Parameterized Control Complexity in Bucklin Voting and in Fallback Voting[1]

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Abstract

We study the parameterized control complexity of Bucklin voting and of fallback voting, a voting system that combines Bucklin voting with approval voting. Electoral control is one of many different ways for an external agent to tamper with the outcome of an election. We show that even though the representation of the votes and the winner determination is different, the parameterized complexity of some standard control attacks is the same. In particular, we show that adding and deleting candidates in both voting systems are W[2]-hard for both the constructive and destructive case, parameterized by the amount of action taken by the external agent. Furthermore, we show that adding and deleting voters in both Bucklin voting and fallback voting are W[2]-hard for the constructive case, parameterized again by the amount of action taken by the external agent, and are in FPT for the destructive case.

1 Introduction

The study of algorithmic issues related to voting systems has become an important topic in contemporary computer science, due to the many applications of deciding between alternatives, or ranking information, in a wide variety of contexts.

Rich questions inevitably arise about the tractability of the election processes, and their susceptibility to manipulation. This paper is about this context of research.

We study the complexity of manipulation of elections based on Bucklin voting, and of *fallback voting*, a voting system that combines Bucklin voting with approval voting.

2 Preliminaries

Many different ways of changing the outcome of an election have been studied with respect to the computational complexity of the strategy, such as *manipulation* [BTT89, BO91, CSL07, HH07, FHHR09b], where a group of voters casts their votes strategically, *bribery* [FHH09, FHHR09a], where an external agent bribes a group of voters in order to change their votes, and *control* [BTT92, HHR07, FHHR09a, HHR09, ENR09, FHHR09b, EPR10], where an external agent—which is referred to as "The Chair"—changes the structure of the election (for example, by adding/deleting/partitioning either candidates or voters).

In this paper, we are concerned with *control issues* for the relatively recently introduced system of *fallback voting* (FV, for short) [BS09] and *Bucklin voting* (BV, for short). A voting system is said to be *immune* against a certain type of control if it is impossible to affect the outcome of the election via that type of control. If a voting system is not immune to a type of control, then it is said to be *susceptible*. When control is possible, the task of exerting control may still be NP-hard. In this case the voting system is said to be *resistant* against that type of control. If the chair's task can be solved in polynomial-time for a type of control then the voting system is said to be *vulnerable* to that type of control.

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We investigate the issues in the framework parameterized complexity. Many voting systems present NP-hard algorithmic challenges. Parameterized complexity is a particularly appropriate framework in many contexts of voting systems because it is concerned with exact results that exploit the structure of input distributions. It is not appropriate in political contexts, for example, to algorithmically determine a winner "approximately". The computational complexity of control problems under the parameterized complexity framework has been studied before. Betzler and Uhlmann [BU08] proved that constructive control by deleting candidates in plurality voting is W[2]-hard with respect to the number of deleted candidates, and destructive control by deleting candidates in plurality voting is W[1]-hard with respect to the number of deleted candidates. They also proved that constructive control by adding/deleting candidates in Copeland voting is W[2]-complete with respect to the number of added/deleted candidates. Recently, Liu et al. [LFZL09] proved that both constructive and destructive control by adding candidates in plurality voting is W[2]-hard with respect to the number of added candidates, constructive control by adding/deleting voters in Condorcet voting is W[1]-hard, constructive control by adding voters in approval voting is W[1]-hard, and constructive control by deleting voters in approval voting is W[2]-hard. In all four voter control results they parameterized by the natural parameterization, i.e., the number of added/deleted voters.

We study Bucklin voting and fallback voting, a voting system that combines Bucklin voting with approval voting. Fallback voting is the natural voting system with an easy winner-determination procedure, that currently has the most resistances for control attacks (19 out of 22) [EPR10].

2.1 Elections and Electoral Control

An election (C,V) consists of a finite set of candidates C and a finite collection of voters V who express their preferences over the candidates in C. A voting system is a set of rules determining the winners of an election. Votes can be represented in different ways, depending on the voting system used. We say that a voter $v \in V$ has a preference *weak order* $\succcurlyeq$ on C, if $\succcurlyeq$ is *transitive* (i.e., for any three distinct candidates $x,y,z \in C$, $x \succcurlyeq y$ and $y \succcurlyeq z$ imply $x \succcurlyeq z$) and *complete* (i.e., for any two distinct candidates $x,y \in C$, either $x \succcurlyeq y$ or $y \succcurlyeq x$). $x \succcurlyeq y$ means that voter v likes x at least as much as y. If ties are excluded in the voters' preference rankings, this leads to a *linear order* or *strict ranking*, denoted by $\succ$. A strict ranking is always antisymmetric (i.e., for any two distinct candidates $x,y \in C$ either $x \succ y$ or $y \succ x$ holds, but not both at the same time) and irreflexive (i.e., for each $x \in C$ the following does not hold: $x \succ x$). In this paper we will write $x \; y$, instead of $x \succ y$.

Definition 2.1. *Let (C,V) be an election with $\|C\| = m$ and $\|V\| = n$. Define the* strict majority threshold *(SMT, for short) as the value $M_t = \lfloor n/2 \rfloor + 1$. In* Bucklin voting *every voter $v \in V$ has to provide a strict ranking.*

The votes of a voter v are represented as a list of all candidates, where the leftmost candidate is v's most preferred candidate, the second candidate from left is v's second most preferred candidate and so on. In our constructions, we sometimes also insert a subset $B \subseteq C$ into such votes, where we assume some arbitrary, fixed order of the candidates in B (e.g., " $c_1 \; B \; c_5$ " means that c_1 is the voter's favourite candidate, c_5 is the voter's most despised candidate and all $b \in B$ are in between these two candidates). Let $score^i_{(C,V)}(c)$ denote the number of voters who rank candidate c on level i or higher in election (C,V). Define the Bucklin score *of candidate c as $score_B(c) = min\{i \,|\, score^i_{(C,V)}(c) \geq M_t\}$, i.e., the smallest level i where the level i score of c is at least as high as the SMT. The candidate with the lowest Bucklin score is the unique Bucklin winner of the election. If there are more than one candidates with a lowest Bucklin score, say i, then each candidate with the highest level i score is the Bucklin winner of the election.*

Note that there always exists a Bucklin winner.

Approval voting, introduced by Brams and Fishburn [BF78, BF83] is not a preference based voting system. In *approval voting* each voter has to vote "yes" or "no" for each candidate and the candidates with the most “yes” votes are the winners of the election. Clearly, approval voting completely ignores preference rankings.

Brams and Sanver [BS09] introduced two voting systems that combine preference-based with approval voting in a sense that each voter has to specify his or her approval vector and in addition has to give a strict ranking for the candidates he or she approved of. One of these systems is fallback voting.

Definition 2.2 ([BS09])**.** *Let (C,V) be an election with $\|C\| = m$ and $\|V\| = n$. Define the strict majority threshold M_t analogously as for BV. Every voter $v \in V$ has to divide the set of candidates C into two subsets $S_v \subseteq C$ indicating that v approves of all candidates in S_v and disapproves of all candidates in $C - S_v$. S_v is called v's* approval strategy*. In addition, each voter $v \in V$ provides also a strict ranking of all candidates in S_v.*

Representation of votes: Let $S_v = \{c_1, c_2, \dots, c_k\}$ for a voter v who ranks the candidates in S_v as follows. $c_1 \succ c_2 \succ \cdots \succ c_k$, where c_1 is v's most preferred candidate and c_k is v's least preferred candidate. We denote the vote v by

$$c_1 \;\; c_2 \;\; \cdots \;\; c_k \;\;|\;\; C - S_v,$$

where the approved candidates to the left of the approval line are ranked from left to the right and the disapproved candidates to the right of the approval line are not ranked and written as a set $C - S_v$.

Let $score_{(C,V)}(c) = \|\{v \in V \mid c \in S_v\}\|$ denote the number of voters who approve of candidate c, and let $score^i_{(C,V)}(c)$ be the level i score of c in (C,V)*, which is the number of c's approvals when ranked on position i or higher.*

Winner determination:

1. *On the first level, only the highest ranked approved candidates (if they exist) are considered in each voters' approval strategy. If there is a candidate $c \in C$ with $score^1_{(C,V)}(c) \geq M_t$ (i.e., $c \in C$ has a strict majority of approvals on this level), then c is the* (unique) level 1 FV winner of the election*, and the procedure stops.*

2. *If there is no level 1 winner, we "fall back" to the second level, where the two highest ranked approved candidates (if they exist) are considered in each voters' approval strategy. If there is exactly one candidate $c \in C$ with $score^2_{(C,V)}(c) \geq M_t$, then c is the* (unique) level 2 FV winner of the election*, and the procedure stops. If there are at least two such candidates, then every candidate with the highest level 2 score is a* level 2 FV winner of the election*, and the procedure stops.*

3. *If we haven't found a level 1 or level 2 FV winner, we in this way continue level by level until there is at least one candidate $c \in C$ on a level i with $score^i_{(C,V)}(c) \geq M_t$, If there is only one such candidate, he or she is the* (unique) level i FV winner of the election*, and the procedure stops. If there are at least two such candidates, then every candidate with the highest level i score is a* level i FV winner of the election*, and the procedure stops.*

4. *If for no $i \leq \|C\|$ there is a level i FV winner, every candidate with the highest $score_{(C,V)}(c)$ is a* FV winner of (C,V) by score.

Note that BV is a special case of FV, where each voter approves of each candidate. Although BV and FV seem to be alike, there are significant differences between them. A voting system is said to be *majority-consistent* if the winner of the election is always the majority winner, whenever one exists. (A majority winner is the candidate who gets ranked first by a strict majority of voters.) Clearly, BV is majority-consistent, if a majority winner exists he or she is also the unique level 1 Bucklin winner of the election. In contrast, FV is not majority-consistent. Consider the following election with three voters and two candidates: $v_1 = \; a \;|\; b \;$, $v_2 = \;|\; b \; a \;$, and $v_3 = \;|\; b \; a \;$. The FV winner of this election is candidate a by score but the majority winner would be candidate b.

We now formally define the computational problems that we study in our paper. In our paper we only consider the unique-winner model, where we want to have exactly one winner. We consider two different control types. In *constructive* control scenarios, introduced by Bartholdi, Tovey, and Trick [BTT92], the chair seeks to make his or her favourite candidate win the election. In a *destructive* control scenario, introduced by Hemaspaandra, Hemaspaandra, and Rothe [HHR07], the chair's goal is to prevent a despised candidate from winning the election. We will only state the constructive cases. The questions in the destructive cases can be asked similarly with the difference that we want the distinguished candidate *not to be* a unique winner.

We first define control via adding a limited number of candidates.

Name Control by Adding a Limited Number of Candidates.
Instance An election $(C \cup D, V)$, where C is the set of qualified candidates and D is the set of spoiler candidates, a designated candidate $c \in C$, and a positive integer k.
Parameter k.
Question Is it possible to choose a subset $D' \subseteq D$ with $||D'|| \leq k$ such that c is the unique winner of election $(C \cup D', V)$?

In the following control scenario, the chair seeks to reach his or her goal by deleting (up to a given number of) candidates.

Name Control by Deleting Candidates.
Instance An election (C, V), a designated candidate $c \in C$, and a positive integer k.
Parameter k.
Question Is it possible to delete up to k candidates (other than c) from C such that c is the unique winner of the resulting election?

Turning to voter control, we first specify the problem control by adding voters.

Name Control by Adding Voters.
Instance An election $(C, V \cup W)$, where V is the set of registered voters and W is the set of unregistered voters, a designated candidate $c \in C$, and a positive integer k.
Parameter k.
Question Is it possible to choose a subset $W' \subseteq W$ with $||W'|| \leq k$ such that c is the unique winner of election $(C, V \cup W')$?

Finally, the last problem we consider, control by deleting voters.

Name Control by Deleting Voters.
Instance An election (C, V), a designated candidate $c \in C$, and a positive integer k.
Parameter k.
Question Is it possible to delete up to k voters from V such that c is the unique winner of the resulting election?

The above defined problems are all natural problems, see the discussions in [BEH$^+$09, BTT92, HHR07, FHHR09a, HHR09].

2.2 Parameterized Complexity

The theory of parameterized complexity offers toolkits for two tasks: (1) the fine-grained analysis of the sources of the computational complexity of NP-hard problems, according to secondary measurements (the *parameter*) of problem inputs (apart from the overall input size n), and (2) algorithmic methods for exploiting parameters that contribute favorably to problem complexity. Formally, a parameterized decision problem is a language $\mathscr{L} \subseteq \Sigma^* \times N$. $\mathscr{L}$ is *fixed-parameter tractable* (FPT) if and only if it can be determined, for input (x,k) of size $n = |(x,k)|$, whether $(x,k) \in \mathscr{L}$ in time $O(f(k)n^c)$, for some computable function f.

A parameterized problem $\mathscr{L}$ *reduces* to a parameterized problem $\mathscr{L}'$ if there (x,k) can be transformed to (x',k') in FPT time so that $(x,k) \in \mathscr{L}$ if and only if $(x',k') \in \mathscr{L}'$, where $k' = g(k)$ (that is, k' depends only on k).

The main hierarchy of parameterized complexity classes is

$$FPT \subseteq W[1] \subseteq W[2] \subseteq \cdots \subseteq W[P] \subseteq XP.$$

$W[1]$ is a strong analog of NP, as the k-Step Halting Problem for Nondeterministic Turing Machines is complete for $W[1]$ under the above notion of parameterized problem reducibility. The k-Clique problem is complete for $W[1]$, and the parameterized Dominating Set problem is complete for $W[2]$. These two parameterized problems are frequent sources of reductions that show likely parameterized intractability. See the Downey-Fellows [DF99] monograph for further background.

2.3 Graphs

Many problems proven to be W[2]-hard are derived from problems concerning graphs. We will prove W[2]-hardness via parameterized reduction from the problem Dominating Set, which was proved to be W[2]-complete by Downey and Fellows [DF99]. Before the formal definition of the Dominating Set problem, we first have to present some basic notions from graph theory.

An *undirected graph* G is a pair $G = (V,E)$, where $V = \{v_1, \ldots, v_n\}$ is a finite (nonempty) set of vertices and $E = \{\{v_i, v_j\} \mid 1 \leq i < j \leq n\}$ is a set of edges.[2] Any two vertices connected by an edge are called *adjacent*. The vertices adjacent to a vertex v are called the *neighbours* of v, and the set of all neighbours of v is denoted by $N[v]$ (i.e., $N[v] = \{u \in V \mid \{u,v\} \in E\}$). The *closed neighbourhood* of v is defined as $N_c[v] = N[v] \cup \{v\}$. The parameterized version of Dominating Set is defined as follows.

Name Dominating Set.
Instance A graph $G = (V,E)$, where V is the set of vertices and E is the set of edges.
Parameter A positive integer k.
Question Does G have a dominating set of size k (i.e., a subset $V' \subseteq V$ with $||V'|| \leq k$ such that for all $u \in V - V'$ there is a $v \in V'$ such that $\{u,v\} \in E$)?

3 Results

Table 1 shows our results on the parameterized control complexity of FV and BV. The FPT results in Table 1 are in parenthesis because the two results for FV are trivially inherited from the classical P results given by Erdélyi and Rothe [ER10], and since BV is a special case of FV, BV inherits the FPT upper bound from FV in both destructive voter cases. We won't prove the W[2]-hardness results for FV, since BV is a special case of FV, FV inherits the W[2]-hardness lower bound from BV in all six cases.

In all of our results we will prove W[2]-hardness by parameterized reduction from the W[2]-complete problem Dominating Set defined in Section 2.3. In these six proofs we will always start from a given Dominating Set instance $(G = (B,E), k)$, where $B = \{b_1, b_2, \ldots, b_n\}$ is the set of vertices with $n > 2$,[3] E the set of

[2] In this paper we will use the symbol V strictly for voters. From the next section on, we will use the symbol B instead of V for the set of vertices in a graph G.

[3] Note that the assumption $n > 2$ can be made without loss of generality, since the problem Dominating Set remains W[2]-complete.

Control by	Fallback Voting		Bucklin	
	Constructive	Destructive	Constructive	Destructive
Adding a Limited Number of Candidates	**W[2]-hard**	**W[2]-hard**	**W[2]-hard**	**W[2]-hard**
Deleting Candidates	**W[2]-hard**	**W[2]-hard**	**W[2]-hard**	**W[2]-hard**
Adding Voters	**W[2]-hard**	**(FPT)**	**W[2]-hard**	**(FPT)**
Deleting Voters	**W[2]-hard**	**(FPT)**	**W[2]-hard**	**(FPT)**

Table 1: Overview of results.

edges in graph G, and $k \leq n$ is a positive integer. In the following constructions, the set of candidates will always contain the set B which means that for each vertex $b_i \in B$ we will have a candidate b_i in our election. We will also refer to candidate set $N_c[b_i]$, which is the set of candidates corresponding to the vertices in G that are in $N_c[b_i]$.

3.1 Candidate Control

Theorem 3.1. *Both constructive and destructive control by adding candidates in BV are* W[2]*-hard.*

Proof. We first prove W[2]-hardness of constructive control by adding candidates. Let $(G = (B,E),k)$ be a given instance of Dominating Set as described above. Define the election (C,V), where $C = \{c,w\} \cup B \cup X \cup Y \cup Z$ with $X = \{x_1, x_2, \ldots, x_{n-1}\}$, $Y = \{y_1, y_2, \ldots, y_{n-2}\}$, $Z = \{z_1, z_2, \ldots, z_{n-1}\}$ is the set of candidates, w is the distinguished candidate, and V is the following collection of $2n+1$ voters:

1. For each i, $1 \leq i \leq n$, there is one voter of the form:

$$N_c[b_i] \;\; X \;\; c \;\; ((B - N_c[b_i]) \cup Y \cup Z \cup \{w\}).$$

2. There are n voters of the form:

$$Y \;\; c \;\; w \;\; (B \cup X \cup Z).$$

3. There is one voter of the form:

$$Z \;\; w \;\; (B \cup X \cup Y \cup \{c\}).$$

Note that candidate w is not a unique Bucklin winner of the election $(C-B,V)$, since only candidates c and w reach the SMT until level n (namely, exactly on level n) with $score^n_{(C-B,V)}(w) = n+1 < 2n = score^n_{(C-B,V)}(c)$ thus, c is the unique level n Bucklin winner of the election $(C-B,V)$. Now, let $C-B$ be the set of qualified candidates and let B be the set of spoiler candidates.

We claim that G has a dominating set of size k if and only if w can be made the unique Bucklin winner by adding at most k candidates.

From left to right: Suppose G has a dominating set of size k. Add the corresponding candidates to the election. Now candidate c gets pushed at least one position to the right in each of the n votes in the first voter group. Thus, candidate w is the unique Bucklin winner of the election, since w is the only candidate on level n who passes the SMT.

From right to left: Suppose w can be made the unique Bucklin winner by adding at most k candidates denoted by B'. By adding candidates from candidate set B, only votes in voter group 1 are changed. Note that candidate c has already a score of n on level $n-1$ in voter group 2 thus, c cannot have any more approvals until level n (else, $score^n_{((C-B)\cup B',V)}(c) \geq n+1$ so, c would tie or beat w on level n). This is possible only if candidate c is pushed in all votes in voter group 1 at least one position to the right. This, however, is possible only if G has a dominating set of size k.

For the W[2]-hardness proof in the destructive case, we have to do minor changes to the above construction, and we will change the roles of candidates c and w[4]. Let $(G=(B,E),k)$ be a given instance of Dominating Set as described above. Define the election (C,V), where $C=\{c,w\}\cup B\cup X\cup Y\cup Z$ with $X=\{x_1,x_2,\dots,x_{n-1}\}$, $Y=\{y_1,y_2,\dots,y_{n-2}\}$, $Z=\{z_1,z_2,\dots,z_{n-2}\}$ is the set of candidates, c is the distinguished candidate, and V is the following collection of $2n+1$ voters:

1. For each i, $1\le i\le n$, there is one voter of the form:

$$N_c[b_i]\ \ X\ \ c\ \ ((B-N_c[b_i])\cup Y\cup Z\cup\{w\}).$$

2. There are n voters of the form:

$$Y\ \ c\ \ w\ \ (B\cup X\cup Z).$$

3. There is one voter of the form:

$$Z\ \ w\ \ c\ \ (B\cup X\cup Y).$$

Note that again only candidates c and w pass the SMT until level n in election $(C-B,V)$, both passing it on level n with $score^n_{(C-B,V)}(w)=n+1<2n+1=score^n_{(C-B,V)}(c)$ thus, c is the unique Bucklin winner of the election $(C-B,V)$. Again, let $C-B$ be the set of qualified candidates and let B be the set of spoiler candidates.

We claim that G has a dominating set of size k if and only if c can be prevented from being a unique Bucklin winner by adding at most k candidates.

From left to right: Suppose G has a dominating set B' of size k. Add the corresponding candidates to the election. Now candidate c gets pushed at least one position to the right in each of the n votes in the first voter group. Thus, on level $n-1$ none of the candidates pass the SMT, and $score^n_{((C-B)\cup B',V)}(c)=n+1=score^n_{((C-B)\cup B',V)}(w)$, i.e., both candidates c and w reach the SMT exactly on level n, and since their level n score is equal, c is not a unique Bucklin winner of the election anymore.

From right to left: Suppose c can be prevented from being a unique Bucklin winner by adding at most k candidates denoted by B'. By a similar argument as in the constructive case, this is possible only if G has a dominating set of size k. ❑

Theorem 3.2. *Both constructive and destructive control by deleting candidates in BV are* W[2]*-hard.*

Proof. We will start with the W[2]-hardness proof in the constructive case. Let $(G=(B,E),k)$ be a given instance of Dominating Set. Define the election (C,V), where $C=\{c,w\}\cup B\cup X\cup Y\cup Z$ with $X=\{x_1,x_2,\dots,x_{n^2-\sum_{i=1}^n ||N_c[b_i]||}\}$, $Y=\{y_1,y_2,\dots,y_{n-1}\}$, $Z=\{z_1,z_2,\dots,z_{n-2}\}$ is the set of candidates, w is the distinguished candidate, and V is the following collection of $2n+1$ voters:

1. For each i, $1\le i\le n$, there is one voter of the form:

$$N_c[b_i]\ \ X_i\ \ w\ \ ((B-N_c[b_i])\cup(X-X_i)\cup Y\cup Z\cup\{c\}),$$

   where $X_i=\{x_{1+(i-1)n-\sum_{j=1}^{i-1}||N_c[b_j]||},\dots,x_{in-\sum_{j=1}^{i}||N_c[b_j]||}\}$.

2. There are $n-1$ voters of the form:

$$Y\ \ c\ \ (B\cup X\cup Z\cup\{w\}).$$

[4] Here, changing the roles of c and w means simply that now not candidate w but c is the distinguished candidate.

3. There is one voter of the form:

$$(Y - \{y_1\})\ \ c\ \ w\ \ (B \cup X \cup Z \cup \{y_1\}).$$

4. There is one voter of the form:

$$Z\ \ w\ \ c\ \ (B \cup X \cup Y).$$

Note that candidate c is the unique level n Bucklin winner of the election (C,V), since only c passes the SMT on level n among all candidates.

We claim that G has a dominating set of size k if and only if w can be made the unique Bucklin winner by deleting at most k candidates.

From left to right: Suppose G has a dominating set $B' \subseteq B$ of size k. Delete the corresponding candidates. Now candidate w gets pushed at least one position to the left in each of the n votes in the first voter group. Since candidate c reaches the SMT on level n and $score^n_{(C-B',V)}(w) = n+2 > n+1 = score^n_{(C-B',V)}(c)$, and no other candidate passes the SMT until level n, candidate w is the unique Bucklin winner of the resulting election.

From right to left: Suppose w can be made the unique Bucklin winner of the election by deleting at most k candidates. Since candidate c already passes the SMT on level n, w has to beat c no later than on level n. This is possible only if candidate w is pushed in all votes in voter group 1 at least one position to the left. This, however, is possible only if G has a dominating set of size k.

For the W[2]-hardness proof in the destructive case in Bucklin, let $(G = (B,E),k)$ be a given instance of Dominating Set. Define the election (C,V), where $C = \{c,w\} \cup B \cup M \cup X \cup Y_1 \cup Y_2 \cup Z_1 \cup Z_2$ with $M = \{m_1, m_2 \ldots, m_k\}$, $X = \{x_1, x_2, \ldots, x_{n^2 - \sum_{i=1}^{n} ||N_c[b_i]||}\}$, $Y_1 = \{y_{1,1}, y_{1,2}, \ldots, y_{1,n-1}\}$, $Y_2 = \{y_{2,1}, y_{2,2}, \ldots, y_{2,k}\}$, $Z_1 = \{z_{1,1}, z_{1,2}, \ldots, z_{1,n-2}\}$, $Z_2 = \{z_{2,1}, z_{2,2}, \ldots, z_{2,n-2}\}$ is the set of candidates, c is the distinguished candidate, and V is the following collection of $2n+1$ voters:

1. For each i, $1 \leq i \leq n$, there is one voter of the form:

$$N_c[b_i]\ \ X_i\ \ w\ \ M\ \ ((B - N_c[b_i]) \cup (X - X_i) \cup Y_1 \cup Y_2 \cup Z_1 \cup Z_2 \cup \{c\}),$$

   where $X_i = \{x_{1+(i-1)n - \sum_{j=1}^{i-1} ||N_c[b_j]||}, \ldots, x_{in - \sum_{j=1}^{i} ||N_c[b_j]||}\}$.

2. There are n voters of the form:

$$Y_1\ \ c\ \ Y_2\ \ (B \cup M \cup X \cup Z_1 \cup Z_2 \cup \{w\}).$$

3. There is one voter of the form:

$$Z_1\ \ w\ \ c\ \ Z_2\ \ (B \cup M \cup X_1 \cup X_2 \cup Y_1 \cup Y_2).$$

Note that candidate c is the unique level n Bucklin winner of the election (C,V), since only c passes the SMT on level n among all candidates.

We claim that G has a dominating set of size k if and only if c can be prevented from being a unique Bucklin winner by deleting at most k candidates.

From left to right: Suppose G has a dominating set $B' \subseteq B$ of size k. Delete the corresponding candidates. Now candidate w gets pushed at least one position to the left in each of the n votes in the first voter group. Since candidate c passes the SMT no earlier than on level n and $score^n_{(C-B',V)}(w) = n+1 = score^n_{(C-B',V)}(c)$, candidate c is not a unique Bucklin winner of the resulting election anymore.

From right to left: Suppose c can be prevented from being a unique Bucklin winner of the election by deleting at most k candidates. Note that deleting one candidate from an election can move the strict majority level of another candidate at most one level to the left. Observe that only candidate w can prevent c from winning the election, since w is the only candidate other than c who passes the SMT until level $n+k$. In election (C,V), candidate w passes the SMT no earlier than on level $n+1$, candidate c not before level n. Candidate w could only prevent c from winning by reaching the SMT no later than on level n. This is possible only if candidate w is pushed in all votes in voter group 1 at least one position to the left. This, however, is possible only if G has a dominating set of size k. ❑

Theorem 3.3. *Both constructive and destructive control by adding and deleting candidates in FV are* W[2]-*hard.*

3.2 Voter Control

Theorem 3.4. *Constructive control by adding voters in BV is* W[2]-*hard.*

Proof. Let $(G=(B,E),k)$ be a given instance of Dominating Set. Define the election $(C,V\cup W)$, where $C=B\cup\{w,x\}\cup Y\cup Z$, with $Y=\{y_1,y_2,\dots,y_{\sum_{i=1}^{n}\|N_c[b_i]\|}\}$, $Z=\{z_1,z_2,\dots,z_{n-1}\}$ is the set of candidates, w is the distinguished candidate, and $V\cup W$ is the following collection of $n+k-1$ voters:

1. V is the collection of $k-1$ registered voters of the form:

$$x\ \ Z\ \ B\ \ w\ \ Y.$$

2. W is the collection of unregistered voters, where for each i, $1\le i\le n$, there is one voter w_i of the form:

$$(B-N_c[b_i])\ \ Y_i\ \ w\ \ x\ \ (N_c[b_i]\cup(Y-Y_i)\cup Z),$$

   where $Y_i=\{y_{(\sum_{j=1}^{i-1}\|N_c[b_j]\|)+1},\dots,y_{\sum_{j=1}^{i}\|N_c[b_j]\|}\}$.

Clearly, x is the level 1 Bucklin winner of the election (C,V).

We claim that G has a dominating set of size k if and only if w can be made the unique Bucklin winner by adding at most k voters from W.

From left to right: Suppose G has a dominating set B' of size k. Add the corresponding voters from set W to the election (i.e., each voter w_i if $b_i\in B'$). Now there are $2k-1$ registered voters, thus the SMT is $M_t=k$. Since until level n only candidate w passes the SMT, namely on level n, w is the unique Bucklin winner of the resulting election.

From right to left: Suppose w can be made the unique Bucklin winner by adding at most k voters (denote these voters by W'). Note that $score^1_{(C,V\cup W')}(x)=k-1$. Since if a candidate passes the SMT on level 1, he or she is the unique winner of the election, $k-1$ cannot be the SMT. This is only possible, if $\|W'\|\ge k-1$. If $\|W'\|=k-1$ then $score^{n+1}_{(C,V\cup W')}(w)=k-1<M_t=k<score^{n+1}_{(C,V\cup W')}(x)=2k-1$. In this case candidate w couldn't be made the unique Bucklin winner of the election. Thus, $\|W'\|=k$. Note that $score^{n}_{(C,V\cup W')}(w)=k>k-1=score^{n}_{(C,V\cup W')}(x)$ and k is also a strict majority. Since we could make w the unique Bucklin winner of the election, none of the candidates in B can be ranked on the first n positions by each voter in W', otherwise there would exist a candidate $b\in B$ with $score^{n}_{(C,V\cup W')}(b)=k$ and b would reach the SMT on a higher level than w. This is only possible if G has a dominating set of size k. ❑

Theorem 3.5. *Constructive control by deleting voters in BV is* W[2]*-hard.*

Proof. To prove W[2]-hardness, we provide again a reduction from Dominating Set. Let $(G = (B,E),k)$ be a given instance of Dominating Set. Define the election (C,V), where $C = \{c,w\} \cup B \cup X \cup Y \cup Z$ with $X = \{x_1,\dots,x_{\sum_{i=1}^{n}\|(B-N_c[b_i])\|}\}$, $Y = \{y_1,\dots,y_{\sum_{i=1}^{n}\|N_c[b_i]\|}\}$, $Z = \{z_1,\dots,z_{(k-1)(n+1)}\}$ is the set of candidates, w is the distinguished candidate, and V is the following collection of $2n+k-1$ voters:

1. For each i, $1 \leq i \leq n$, there is one voter v_i of the form:
$$N_c[b_i] \;\; c \;\; X_i \;\; ((B - N_c[b_i]) \cup (X - X_i) \cup Y \cup Z) \;\; w,$$
where $X_i = \{x_{1+\sum_{j=1}^{i-1}\|(B-N_c[b_j])\|},\dots,x_{\sum_{j=1}^{i}\|(B-N_c[b_j])\|}\}$.
2. For each i, $1 \leq i \leq n$, there is one voter of the form:
$$(B - N_c[b_i]) \;\; Y_i \;\; w \;\; (N_c[b_i] \cup X \cup (Y - Y_i) \cup Z \cup \{c\},$$
where $Y_i = \{y_{1+\sum_{j=1}^{i-1}\|N_c[b_j]\|},\dots,y_{\sum_{j=1}^{i}\|N_c[b_j]\|}\}$.
3. There are $k-1$ voters of the form:
$$c \;\; Z_i \;\; (B \cup X \cup Y \cup (Z - Z_i)) \;\; w,$$
where $Z_i = \{z_{(i-1)(n+1)+1},\dots,z_{i(n+1)}\}$.

Note that since candidate w reaches the SMT only on the last level, he or she is not the unique Bucklin winner of the election.

We claim that G has a dominating set of size k if and only if w can be made the unique Bucklin winner by deleting at most k voters.

From left to right: Suppose G has a dominating set B' of size k. Delete the corresponding voters from the first voter group (i.e., each voter v_i if $b_i \in B'$). Let V' denote the new set of voters. Now on level $n+1$ only candidate w passes the SMT, namely with $score^{n+1}_{(C,V')}(w) = n = M_t$. Thus, w is the unique Bucklin winner of the resulting election.

From right to left: Suppose w can be made the unique Bucklin winner by deleting at most k voters. Observe that deleting less than k voters would make it impossible for candidate w to be the unique winner of the election. In that case the SMT $M_t > n$ and since w is ranked last place in all votes except of n votes, he would reach the SMT on the last level thus, would not be the unique Bucklin winner of the election. Clearly, w has to win the election on level $n+1$. Now, since for all i with $1 \leq i \leq n$ $score^{n+1}_{(C,V)}(b_i) = n = score^{n+1}_{(C,V)}(w)$, each b_i had to loose at least one point on the first $n+1$ levels. Obviously, we cannot delete voters from the second voter group, else candidate w wouldn't reach the SMT on level $n+1$. So the k voters were deleted from the first voter group. Since each candidate b_i has lost at least one point, this is only possible if G has a dominating set of size k. ❑

Theorem 3.6. *Both constructive control by adding and deleting voters in FV are* W[2]*-hard.*

4 Conclusions and Open Questions

In this paper we have studied the parameterized complexity of the control problems for the recently proposed system of *fallback voting* and of *Bucklin voting*, parameterized by the amount of action taken by the chair.

In the case of constructive control, all of the problems are W[2]-hard. A natural question to investigate is whether these problems remain intractable when parameterized by both the amount of action and some other measure. We have shown that all four problems of constructive and destructive control by adding or deleting candidates are hard for W[2]. What is the complexity when the parameter is both the amount of action and the number of voters? We have also shown that both constructive control by adding and deleting voters are hard for W[2] in both fallback voting and Bucklin voting, and that both destructive control by adding and deleting voters are in FPT in both fallback voting and Bucklin voting. What is the complexity of constructive control parameterized by both the amount of action and the number of candidates?

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