Optimal Partitions in Additively Separable Hedonic Games

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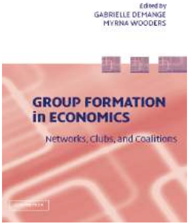
PREFERENCE AGGREGATION IN MULTIAGENT SYSTEMS

Coalition formation

“Coalition formation is of fundamental importance in a wide variety of social, economic, and political problems, ranging from communication and trade to legislative voting. As such, there is much about the formation of coalitions that deserves study.”

(A. Bogomolnaia and M. O. Jackson. The stability of hedonic coalition structures. Games and Economic Behavior. 2002.)

Coalition formation



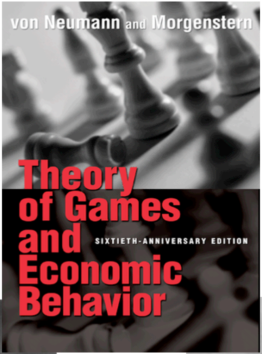
Hedonic Games

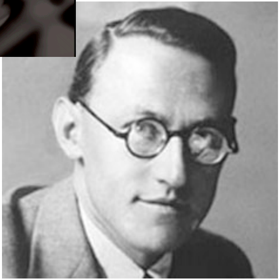
A **hedonic game** is a pair $(N, \mathcal{P})$ where N is a set of players and $\mathcal{P}$ is a **preference profile** which specifies for each player $i \in N$ the preference relation $\succsim_i$, a reflexive, complete and transitive binary relation on set $\mathcal{N}_i = \{S \subseteq N \mid i \in S\}$.

A **partition** π is a partition of players N into disjoint coalitions

A player's appreciation of a coalition structure (partition) only depends on the coalition he is a member of and not on how the remaining players are grouped.

Coalition formation games literature



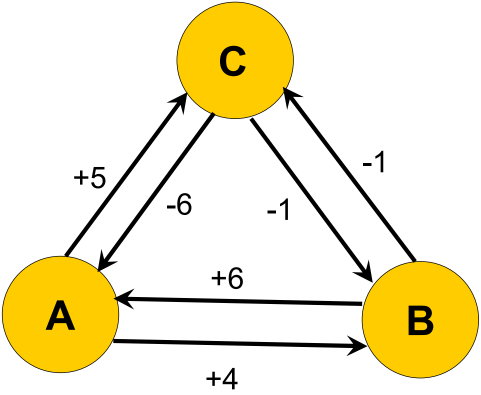


Hedonic coalition formation games literature

- J. H. Dreze and J. Greenberg. Hedonic coalitions: Optimality and stability. Econometrica, 1980.
- S. Banerjee, H. Konishi, and T. Sonmez. Core in a simple coalition formation game. Social Choice and Welfare, 2001.
- A. Bogomolnaia and M. O. Jackson. The stability of hedonic coalition structures. Games and Economic Behavior. 2002.
- K. Cechlárová. Stable partition problem. Encyclopedia of Algorithms. 2008.
- E. Elkind and M. Wooldridge. Hedonic coalition nets. AAMAS, 2009.
- S. C. Sung and D. Dimitrov. Computational complexity in additive hedonic games. European Journal of Operational Research, 2010.

Additively Separable Hedonic Games (ASHGs)

In **additively separable hedonic games (ASHGs)**, a player i gets value $v_i(j)$ for player j being in the same coalition as i and if i is in coalition $S \in \mathcal{N}_i$, then i gets utility $\sum_{j \in S \setminus \{i\}} v_i(j)$.



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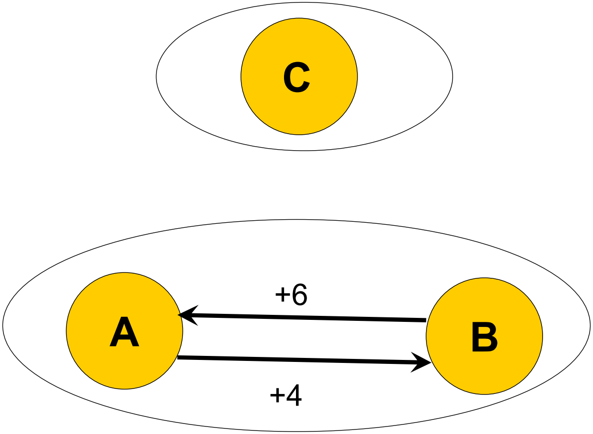

Preference restrictions

A preference profile is

- **symmetric** if $v_i(j) = v_j(i)$ for any two players $i, j \in N$

  "Players like/dislike each other with the same intensity."

- **strict** if $v_i(j) \neq 0$ for all $i, j \in N$ such that $i \neq j$.

  " No one is indifferent about another player."

Optimal and fair partitions

The different notions of fair or optimal partitions are defined:

- The **utilitarian social welfare** of a partition is defined as the sum of individual utilities of the players
- The **egalitarian social welfare** is given by the utility of the player that is worst off
- A partition π of N is **Pareto optimal** if there exists no partition π' of N in which each player is as happy and one player is strictly happier
- **Envy-freeness** is a notion of fairness. In an **envy-free** partition, no player has an incentive to replace another player.
- A partition is **individually rational** if each player can do as well as by being alone.

Computational Problems

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OPTIMALITY: Given $(N, \mathcal{P})$ and a partition π of N, is π optimal?
EXISTENCE: Does an optimal partition for a given $(N, \mathcal{P})$ exist?
SEARCH: If an optimal partition for a given $(N, \mathcal{P})$ exists, find one.

Maximizing Utilitarian welfare

Theorem

Computing a maximum utilitarian partition is NP-hard in the strong sense even with symmetric and strict preferences.

Proof idea

- *Reduction from* MaxCut*.*
- *Can also be shown to be equivalent to a problem in correlation clustering [Bansal, Blum, and Chawla. Correlation clustering. Machine Learning. 2002.]*

Maximizing Egalitarian Welfare

Theorem

Computing a maximum egalitarian partition is NP-hard in the strong sense.

Proof idea

Reduction from MaxMinMachineCompletionTime*.*

Maximizing Egalitarian Welfare

Theorem

Computing a maximum egalitarian partition is NP-hard in the strong sense.

Proof idea

Reduction from MAXMINMACHINECOMPLETIONTIME*.*

Conjecture: Remains hard if preferences are strict and symmetric.

Computing PO partitions

Theorem

For strict preferences, a Pareto optimal partition can be computed in polynomial time.

- Greedy approach - Serial Dictatorship
- May output a partition which is not **individually rational** (where each player gets at least zero utility)

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What if we consider not necessarily strict preferences and want individual rationality (IR)?

Computing PO partitions

Theorem

For not necessarily strict preferences, computing a PO+IR partition is weakly NP-hard.

Proof idea

Reduction from SubsetSum

Computing PO partitions

Theorem

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Proof idea

Reduction from SUBSETSUM

Questions:

- Complexity of computing PO+IR partitions for strict preferences
- Complexity of computing PO partitions for not necessarily strict preferences

Verifying PO partitions

Theorem

The problem of checking whether a partition is Pareto optimal is coNP-complete in the strong sense, even if preferences are symmetric and strict.

Proof idea

Reduction from E3C (EXACT-3-COVER)*.*
E3C (EXACT-3-COVER)*:*
INSTANCE: A pair (R, S)*, where* $R = \{1, \ldots, r\}$ *is a set and* S *is a collection of subsets of* R *such that* $|R| = 3m$ *for some positive integer* m *and* $|s| = 3$ *for each* $s \in S$*.*
QUESTION: Is there a sub-collection $S' \subseteq S$ *which is a partition of* R*?*

Verifying PO partitions

Reduction from E3C to Verifying PO for ASHG.

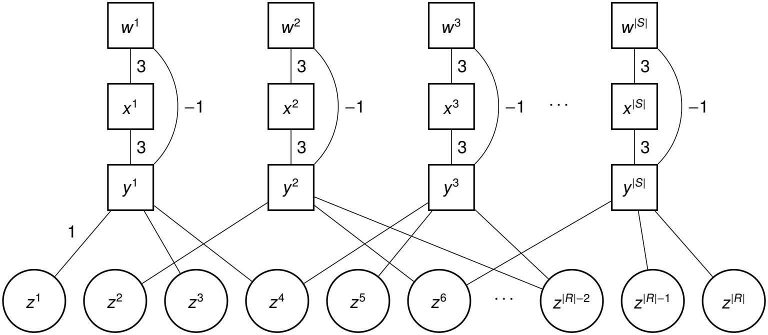

Figure: A graph representation of an ASHG derived from an instance of E3C. The (symmetric) utilities are given as edge weights. Some edges and labels are omitted: All edges between any y^s and z^r have weight 1 if $r \in s$. All $z^{r'}, z^{r''}$ with $r' \neq r''$ are connected with weight $\frac{1}{|R|-1}$. All other edges missing for a complete undirected graph have weight -4.

Envy-freeness

- Pareto optimal and welfare maximizing partitions exist (by definition).
- Take any partition π'. If it is PO, we are done. If not, take another π' which Pareto dominates π. There can only be a finite number of Pareto improvements.
- What about envy-free partitions?

Reminder: In an ***envy-free*** *partition, no player has an incentive to replace another player.*

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- Pareto optimal and welfare maximizing partitions exist (by definition).
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  Reminder: In an ***envy-free*** *partition, no player has an incentive to replace another player.*

- Envy-freeness + individual rationality can be trivially achieved.
- What if we want to satisfy other properties along with envy-freeness?

Envy-free + Nash-stable

Nash stable partition: no player has an incentive to leave his coalition.

Theorem

For symmetric preferences, checking whether there exists a partition which is both envy-free and Nash stable is NP-complete in the strong sense.

Proof idea

Reduction from E3C (EXACT-3-COVER)*.*

Equivalent theorems for other notions of individual-based stability.

Envy-free+PO

Theorem

Checking whether there exists a partition which is both Pareto optimal and envy-free is Σ_2^p-complete.

Proof idea

Reduction from a problem in
(de Keijzer, Bouveret, Klos, and Zhang. On the complexity of efficiency and envyfreeness in fair division of indivisible goods with additive preferences. ADT 2009.)

Conclusions

Results:

- Maximizing utilitarian welfare is strongly NP-hard even for strict and symmetric preferences
- Maximizing egalitarian welfare is strongly NP-hard
- Computing PO partitions is easy for strict preferences
- Computing PO+IR is weakly NP-hard
- Verifying PO is strongly NP-hard even for symmetric and strict preferences
- Checking existence of an Envy-free+PO partition is Σ_2^p-complete
- Checking existence of an Envy-free+Nash-stable partition is strongly NP-hard

Conclusions

Take-home message:

- Computing optimal partitions is computationally hard in general
- Satisfying envy-freeness along with other properties is not feasible in general
- Verifying can be harder than searching! (example of Pareto optimality for strict preferences)
- Using strict preferences makes some problems much easier