# Three Hierarchies of Simple Games Parameterized by "Resource" Parameters

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joint work with
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# Simple Games

The set $P = \{1, 2, \ldots, n\}$ denotes the set of players.

## Definition

A simple game is a pair $G = (P, W)$, where $W$ is a subset of the power set $2^P$, different from $\emptyset$, which satisfies the monotonicity condition:

*if $X \in W$ and $X \subset Y \subseteq P$, then $Y \in W$.*

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*if* $X \in W$ *and* $X \subset Y \subseteq P$, *then* $Y \in W$.

Coalitions from $W$ are called winning. We also denote

$$L = 2^P \setminus W$$

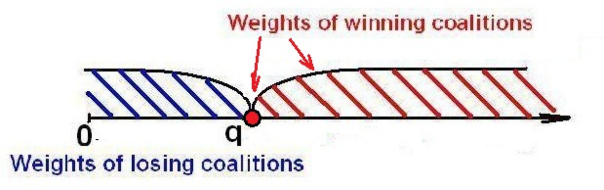
and call coalitions from $L$ losing.

# Weighted Majority Games

## Definition

A simple game $G$ is called a weighted majority game if there exists a weight function $w\colon P \to \mathcal{R}^+$, where $\mathcal{R}^+$ is the set of all non-negative reals, and a real number $q$, called quota, such that

$$X \in W \Longleftrightarrow \sum_{i \in X} w_i \geq q.$$



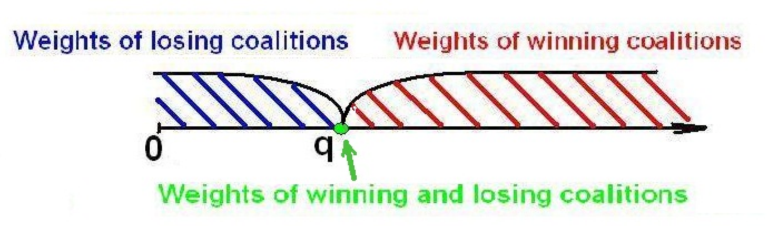
# Rough weights

## Definition

A simple game $G$ is called roughly weighted if there exists a weight function $w\colon P \to \mathcal{R}^+$, not identically equal to zero, and a non-negative real number $q$, called quota, such that for $X \in 2^P$

$$\sum_{i \in X} w_i > q \Longrightarrow X \in W,$$

$$\sum_{i \in X} w_i < q \Longrightarrow X \in L.$$



# Trading transform

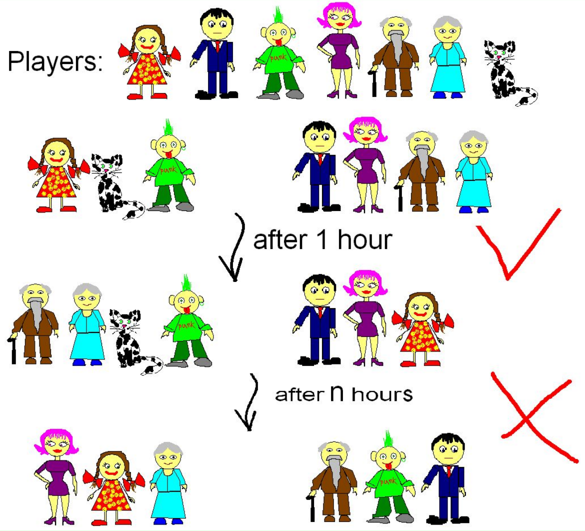
**Definition**

The sequence of coalitions

$$\mathcal{T} = (X_1, \ldots, X_j; Y_1, \ldots, Y_j)$$

is called a trading transform if the coalitions $X_1, \ldots, X_j$ can be converted into the coalitions $Y_1, \ldots, Y_j$ by rearranging players.

# Example of a trading transform



# A criterion of rough weightedness

**Theorem (Gvozdeva-Slinko, 2009)**

*A game $G$ is roughly weighted if for no $j$ there exists a trading transform of the form*

$$\mathcal{T} = (X_1, \ldots, X_j, P; Y_1, \ldots, Y_j, \emptyset), \qquad (\star)$$

*where $X_1, \ldots, X_j$ are winning and $Y_1, \ldots, Y_j$ are loosing*

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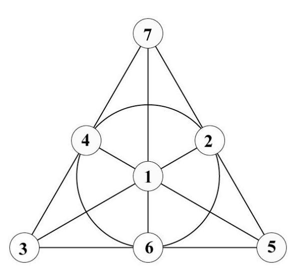
A trading transform of the form $(\star)$ is called a potent.

# The Fano plane game




We take $P = \{1, 2, \ldots, 7\}$ and the lines $X_1, \ldots, X_7$ as minimal winning coalitions:

$\{1,2,3\}, \{1,4,5\}, \{1,6,7\}, \{2,5,7\},$
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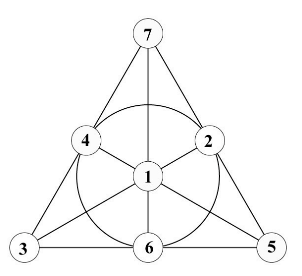


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Then the sequence

$$\mathcal{T} = (X_1, \ldots, X_7, P; X_1^c, \ldots, X_7^c, \emptyset)$$

shows the absence of rough weights.

# The $\mathcal{A}$-Hierarchy

### Definition

Let $q$ be a rational number. A game $G$ belongs to the class $\mathcal{A}_q$ of $\mathcal{A}$-hierarchy if $G$ possesses a potent certificate of nonweightedness

$$(X_1, \ldots, X_m; Y_1, \ldots, Y_m),$$

with $\ell$ grand coalitions among $X_1, \ldots, X_m$ and $\ell$ empty coalitions among $Y_1, \ldots, Y_m$, such that $q = \ell/m$. If $\alpha$ is irrational,we set $\mathcal{A}_\alpha = \bigcap_{q<\alpha} \mathcal{A}_q$.

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The parameter of the $\mathcal{A}$-Hierarchy reflects the balance of power between small and large coalitions; the larger $\alpha \in (0, \frac{1}{2})$ the more powerful some of the small coalitions are.

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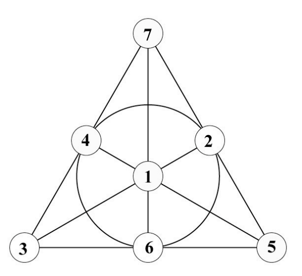


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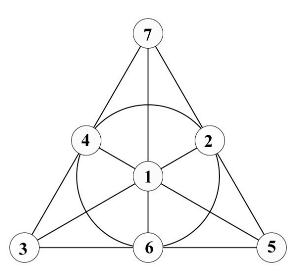


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shows the absence of rough weights.

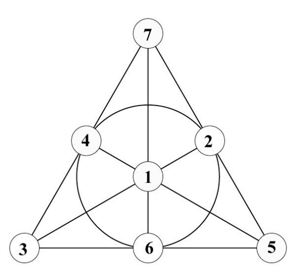
The Fano plane game belongs to the class $\mathcal{A}_{1\backslash 8}$.

# The $\mathcal{A}$-Hierarchy

**Theorem**

*If* $0 < \alpha < \beta < \frac{1}{2}$*, then* $\mathcal{A}_\alpha \supsetneq \mathcal{A}_\beta$*.*

# Two thresholds for the Fano plane game

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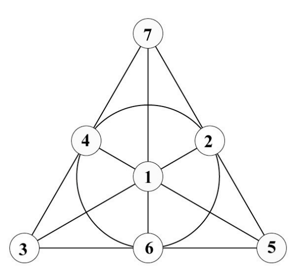


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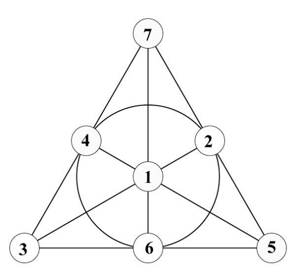
Then we can assign weight one to every player and tell, that

- Coalitions whose weight is $> 4$ are winning.

# Two thresholds for the Fano plane game

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- Coalitions whose weight is $< 3$ are losing.

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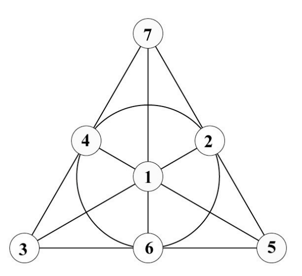


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Then we can assign weight one to every player and tell, that

- Coalitions whose weight is $> 4$ are winning.
- Coalitions whose weight is $< 3$ are losing.
- Coalition whose weight is 3 is winning if it is a line.
- Coalition whose weight is 4 is winning if it contains a line.

# $\mathcal{B}$-Hierarchy

## Definition

A simple game $G = (P, W)$ belongs to $\mathcal{B}_k$ if there exist real numbers $0 < q_1 \leq q_2 \leq \ldots \leq q_k$, called thresholds, and a weight function $w \colon P \to \mathbb{R}^{\geq 0}$ such that

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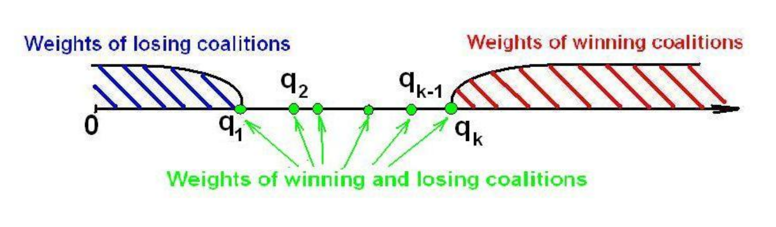

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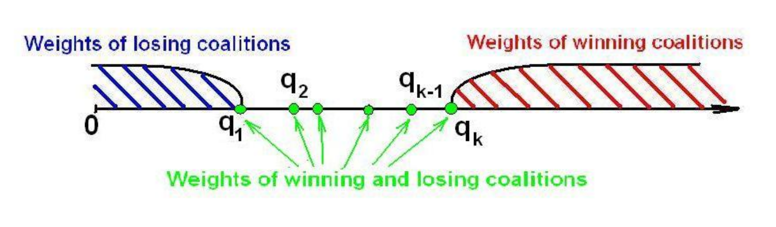

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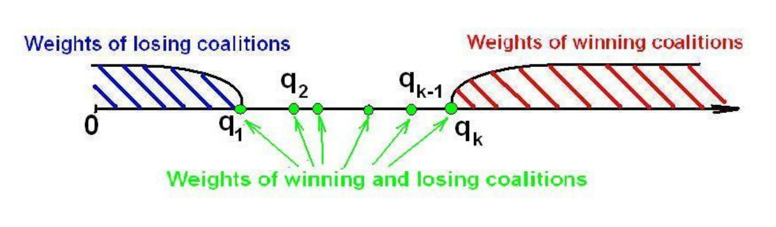

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**Theorem**

*For every natural number $k \in \mathbb{N}^+$, there exists a game in $\mathcal{B}_{k+1} \setminus \mathcal{B}_k$.*

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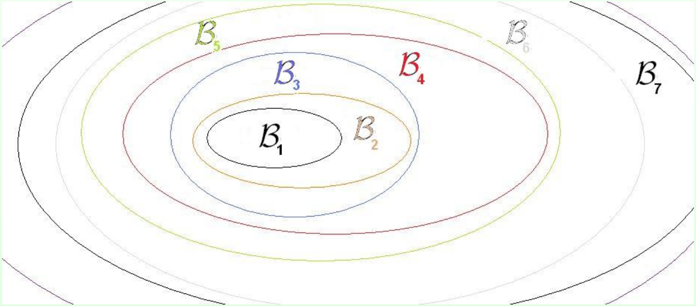


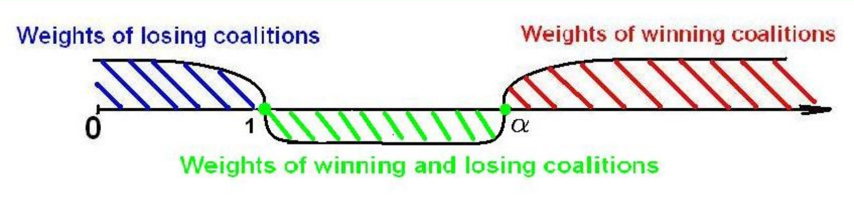
# A faculty vote

If neither side controls more than a $2/3$ majority (calculated in faculty members or their grant dollars), then the Dean would decide the outcome as he wished.

# $\mathcal{C}$-Hierarchy

## Definition

We say that a simple game $G = (P, W)$ is in the class $\mathcal{C}_\alpha$, $\alpha \in \mathbb{R}^{\geq 1}$, if there exists a weight function $w\colon P \to \mathbb{R}^{\geq 0}$ such that for $X \in 2^P$ the condition $w(X) > \alpha$ implies that $X$ is winning, and $w(X) < 1$ implies $X$ is losing.



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This game is in the class $\mathcal{C}_{\frac{2\backslash 3}{1\backslash 3}} = \mathcal{C}_2$

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**Proposition**

*If* $\alpha \leq \beta$*, then* $\mathcal{C}_\alpha \subseteq \mathcal{C}_\beta$*.*

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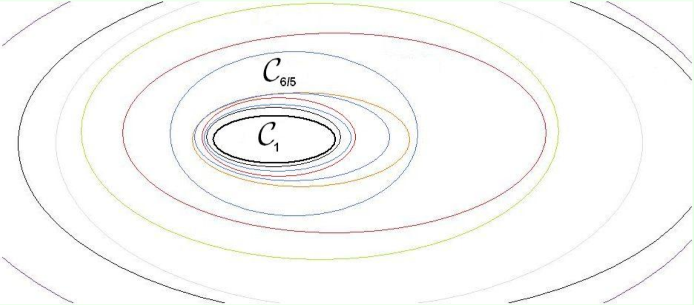
**Proposition**

*Let c and d be natural numbers with $1 < d < c$. Then there is a simple game G that is in $\mathcal{C}_{c/d}$, but that for each $\alpha < c/d$ is not in $\mathcal{C}_\alpha$.*

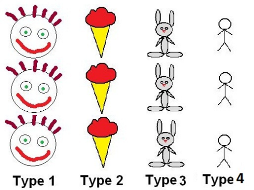
# $\mathcal{C}$-Hierarchy

**Theorem**

*For each* $1 \leq \alpha < \beta$, *it holds that* $\mathcal{C}_\alpha \subsetneq \mathcal{C}_\beta$.



# Idea of the proof




Define a game $G = (P, W)$, where $P = \{1, 2, \ldots, 12\}$. We have 4 types of players with 3 players in each type.

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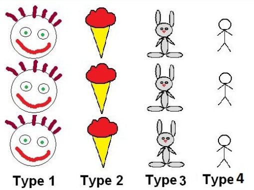


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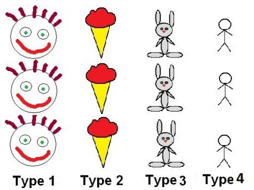
A winning coalition is any set with more than 5 players and also any set having 3 players of the same type.

# Idea of the proof

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Assign weight $\frac{1}{3}$ to every player, then $G$ belongs to $\mathcal{C}_{\frac{4}{3}}$.

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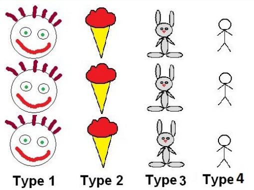


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Assume, that $G \in \mathcal{C}_{\alpha}$ for some $\alpha < \frac{4}{3}$, i.e., there is a weight function realizing this interval.

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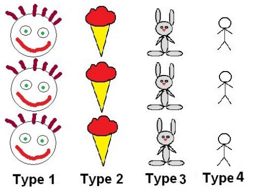


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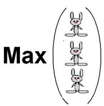


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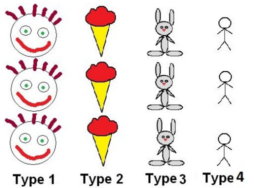


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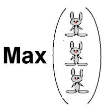
Let 

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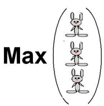
The weight of 

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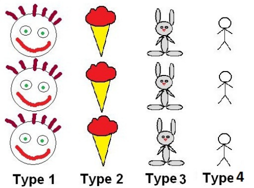


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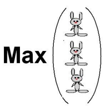
Let 

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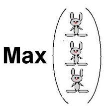
The weight of 

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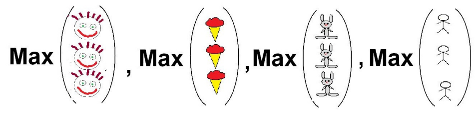

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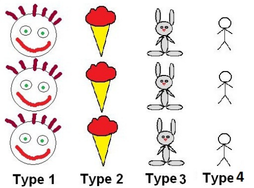


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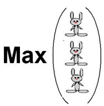
Let 

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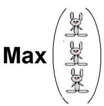
The weight of 

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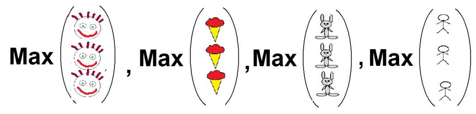
$Y$ is losing and it has weight $w(Y) \geq \frac{4}{3}$.



# Degrees of Roughness of Games with Small Number of Players

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The n-th spectrum is the set of all values of $\alpha$, such that there is a simple game with *n* players in $\mathcal{C}_\alpha$ which does not belong to $\mathcal{C}_\beta$ for all $\beta < \alpha$.

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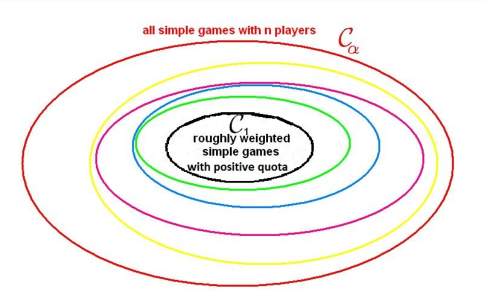

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The n-th spectrum is the set of all values of $\alpha$, such that there is a simple game with *n* players in $\mathcal{C}_\alpha$ which does not belong to $\mathcal{C}_\beta$ for all $\beta < \alpha$.

## Proposition

$Spec(5) = \{1, \frac{6}{5}, \frac{7}{6}, \frac{8}{7}, \frac{9}{8}\}$.

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## Proposition

*The 6th spectrum Spec*(6) *is the set*

$$\left\{1, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \frac{7}{6}, \frac{8}{7}, \frac{9}{7}, \frac{9}{8}, \frac{10}{9}, \frac{11}{9}, \frac{11}{10}, \frac{12}{11}, \frac{13}{10}, \frac{13}{11}, \frac{13}{12}, \frac{14}{11}, \frac{14}{13}, \frac{15}{13}, \frac{15}{14}, \frac{16}{13}, \frac{16}{15}, \frac{17}{13}, \frac{17}{14}, \frac{17}{15}, \frac{17}{16}, \frac{18}{17}\right\}$$

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Thanks!