Complexity consideration on the existence of strategy-proof social choice functions

Koji TAKAMIYA

Dept Economics, Niigata Univ, Japan

COMSOC-2010

Sep 16, 2010

Summary

Objective To examine, in **private goods economies**, the difficulty of deciding the existence of a social choice function which is **strategy-proof, Pareto efficient and individually rational** .

Result In a certain model of private goods exchange, it is *NP*-**hard** to decide the existence of such functions.

Plan of this talk

1 Motivation

2 Setting

3 Results

1 Motivation

Basic observation.

As general tendencies...

- There is incompatibility between *strategy-proofness* and other desirable properties.
- Not only there is incompatibility, it is often *difficult* to prove there is indeed incompatibility.

For example,

The well-known Gibbard-Satterthwaite theorem —
Strategy-proofness vs *non-dictatorship plus non-imposition*

In private goods economies,

the conflict between

Strategy-proofness vs *Pareto efficiency plus individual rationality*

is often observed.

And this is also often difficult to prove the incompatilibity.

Some histories.

Example 1. Model of the classical pure exchange economy

In 1972, Hurwicz proved for the classical pure exchange economy with *2-agents and 2-goods*, there is no s.c.f which is strategy-proof, Pareto efficient and individually rational.

Hurwicz conjectured this non-existence proposition holds for the general n*-agents* m*-goods* case.

In 2002, Serizawa solved this problem affirmative.

Example 2. Matching models

By the early 1980's, it has been proved that

on the full strict preference domain,

(i) in the *marriage problem* (Gale and Shapley, 1962), there is *no s.c.f.* which is strategy-proof and core stable (which implies Pareto efficiency and individual rationality) (Roth, 1982a); and

(ii) in the *housing market* (Shapley and Scarf, 1974), the core stable rule is is a *s.c.f.* which is strategy-proof, Pareto efficient and individually rational. (in this model, core stable allocation is unique for each preference profile.)

There is a sharp contrast!

Example 2. (cont'd)

(iii) Sönmez (1999) and Takamiya (2003) have revealed the "source" of this sharp contrast.

(a) Sönmez introduced *the generalized model of indivisible goods exchange*, which contains both the *marriage problem* and the *housing market* as special cases, and he proved that on the full strict preference domain, *if there is a s.c.f. which is strategy-proof, Pareto efficient and individually rational, then the set of core stable allocations is a singleton or empty for each preference profile, and this s.c.f chooses the core stable allocation whenever it exists.*

Example 2. (cont'd)

(b) Takamiya gave a conditional converse of Sönmez's result: on the full strict preference domain, *if the set of core stable allocations is a singleton for each preference profile, then the core stable rule is strategy-proof, Pareto efficient and individually rational.*

(iv) But it is still not fully understood under what conditions core stable allocations are unique.

Experience tells us that in models of private goods economies, it is more or less difficult to decide if there is a s.c.f. which is strategy-proof, Pareto-efficient and individually rational.

Objective of this study. To examine the idea that *in models of private goods economies, it is "difficult" to decide if there exists a s.c.f. which is strategy-proof, Pareto efficient and individually rational.*

That is, to give a formal model to our conventional wisdom.

Method.

- Set up a simple economic model for our own analytical purpose.
- Capture the concept of "difficulty" by *computational complexity* notions.

Nature of this study.

Different from most lines of research in Computational Social Choice, which uses computational complexity notions to study the difficulty in execution of social choice procedures or behavior of agents in social choice systems.

In contrast, this study employs computational complexity notions to study the difficulty (traditional) social choice theorists face with in their research.

“Computational study on traditional social choice research”

Significance. Theorists’ experience is worth investigating. Theorists’ knowledge determines what is possible in actual social choice systems.

2 Setting

Economic model

- n agents have initial allocations of indivisible objects.
- Agents re-allocate the objects.
  Every object needs to be allocated to someone. (*No free disposal*)

Economic model (cont'd)

- *Preferences.*

  Each agent evaluates the objects allocated to him/her by the sum of his/her personal "value" of each object. (I.e. He/She has an additive utility function.)

  Every value is an integer (positive or non-positive).

$$\text{agent } i'\text{s utility} = \sum_{x \in \text{ Objects allocated to } i} i'\text{s value of } x.$$

Economic model (cont'd)

- *Feasible allocations.*

  Each agent faces a consumption constraint:

  Each agent has his/her personalized " weight" of each object, and his/her "capacity".

  The sum of the weights of the objects allocated to him/her must not be exceed his/her capacity.

$$\sum_{x \in \text{ Objects allocated to } i} i'\text{s weight of } x \;\leq\; i'\text{s capacity}.$$

  Every weight and every capacity is an non-negative integer.

Upside/downside of the model

- Upside.
  - A special case of the well-known general model of indivisible goods exchange by Sönmez (1999) mentioned in Example 2.
  - Related to the recent "*matching market design*" literature: The model contains *house allocation problem*, which is related to *kidney exchange problem*.
- Downside. Too large.
  The model contains many instances with little economic meaning.

“Nice” allocations

- *Individual rationality.* Allocation x is **individually rational** if for each agent i,

$$i'\text{s utility from } x \geq i'\text{s utility from his/her initial allocation.}$$

- Pareto efficiency. Allocation x is **Pareto efficient** if there is no other allocation y such that

$$\text{(i)}\ \ \forall\text{agent } i, \quad i'\text{s utility from } x \geq i'\text{s utility from } y, \text{ and}$$

$$\text{(ii)}\ \ \exists\text{agent } j, \ \ j'\text{s utility from } x > j'\text{s utility from } y.$$

Mechanism design problem

Let us consider the mechanism design problem in this model...

- Each agent i's utility function is determined by his/her type θ^i.
- The set of agent i's all possible types is Θ^i.
  Denote
  $$\Theta := \Theta^1 \times \Theta^2 \times \ldots \times \Theta^n.$$
- Social choice function is a function
  $$f : \Theta \to \text{the set of feasible allocations.}$$

Three properties of social choice functions

- *Strategy-proofness.* Let i be an agent and $\theta \in \Theta$. Then we say that i **manipulates** f at θ if for some $\tilde{\theta}^i \in \Theta^i$,

$$u^i(f(\theta^{-i}, \tilde{\theta}^i), \theta^i) > u^i(f(\theta^{-i}, \theta^i), \theta^i),$$

  f is called **strategy-proof** if for any agent i, i cannot manipulate f at any $\theta \in \Theta$.

Three properties of social choice functions (cont'd)

- *Individual rationality.* Let us call f **individually rational** if for any $\theta \in \Theta$, x is individually rational at θ.
- *Pareto efficiency.* Let us call f **Pareto efficient** if for any $\theta \in \Theta$, x is Pareto efficient at θ.

Computational problem

Given all the above, we consider the following computational problem...

Let a positive integer $\bar{n}$ be given.

NAME: $\mathrm{SP} + \mathrm{IR} + \mathrm{PE}(\bar{n})$

Computational problem

Let a positive integer $\bar{n}$ be given.

NAME: $\mathrm{SP}+\mathrm{IR}+\mathrm{PE}(\bar{n})$

INSTANCE: *A specification of the mechanism design problem:*

(i) The set of agents with the number of agents equal to $\bar{n}$*,*

(ii) The set of objects, (iii) The list of initial allocation,

(iv) Each agent's capacity and list of weights of objects,

(v) Each agent's type space,

(vi) Each agent's list of values of objects for each of his/her type.

Computational problem

Let a positive integer $\bar{n}$ be given.

NAME: $\mathrm{SP} + \mathrm{IR} + \mathrm{PE}(\bar{n})$

INSTANCE: A specification of the mechanism design problem.

QUESTION: *Does there exist a social choice function for this specification which is strategy-proof, individually rational and Pareto efficient?*

3 Results

Theorem 1 (Main theorem)

$\mathrm{SP} + \mathrm{IR} + \mathrm{PE}(\bar{n})$ *is* $\mathcal{NP}$*-hard if* $\bar{n} \geq 4$*.*

To underscore the significance of Theorem 1...

Theorem 2

Let any instance be given.

And let us pick up any two of the three properties,

strategy-proofness, individual rationality and Pareto efficiency.

Then there exists a social choice function which satisfies these two properties.

To underscore the significance of Theorem 1...

We are interested in deciding the *existence* of social choice functions with these properties, not in deciding if a *given* social choice function satisfies these properties.

Theorem 3

It is an $\mathcal{NP}$-hard problem to decide if a social choice function is Pareto efficient. (But it is computationally trivial to decide if there exists a function which is Pareto efficient.)

Proof of Theorem 2.

- Strategy-proof and individually rational:
  Constant function.

- Individually rational and Pareto efficient:
  Always exists.

- Strategy-proof and Pareto efficient:
  Serial dictatorships.

Proof of Theorem 1 (sketch)

The proof is done in *two steps*.

It suffice to prove the case where $\bar{n} = 4$ because one can increase $\bar{n}$ by adding dummy agents.

Step 1 (i) Construct an instance E with three agents which mimics the "*voting problem*" with two agents and three candidates.

(ii) Then by the *Gibbard-Satterthwaite Theorem*, any strategy-proof and Pareto efficient function is dictatorial. But no dictatorial function in this instance can be individually rational. Thus the three properties cannot be satisfied at the same time for this instance E.

Step 2 We make use of reduction from the PARTITION problem.

NAME: $\mathrm{PARTITION}$

INSTANCE: A finite set $A = \{a_1, a_2, \ldots, a_p\}$ and a function $s : A \to \mathbb{N}$.

QUESTION: Does there exist a partition $\{A_1, A_2\}$ of A such that $\sum_{a \in A_1} s(a) = \sum_{a \in A_2} s(a)$.

Given an instance of PARTITION (A, s), we construct an instance with four agents such that:

(i) if it is possible to split A into A_1 and A_2 with $\sum_{A_1} s(a) = \sum_{A_2} s(a)$, then there is only one allocation which is the unique Pareto efficient and individually rational allocation for each preference profile. Thus the s.c.f. which chooses this allocation is strategy-proof.

(ii) if it is not possible to do that, then the situation is identical with the instance E given in Step 1. There is no s.c.f. which is strategy-proof, Pareto efficient and individually rational.

Conclusions

- This study (hopefully) has proposed a new line of research, a computational research on social choice research itself.
- Our model is more or less artificial, and leaves room for improvement.