# Pareto Stable Assignment

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# Stable marriage

- Marriage model (Gale-Shapley, 1954)
  - $n$ men, $n$ women
  - Strict, complete, preferences
  - *Stable matching*: No blocking pair $(m, w)$ where both prefer the other to current partner

# Relaxing the Marriage Model

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- Ties and incomplete lists:
  - No strict ordering: $m$ can be indifferent amongst $w$
  - Unacceptable partners: Need not rank all $w$

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# Stable Matching with Indifferences

- Man/woman-optimal matching not well-defined; max-cardinality matching is NP-hard
- With indifferences, stable matchings need not be Pareto optimal:
  - 2 men, 2 women
  - $i_1$ strictly prefers $j_1$ to $j_2$
  - Remaining nodes indifferent amongst all partners
  - $(i_1, j_2), (i_2, j_1)$ is a stable matching
  - But not Pareto optimal:
    - Reassign $i_1 \to j_1$, $i_2 \to j_2$

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  - Strict refinement of stability
  - 'Locally' optimal amongst stable matchings
- Other solution concepts
  - Strong & super-stable; maximum size & weight stable matchings
  - Existence or computability issues

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- Can we find one?
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  - This talk: Many-to-many matching

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  - $x_{ij} \geq 0$ is number of units assigned between $i \in A$ and $j \in B$
  - $\sum_j x_{ij} \leq c_i$; $\sum_i x_{ij} \leq c_j$

# Preference Model

- $i \in A$ has preference list $P_i$ ranking neighbors $\{j \in B : (i, j) \in E\}$
- $P_i$ can have ties, be incomplete
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- $P_i$ can have ties, be incomplete
  - $P_i = ([j_1, j_2], [j_3, j_5])$
- Preferences over *sets* of partners: Partial order induced by $P_i$
  - Erdil-Ergin (AER'07)

- $X = (x_{ij})$ is *pairwise stable* if there is no pair $(i, j)$ such that
  - Both $i, j$ have leftover capacity, or
  - $i$ has leftover capacity and there is $i'$, $x_{i'j} > 0$ with $i \succ_j i'$ (or vice-versa), or
  - There are $i'$ and $j'$, $x_{ij'} > 0$ and $x_{i'j} > 0$, such that $i \succ_j i'$ and $j \succ_i j'$

# Pareto-stable Assignments

- Pareto optimality
  - Level function $L_i(j) \in \{1, \ldots, n\}$: Ranking of $j$ in $i$'s preference list
  - For $X = (x_{ij})$, $x_i(\alpha) = \sum_{j:\ L_i(j) \leq \alpha} x_{ij}$
  - $Y$ Pareto-dominates $X = (x_{ij})$ if:
    - $y_i(\alpha) \geq x_i(\alpha)$ for all $i, \alpha$
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- Pareto stability: Pairwise stability + Pareto optimality

- 'Augmenting path': $m_0 \rightarrow w_1 \xrightarrow{X} m_1 \rightarrow w_2 \xrightarrow{X} m_3 \ldots \rightarrow w_{k+1}$
  - $m_0$ and $w_{k+1}$ have leftover capacity
  - $x_{m_\ell w_\ell} > 0$
  - $m_\ell$ weakly prefers $w_{\ell+1}$ to $w_\ell$ & $w_\ell$ weakly prefers $m_{\ell-1}$ to $w_\ell$

# Preliminaries

- 'Augmenting path': $m_0 \longrightarrow w_1 \stackrel{X}{\longrightarrow} m_1 \longrightarrow w_2 \stackrel{X}{\longrightarrow} m_3 \ldots \longrightarrow w_{k+1}$
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- 'Augmenting cycle': $m_0 \rightarrow w_1 \xrightarrow{X} m_1 \rightarrow w_2 \xrightarrow{X} m_3 \ldots \rightarrow m_0$
  - At least one strict improvement
- $X$ is Pareto optimal iff no augmenting paths and cycles

- Matching: Unit capacity $c_i = 1$
- Suppose $X$ is stable
- Reassigning via augmenting path or cycle of $X$ preserves stability

# How to find 'augmenting paths' and 'cycles' in $X$?

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- Construct directed bipartite graph on $M$
  - Forward edges are *"weak improvement"* edges wrt $X$
  - Backward edges: Pairings in $X$
- Finding Pareto improvements
  - Augmenting paths: Introduce source, sink linking to unmatched nodes in $(A, B)$
  - Cycles: One subgraph for each node with only strict improvement edges

# Moving on: Assignment

- Multiunit demand: $c_i \geq 1$
- *Concept of improvement edges for a node is not well defined*
  - Nodes have multiple partners
  - Edge can be improvement for some partners but not others
    - Suppose $i$ ($c_i = 2$) is matched to $j_1, j_3$
    - $j_1 \succ_i j_2 \succ_i j_3$
    - $(i, j_2)$ is improvement relative to $(i, j_3)$, but *not* relative to $(i, j_1)$

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- Make $c_i$ copies of node $i$
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- But...
- New graph size $\sum_i c_i + \sum_j c_j$
  - Exponential in input!

# Algorithm Idea I: Defining Improvement Edges

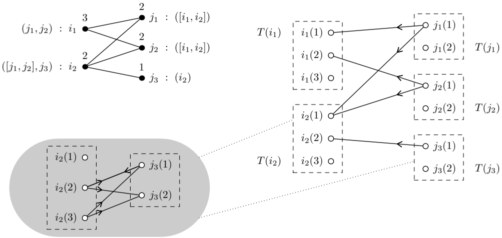
- Construct augmented graph $G$ from original $M$
- Node structure in $G$: Create a copy of a node for *each level in its preference list*
    - Number of levels in preference list cannot exceed number of nodes

# Algorithm Idea I: Defining Improvement Edges

- Construct augmented graph $G$ from original $M$
- Node structure in $G$: Create a copy of a node for *each level in its preference list*
  - Number of levels in preference list cannot exceed number of nodes
- Edge Structure in $G$: Every augmenting path or cycle in any assignment $X$ maps to feasible path or cycle in $G$

# The Augmented Graph $G$

- $G$ depends only on preferences $P_i$, independent of $X$




- Size of $G$ polynomial in $n = |M|$; *independent* of $c_i$

# Algorithm Idea II: Eliminating Pareto Improvements

- Start from arbitrary stable assignment $X$
- Construct sequence of networks $H$ to remove all Pareto improvements:
  - $H$ based on $G$; capacities based on assignment $X$
  - Increases in flow preserve stability; correspond to Pareto improvements

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- No remaining Pareto improvements after all networks have been executed once

# Polynomial-time Algorithm for Pareto-Stability

- *Identifying* Pareto improvements in polynomial time
  - Create one copy for each preference level, rather than each unit of capacity
- *Removing* all Pareto improvements in polynomial time (convergence)
  - Reassignments *do not re-introduce* Pareto-improvements
  - Happens for slightly different construction of sequence of networks!

# Summary

- Many-to-many two-sided matching market
  - Ties and incomplete preference lists
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- Many-to-many two-sided matching market
  - Ties and incomplete preference lists
  - Nodes on *both* sides have multi-unit capacities
  - Multiple edges between pairs
- Strongly polynomial time algorithm to find Pareto-optimal, pairwise-stable assignment

# Future directions

- What's next?
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  - Work in progress: Algorithm for edge capacities, other models of set preferences
  - Broadly: What are limits of existence and computation of solution concepts in many-to-many matching?