 7-24

Celen B, Kariv S (2004) Distinguishing informational cascades from herd behavior in the laboratory, *American Economic Review* 94(3), 484-497

Cialdini RB (2001) *Influence: Science and practice* (4th ed.), Boston: Allyn & Bacon

Clifford P, Sudbury A (1973) A model for spatial conflict, *Biometrika* 60(3), 581-588

Conlisk J (1976) Interactive Markov chains, *Journal of Mathematical Sociology* 4, 157-185

Conlisk J (1978) A stability theorem for an interactive Markov chain, *Journal of Mathematical Sociology* 6, 163-168

Conlisk J (1980) Costly optimizers versus cheap imitators, *Journal of Economic Behavior and Organization* 1, 275-293

Literature (7/19)

Conlisk J (1992) Stability and monotonicity for interactive Markov chains, *Journal of Mathematical Sociology* 17, 127-143

DeGroot MH (1974) Reaching a consensus, *Journal of the American Statistical Association* 69, 118-121

DeMarzo PM (1992) Coalitions, leadership, and social norms: The power of suggestion in games, *Games and Economic Behavior* 4, 72-100

DeMarzo P, Vayanos D, Zwiebel J (2003) Persuasion bias, social influence, and unidimensional opinions, *Quarterly Journal of Economics* 118, 909-968

Deutsch M, Gerard HB (1955) A study of normative and informational social influences upon individual judgment, *Journal of Abnormal and Social Psychology* 51, 629-636

Drazen A (2000) *Political Economy in Macroeconomics*, Princeton: Princeton University Press

Ellison G (1993) Learning, local interaction, and coordination, *Econometrica* 61, 1047-1072

Ellison G, Fudenberg D (1993) Rules of thumb for social learning, *Journal of Political Economy* 101(4), 612-643

Ellison G, Fudenberg D (1995) Word-of-mouth communication and social learning, *Journal of Political Economy* 111(1), 93-125

Estlund DM (1994) Opinion leaders, independence, and Condorcet's jury theorem, *Theory and Decision* 36, 131-162

French J (1956) A formal theory of social power, *Psychological Review* 63(3), 181-194

Friedkin NE (1999) Choice shift and group polarization, *American Sociological Review* 64, 856-875

Friedkin NE, Cook KS (1990) Peer group influence, *Sociological Methods and Research* 19(1), 122-143

Friedkin NE, Johnsen EC (1990) Social influence and opinions, *Journal of Mathematical Sociology* 15, 193-206

Friedkin NE, Johnsen EC (1997) Social positions in influence networks, *Social Networks* 19, 209-222

Friedkin NE, Johnsen EC (1999) Social influence networks and opinion change, *Advances in Group Processes* 16, 1-29

Gächter S, Renner E (2005) Leading by example in the presence of free rider incentives, University of Nottingham, CeDEx Discussion paper

Gächter S, Renner E (2006) Leaders as belief managers for pro-social behavior, University of Nottingham, Mimeo

Gale D, Kariv S (2003) Bayesian learning in social networks, *Games and Economic Behavior* 45(2), 329-346

Golub B, Jackson MO (2010) Naïve learning in social networks and the wisdom of crowds, *American Economic Journal: Microeconomics* 2(1), 112-149

Grabisch M, Rusinowska A (2009) Measuring influence in command games, *Social Choice and Welfare* 33, 177-209

Grabisch M, Rusinowska A (2010a) A model of influence in a social network, *Theory and Decision* 69, 69-96

Literature (10/19)

Grabisch M, Rusinowska A (2010b) Different approaches to influence based on social networks and simple games, In van Deemen A & Rusinowska A (Eds.) *Collective Decision Making: Views From Social Choice and Game Theory*, Series Theory and Decision Library C, pages 185-209, Springer

Grabisch M, Rusinowska A (2010c) A model of influence with an ordered set of possible actions, *Theory and Decision*, Forthcoming

Grabisch M, Rusinowska A (2010d) Influence functions, followers and command games, *Games and Economic Behavior*, Forthcoming

Grabisch M, Rusinowska A (2010e) A model of influence with a continuum of actions, GATE Working Paper, 2010-04

Grabisch M, Rusinowska A (2010f) Iterating influence between players in a social network, Mimeo

Granovetter M (1978) Threshold models of collective behavior, *American Journal of Sociology* 83, 1420-1443

Grossman GM, Helpman E (2001) *Special Interest Politics*, Cambridge: MIT Press

Hammond TH, Thomas P (1990) Invisible decisive coalitions in large hierarchies, *Public Choice* 66, 101-116

Harary F (1959) Status and contrastatus, *Sociometry* 22, 23-43

Hermalin BE (1998) Toward an economic theory of leadership: Leading by example, *American Economic Review* 88, 1188-1206

Hoede C, Bakker R (1982) A theory of decisional power, *Journal of Mathematical Sociology* 8, 309-322

Hogan K (2005) *The Science of Influence: How to Get Anyone to Say "Yes" in 8 Minutes or Less!*, John Wiley & Sons

Holley RA, Liggett TM (1975) Ergodic theorems for weakly interacting infinite systems and the voter models, *Annals of Probability* 3, 643-663

Hoyer WD, Stockburger-Sauer NE (2007) A comparison of antecedents and consequences of market mavens and opinion leaders, In Bayön T, Herrmann A, Huber F, Hammerschmidt M, Stockburger-Sauer N (Eds.) *Vielfalt und Einheit in der Marketingwissenschaft - Ein Spannungsverhltnis. Gabler*, pp. 215-236

Literature (12/19)

Hu X, Shapley LS (2003a) On authority distributions in organizations: controls, *Games and Economic Behavior* 45, 153-170

Hu X, Shapley LS (2003b) On authority distributions in organizations: equilibrium, *Games and Economic Behavior* 45, 132-152

Isbell JR (1958) A class of simple games, *Duke Mathematical Journal* 25, 423-439

Jackson MO (2008) *Social and Economic Networks*, Princeton University Press

Jones EE (1964) *Ingratiation: A Social Psychological Analysis*, New York, Appleton-Century-Crofts

Jones SRG (1984) *The Economics of Conformism*, Oxford: Blackwell

Katz L (1953) A new status index derived from sociometric analysis, *Psychometrika* 18, 39-43

Katz E, Lazarsfeld PF (1955) *Personal Influence: The Part Played by People in the Flow of Mass Communication*, Free Press

Katz ML, Shapiro C (1986) Technology adoption in the presence of network externalities, *Journal of Political Economy* 94, 822-841

Kelman H (1958) Compliance, identification, and internalization: Three processes of attitude change, *Journal of Conflict Resolution* 1, 51-60

Koller D, Milch B (2003) Multi-agent influence diagrams for representing and solving games, *Games and Economic Behavior* 45, 181-221

Koster M, Lindner I, Napel S (2010) Voting power and social interaction, SING7 Conference, Palermo, July 7-9

Krackhardt D (1987) Cognitive social structures, *Social Networks* 9, 109-314

Krause U (2000) A discrete nonlinear and nonautonomous model of consensus formation, In Elaydi S, Ladas G, Popenda J, Rakowski J (Eds.) *Communications in Difference Equations*, Amsterdam: Gordon and Breach

Kumru C, Vesterlund L (2005) The effects of status on voluntary contribution, University of Pittsburg, Mimeo

Literature (14/19)

Latane B (1981) The psychology of social impact, *American Psychologist* 36, 343-365

Latane B, Bourgeois MJ (2001) Successfully simulating dynamic social impact: Three levels of prediction, In Forgas & Williams (Eds.), *Social influence: Direct and indirect processes*, Philadelphia: Psychology Press

Lazarsfeld PF, Berelson B, Gaudet H (1944) The People's Choice - How the Voter Makes up his Mind in a Presidential Campaign, Columbia University Press

Lehoczky JP (1980) Approximations for interactive Markov chains in discrete and continuous time, *Journal of Mathematical Sociology* 7, 139-157

List JA, Lucking-Reiley D (2002) The effects of seed money and refunds on charitable giving: Experimental evidence from a university capital campain, *Journal of Political Economy* 110, 215-233

Lopez-Pintado D (2008) Diffusion in complex social networks, *Games and Economic Behavior* 62, 573-590

Lorenz J (2005) A stabilization theorem for dynamics of continuous opinions, *Physica A* 355, 217-223

McKelvey W, Kerr NH (1988) Differences in conformity among friends and strangers, *Psychological Reports* 62, 759-762

Meidinger C, Villeval M-C (2002) Leadership in teams: Signaling or reciprocating?, GATE Working Paper, 2002-01

Milgrom PR, Roberts DJ (1988) An economic approach to influence activities and organizational responses, *American Journal of Sociology* 94, 154-179

Moxnes E, van der Heijden E (2003) The effect of leadership in a public bad experiment, *Journal of Conflict Resolution* 47, 773-795

Persson T, Tabellini G (2000) *Political Economics*, Cambridge: MIT Press

Potters J, Sefton M, Vesterlund L (2005) After you - endogenous sequencing in voluntary contribution games, *Journal of Public Economics* 89, 1399-1419

Potters J, Sefton M, Vesterlund L (2007) Leading-by-example and signaling in voluntary contribution games: an experimental study, *Economic Theory* 33, 169-182

Literature (16/19)

Potters J, Sloof R (1996) Interest groups: Z survey of empirical models that try to assess their influence, *European Journal of Political Economy* 12, 403-442

Prendergast C (1993) A theory of "Yes Men", *American Economic Review* 83(4), 757-770

Robin S, Rusinowska A, Villeval MC (2010) A model of ingratiation: an experimental study, Mimeo

Rusinowska A (2008) On the not-preference-based Hoede-Bakker index, In Petrosjan L & Mazalov V (Eds.) *Game Theory and Applications*, Vol. XIII, Chapter 9, pp. 127-141, Nova Science Publishers, Inc. New York

Rusinowska A (2009) The Hoede-Bakker index modified to the Shapley-Shubik and Holler-Packel indices, *Group Decision and Negotiation* (forthcoming)

Rusinowska A, de Swart H (2006) Generalizing and modifying the Hoede-Bakker index, In de Swart et al. (Eds.) *Theory and Applications of Relational Structures as Knowledge Instruments*, No 2. Springer's Lecture Notes in Artificial Intelligence LNAI 4342, pp. 60-88, Springer, Heidelberg, Germany

Rusinowska A, de Swart H (2007) On some properties of the Hoede-Bakker index, *Journal of Mathematical Sociology* 31(4), 267-293

Scharfstein D, Stein J (1990) Herd behavior and investment, *American Economic Review* 80, 465-479

Schmidt G, Ströhlein T (1993) *Relations and Graphs, Discrete Mathematics for Computer Scientists*, EATCS Monographs on Theoretical Computer Science, Springer

Seneta E (1973) *Non-Negative Matrices*, New York: John Wiley and Sons

Shang J, Croson R (2007) The impact of social comparisons on non-profit fund-raising, *Research in Experimental Economics*, forthcoming

Shapley LS (1994) A Boolean model of organization authority based on the theory of simple games, Mimeo

Sherif M (1936) *The psychology of social norms*, New York: Harper Collins

Literature (18/19)

Sloof R (1998) *Game-Theoretic Models of the Political Influence of Interest Groups*, Boston: Kluwer

de Swart H, Berghammer R, Rusinowska A (2009) Computational Social Choice Using Relation Algebra and RelView, In Berghammer et al. (Eds.), *RelMiCS / AKA 2009*, Lecture Notes in Computer Science (LNCS) 5827, pp. 13-28, Springer

de Swart H, Orlowska E, Schmidt G, Roubens M (2003) (Eds.) *Theory and Applications of Relational Structures as Knowledge Instruments*, LNCS 2929, Springer

de Swart H, Orlowska E, Schmidt G, Roubens M (2006) (Eds.) *Theory and Applications of Relational Structures as Knowledge Instruments II*, LNAI 4342, Springer

Tchantcho B, Diffo Lambo L, Pongou R, Mbama Engoulou B (2008) Voters' power in voting games with abstention: Influence relation and ordinal equivalence of power theories, *Games and Economic Behavior* 64, 335-350

Troldahl VC (1966) A field test of a modified "Two-step flow of communication" model, *Public Opinion Quarterly* 30, 609-623

Literature (19/19)

Tsuji R (2002) Interpersonal influence and attitude change toward conformity in small groups: A social psychological model, *Journal of Mathematical Sociology* 26, 17-34

van Winden F (2004) Interest group behavior and influence, In Rowley CK & Schneider F (Eds.), *The Encyclopedia of Public Choice*, pp. 118-129, Springer

Urberg KA, Degirmencioglu SM, Pilgrim C (1997) Close friend and group influence on adolescent cigarette smoking and alcohol use, *Developmental Psychology* 33, 834-844

Vaughan GM, Hogg HA (2008) *Introduction to social psychology* (5th ed.), French Forest NSW, Australia: Pearson Education

Vesterlund L (2003) Informational value of sequential fund-raising, *Journal of Public Economics* 87, 627-657

Wasserman S, Faust K (1994) *Social Network Analysis: Methods and Applications*, Cambridge University Press, Cambridge

Wilson RK, Rhodes CM (1997) Leadership and credibility in n-person coordination games, *Journal of Conflict Resolution* 41(6), 767-791